Does portfolio resampling really improve out-of-sample performance? evidence from the Brazilian market.

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Abstract

Markowitz optimization plays an important role in modern portfolio theory. However, it is well-known that Markowitz optimization is highly affected by the estimation error of the mean vector and covariance matrix, resulting in extreme and/or unrealistic portfolio weights, lacks of diversification and poor out-of-sample performance. To deal with this issue, Michaud and Michaud (1998) proposed a heuristic portfolio resampling approach which can deliver more diversified and better out-of-sample portfolio performance in practice. In this paper, we assess the performance of the Michaud and Michaud (1998) portfolio resampling approach in the Brazilian context and also introduce a new portfolio resampling scheme called factor-based portfolio resampling, which takes advantage of the factor structure of stock returns. The results suggest that portfolio resampling can be an easy to implement alternative to increase portfolio diversification, reduce transaction costs and improve out-of-sample performance in the Brazilian context.

Keywords: Bootstrapping; Covariance Matrix; Diversification; Efficient Frontier; Markowitz; Portfolio Allocation.
JEL classification: C15; C58; C63; G11.

1 Introduction

Modern portfolio theory brought a new way to deal with how investment decisions should be made. In accordance with this, the efficient frontier (Markowitz, 1952, 1959) refers to a set of portfolios where rational risk-averse investors should maximize their expected

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return for a given level of risk or, equivalently, minimize their portfolio risk for a given level of expected return (Meucci, 2007). To this end, the optimal portfolio weights are selected by using the mean vector and covariance matrix in a quadratic optimization problem, a procedure also known as Markowitz optimization.

Markowitz optimization has gained popularity among both academics and practitioners, being a widely used tool for portfolio management as well as an interesting and active topic in empirical and theoretical research. Despite the brilliance and elegance of Markowitz's formulation, Markowitz optimization requires that the mean vector and the covariance matrix of assets returns be known, a requirement that never holds in practice. Therefore, in real applications, those quantities are estimated from the data, thus implying an estimation error.

Estimation error has consequences in portfolio allocation. In fact, it is well known that Markowitz optimization is highly affected by the estimation error of the mean vector and covariance matrix, resulting in extreme and/or unrealistic portfolio weights, a lack of diversification and poor out-of-sample performance (Michaud, 1989; Becker et al., 2015; Huang and Yu, 2020). Thus, to overcome these problems, two main approaches have being proposed in the literature: (a) improving the covariance matrix estimation [1] and (b) taking into account parameter uncertainty.

Improving the covariance matrix estimation has usually been handled by using robust, shrinkage and/or sophisticated time series methods, the benefits of which are improvements well established in the literature (see, for instance, Oliveira and Valls Pereira, 2018; De Nard et al., 2021; Hirukawa, 2021; Trucíos et al., 2021; Ledoit and Wolf, 2022, for some recent references). On the other hand, parameter uncertainty is usually addressed by Bayesian or, so-called, portfolio resampling, the latter of which is the focus of this paper. For readers interested in Bayesian methods to deal with parameter uncertainty in a portfolio allocation context, we refer them to Markowitz and Usmen (2003), Harvey et al. (2010), Anderson and Cheng (2016), and Bauder et al. (2021), among others.

[1] Markowitz optimization requires both the mean vector and covariance matrix. However, it is well known that the mean vector is very difficult to estimate with any accuracy (Merton (1980)), hence covariance matrix estimation is the main focus of research nowadays.

Portfolios resampling (Michaud and Michaud, 1998) is an easy to implement procedure based on the bootstrapping idea (Efron and Tibshirani, 1994) and was proposed to handle estimation error without relying on difficult or complex methods. The Michaud procedure tries to minimize the influence of estimation error on portfolio selection (Becker et al., 2015). Nevertheless, portfolios resampling has been the subject of several controversies and criticisms and its usefulness to improve the out-of-sample portfolio performance still remains unclear in the literature.

Theoretical investigations of Michaud portfoio resampling are provided by Scherer (2002) and Wolf (2004), whose main findings reveal that, in the unrestricted case, portfolio resampling and classical Markowitz optimization show similar performance and no improvement is observed through the use of portfolio resampling over classical Markowitz optimization. However, if constraints are included, such as, for instance, no short-selling constraints, portfolio resampling can deliver more diversified portfolios and better out-of-sample performance.

Unfortunately, empirical applications do not always show significant improvements of portfolio resampling over classical Markowitz optimization, and it is thus difficult to decide whether to use portfolio resampling or not. For instance, Wolf (2004) and Fernandes et al. (2012) favour the resampling technique over classical Markowitz optimization while Kohli (2005), Scherer (2006) and Delcourt and Petitjean (2011) observe no significant superiority of portfolio resampling over classical Markowitz optimization.

That evidence reflects the fact that the usefulness of portfolio resampling is still unclear in the literature, being thus making empirical comparisons of great interest for a better understanding about this approach. In this sense, we use data from the Brazilian stock market, a data set never used before in a portfolio resampling context, to perform an out-of-sample comparison between portfolio resampling and classical Markowitz optimization. Additionally, we propose and alternative portfolio resampling technique that takes advantage of the factor structure of stock returns, and we evaluate whether this new portfolio resampling can improve the results obtained by Michaud portfolio resampling.

The contribution of this paper is twofold. First, we compare the out-of-sample performance of classical Markowitz optimization versus portfolio resampling using recent data from the Brazilian (emerging) stock market. Second, motivated by the idea of the factor structure present in stock returns, we propose a factor-based portfolio resampling that can be used with both observable and unobservable factors.

The rest of the paper is organised as follows. Section 2 briefly introduces the efficient frontier while Section 3 describes Michaud portfolio resampling as well as our resampling proposal. Section 4 introduces measures used to evaluate out-of-sample performance. Section 5 presents the empirical application and Section 6 concludes the paper.

2 Efficient Frontier

The efficient frontier (Markowitz, 1952, 1959) postulates that rational risk-averse investors should maximize their expected return for a given level of risk or, equivalently, minimize their portfolio risk for a given level of expected return (Meucci, 2007). Thus, any portfolio with the same expected return but with higher risk is an inefficient (worse) decision.

Determining the optimal portfolio allocation relies on finding the optimal solution in a quadratic optimization problem. Let us consider a set of N risky assets with a covariance matrix and mean vector denoted by Σ and μ, respectively. Let $\omega = (\omega_1, \cdots, \omega_N)'$ be a column vector with elements ω_i representing the fraction of wealth invested in asset i. Thus, the portfolio at the efficient frontier can be obtained by the solution to the following optimization problem

$$\textit{Max:} \quad \omega'\mu - \frac{\lambda}{2}\omega'\Sigma\omega, \tag{1}$$

subject to $\sum_{i=1}^{N} \omega_i = 1$ and $\omega_i \geq 0 \quad \forall i = 1, \cdots, N$, where $\lambda > 0$ is a pre-specified risk-aversion parameter.

The non-negativeness restriction is widely known as no short-selling constraint and is not necessary for the efficient frontier problem. However, due to usual investor preferences and/or financial institution restrictions, it is commonly included in the quadratic optimization problem. Note that if λ is large, then the investor is highly averse to risk and the optimization problem (1) is equivalent to

$$Min: \quad \omega^{'}\Sigma\omega,$$

subject to the same restrictions given in (1). This portfolio is known as the (global) minimum variance portfolio.

If additionally we consider a risk-free asset (R_f), the portfolio that maximizes the Sharpe ratio is obtained by the solution to

$$Max: \quad \frac{\omega^{'}\mu - R_f}{\sqrt{\omega^{'}\Sigma\omega}}.$$

This portfolio is also at the efficient frontier and is known as the tangency portfolio.

Markowitz optimization started a new era in portfolio theory, with Markowitz himself being considered the father of modern portfolio theory. However, the main drawback of this approach is the requirement of the true mean vector and covariance matrix, both quantities that are never known in practice. Therefore, in empirical applications, those quantities are estimated from the data, implying estimation error which strongly impacts the optimal solution (DeMiguel et al., 2009; Michaud and Michaud, 1998; Delcourt and Petitjean, 2011; De Prado, 2016).

3 Portfolio Resampling

The estimation error coming from replacing the true covariance matrix and mean vector with theirs estimated sample versions has consequences in the portfolio allocation. In fact, it is well known that Markowitz optimization is highly affected by the estimation error of the mean vector and covariance matrix, resulting in extreme and/or unrealistic portfolio weights, a lack of diversification and poor out-of-sample performance (Michaud, 1989; Becker et al., 2015; Huang and Yu, 2020). One way to overcome those problems is through portfolio resampling (Michaud and Michaud, 1998), a procedure proposed to handle estimation error without relying on difficult or complex procedures.

The portfolio resampling proposed by Michaud and Michaud (1998) is briefly described in Section 3.1 and a new portfolio resampling scheme called factor-based portfolio resampling is introduced in Section 3.2.

3.1 Michaud portfolio resampling

Let R_T be a matrix of monthly returns of dimension $T \times p$, where T stands for the sample size and p for the number of assets considered and let r_i be the corresponding row i of matrix R_T. Michaud portfolio resampling can be summarised as follows:

- **Step 1:** For a given R_T, calculate its sample mean vector and covariance matrix. Denote these values as $\hat{\mu}$ and $\hat{\Sigma}$, respectively.

- **Step 2:** Take T draws from $N_p(\hat{\mu}, \hat{\Sigma})$, a p-dimensional multivariate Normal distribution with mean vector $\hat{\mu}$ and covariance matrix $\hat{\Sigma}$. Denote the resulting matrix as R_T^b.

- **Step 3:** Estimate the mean vector and covariance matrix of R_T^b by their sample versions ($\hat{\mu}^b$ and $\hat{\Sigma}^b$) and then calculate the optimal portfolio weights.

- **Step 4:** Repeat Steps 2 and 3 B times. The final optimal portfolio weights are obtained by averaging the optimal portfolio weights obtained in each iteration of Step 3.

The algorithm described in steps 1–4 uses a parametric bootstrapping, hence the commonly used name of Michaud parametric portfolio resampling. Instead of using a parametric bootstrapping approach as described in steps 1 and 2, Michaud and Michaud (1998) also proposes using a non-parametric bootstrapping approach, which means to resample with replacement directly from past vector returns. In practice, both approaches yield very similar results (Michaud and Michaud, 1998; Wolf, 2004).

Modifications of Michaud's algorithm were proposed by Wolf (2004) and Huang and Yu (2020) and include using shrinkage procedures to estimate the covariance matrix and/or using alternative procedures for forecasting returns. However, in this paper we only focus on Michaud's original proposal, which uses the sample mean vector and covariance matrix, since the focus is on keepping the simplicity of the procedure. We will leave the aforementioned modified versions for further research.

3.2 Factor-based portfolio resampling

Since the CAPM (Lintner, 1965; Sharpe, 1964) there have been several attempts to use a factor structure to explain and better estimate stock return and its covariance matrix (see, for instance, Pettenuzzo et al., 2014; Trucíos et al., 2019a; Giovannelli et al., 2021; Cai et al., 2022, for some recent references). In order to use this factor structure, we propose a resampling scheme that is conditional on factors. The procedure is also based on the bootstrapping idea and can be summarised in the following four steps:

- **Step 1:** Let F_t be the observed or unobserved factors. Run a multivariate linear regression of the form

$$r_t = \alpha + \beta F_t + \epsilon_t, \quad t = 1, \cdots, T,$$

  and obtain the estimates $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\epsilon}$, as well as $\hat{\Sigma}_F$ and $\hat{\Sigma}_\epsilon$, the estimates of the covariance matrices of F and ϵ, respectively.

- **Step 2:** Generate a new matrix resampling monthly returns

$$r_t^b = \hat{\alpha} + \hat{\beta} F_t + \epsilon_t^b, \quad t = 1, \cdots, T,$$

  where $\epsilon_i^b \sim N_p(0, \hat{\Sigma}_\epsilon)$. Denote as R_T^b the corresponding $T \times p$ matrix of resampled monthly returns.

- **Step 3:** Using R_T^b, estimate the mean vector and covariance matrix through $\hat{\mu}^b = \hat{\alpha}^b + \hat{\beta}^b \bar{F}$ and $\hat{\Sigma}^b = \hat{\beta}^{b\prime} \hat{\Sigma}_F \hat{\beta}^b + \hat{\Sigma}_{\epsilon^b}^b$, where $\hat{\alpha}^b, \hat{\beta}^b$ and $\hat{\Sigma}_{\epsilon^b}^b$ are the estimated versions of $\hat{\alpha}, \hat{\beta}$ and $\hat{\Sigma}_\epsilon$, respectively. Then, calculate the optimal portfolio weights.

- **Step 4:** Repeat steps 2 and 3 B times. The final optimal portfolio weights are obtained by averaging the optimal portfolios weights obtained in each iteration of Step 3.

As in Section 3.1, we used a parametric bootstrapping approach. Replacing the parametric bootstrapping with non-parametric bootstrapping is straightforward; instead of sampling from a multivariate normal distribution in Step 2 we can directly sample with replacement from $\hat{\epsilon}$. The version described in the algorithm is called Factor-Based Parametric while the modified (non-parametric) version is called Factor-Based Non-Parametric. Both alternatives are implemented in the empirical application.

Note that if there is evidence of returns predictability, instead of using the factor structure defined in Step 1, we can use $r_{t+1} = \alpha + \beta F_t + \epsilon_{t+1}, \quad t = 1, \cdots, T-1$ to obtain a better forecast of expected returns and modify the algorithm accordingly. To make both resampling approaches as comparable as possible, we use the factor structure described in the algorithm and estimate the expected return as described in Step 3.

On the other hand, it is worth mentioning that the factors F can be observed (market factor, Fama-French, Fama-French-Carhart , etc.) or unobserved, in which case they are obtained by applying, for instance, principal components analysis in R_T as in our empirical application. However, there are several other dimension reduction techniques that could be also used (see, for instance; Hu and Tsay, 2014; Peña and Yohai, 2016; Forni et al., 2017; Trucíos et al., 2019a).

Before ending this section, it is important to point out that both resampling procedures implicitly assume that returns are homoscedastic and not serially correlated. These characteristics do not occur in daily or higher frequency returns but there is evidence that monthly returns have little or no GARCH effects (Galea et al., 2010; Kan and Zhou, 2017), which is the reason why empirical applications of portfolio resampling usually use monthly data (Fletcher and Hillier, 2001; Scherer, 2002; Markowitz and Usmen, 2003; Wolf, 2004; Delcourt and Petitjean, 2011; Becker et al., 2015; Huang and Yu, 2020). If the focus is on daily or higher frequency returns, alternative resampling procedures similar to the ones proposed by Fresoli and Ruiz (2016) or Trucíos et al. (2018) could be used.

4 Out-of-Sample Performance

The out-of-sample portfolio performance is evaluated through economic measures as in Gambacciani and Paolella (2017), Raffinot (2017), and Trucíos et al. (2019b), among others. The measures used are common in the financial econometric literature and are briefly describe below.

- **AV**: Annualised average return. This is given by $12 \times \bar{R}_p$, where $\bar{R}_p$ is the sample mean of the out-of-sample realised portfolio returns. The larger the AV is, the better the portfolio performance.

- **SD**: Annualised standard deviation. This is given by $\sqrt{12} \times \hat{\sigma}_p$, where $\hat{\sigma}_p$ is the sample standard deviation of the out-of-sample realised portfolio returns. The smaller the SD is, the better the portfolio performance.

- **SR**: Annualised Sharpe ratio. This is given by $\sqrt{12} \times SR$, where SR is the well-known Sharpe ratio (Sharpe, 1975), which is a risk-adjusted performance measure defined by
$$SR = \frac{\bar{R}_p - \bar{R}_f}{\hat{\sigma}_{p-f}},$$
where $\bar{R}_f$ is the average risk-free rate and $\hat{\sigma}_{p-f}$ is the estimated standard deviation of the excess returns. The higher the annualised Sharpe ratio is, the better the portfolio performance. Here we use a risk-free rate of 0.5%.

- **ASR**: Annualised adjusted Sharpe ratio. This is given by $\sqrt{12} \times ASR$, where ASR is the adjusted Sharpe ratio (Pézier and White, 2008), a performance measure that penalises the Sharpe ratio by negative skewness and an excess of kurtosis. It is given by
$$ASR = SR\Big[1 + (\frac{\mu_3}{6})SR - (\frac{\mu_4 - 3}{24})SR^2\Big],$$
where μ_3 and μ_4 stand for the skewness and kurtosis of the realised out-of-sample portfolio returns. The higher the annualised adjusted Sharpe ratio is, the better the performance.

- **SoR**: Annualised Sortino ratio. This is given by $\sqrt{12} \times SoR$, where SoR is the Sortino ratio (Sortino and Van Der Meer, 1991), another risk-adjusted performance measure. It is defined by

$$SoR = \frac{\bar{R}_p}{\sqrt{\frac{\sum_{i=1}^{K} Min(0, R_{p,i} - MAR)^2}{K}}},$$

where K is the size of the out-of-sample period and MAR is the minimum accepted return, which in this paper is equal to the monthly risk free rate (0.5%). The higher the annualised Sortino ratio is, the better the portfolio performance.

- **TO**: Average portfolio turnover. This measures the impact of transaction costs on portfolio performance per rebalancing, on average. It is given by

$$TO = \frac{1}{K-1} \sum_{i=2}^{K} \sum_{j=1}^{N} |\hat{\omega}_{i,j} - \hat{\omega}_{i,j}^{+}|,$$

where $\hat{\omega}_{i-1}^{+}$ stands the portfolio weights obtained in the window $i-1$ updated at time i before rebalancing to $\hat{\omega}_i$. Lower values of TO indicate smaller impacts of transaction costs on portfolio performance.

- **SSPW**: Sum of squared portfolio weights. This is a measure of portfolio diversification proposed by Goetzmann and Kumar (2008) and is given by

$$SSPW = \frac{1}{K} \sum_{i=1}^{K} \sum_{j=1}^{N} \hat{\omega}_{i,j}^2,$$

lower values of SSPW indicate higher levels of diversification.

5 Empirical Study

5.1 Data

Our dataset corresponds to monthly simple returns of Brazilian stocks spanning from January 2000 to February 2022 and used in the composition of the IBrx-100 Index[2]. Only stocks traded in the whole sample period and reporting no serial autocorrelation of returns and squared returns according to the Ljung-Box test were considered, resulting in a panel of 16 assets and 266 months.

All analyses were performed in the R software (R Core Team, 2021). For reproduction purposes the codes to replicate the empirical application are freely available in our GitHub repository `https://github.com/ctruciosm/ResamplingBRPortfolios`. All stock price data were downloaded from Economatica.

The monthly returns of the 16 assets are displayed in Figure 1 and their descriptive statistics, as well as the month where minimum and maximum monthly returns were observed, are reported in Table 1.

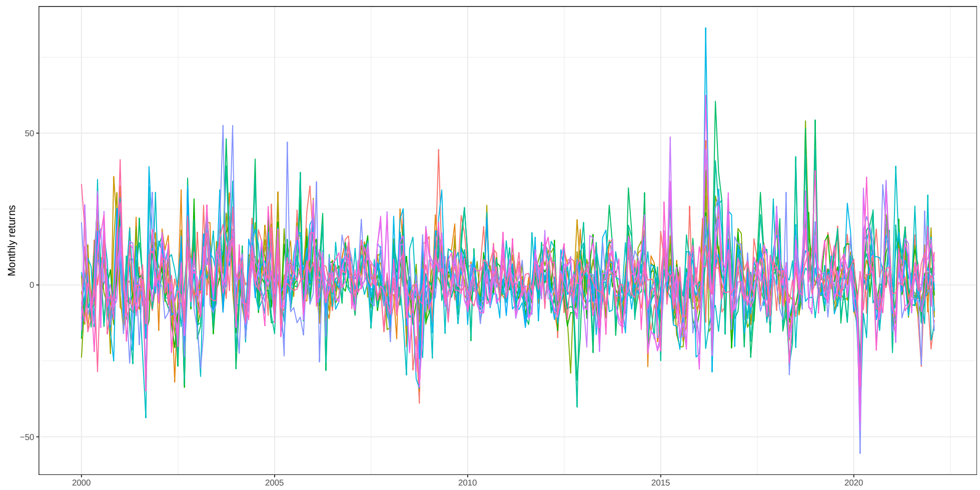

Figure 1: Monthly returns of 16 stocks in the Brazilian stock market over the full sample period.

All stocks have positive mean monthly returns, a kurtosis larger than 3 and almost all of them report positive skewness. The null hypothesis of normality is rejected in all

[2]The IBrx-100 index covers the 100 most liquid assets in the Brazilian stock market. Only stocks listed in the IBrx-100 Index on March 2, 2022 were considered.

Table 1: Descriptive statistics of monthly returns of 16 stocks in the Brazilian stock market over the full sample period

	Min.	Max.	Mean	S. Dev.	Skewness	Kurtosis	JB p-value	Min month	Max month
ALPA4	-38.944	44.653	2.631	10.971	0.083	4.204	0.000	Oct. 2008	April 2009
BBAS3	-40.182	47.544	2.168	12.030	0.162	4.545	0.000	March 2020	March 2016
BBDC3	-31.031	35.733	1.723	9.634	0.410	3.675	0.002	March 2020	Nov. 2000
BBDC4	-31.927	30.425	1.728	9.780	0.201	3.489	0.108	March 2020	Dez. 2000
CMIG4	-36.175	54.050	1.603	10.614	0.365	5.612	0.000	March 2020	Oct. 2018
CPLE6	-33.735	28.677	1.403	10.001	0.013	3.225	0.753	Sep. 2002	Sep. 2005
ELET3	-32.960	60.500	1.629	14.603	0.856	4.810	0.000	Sep. 2002	June 2016
ELET6	-40.248	44.480	1.582	13.046	0.441	4.196	0.000	Nov. 2012	Jan. 2019
EMBR3	-43.769	39.135	1.179	11.641	0.043	5.092	0.000	Sep. 2001	Feb. 2021
GGBR4	-40.533	84.703	2.078	13.110	0.867	8.605	0.000	March 2020	March 2016
ITSA4	-26.856	31.469	1.805	8.632	0.045	3.548	0.181	March 2020	Oct. 2002
LIGT3	-55.509	52.560	0.480	13.488	0.444	5.324	0.000	March 2020	Sep. 2003
PETR3	-47.919	48.747	1.780	12.062	0.230	4.779	0.000	March 2020	April 2015
PETR4	-44.791	62.451	1.742	11.996	0.385	5.802	0.000	March 2020	March 2016
SBSP3	-34.925	37.683	1.522	10.518	-0.098	4.221	0.000	Sep. 2001	Jan. 2019
VIVT3	-28.623	41.262	1.447	7.673	0.695	6.955	0.000	June 2000	Jan. 2001

but the BBDC4, CPLE6 and ITSA4 stocks. The most volatile asset is ELET3 with an annualised standard deviation of 50.58% ($\sqrt{12} \times 14.603$) while the least volatile is VIVT3 with an annualised standard deviation of 26.58%. Except for ALPA4, CPLE6, ELET3, ELET6, EMBR3, SBSP3 and VIVT3, the smallest monthly returns for each asset all occur in March 2020, when the World Health Organization declared the SARS-COVID-19 virus pandemic. The largest monthly return corresponds to GGBR4 (Gerdau) in March 2016. Unreported results reveal that PETR3 and PETR4 are the most correlated assets (0.97)[3] while ELET3 and EMBR3 are the least correlated ones (0.13).

The asset returns described above are used to compare the out-of-sample portfolio performance for the procedures described in Section 3 versus classical Markowitz portfolio optimization. The minimum variance, tangency and mean-variance portfolio with risk-averse parameter $\lambda = 2$ were considered in our comparison[4]. Following Scherer (2002) and Wolf (2004) who argue that portfolio resampling can be useful when constraints are included in the portfolio weights, no short-selling constraints ($\omega_i \geq 0 \quad \forall i$) were considered in all cases.

[3]This is expected because PETR3 and PETR4 are, respectively, the ordinary and preference shares of PETROBRÁS.

[4]Our choice of the risk-averse parameter is purely arbitrary. Other values of λ were also analysed, obtaining similar results, and are presented in the Supplementary Material.

5.2 Out-of-sample results

The out-of-sample analysis was performed using a rolling window approach. As commonly used in practice (Wolf, 2004; DeMiguel et al., 2009; Becker et al., 2015; Huang and Yu, 2020), we used window sizes of $T = 60$ and 120 which corresponds to five and ten years of monthly data with corresponding out-of-sample periods of 206 and 146 months, respectively. For each window, the mean vector and covariance matrix were estimated and used to select the optimal month-ahead portfolios. We then moved the window forward a month and repeated the estimation and portfolio allocation processes. The same procedure was carried on until no more data were available.

To evaluate the out-of-sample performance, the portfolio measures described in Section 4 were used. To verify which procedures outperform classical Markowitz optimization in terms of out-of-sample standard deviation and Sharpe ratio, the bootstrapping tests of Ledoit and Wolf (2008) and Ledoit and Wolf (2011) were applied.

The factor-based portfolio resampling of Section 3.2 was implemented using unobservable factors (extracted by PCA) and one observable factor, the market factor, in which case the Ibovespa index was used. In each window, the procedure of Bai and Ng (2002) was used to determine the number of principal components. For the sake of comparison, we also included the Markowitz optimization when both mean vector and covariance matrix are estimated via factor models with observable and unobservable factors. Those procedures are denoted as Markowitz-Ibov and Markowitz-PCA, respectively.

Table 2 reports the out-of-sample performance of the portfolios constructed using both window sizes. The top panel reports the results for the minimum variance portfolio, the middle panel reports the results for the tangency portfolio and the bottom panel reports the results for the mean-variance portfolio with risk-aversion parameter $\lambda = 2$. The best procedure according to each performance measure is reported in bold. For the out-of-sample standard deviation and Sharpe ratio, the shaded cells indicate procedures with superior out-of-sample performance than classical Markowitz optimization at the 5% of significance level. Figures 2 and 3 display the boxplots of the SSPW and turnover obtained by all procedures considered.

Table 2: Out-of-sample performance measures of the minimum variance (top panel), tangency (middle panel) and mean-variance (bottom panel) portfolios. AV, SD, SR, ASR, SO, TO and SSPW stand for the annualised average, standard deviation, Sharpe ratio, Adjusted Sharpe ratio, Sortino ratio, average turnover and average sum of squared portfolio weights, respectively. The best procedure according to each performance measures is shown in bold. Shaded cells in the SD and SR columns indicate procedures superior to the benchmark (classical Markowitz optimization) at the 5% significance level according to the tests of Ledoit and Wolf (2008) and Ledoit and Wolf (2011).

Portfolio	T	Method	AV	SD	SR	ASR	SO	TO	SSPW
Minimum Variance Portfolio	$T = 60$	Markowitz	13.8148	19.2097	0.4068	0.4075	0.5212	0.1246	0.2710
		Michaud Parametric	13.6826	19.1046	0.4021	0.4023	0.5175	0.1123	0.2414
		Michaud Non-Parametric	**13.8297**	19.0885	**0.4102**	**0.4101**	**0.5281**	0.1117	**0.2357**
		Markowitz-PCA	13.8201	19.2060	0.4072	0.4079	0.5217	0.1243	0.2687
		Factor-Based Parametric PCA	13.6649	**19.0740**	0.4018	0.4020	0.5174	**0.1114**	0.2400
		Factor-Based Non-Parametric PCA	13.6101	19.1220	0.3980	0.3981	0.5128	0.1131	0.2420
		Markowitz-Ibov	13.8252	19.2064	0.4074	0.4081	0.5221	0.1244	0.2687
		Factor-Based Parametric Ibov	13.6701	19.0945	0.4017	0.4018	0.5173	0.1122	0.2399
		Factor-Based Non-Parametric Ibov	13.6258	19.1180	0.3989	0.3989	0.5140	0.1126	0.2421
	$T = 120$	Markowitz	10.5339	18.0484	0.2512	0.2503	0.3418	0.0873	0.2741
		Michaud Parametric	10.8868	**18.0124**	0.2713	0.2702	0.3667	0.0819	0.2565
		Michaud Non-Parametric	**11.0921**	18.0495	**0.2821**	**0.2809**	**0.3805**	**0.0814**	**0.2493**
		Markowitz-PCA	10.5422	18.0635	0.2515	0.2505	0.3420	0.0874	0.2727
		Factor-Based Parametric PCA	10.8601	18.0499	0.2693	0.2681	0.3645	0.0832	0.2551
		Factor-Based Non-Parametric PCA	10.8684	18.0312	0.2700	0.2689	0.3653	0.0829	0.2559
		Markowitz-Ibov	10.5316	18.0579	0.2509	0.2500	0.3414	0.0874	0.2728
		Factor-Based Parametric Ibov	10.8935	18.0320	0.2714	0.2703	0.3668	0.0823	0.2553
		Factor-Based Non-Parametric Ibov	10.8930	18.0348	0.2713	0.2702	0.3670	0.0824	0.2564
Tangency Portfolio	$T = 60$	Markowitz	14.1675	25.7408	0.3173	0.3158	0.4247	0.3063	0.3111
		Michaud Parametric	14.4940	**23.1957**	0.3662	0.3632	0.4904	0.1859	0.1641
		Michaud Non-Parametric	14.5269	23.2352	0.3670	0.3642	0.4912	**0.1856**	0.1690
		Markowitz-PCA	14.1827	25.7224	0.3181	0.3166	0.4259	0.3058	0.3092
		Factor-Based Parametric PCA	**14.7765**	23.3383	**0.3761**	**0.3730**	**0.5019**	0.1867	**0.1601**
		Factor-Based Non-Parametric PCA	14.7172	23.3966	0.3726	0.3695	0.4989	0.1860	0.1632
		Markowitz-Ibov	14.1698	25.7093	0.3178	0.3162	0.4255	0.3057	0.3095
		Factor-Based Parametric Ibov	14.5000	23.2752	0.3652	0.3622	0.4892	0.1903	0.1606
		Factor-Based Non-Parametric Ibov	14.5059	23.2820	0.3653	0.3623	0.4900	0.1883	0.1646
	$T = 120$	Markowitz	10.4610	23.0275	0.1937	0.1926	0.2808	0.2482	0.2870
		Michaud Parametric	11.7595	21.6104	0.2665	0.2642	0.3770	0.1410	0.1483
		Michaud Non-Parametric	11.8847	21.6134	0.2723	0.2698	0.3848	0.1398	0.1526
		Markowitz-PCA	10.4501	23.0133	0.1934	0.1923	0.2803	0.2483	0.2859
		Factor-Based Parametric PCA	11.6334	21.6758	0.2599	0.2577	0.3679	0.1438	**0.1454**
		Factor-Based Non-Parametric PCA	11.8034	21.6963	0.2675	0.2652	0.3780	0.1410	0.1478
		Markowitz-Ibov	10.4481	23.0137	0.1933	0.1922	0.2801	0.2484	0.2860
		Factor-Based Parametric Ibov	**11.9551**	21.6067	**0.2756**	**0.2732**	**0.3883**	**0.1386**	0.1464
		Factor-Based Non-Parametric Ibov	11.7093	**21.5954**	0.2644	0.2622	0.3732	0.1414	0.1495
Mean-Variance Portfolio ($\lambda = 2$)	$T = 60$	Markowitz	13.6975	19.1899	0.4011	0.4019	0.5142	0.1252	0.2686
		Michaud Parametric	13.5379	19.0986	0.3947	0.3948	0.5089	0.1123	0.2392
		Michaud Non-Parametric	13.6107	19.0864	0.3987	0.3986	0.5149	0.1124	**0.2334**
		Markowitz-PCA	13.6992	19.1846	0.4013	0.4020	0.5146	0.1248	0.2663
		Factor-Based Parametric PCA	13.5049	19.0948	0.3930	0.3931	0.5070	0.1128	0.2376
		Factor-Based Non-Parametric PCA	13.4678	19.1146	0.3907	0.3907	0.5045	0.1133	0.2397
		Markowitz-Ibov	**13.7089**	19.1846	**0.4018**	**0.4025**	**0.5152**	0.1249	0.2663
		Factor-Based Parametric Ibov	13.5290	**19.0823**	0.3946	0.3946	0.5091	**0.1117**	0.2377
		Factor-Based Non-Parametric Ibov	13.4892	19.1229	0.3916	0.3916	0.5056	0.1125	0.2399
	$T = 120$	Markowitz	10.4761	17.9865	0.2489	0.2479	0.3393	0.0877	0.2712
		Michaud Parametric	10.8657	17.9765	0.2707	0.2695	0.3667	0.0826	0.2533
		Michaud Non-Parametric	**11.0548**	17.9802	**0.2811**	**0.2798**	**0.3801**	**0.0819**	**0.2463**
		Markowitz-PCA	10.4855	18.0019	0.2492	0.2482	0.3397	0.0878	0.2698
		Factor-Based Parametric PCA	10.8207	17.9972	0.2679	0.2667	0.3631	0.0824	0.2523
		Factor-Based Non-Parametric PCA	10.8881	17.9788	0.2719	0.2707	0.3685	**0.0819**	0.2531
		Markowitz-Ibov	10.4752	17.9964	0.2487	0.2477	0.3391	0.0878	0.2699
		Factor-Based Parametric Ibov	10.8448	**17.9698**	0.2696	0.2684	0.3654	0.0825	0.2525
		Factor-Based Non-Parametric Ibov	10.8007	17.9878	0.2669	0.2657	0.3620	0.0827	0.2534

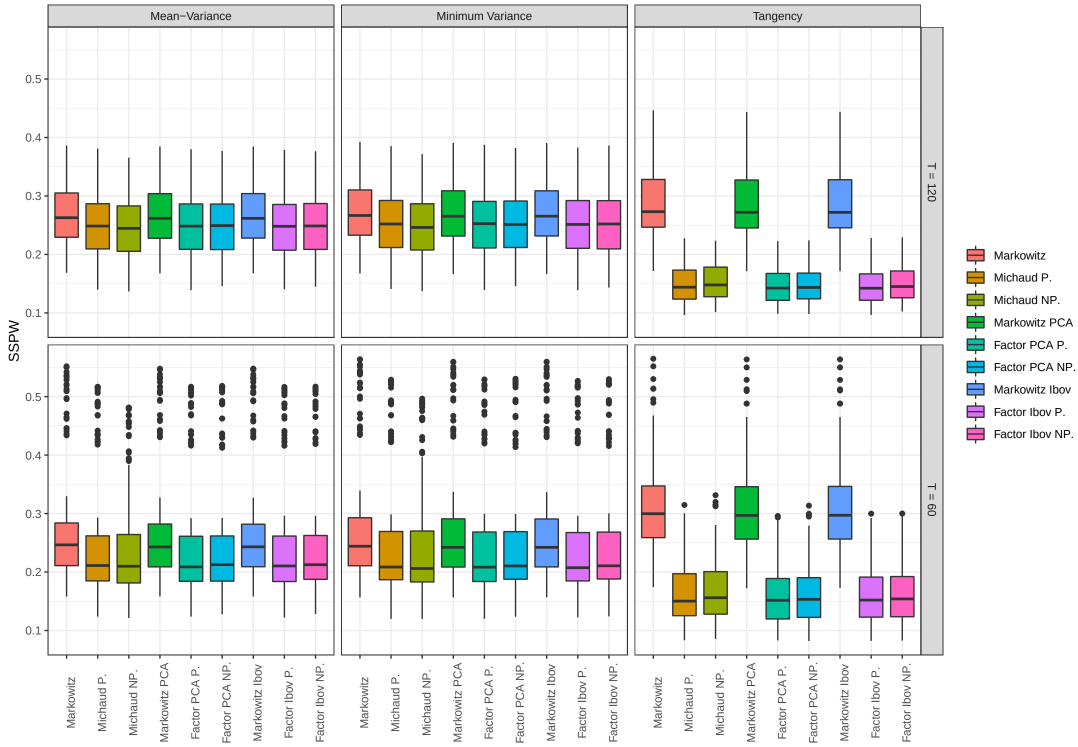

Figure 2: Boxplots of SSPW in the out-of-sample period.

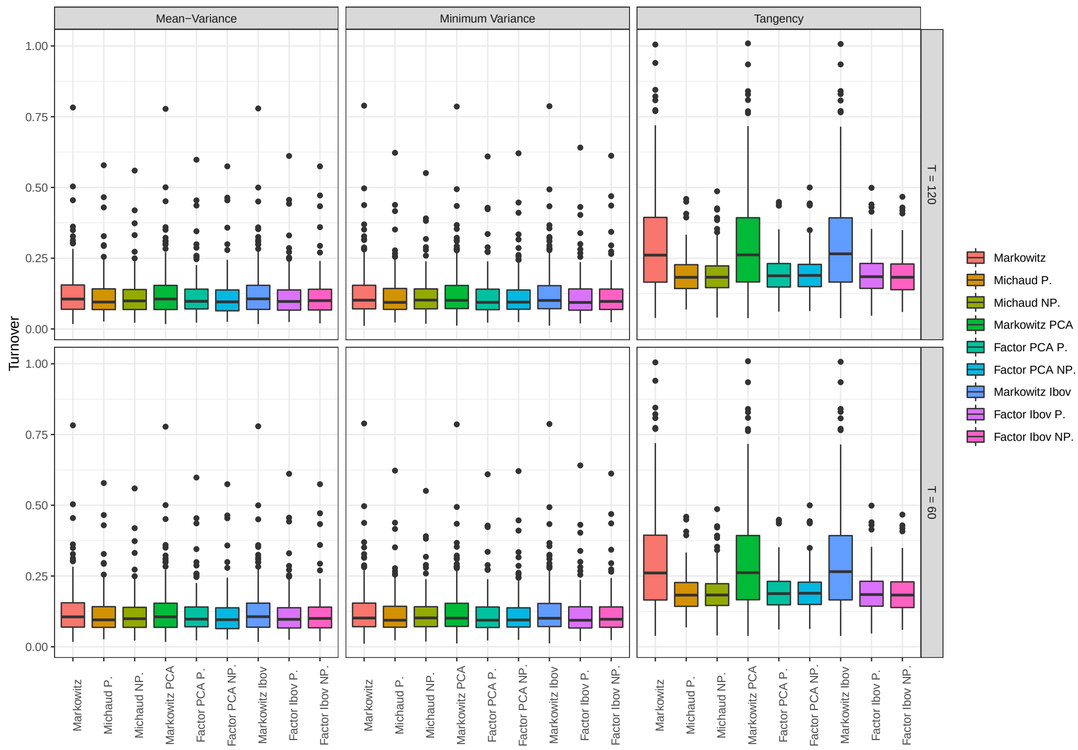

Figure 3: Boxplots of the turnover in the out-of-sample period.

For the minimum variance portfolios, no procedure outperforms classical Markowitz optimization in terms of out-of-sample SD and SR according to the tests of Ledoit and Wolf (2008) and Ledoit and Wolf (2011). For the AV, SO and ASR out-of-sample measures, the differences are small between all procedures and no improvement is observed with respect to classical Markowitz optimization. In terms of diversification (Table 2), Markowitz optimization (classical and factor-based models) perform slightly worse than the resampled portfolios, reporting larger SSPW values.

For the mean-variance portfolio, the results in Table 2 and Figures 2 and 3 report that, regardless of the sample size used, slightly better performance is observed in terms of SSPW with the resampled portfolios, implying more diversified portfolios. For the other measures considered, all procedures perform similarly and no evidence is observed in favour of portfolio resampling.

For the tangency portfolio, the results are slightly different than those obtained in the other two cases. All procedures, except Markowitz-PCA for $T = 60$, outperform classical Markowitz optimization in terms of out-of-sample standard deviation according to the test of Ledoit and Wolf (2011). For the other performance measures the differences between resampling and Markowitz optimization procedures is larger than for the other two kinds of portfolios (minimum variance and mean-variance) and, in particular, there are interesting gains in terms of diversification (SSPW) and transaction cost (turnover) when portfolio resampling is used, and so it is preferable.

Since almost all procedures in the tangency portfolios outperform the classical Markowitz optimization in terms of out-of-sample standard deviation, we went one step further and perform a pairwise comparison. The p-values obtained by using the bootstrapping test of Ledoit and Wolf (2011) are reported in Table 3 for $T = 60$ (top panel) and $T = 120$ (bottom panel). The pairwise comparison tested whether the procedures placed in the columns are statistically superior (smaller standard deviation) than the procedures placed in the rows. For $T = 120$, all resampling procedures outperform Markowitz optimization (classical, PCA-based and Ibov-based), Factor-Based Ibov (parametric and non-parametric) outperform Factor-Based Non-Parametric PCA but do not outperform

Michaud portfolios. For $T = 60$, once again all resampling procedures outperform Markowitz optimization (classical, PCA-based and Ibov-based). Additionally, Michaud Parametric outperforms Factor-Based PCA (in its parametric and non-parametric version) but does not outperforms Factor-Based Ibov. By they turn, Factor-Based Ibov (parametric and non-parametric) outperform Factor-Based Non-Parametric PCA but do not outperform Michaud portfolios.

Table 3: Pairwise out-of-sample portfolio standard deviation in tangency portfolios. Shaded cells indicate the procedures in the columns are superior to the procedures in the rows at the 5% of significance level.

		Classical Markowitz	Michaud Parametric	Michaud Non-Parametric	Markowitz-PCA	Factor-Based Parametric PCA	Factor-Based Non-Parametric PCA	Markowitz-Ibov	Factor-Based Parametric Ibov	Factor-Based Non-Parametric Ibov
$T = 60$	Classical Markowitz	-	0.0002	0.0002	0.1782	0.0002	0.0006	0.0122	0.0002	0.0002
	Michaud Parametric	1.0000	-	0.6069	1.0000	0.9784	0.9862	1.0000	0.9232	0.9242
	Michaud Non-Parametric	1.0000	0.4005	-	1.0000	0.8068	0.9146	1.0000	0.6367	0.7331
	Markowitz-PCA	0.8288	0.0002	0.0002	-	0.0002	0.0002	0.0308	0.0002	0.0002
	Factor-Based Parametric PCA	1.0000	0.0178	0.2000	1.0000	-	0.8708	1.0000	0.0640	0.1892
	Factor-Based Non-Parametric PCA	1.0000	0.0136	0.0958	1.0000	0.1298	-	0.9998	0.0102	0.0264
	Markowitz-Ibov	0.9902	0.0002	0.0002	0.9718	0.0004	0.0004	-	0.0002	0.0002
	Factor-Based Parametric Ibov	1.0000	0.0780	0.3451	0.9998	0.9392	0.9908	1.0000	-	0.5163
	Factor-Based Non-Parametric Ibov	1.0000	0.0774	0.2739	1.0000	0.8146	0.9752	1.0000	0.4849	-
$T = 120$	Classical Markowitz	-	0.0212	0.0228	0.0230	0.0274	0.0284	0.0240	0.0246	0.0206
	Michaud Parametric	0.9774	-	0.5343	0.9766	0.8220	0.9392	0.9776	0.4769	0.4151
	Michaud Non-Parametric	0.9788	0.4931	-	0.9722	0.7271	0.8410	0.9740	0.4541	0.4045
	Markowitz-PCA	0.9798	0.0222	0.0300	-	0.0316	0.0258	0.5591	0.0194	0.0224
	Factor-Based Parametric PCA	0.9742	0.1854	0.2851	0.9694	-	0.5607	0.9740	0.0654	0.1172
	Factor-Based Non-Parametric PCA	0.9710	0.0612	0.1572	0.9728	0.4587	-	0.9750	0.0104	0.0232
	Markowitz-Ibov	0.9754	0.0238	0.0260	0.4349	0.0312	0.0252	-	0.0232	0.0170
	Factor-Based Parametric Ibov	0.9780	0.5259	0.5441	0.9754	0.9348	0.9846	0.9798	-	0.3999
	Factor-Based Non-Parametric Ibov	0.9764	0.5847	0.6071	0.9810	0.8748	0.9784	0.9810	0.6033	-

Overall, portfolio resampling and classical Markowitz optimization report similar results for the minimum variance and mean-variance ($\lambda = 2$) portfolios with a slightly smaller turnover and SSPW. For the tangency portfolio, portfolio resampling provides an interesting improvement in the out-of-sample performance and also implies smaller transaction costs and more diversified portfolios, thus being then a preferable alternatives.

6 Conclusions

In this paper, we deal with portfolio resampling, an easy to implement way to handle estimation error in the portfolio construction without relying on difficult and complex methods. An empirical out-of-sample comparison between classical Markowitz optimization and portfolio resampling is performed using data from the Brazilian stock market, a market characterized by its higher volatility and lower liquidity than developed markets.

The main results are that in minimum variance and mean-variance portfolios, portfolio resampling brought no improvements over classical Markowitz optimization with the results being slightly better but not statistically significant. However, the results for the tangency portfolio show that portfolio resampling can improve the out-of-sample portfolio performance and also imply smaller transaction cost and more diverse portfolios.

Another well-known approach to deal with estimation error is through Bayesian analysis. Although we have not used that methods here, we recognize its importance and usefulness to deal with the aforementioned issue. Instead, we focused on portfolio resampling since this approach could be more appealing and easy to implement for practitioners from various backgrounds, including non-technical ones.

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Supplementary Material Appendix for Does portfolio resampling really improve out-of-sample performance? evidence from the Brazilian market.

In the material it is presented the out-of-sample performance measures for two other risk-averse parameters, $\lambda = 5$ and $\lambda = 10$. As poninted out in section 5.1 the results of table (2) remain the same.

Table 4: Out-of-sample performance measures of the minimum variance (top panel) with $\lambda = 5$, and minimum-variance (bottom panel) portfolios with $\lambda = 10$. AV, SD, SR, ASR, SO, TO and SSPW stand for the annualised average, standard deviation, Sharpe ratio, adjusted Sharpe ratio, Sortino ratio, average turnover and average sum of squared portfolio weights, respectively. The best procedure according to each performance measures is shown in bold.

		Method	AV	SD	SR	ASR	SO	TO	SSPW
Minimum-Variance Portfolio ($\lambda = 5$)	$T = 60$	Markowitz	13.7657	19.1979	0.4045	0.4052	0.5184	0.1245	0.2700
		Michaud Parametric	13.6290	19.0997	0.3994	0.3996	0.5143	0.1124	0.2405
		Michaud Non-Parametric	13.7706	19.0845	**0.4072**	**0.4071**	**0.5245**	0.1117	**0.2349**
		Markowitz-PCA	13.7692	19.1946	0.4048	0.4055	0.5188	0.1243	0.2677
		Factor-Based Parametric PCA	13.6138	**19.0680**	0.3993	0.3994	0.5144	**0.1114**	0.2391
		Factor-Based Non-Parametric PCA	13.5560	19.1187	0.3952	0.3953	0.5095	0.1131	0.2411
		Markowitz-Ibov	**13.7750**	19.1941	0.4051	0.4058	0.5192	0.1243	0.2677
		Factor-Based Parametric Ibov	13.6161	19.0882	0.3990	0.3991	0.5141	0.1121	0.2391
		Factor-Based Non-Parametric Ibov	13.5683	19.1155	0.3959	0.3960	0.5105	0.1127	0.2413
	$T = 120$	Markowitz	10.5119	18.0230	0.2503	0.2494	0.3409	0.0874	0.2729
		Michaud Parametric	10.8637	**17.9894**	0.2704	0.2693	0.3658	0.0819	0.2554
		Michaud Non-Parametric	**11.0638**	18.0245	**0.2809**	**0.2797**	**0.3793**	**0.0814**	**0.2482**
		Markowitz-PCA	10.5217	18.0379	0.2507	0.2498	0.3413	0.0876	0.2716
		Factor-Based Parametric PCA	10.8346	18.0263	0.2682	0.2671	0.3634	0.0832	0.2540
		Factor-Based Non-Parametric PCA	10.8505	18.0084	0.2693	0.2682	0.3647	0.0828	0.2547
		Markowitz-Ibov	10.5118	18.0323	0.2502	0.2493	0.3407	0.0876	0.2716
		Factor-Based Parametric Ibov	10.8713	18.0079	0.2705	0.2694	0.3660	0.0822	0.2542
		Factor-Based Non-Parametric Ibov	10.8706	18.0118	0.2704	0.2693	0.3662	0.0824	0.2552
Minimum-Variance Portfolio($\lambda = 10$)	$T = 60$	Markowitz	13.7884	19.2025	0.4056	0.4063	0.5197	0.1246	0.2705
		Michaud Parametric	13.6558	19.1014	0.4008	0.4009	0.5159	0.1123	0.2410
		Michaud Non-Parametric	**13.8001**	19.0857	**0.4087**	**0.4086**	**0.5263**	0.1117	**0.2353**
		Markowitz-PCA	13.7937	19.2000	0.4059	0.4066	0.5202	0.1243	0.2682
		Factor-Based Parametric PCA	13.6392	**19.0703**	0.4006	0.4007	0.5159	**0.1114**	0.2396
		Factor-Based Non-Parametric PCA	13.5830	19.1197	0.3966	0.3967	0.5112	0.1131	0.2416
		Markowitz-Ibov	13.7991	19.1997	0.4062	0.4069	0.5206	0.1243	0.2682
		Factor-Based Parametric Ibov	13.6430	19.0907	0.4004	0.4005	0.5157	0.1121	0.2395
		Factor-Based Non-Parametric Ibov	13.5970	19.1160	0.3974	0.3975	0.5123	0.1127	0.2417
	$T = 120$	Markowitz	10.5225	18.0354	0.2508	0.2498	0.3413	0.0874	0.2735
		Michaud Parametric	10.8750	**18.0007**	0.2708	0.2697	0.3663	0.0819	0.2559
		Michaud Non-Parametric	**11.0777**	18.0368	**0.2815**	**0.2803**	**0.3799**	**0.0814**	**0.2488**
		Markowitz-PCA	10.5316	18.0504	0.2511	0.2501	0.3416	0.0875	0.2721
		Factor-Based Parametric PCA	10.8471	18.0379	0.2687	0.2676	0.3639	0.0832	0.2546
		Factor-Based Non-Parametric PCA	10.8593	18.0196	0.2697	0.2685	0.3650	0.0829	0.2553
		Markowitz-Ibov	10.5212	18.0448	0.2506	0.2496	0.3410	0.0875	0.2722
		Factor-Based Parametric Ibov	10.8821	18.0197	0.2709	0.2698	0.3664	0.0822	0.2547
		Factor-Based Non-Parametric Ibov	10.8816	18.0231	0.2709	0.2697	0.3666	0.0824	0.2558