Heterogeneous Beliefs, Asset Prices, and Business Cycles*

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Abstract

This paper develops a complete-market production economy with heterogeneous beliefs about TFP growth. Hiring occurs before TFP is known and is, therefore, risky (operational leverage). The firm's discount factor depends on a wealth-weighted average of investors' beliefs. Waves of optimism influence asset markets and percolate to labor hiring, tying together the equity premium, equity volatility, and labor volatility puzzles. A taxonomy of belief systems shows that only extrapolative beliefs amplify the volatility of asset prices and hours and lead to risk build up. Disciplined by survey data, the model matches asset-pricing and business-cycle moments, highlighting how heterogeneous beliefs can be a direct driver of aggregate fluctuations.

KEYWORDS: Heterogeneous Beliefs; Business Cycles; Asset Prices; Speculation.

JEL CLASSIFICATION: D84, E32, E44, E71, G41.

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1 Introduction

One of the oldest and most prominent economic narratives holds that swings in investor beliefs drive asset prices and business cycles. Sprague (1910), Fisher (1933), Keynes (1936), Minsky (1986), Kindleberger (1996), and Shiller (2000), all feature a common theme: fueled by leverage, a wave of excessive optimism drives up share prices, spurring excessive firm investment and hiring. Eventually, disappointing earnings burst asset prices on Wall Street with a blast wave felt all the way onto Main Street.

Formalizing this narrative, Harrison and Kreps (1978) and Scheinkman and Xiong (2003) show how asset prices can be propelled by excessive optimism, emphasizing that belief heterogeneity engenders amplification through leverage. A virtue of these formalizations is their subtle departure from strict rational expectations: agents fully understand the workings of the economy but differ in their optimism. Yet, it remains unclear how—and by how much—waves of optimism percolate into the real economy.

This paper studies a benchmark economy where beliefs are a direct source of business cycles. Prices are flexible, there are no financial frictions, and markets are complete. The only link between beliefs and output is that hiring labor is risky. Because hiring is a business risk, hours fluctuations are driven exclusively by investor beliefs about earnings growth and their risk appetite.

Also motivated by these narratives, a growing body of literature studies how beliefs spill over to the real economy by aggravating demand externalities or financial constraints. However, to our knowledge, we do not have a frictionless benchmark economy where beliefs induce fluctuations in an otherwise frictionless setting. Developing such a benchmark is important: First, by abstracting away market imperfections, we can provide a qualitative and quantitative assessment of the direct effects of beliefs on the business cycle. Second, we can derive general principles that carry over to environments where market imperfections amplify these direct effects. Finally, a frictionless benchmark serves as an ideal target that policy interventions to mitigate market imperfections should strive to replicate. After all, policy should not care about asset prices per se, but only inasmuch as these create inefficiencies.

The Model. Households differ in their beliefs about the evolution of TFP. Given their distinct views, they actively trade firm shares, affecting asset prices. Households supply labor with preferences that eliminate wealth effects, as in Greenwood, Hercowitz and Huffman (1988, henceforth GHH).

The key feature is that heterogeneity in household beliefs shapes the firm's labor demand because hiring is risky. As in Burnside, Eichenbaum and Rebelo (1993), labor is chosen before the shock realizations. This timing induces an operational leverage channel whereby hiring becomes a risky investment, as it can add cost pressure to firms, which also occurs in labor search models.[1] This feature is critical to linking business-cycle fluctuations to asset-price fluctuations. We focus on the firm's investment in forming a workforce, as opposed to physical capital formation, because it is understood that, absent additional frictions, fluctuations in capital investment cannot be a source of business cycles; instead, what is needed are movements in the labor wedge (Chari, Kehoe and McGrattan, 2007).[2] Moreover, the operational leverage channel leads to testable implications regarding the correlation of returns and the labor share that is consistent with data, (e.g., see Donangelo, Gourio, Kehrig and Palacios, 2019).

Belief heterogeneity is essential for generating leverage and turnover dynamics, both recurrent themes in optimism-wave narratives. Because of heterogeneous beliefs, households take long or short positions in the firm. Optimistic households increase leverage, amplifying their risk exposure. Leverage creates internal propagation by magnifying differences in investors' risk exposures: optimistic investors accumulate relatively more wealth when positive shocks occur, while pessimists gain during downturns.

These dynamics produce internal propagation as wealth-weighted beliefs jointly affect asset prices and output through the operational leverage channel. When wealth-weighted beliefs become more optimistic, increased demand for risky assets pushes stock prices upward, compresses risk premia, and encourages hiring. Thus, as investors' risk appetite increases, discount rates decline, prompting more hiring by a firm attempting to raise its market value. Through these dynamics, the *equity premium*, the *equity volatility*, and *labor volatility* puzzles are tied together into a single volatility puzzle.

Theoretical Results. Despite featuring Epstein-Zin preferences and endogenous labor supply, the model remains highly tractable due to a convenient "as if" property portable to other settings. Specifically, the equilibrium is characterized as if the economy were an endowment economy, with a common stochastic discount factor (SDF). Importantly, the firm makes risky hiring decisions using this common SDF to weigh future states, thereby linking the SDF directly to a labor-demand factor (LDF) that ultimately determines employment. Crucially, we impose no assumptions regarding the firm's direct knowledge

[1] We reinterpret this form of *investment in labor* as a simplified version of the investment in hiring that happens in labor-search models. Recall that in labor-search models, firms incur sunk costs to hire workers.

[2] Investment is small relative to the capital stock in a business-cycle model. Thus, fluctuations in investment do not meaningfully impact the production possibility frontier.

of investor beliefs; the firm only needs to understand how its hiring decisions affect its market value.

In the special case of log utility, the model admits an analytic solution, enabling us to articulate a clear taxonomy of belief systems. Specifically, we classify beliefs into two broad categories, each with two sub-categories. The first category comprises *rank-preserving* beliefs, subdivided into optimistic and pessimistic beliefs. The second category includes *rank-alternating* beliefs, subdivided into extrapolative and intrapolative beliefs. This taxonomy allows us to derive general properties regarding how each belief structure affects business-cycle dynamics.

We identify several principles regarding the amplification properties under different belief systems. First, we show that only extrapolative beliefs amplify business-cycle fluctuations across all states of the economy relative to rational expectations. Amplification occurs because extrapolation boosts the firm's discounting of risk states during booms but depresses it during busts, a property unique to extrapolative beliefs.

Second, we uncover that only extrapolative beliefs engender *risk build-up*, i.e., longer booms lead to deeper busts. This phenomenon, also exclusive to extrapolative belief systems, occurs because extrapolative households consistently accumulate wealth throughout boom phases. When the economy transitions into a downturn, extrapolative agents remain wealthier while simultaneously becoming the most pessimistic investors. The reversal in the wealthiest households' optimism significantly amplifies the economy's rift.

The taxonomy of beliefs highlights the importance of belief heterogeneity for understanding macro-finance dynamics. Empirical evidence from López-Salido, Stein and Zakrajšek (2017) and Krishnamurthy and Muir (2017) indicates that risk premia are low during credit booms preceding severe crashes.[3] In contrast, standard macro-finance models without belief heterogeneity predict precisely the opposite (e.g., Brunnermeier and Sannikov, 2014; He and Krishnamurthy, 2013). Furthermore, our analysis helps distinguish models driven by heterogeneous beliefs from those driven by heterogeneous risk aversion (e.g., Panageas, 2020), a challenge for the literature. Our results clarify that heterogeneity in risk aversion cannot generate risk build-up germane to extrapolative belief systems. Beyond these results, we further demonstrate that optimism-driven business cycles are consistent with observed stock-market turnover fluctuations in the model's two-state version.

Quantitative Results. To complement our theoretical insights, we provide a quantitative evaluation. Given that aside from beliefs, our framework follows canonical assump-

[3] See also Gennaioli and Shleifer (2018) and Krishnamurthy and Li (2025).

tions, we adopt a standard calibration approach for preferences and the process for TFP.

To discipline the calibration of beliefs, we utilize survey data on earnings forecasts (from the I/B/E/S) to estimate a belief process that explicitly captures household heterogeneity, thus representing various empirically plausible belief systems. Specifically, we specify a continuous Markov process for actual productivity growth and subjective beliefs about it. Actual log productivity growth follows an estimated iid process. In contrast, subjective beliefs allow households to over- or under-react to TFP, captured by an overreaction parameter. We calibrate this overreaction parameter to match the return extrapolation observed in survey data. We further introduce a non-fundamental "sentiment shock." This shock allows for rank-preserving beliefs and generates additional fluctuations in expectations. The volatility of this sentiment shock is calibrated to match the empirically observed ratio between the volatility of survey-based dividend growth expectations relative to its objective counterpart.

Our quantitative analysis underscores the importance of belief heterogeneity by following a progression. We begin with a stripped-down model featuring a representative rational investor without hiring risk. As is known, this baseline economy fails to generate realistic equity premiums or stock market volatility. Introducing operational leverage—where hiring occurs before TFP realizations—successfully matches the observed volatility of dividend growth. However, labor demand remains essentially constant under rational expectations, resulting in no employment volatility.

We then incorporate subjective beliefs calibrated to match our empirical targets. While subjective beliefs alone generate realistic labor fluctuations due to extrapolation, they still fail to produce an adequate equity premium or sufficient asset-price volatility. Crucially, when belief heterogeneity is introduced—allowing rational and extrapolative investors to interact—the additional fluctuation in wealth allows the model to successfully match the magnitude and volatility of equity returns and substantial volatility in employment. These results emphasize the pivotal role of heterogeneous beliefs in jointly explaining key unconditional asset pricing and labor-market moments.

Beyond matching unconditional moments, we also explain how the model can reconcile the Campbell-Shiller decomposition, the lack of predictability in subjective beliefs, and the conditional correlation between labor income and returns. The section demonstrates the model's ability to rationalize business cycle and asset price moments by incorporating beliefs disciplined by survey data.

Literature Review. This paper connects to the literature initiated by Kydland and Prescott (1982) and Mehra and Prescott (1985), who used complete-market benchmarks to analyze business cycle fluctuations and asset prices in settings where these two dimensions do not feed into each other. A recent strand emphasizes the role of fluctuating risk premia in driving real cycles, particularly highlighting the riskiness of labor hiring (e.g., Hall, 2017; Borovička and Borovičková, 2019; Di Tella and Hall, 2019; Kehoe, Lopez, Midrigan and Pastorino, 2019). We bridge this channel with the literature on heterogeneous beliefs and asset pricing.

The literature on heterogeneous beliefs is extensive. Early seminal contributions link belief heterogeneity to speculative behavior and bubbles (Harrison and Kreps, 1978; Scheinkman and Xiong, 2003), a connection further developed by studies examining its interaction with financial constraints (e.g., Geanakoplos, 2003; Fostel and Geanakoplos, 2008; Geanakoplos, 2010; Simsek, 2013a; Iachan, Nenov and Simsek, 2019; Barlevy, 2014, 2022). Closely related papers explore how wealth redistribution driven by heterogeneous beliefs affects asset-price dynamics (Detemple and Murthy, 1994; Xiong and Yan, 2009; Kubler and Schmedders, 2012; Martin and Papadimitriou, 2022). Our paper contributes by explicitly linking these channels to the real economy through the labor market.

Our emphasis on extrapolative beliefs connects us closely to models featuring diagnostic expectations (Gennaioli and Shleifer, 2010; Bianchi, Ilut and Saijo, 2024), and more generally, to the literature on subjective beliefs and macroeconomic dynamics (e.g., Eusepi and Preston, 2011; Angeletos, Collard and Dellas, 2018; Bordalo, Gennaioli and Shleifer, 2018a; Bhandari, Borovička and Ho, 2019). Relatedly, Adam and Merkel (2019) shows how homogeneous extrapolative beliefs simultaneously explain stock market and business cycle fluctuations, though abstracting from heterogeneity. Other models examine different mechanisms connecting speculative bubbles to real activity, typically emphasizing capital rather than labor investment.[4]

Recent work further explores how heterogeneous beliefs interact with aggregate demand externalities, highlighting how changes in wealth distribution can amplify demand-driven recessions (Caballero and Simsek, 2020a,b; Guerreiro, 2022; Caramp and Silva, 2024). Simsek (2021) provides a comprehensive review of the belief heterogeneity and business cycle literature. Relative to this rich literature, our analysis focuses on the labor demand channel instead of an aggregate demand channel.

Roadmap. The next section presents three empirical facts about investor beliefs documented in the literature to motivate our analysis. Subsequently, we present the theoretical model followed by the quantitative evaluation. Finally, we conclude.

[4]See also, Gilchrist, Himmelberg and Huberman (2005), Bolton, Scheinkman and Xiong (2006), Panageas (2005), and Buss, Dumas, Uppal and Vilkov (2016).

2 Three motivating facts

This section discusses three facts about subjective expectations that motivate our model.

Fact 1. Expectations are volatile. As Shiller (1981) famously observed, stock dividends and earnings are too stable to explain the large swings in asset prices. In contrast, recent survey evidence shows that *subjective expectations* of future cash flows are highly volatile, offering a potential explanation for the stock market's excess volatility (Bordalo, Gennaioli, La Porta and Shleifer, 2020a; De La O and Myers, 2021).

We illustrate the magnitude of the difference in the volatility of subjective and objective expectations by constructing an "objective" forecast—based on the statistical properties of dividends—against a survey-based "subjective" forecast. Specifically, we estimate an AR(1) process for quarterly aggregate dividend and earnings growth. The predicted one-year-ahead dividend growth from this AR(1) serves as our measure of objective expectations, $\mathbb{E}^{obj}[\Delta d_{t,t+4}]$. We then compute the fraction of the total variance in realized dividend growth expectations relative to the variance of realized dividend growth. We construct an analogous measure for subjective expectations, $\mathbb{E}^{sub}[\Delta d_{t,t+4}]$, by using the survey of analyst expectations from I/B/E/S. The differences in volatility are significant:

$$\frac{\mathrm{Var}\big[\mathbb{E}^{obj}[\Delta d_{t,t+4}]\big]}{\mathrm{Var}[\Delta d_{t,t+4}]} = 0.05, \qquad \frac{\mathrm{Var}\big[\mathbb{E}^{sub}[\Delta d_{t,t+4}]\big]}{\mathrm{Var}[\Delta d_{t,t+4}]} = 0.70.$$

Whereas objective expectations represent a negligible fraction of the total variation in dividend growth, subjective expectations are an order of magnitude more volatile.[5]

Fact 2. Expectations are heterogeneous. Fact 1 provides evidence of deviations from full-information rational expectations (FIRE). Our business cycle model emphasizes how heterogeneity in these expectations leads to business cycle amplification through asset markets. Critical to our theory is the presence of heterogeneity in beliefs. Concrete evidence is surveyed in Nagel and Xu (2023) reporting subjective return expectations across individual investors, CFOs, and professional forecasters. Table 1 summarizes their findings: The first column shows that realized future returns are positively predicted by the dividend-price ratio (a standard return predictor). In contrast, past returns do not appear significantly. In contrast, regressions for subjective expected returns reveal heterogeneous beliefs: individual investors place a positive and significant weight on past returns (con-

[5]The small fraction of variation explained by objective expectations is consistent with the estimated autocorrelation of cash flows in, for example, Bansal and Yaron (2004).

Dependent var.:	**Objective**	**Individual**	**CFO**	**Professional**
Predictors				
D/P	5.83	-0.00	-0.41	0.42
(p-value)	(0.00)	(1.00)	(0.31)	(0.73)
R^e_{past}	0.36	0.87	0.30	-2.78
(p-value)	(0.71)	(0.02)	(0.29)	(0.01)

Table 1: Subjective return expectations regressions by type of investor.

sistent with extrapolation), professional forecasters behave in a contrarian way (a negative coefficient on past returns indicates interpolation), and CFOs lie somewhere in between. These results underscore the importance of heterogeneity in expectations among different agents.[6]

For our paper, a key question is whether expectations correlate with actual trading behavior. Existing evidence suggests that yes: Greenwood and Shleifer (2014) shows that subjective expectations correlate with mutual fund flows, and Giglio, Maggiori, Stroebel and Utkus (2021) documents a relationship between beliefs and portfolio choices.[7]

Fact 3. Expectations correlate with hiring decisions. The previous two facts focus on *subjective expectations* among groups of investors. Toward a business cycle theory, a natural follow-up question is whether these expectations are reflected in *real* economic outcomes. Indeed, Gennaioli, Ma and Shleifer (2016) shows that CFO expectations predict corporate capital investment, and Armona, Fuster and Zafar (2019) finds that expectations about home price appreciations influence construction. In this spirit, we present related evidence: investor expectations also predict firm-level *employment* decisions. To demonstrate this, we use two labor-related measures from Compustat Annual: the realized growth in staff expenses (payroll) and the employee count (workers). These two measures are standardized using the same procedure we apply to our earnings data.

We follow the approach of Gennaioli et al. (2016): we regress realized employment growth on the firm-level earnings growth expectations drawn from I/B/E/S. We control for past 12-month stock returns (constructed using individual stock prices from CRSP) and contemporaneous returns, clustering standard errors at the firm level. By including

[6]See also Atmaz, Gulen, Cassella and Ruan (2023) for evidence of both extrapolative and contrarian behavior. Greenwood and Shleifer (2014) provides further evidence of extrapolative expectations from past stock return, also associated with the cyclicality of credit and leverage (López-Salido et al., 2017).

[7]In Appendix C.1, we construct a measure of heterogeneity among professional forecasters and show that this measure of disagreement correlates with stock market turnover, particularly during recessions.

Table 2: I/B/E/S Expectations and Labor

Dependent Variable:		*Payroll*			Number of workers	
Model:	(1)	(2)	(3)	(4)	(5)	(6)
(Intercept)	0.388	0.274	0.364	0.392	0.143	0.170
	(0.247)	(0.223)	(0.241)	(0.097)	(0.195)	(0.052)
lag earning growth expectation	0.133***	0.135***	0.136***	0.061***	0.062***	0.067***
	(0.043)	(0.043)	(0.043)	(0.014)	(0.014)	(0.014)
12-month lag return		0.474***			0.535***	
		(0.119)			(0.054)	
12-month return			0.234*			0.326***
			(0.125)			(0.043)
Fit statistics						
Observations	1797	1797	1797	1797	1797	1797
R2	0.081	0.090	0.083	0.081	0.090	0.083
Adjusted R2	0.038	0.046	0.039	0.038	0.046	0.039

Newey-West standard-errors in parentheses (4 lags)
*Signif. Codes: ***: 0.01, **: 0.05, *: 0.1*

past returns, we aim to capture systematic shocks that might otherwise confound the relationship between expectations and subsequent employment growth.

Table 2 shows that earnings growth expectations predict realized employment growth in both total payroll and the number of workers. In light of Facts 1 and 2—namely, that earnings growth expectations exhibit greater volatility than actual outcomes and that CFOs behave differently from individual investors—these results provide reassuring evidence that firm employment decisions correlate with investor expectations.

Naturally, this correlation alone does not establish causality: it may simply reflect that the surveyed analysts' information overlaps with managers' private signals, even if investors do not directly influence managers. However, Gennaioli et al. (2016) finds that investor expectations continue to affect capital investment even after controlling for managers' survey responses, suggesting a meaningful role for such external perceptions. Unlike Gennaioli et al. (2016), we cannot control for CFO expectations, as we lack sufficient data matches for the employment variables. Motivated by the three facts, we proceed to the model.

3 Model

3.1 Environment

We consider a two-state complete-markets economy with time indexed by $t \in \{0, 1, ..., \}$. The economy is populated by heterogeneous households that differ in their beliefs regarding TFP growth. Households hold (or issue) risk-free bonds and hold (or short-sell) shares of a single representative firm. Differences in beliefs induce a desire to lever up. The firm hires labor one period in advance before TFP is realized. Hiring in advance links labor demand with asset pricing.

The exogenous state. Total factor productivity A_t grows according to a two-state Markov process:

$$\frac{A_{t+1}}{A_t} = x_{t+1}, \tag{1}$$

where $x_{t+1} \in \{x_L, x_H\}, 0 < x_L < x_H$. Transition probabilities from state s to s' are denoted by $\{p_{ss'}\}$.

The firm. The representative firm produces a final good according to $A_{t+1}h_{t+1}^{\alpha}$, where labor h_{t+1} is hired in period t, prior to the realization of x_{t+1}. While firms hire and contract the wage W_{t+1} one period ahead, the wage bill is paid when production is finished.

The firm takes hours at the initial date h_0 as given and hires labor in subsequent periods to maximize its value using a stochastic discount factor (SDF), $\Lambda_{t,t+1}$:

$$Q_t = \max_{h_{t+1}} \mathbb{E}_t \left[\Lambda_{t,t+1} \left(\pi_{t+1} + Q_{t+1} \right) \right]. \tag{2}$$

Q_t denotes the firm value and $\pi_{t+1} \equiv A_{t+1}h_{t+1}^{\alpha} - W_{t+1}h_{t+1}$ denotes the profit (dividend). Expectations are taken with respect to the transition probabilities $\{p_{ss'}\}$ and weighted by $\Lambda_{t,t+1}$. Because markets are complete, state prices are unique. This pins down a unique SDF for any fixed set of beliefs. Hence, there is unanimity regarding the firm's objective among shareholders, as discussed in Section 4. Beliefs affect employment decisions through their impact on the SDF.

Households. There is a finite number of infinite-lived households, indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$ with masses $\{\mu_i\}$, $\sum_i \mu_i = 1$. Household i derives utility from consumption $C_{i,t}$ and disutility from working $h_{i,t}$. They have Epstein-Zin preferences over a GHH consumption-labor composite:

$$V_{i,t} = (1-\beta)U\left(C_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu}\right) + \beta U\left(\mathcal{V}_{i,t}\right), \tag{3}$$

where $V_{i,t}$ denotes the utility level, β the discount factor, and ξ_t controls the labor disutility. $\mathcal{V}_{i,t}$ is the certainty-equivalent of future utility, $\mathcal{V}_{i,t} = \Psi^{-1}\left(\mathbb{E}_{i,t}\left[\Psi\left(U^{-1}(V_{i,t+1})\right]\right)$.

The labor disutility coefficient is indexed by *lagged* productivity, $\xi_t = \xi A_{t-1}$ and acts as a long-run wealth effect—as in Jaimovich and Rebelo (2009), this ensures that hours are stationary. We adopt the functional forms: $U(C) = \frac{C^{1-1/\psi}-1}{1-1/\psi}$ and $\Psi(Z) = \frac{Z^{1-\gamma}-1}{1-\gamma}$, where γ controls risk aversion and ψ the elasticity of intertemporal substitution (EIS).[8] As $U(\cdot)$ is defined over positive values, *net consumption*, $C_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu}$, must be positive.

Household i has beliefs $\{p_{ss'}^i\}$ regarding TFP growth x_{t+1} from state s to s' and forms an expectation $\mathbb{E}_{i,t}$ accordingly. Households are dogmatic, as in Chen, Joslin and Tran (2012) and Simsek (2013b): they *agree to disagree* and do not learn from the views of others. Beliefs about productivity translate into beliefs about earnings and asset prices. Their differences are settled through financial trades.

Household i chooses consumption $C_{i,t}$, hours $h_{i,t}$, firm shares $S_{i,t}$, and risk-free bonds $B_{i,t}$ to maximize (3) subject to a flow budget constraint

$$C_{i,t} + Q_t S_{i,t} + B_{i,t} = R_{e,t} Q_{t-1} S_{i,t-1} + R_{b,t} B_{i,t-1} + W_t h_{i,t}. \tag{4}$$

We denote human wealth by:

$$\mathcal{H}_{i,t} = \mathbb{E}_t\left[\sum_{k=1}^{\infty} \Lambda_{t,t+k}\left(W_{t+k} h_{i,t+k} - \xi_{t+k}\frac{h_{i,t+k}^{1+\nu}}{1+\nu}\right)\right]. \tag{5}$$

Human wealth is the present discounted value of future *net labor income*. Net labor income equals the labor earnings minus labor disutility. The present value is discounted using the SDF $\Lambda_{t,t+k} = \prod_{j=1}^{k} \Lambda_{t+j-1,t+j}$. The SDF is the same as the one used to value firms.

Households face a natural borrowing limit: $R_{e,t}Q_{t-1}S_{i,t-1} + R_{b,t}B_{i,t-1} + W_t h_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu} \geq -\mathcal{H}_{i,t}$, given $S_{i,-1}$ and $B_{i,-1}$, where $R_{b,t}$ denotes the return on the risk-free bond and $R_{e,t} = \frac{Q_t+\pi_t}{Q_{t-1}}$ the return on equity.[9]

[8]CRRA preferences correspond to $\psi = \gamma^{-1}$. Given the endogenous labor supply, γ controls but does not coincide with the risk aversion for lotteries on financial wealth (see, e.g., Swanson 2018).

[9]This borrowing limit corresponds to the maximum households can borrow without violating the non-negativity of net consumption and is therefore never binding.

Example I: Diagnostic expectations. We do not impose restrictions on beliefs. In particular, our formulation is flexible enough to capture different belief dynamics, including extrapolation or underreaction. For example, consider the case of *heterogeneous diagnostic expectations*. Diagnostic expectations correspond to the following belief structure:

$$p^i_{sH} = \underbrace{p_{sH}}_{\text{rational beliefs}} \times \underbrace{\left(\frac{p_{sH}}{p_{-sH}}\right)^{\theta_i}}_{\text{belief distortion}} \times C,$$

where C is a normalizing constant, p_{sH} and p_{-sH} denote the probability of being in the high state next period if the current state is s and not s, respectively, and θ_i is a parameter controlling the degree of diagnosticity. If productivity growth is persistent under rational beliefs, i.e., $p_{HH} > p_{LH}$, then under diagnostic expectations, households overreact to news: households believe it is more likely they would remain in the high (low) state after switching to the high (low) state. If $p_{ss} = \frac{1+\rho}{2}$, as in Mehra and Prescott (1985), then $p^i_{ss} = \frac{1+\rho_i}{2}$, where $\rho_i > \rho$, and the (endogenous) subjective persistence parameter ρ_i is a function of θ_i.

Example II: Optimism and pessimism. Diagnostic expectations capture a form of extrapolation: households are optimistic in the boom and pessimistic in the bust. Alternatively, we could have a persistent degree of optimism/pessimism: $p^i_{sH} = p_{sH} + \Delta_i$ where households with $\Delta_i > 0$ would be optimistic at all states.

Both the diagnostic expectations and persistent optimism formulations are one-parameter belief specifications. In general, beliefs can depend on two parameters, capturing a combination of these two cases: $p^i_{HH} = (1+\rho_i)/2 + \Delta_i$ and $p^i_{LL} = (1+\rho_i)/2) - \Delta_i$.[10]

SDF and Equilibrium. The SDF can be inferred from the process of asset returns through no-arbitrage conditions:

$$1 = \mathbb{E}_t\left[\Lambda_{t,t+1}\frac{\pi_{t+1}+Q_{t+1}}{Q_t}\right], \qquad 1 = \mathbb{E}_t\left[\Lambda_{t,t+1}R_{b,t+1}\right]. \tag{6}$$

A competitive equilibrium is defined next.

Definition 1 (Competitive equilibrium)**.** *Given initial bond holdings and shares $\{B_{i,-1}, S_{i,-1}\}_{i=1}^I$ and hours h_0, a competitive equilibrium is a set of stochastic process for quantities $\{\{C_{i,t}, h_{i,t}, B_{i,t}, S_{i,t}\}_{i=1}^I, h_t\}$ and prices $\{W_t, R_{b,t}, Q_t\}$ such that*

[10]Notice that diagnostic beliefs coincide with rational beliefs in the case productivity growth is iid, $p_{sH} = p_{-sH}$. Our formulation allows for an arbitrary persistence under subjective beliefs, even in this case.

(i) $\{h_{t+1}\}$ *maximizes* (2) *given wages* W_t *and the SDF* $\Lambda_{t,t+1}$.

(ii) $\{C_{i,t}, h_{i,t}, B_{i,t}, S_{i,t}\}$ *maximizes* (3) *subject to* (4) *given prices, for* $i \in \mathcal{I}$.

(iii) Markets for goods, labor, bonds, and shares clear

$$\sum_{i=1}^{I} \mu_i C_{i,t} = A_t h_t^{\alpha}, \qquad \sum_{i=1}^{I} \mu_i h_{i,t} = h_t, \qquad \sum_{i=1}^{I} \mu_i B_{i,t} = 0, \qquad \sum_{i=1}^{I} \mu_i S_{i,t} = 1.$$

We proceed to a characterization.

4 Characterization

We now present a recursive representation of the Markov equilibrium in the exogenous state s and an aggregate endogenous state variable X, to be revealed below. The law of motion of X is given by a function χ to be solved for, i.e., $X' = \chi(X, s, s')$. All aggregate variables are functions of X and s, e.g. $R_{e,t+1} = R_e(X_t, s_t, s_{t+1})$.

The household problem features portfolio and labor-supply choices. In general, this combination complicates obtaining closed-form expressions.[11] Under complete markets and GHH preferences, there is an *as-if result*: the household problem can be recast into the portfolio problem of an endowment economy without reference to labor decisions.

We proceed as follows: First, we reduce the investor's problem into a consumption-savings problem without labor. We then obtain explicit expressions for consumption and portfolio choices under the alternative representation. Finally, we recover the original economy's labor, consumption, and asset prices. All the proofs are in the Appendix.

An equivalent household problem Toward obtaining our as-if result, we observe that under GHH preferences, there are no wealth effects. As a result, all households have the labor supply (derived from the first-order condition for their problems):

$$h_{i,t} = h_t \equiv \left(W_t / \xi_t\right)^{\frac{1}{\nu}}. \tag{7}$$

We define human wealth, $\mathcal{H}_t$, as an asset whose dividend is labor income minus the consumption value of labor disutility, $W_{t+1}h_{t+1} - \xi_{t+1}h_{t+1}^{1+\nu}\left(1+\nu\right)^{-1}$. Since labor is common across agents, so is human wealth, so there is no need for an agent-specific index.

[11] With homothetic preferences and no labor supply, the coefficient of relative risk aversion is independent of wealth. The model with labor supply is analogous to settings with background risk.

The return to human wealth is:

$$R_{h,t+1} \equiv \frac{W_{t+1}h_{t+1} - \xi_{t+1}\frac{h_{t+1}^{1+\nu}}{1+\nu} + \mathcal{H}_{t+1}}{\mathcal{H}_t}.$$

With these objects, we recast the households' flow budget constraint:

$$\tilde{C}_{i,t} + Q_t S_{i,t} + B_{i,t} + \mathcal{H}_t = R_{e,t}Q_{t-1}S_{i,t-1} + R_{b,t}B_{i,t-1} + R_{h,t}\mathcal{H}_{t-1} \equiv N_{i,t} \tag{8}$$

where $\tilde{C}_{i,t} \equiv C_{i,t} - \xi_t \frac{h_t^{1+\nu}}{1+\nu}$ defines *net consumption* and $N_{i,t}$ defines the individual's *total wealth*. The investor's total wealth, $N_{i,t}$, is the sum of financial and human wealth. Total wealth funds the terms on the left-hand side: current net consumption and the future holdings of stocks, bonds, and human wealth.

The household's objective is to maximize net consumption. From the modified budget constraint, (8), it is as if households hold portfolios of bonds and two risky assets: stocks and human wealth. However, because the returns to stocks and human wealth are correlated, only the exposure to risk in total wealth matters, regardless of the portfolio composition. We can then work with the sum of stocks and human wealth as a single risky asset, which we call the *surplus claim*. The price of the surplus claim, is $A_{t-1}P_t$, where P_t is given by

$$P_t = \mathbb{E}_t \left[\sum_{k=0}^{\infty} \frac{\Lambda_{t,t+k}}{A_{t-1}} \left(A_{t+k} h_{t+k}^{\alpha} - \xi_{t+k} \frac{h_{t+k}^{1+\nu}}{1+\nu} \right) \right].$$

The dividend of the surplus claim is the social surplus: the sum of output minus labor disutility—both measured in terms of goods. We denote the return on the surplus claim by $R_r(X, s, s')$. The household's problem can be written in terms of net consumption and the surplus claim:

Problem 1. *The modified household's problem is:*

$$V_i(N, X, s) = \max_{\tilde{C}_i, \omega_i} (1-\beta) U\left(\tilde{C}_{i,t}\right) + \beta U\left(\mathcal{V}_i(N, X, s)\right), \tag{9}$$

subject to:

$$N' = R_{i,n}(X, s, s')(N - \tilde{C}_i), \qquad R_{i,n}(X, s, s') = (1-\omega_i)R_b(X, s) + \omega_i R_r(X, s, s') \tag{10}$$

where $\mathcal{V}_i(N, X, s) = \Psi^{-1}\left(\mathbb{E}_i\left[\Psi\left(U^{-1}(V_i(N', X', s')\right) | N, X, s\right]\right)$, $N' \geq 0$.

In this problem, the household only chooses net consumption $\tilde{C}_i$ and his holdings (risk exposure) of the surplus claim ω_i. We recover $\{C_{i,t}, h_{i,t}, S_{i,t}, B_{i,t}\}$ in the original problem through:

$$W_t = \xi_t h_t^{\nu}, \quad C_{i,t} = \tilde{C}_{i,t} + \frac{\xi_t h_t^{1+\nu}}{1+\nu},$$

$$Q_t S_{i,t} = \frac{\omega_{i,t}}{\omega_{e,t}}(N_{i,t} - \tilde{C}_{i,t}) - \frac{\omega_{h,t}}{\omega_{e,t}}\mathcal{H}_t, \quad B_{i,t} = N_{i,t} - \mathcal{H}_t - Q_t S_{i,t} - \tilde{C}_{i,t},$$

where $\omega_{k,t}$ satisfies $R_{k,t} = \omega_{k,t} R_{r,t} + (1 - \omega_{k,t}) R_{b,t}$, for $k \in \{h_i, e\}$.

Complete markets and GHH preferences are key to this as-if result. With complete markets, a combination of bonds and the surplus claim yields the same payoffs as human wealth. Thus, the investor's problem is akin to a problem where human wealth is traded and could be sold at time zero, as any other financial asset.

Euler equations and portfolio share. As any model with homothetic preferences and linear budget sets, the solution to the modified household problem admits aggregation and portfolio separation. The characterization of the modified problem is standard, and it is provided in Appendix A.2.

The Euler equations for investor $i \in \mathcal{I}$ are given by:

$$1 = \mathbb{E}_i \left[\Lambda_i(X, s, s') R_j(X, s, s') \right], \tag{11}$$

for $j \in \{r, b\}$, where $\Lambda_i(X, s, s')$ denotes the investor's SDF.

These Euler equations yield the portfolio weights ω_i. Since (11) holds for every household, there are an equal number of equations as states for each. Hence, all agents discount payoffs in different states by the same amount, regardless of their beliefs.

The no-arbitrage conditions (6) coincide with the Euler equations if we replace individual beliefs and discount factors with objective probabilities and an endogenous economy-wide SDF. Any individual SDF can be recovered from:

$$\Lambda_i(X, s, s') = \frac{p_{ss'}}{p^i_{ss'}} \Lambda(X, s, s').$$

Thus, i's SDF is the economy-wide SDF, scaled by the ratio of objective to subjective probabilities. In turn, given the objective probabilities, $\Lambda(X, s, s')$ can be recovered from observed asset prices, by inverting the no-arbitrage conditions (6),

$$\Lambda(X, s, s') = \frac{1}{p_{ss'}} \frac{|R^e_r(X, s, -s')|}{\Delta R_r(X, s)}.$$

The SDF depends on the excess return of the risky asset, $R^e_r(X, s, s') \equiv$

$R_r(X,s,s')/R_b(X,s) - 1$, relative to the difference in realized returns $\Delta R_r(X,s) \equiv R_r(X,s,H) - R_r(X,s,L)$, which is proportional to the volatility of returns.[12]

So far we have observed that investors agree on the value of one unit of consumption state by state, despite disagreeing on their likelihood. For this, investors must be differently exposed to risk. For instance, for a log-utility investor, the portfolio share is approximately given by Merton (1969) formula derived in a continuous-time setting:[13]

$$\omega_i(X,s) \approx \frac{\mathbb{E}_i[R_r^e(X,s,s')]}{Var_i[R_r(X,s,s')]}. \tag{12}$$

Since the expected excess return $\mathbb{E}_i[R_r^e(X,s,s')]$ increases with p_{sH}^i, more optimistic investors hold larger risky surplus-claim positions. Heterogeneity in beliefs, thus, translates into heterogeneity in portfolio shares. As a result, agents are differently exposed to risk, and their wealth shares fluctuate, opening a feedback into the economy-wide SDF. Next, we link this SDF to the firm's hiring decisions.

Firm's problem. The firm's first-order condition is:

$$\mathbb{E}_t\left[\Lambda_{t,t+1}\left(\alpha A_{t+1}h_{t+1}^{\alpha-1} - W_{t+1}\right)\right] = 0.$$

Given the timing of hires, the marginal product of labor will typically deviate from the wage, i.e., there is ex-post a non-zero labor wedge. The firm's labor demand satisfies the condition:

$$\alpha\mathcal{L}_t h_{t+1}^{\alpha-1} = w_{t+1}, \tag{13}$$

where $w_{t+1} \equiv W_{t+1}/A_t$ denotes a TFP-detrended wage. The term $\mathcal{L}_t$ acts as a shifter of the labor demand curve. Hence, we dub the risk-neutral expectation of productivity growth, $\mathcal{L}_t$, *the labor demand factor* (LDF):

$$\mathcal{L}_t = \mathbb{E}_t\left[\frac{\Lambda_{t,t+1}}{\mathbb{E}_t[\Lambda_{t,t+1}]}x_{t+1}\right].$$

Note that $\frac{p_{ss'}\Lambda_{t,t+1}}{\mathbb{E}_t[\Lambda_{t,t+1}]}$ is the risk-neutral probability at time t. Hence, the LDF is consistent with a zero expected labor wedge under the risk-adjusted expectations.

Recall that at t, the firm chooses future labor, h_{t+1}. Thus, current labor h_t depends on the lagged LDF. Therefore, $\mathcal{L}_{t-1}$ is an endogenous aggregate state variable with law of

[12] Notice that $[R_r(X,s,s') - R_b(X,s)]/\Delta R_r(X,s) = [R_e(X,s,s') - R_b(X,s)]/\Delta R_e(X,s)$, so we can use stocks or the surplus claim interchangeably.

[13] In Appendix A.3 we solve for $\omega_i(X,s)$ in the general case and derive this approximation.

motion:

$$\mathcal{L}'(X,s) = \frac{p_{sL}\Lambda(X,s,L)}{p_{sL}\Lambda(X,s,L) + p_{sH}\Lambda(X,s,H)} x_L + \frac{p_{sH}\Lambda(X,s,H)}{p_{sL}\Lambda(X,s,L) + p_{sH}\Lambda(X,s,H)} x_H.$$

Labor Market Equilibrium. We combine the labor supply and demand schedules, equations (7) and (13), to obtain the labor market equilibrium:

$$h(\mathcal{L}) = \left(\frac{\alpha\mathcal{L}}{\xi}\right)^{\frac{1}{1+\nu-\alpha}}, \qquad w(\mathcal{L}) = \xi\left(\frac{\alpha\mathcal{L}}{\xi}\right)^{\frac{\nu}{1+\nu-\alpha}}. \tag{14}$$

Given that hours and wages are determined solely by the LDF, realized profits also depend on the LDF and realized productivity growth:[14]

$$\pi(\mathcal{L},s) = x_s\left(\frac{\alpha\mathcal{L}}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\left[1 - \alpha\frac{\mathcal{L}}{x_s}\right],$$

where $\pi(\mathcal{L},s)$ denotes the firm's profits in state s detrended by lagged TFP. Because labor is chosen in advance, profits may be negative unless $x_L > \alpha x_H$.[15]

Relating the LDF to the SDF. Let $\mathbb{E}_s[z_{s'}]$ and $\sigma_s[z_{s'}]$ respectively denote the mean and the standard deviation of a variable $z_{s'}$ conditional on s under the objective probabilities $p_{ss'}$. The next proposition presents a convenient representation of the LDF:

Proposition 1. *The risk-neutral expectation of next period productivity growth is given by*

$$\mathcal{L}'(X,s) = \mathbb{E}_s[x_{s'}] - \underbrace{\frac{\mathbb{E}_s[R_r^e(X,s,s')]}{\sigma_s[R_r^e(X,s,s')]}}_{\textit{price of risk}} \underbrace{\sigma_s[x_{s'}]}_{\textit{quantity of risk}}. \tag{15}$$

where the Sharpe ratio is

$$\frac{\mathbb{E}_s[R_r^e(X,s,s')]}{\sigma_s[R_r^e(X,s,s')]} = \frac{\sigma_s[\Lambda(X,s,s')]}{\mathbb{E}_s[\Lambda(X,s,s')]}.$$

Proposition 1 shows that the LDF depends on two components: First, a *quantity of risk*, the underlying risk in productivity growth, $\sigma_s[x_{s'}]$. Second, a *price of risk*, the required market compensation per quantity of risk. This price of risk is proportional to the volatility of the SDF. Thus, the LDF and the SDF are intimately connected.

[14] As hours and wages depend only on $\mathcal{L}$, we simplify notation and write $h(\mathcal{L})$ and $w(\mathcal{L})$ instead of the more general notation $h(X,s)$ and $w(X,s)$. Similarly, we write profits as $\pi(\mathcal{L},s)$ instead of $\pi(X,s)$.

[15] Given that the highest possible value for $\mathcal{L}$, the last period expected TFP growth under the risk-neutral measure is x_H, profits are positive as long as $x_s > \alpha\mathcal{L}$. The condition $x_L > \alpha x_H$ guarantees this is the case.

The labor market equilibrium showcases that variation in the SDF, which provokes variations in the LDF, also provokes fluctuations in labor. This feature connects the equity premium, equity volatility, and labor volatility puzzles. Intuitively, as investors become more willing to bear risk, expected excess returns are low, and risk-adjusted probabilities put more weight on higher TFP growth. As a result, the firm takes greater risks by hiring more. As we know from asset-pricing—e.g. Hansen and Jagannathan (1991) and Cochrane and Hansen (1992)—not only is the volatility of the SDF relatively large, but it is associated with substantial movements in expected returns—see e.g. Cochrane (2011). Therefore, success in generating large employment fluctuations is tied to obtaining a large and volatile equity premium because the labor volatility and equity volatility puzzles are the same puzzle here.

The Operating Leverage Channel. If labor could be contracted after productivity is realized, the labor share would be constant as in a standard neoclassical setting. Thus, firms would reduce their labor demand in response to a negative productivity shock such that revenues, wages, and profits would all fall proportionally. In contrast, the timing of hiring plays an essential role in shaping the dividend process and the connection between asset prices and labor volatility in our setting. Here, dividend (or profit) growth is:

$$\frac{x_s \pi(\mathcal{L}', s')}{\pi(\mathcal{L}, s)} = x_s \frac{x_{s'} - \alpha \mathcal{L}'}{x_s - \alpha \mathcal{L}} \left(\frac{\mathcal{L}'}{\mathcal{L}}\right)^{\frac{\alpha}{1+\nu-\alpha}}.$$

We can observe, by setting $\alpha = 0$, i.e., in the endowment limit, dividend growth follows the productivity growth. Hence, dividend volatility coincides with aggregate consumption volatility. By contrast, dividends are endogenously riskier than aggregate consumption.

The fact that quasi-fixed factors amplify the volatility of profits is known as the *operating leverage channel*.[16] Formally, the conditional volatility of dividend growth can be written as

$$\underbrace{\sigma_s \left[\frac{x_s \pi(\mathcal{L}', s')}{\pi(\mathcal{L}, s)}\right]}_{\substack{\text{dividend growth} \\ \text{volatility}}} = \underbrace{\frac{x_s}{x_s - \alpha \mathcal{L}}}_{\substack{\text{operating} \\ \text{leverage}}} \times \underbrace{\sigma_s \left[\frac{x_{s'} h(\mathcal{L}')^{\alpha}}{h(\mathcal{L})^{\alpha}}\right]}_{\substack{\text{consumption growth} \\ \text{volatility}}}. \tag{16}$$

The literature defines operating leverage as the ratio of revenues minus current variable costs—zero here—to profits. This corresponds to the term $\frac{x_s h(\mathcal{L})^{\alpha}}{x_s h(\mathcal{L})^{\alpha} - w(\mathcal{L}) h(\mathcal{L})} = \frac{x_s}{x_s - \alpha \mathcal{L}} > 1$. Equation (16) shows how operating leverage endogenously makes dividends more

[16] The relationship between risk and operating leverage initially appears in Lev (1974).

volatile than consumption, something that does not occur in the endowment limit but is consistent with the evidence (see, e.g., Campbell 2003).

Operational leverage not only increases the volatility of dividend growth, but also induces mean reversion in dividends. Notice that even if TFP growth is iid, and the LDF is constant, expected dividend growth is time varying:

$$\mathbb{E}\left[\frac{x_s \pi(\mathcal{L}', s')}{\pi(\mathcal{L}, s)}\right] = x_s \frac{\mathbb{E}\left[x_s\right] - \alpha\mathcal{L}}{x_s - \alpha\mathcal{L}}.$$

Thus, upon a bad realization of TFP, and therefore dividends, we expect dividends to recover.

Given that operating leverage is increasing in $\mathcal{L}$, this channel links the volatility of dividends to the LDF, which ultimately responds to fluctuations in beliefs. This feature is important to generate an endogenous risk buildup from waves of optimism.

The timing of hires here is a simple way of capturing this operating leverage channel. It delivers labor costs that are smoother than productivity and a countercyclical firm-level labor share, which are necessary conditions for the operating leverage channel.[17] Donangelo et al. (2019) provides direct evidence that these two conditions are observed in cross-sectional and time-series data. Moreover, they show that the sensitivity of profits to productivity shocks is increasing in the labor share, consistent with the mechanism embedded in Equation (16).

Discussion: the firm's objective. In any model where labor is chosen in advance, hiring is risky. Hence, we must specify the SDF used by firms. This raises the question of what an appropriate SDF should be. Under complete markets, *any* pair of beliefs/SDF that correctly price stocks and bonds leads to the same firm value. In turn, the firm value is maximized using the firm's first-order condition under the corresponding economy-wide SDF. Regardless of beliefs, every shareholder agrees on the hiring decision that maximizes the firm's value.

While we adopted the convention that firms compute expectations using objective probabilities, any managerial belief would deliver the same labor choice if its objective is to maximize shareholder value.[18]

[17] Labor being a (quasi)-fixed factor, as in our setting, is unnecessary for the operating leverage channel to be active. A similar mechanism is operative in models with implicit contracts (Danthine and Donaldson, 2002), labor adjustment costs (Belo, Lin and Bazdresch, 2014), and wage rigidities (Favilukis and Lin, 2016).

[18] How to determine the firms' objectives away from complete markets is still a matter of discussion, e.g., see Geanakoplos, Magill, Quinzii and Dreze (1990).

Markov equilibrium In addition to $\mathcal{L}$, the *wealth distribution* is also an endogenous state variable. We define the share of the wealth of investor $i \in \mathcal{I}$ as:

$$\eta_{i,t} \equiv \frac{\mu_i N_{i,t}}{\sum_{j=1}^{I} \mu_j N_{j,t}},$$

which evolves according to

$$\eta_i'(X,s,s') = \frac{\eta_i R_{i,n}(X,s,s')(1-c_i(X,s))}{\sum_{j=1}^{I} \eta_j R_{j,n}(X,s,s')(1-c_j(X,s))}. \tag{17}$$

Given $\mathcal{L}$ and $\{\eta_i\}_{i=1}^{I-1}$, and the realization of TFP growth, we can characterize all aggregate variables. We stack the endogenous state variables in $X \equiv \left(\mathcal{L}, \{\eta_i\}_{i=1}^{I-1}\right)$ and define a Markov equilibrium in (X,s).

Definition 2 (Markov Equilibrium)**.** *A Markov equilibrium in (X,s), with a law of motion for X given by (15) and (17), is the set of functions: price of surplus claim $P(X,s)$, interest rate $R_b(X,s)$, labor hours $h(\mathcal{L})$, wages $w(\mathcal{L})$, wealth multiplier $v_i(X,s)$ and policy functions $(c_i(X,s), \omega_i(X,s))$, for $i \in \{1,\ldots,I\}$, such that: (I) The value function satisfies (9). The consumption-wealth ratio is given by (A.10) in Appendix A.2, and the portfolio share is given by (A.24) in Appendix A.3. (II) Hours and wages satisfy (14). (III) The goods and the risky asset markets clear:*

$$\sum_{i=1}^{I} \eta_i c_i(X,s) = \frac{x_s h(\mathcal{L})^{\alpha} - \xi \frac{h(\mathcal{L})^{1+\nu}}{1+\nu}}{P(X,s)}, \qquad \sum_{i=1}^{I} \tilde{\eta}_i(X,s)\omega_i(X,s) = 1, \tag{18}$$

where $\tilde{\eta}_i(X,s) \equiv \frac{\eta_i(1-c_i(X,s))}{\sum_{j=1}^{I} \eta_j (1-c_j(X,s))}$.

For the rest of the paper, we work directly with this Markovian representation.

5 Analytic Solution: Log Utility Case ($\psi = \gamma = 1$)

Here, we consider log utility. This case is special because, as we show next, the LDF is only a function of market beliefs.

For this case, we derive a law of motion for market beliefs, which renders sharp predictions about how belief heterogeneity amplifies business cycles. We return to a general class of preferences in the quantitative evaluation.

A Demand-Supply Representation. Under log utility, the consumption-wealth ratio and portfolio shares are given by:

$$c_i(X) = 1 - \beta, \qquad \omega_i(X,s) = \frac{1}{\Delta R_r(X,s)} \left[\frac{p^i_{sH}}{p_{sH}\Lambda(X,s,H)} - \frac{p^i_{sL}}{p_{sL}\Lambda(X,s,L)} \right].$$

A constant consumption-wealth ratio implies that the price-dividend ratio of the surplus claim is constant, as commonly found in settings with log utility. Notice that this will not be the case for the equity claim, given dividends are more volatile than the surplus.[19]

Combined with the market clearing conditions, (18), they yield the risk-free rate and the risk premium:

Proposition 2 (Risk-free rate and risk premium)**.** *Given $\mathcal{L}'(X,s)$*

(i) The risk-free rate is

$$R_b(X,s) = \left(1 - \frac{\alpha}{1+\nu}\right) \frac{x_s}{\beta} \frac{\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} \mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{19}$$

(ii) The conditional risk premium is given by

$$\mathbb{E}_s[R^e_r(X,s,s')] = \frac{1}{1-\frac{\alpha}{1+\nu}} \frac{\mathbb{E}_s[x_{s'}] - \mathcal{L}'(X,s)}{\mathcal{L}'(X,s)}. \tag{20}$$

That is, $\mathbb{E}_s[R^e_r(X,s,s')]$ is decreasing in $\mathcal{L}'(X,s)$.

Unlike when hiring occurs after productivity is realized, here, labor demand and asset prices are determined jointly. Proposition 2 shows that the risk-free rate and the risk premium can be deduced from the current productivity growth, x_s, and the lagged and current period's LDF, $\mathcal{L}$ and $\mathcal{L}'(X,s)$. The risk free rate, $R_b(X,s)$, is increasing in $\mathcal{L}'(X,s)$ and decreasing in x_s whereas the risk premium $\mathbb{E}_s[R^e_r(X,s,s')]$ moves in the opposite direction of $\mathcal{L}'(X,s)$. The formula reveals that ceteris paribus, periods of low-risk premia are associated with high labor demand. Of course, the LDF is endogenous; it is a function of market beliefs.

Next, we solve for the equilibrium LDF by translating the market clearing conditions into a demand and supply system, where the quantity variable is the risk in the

[19] The surplus and the dividend are differentially exposed to aggregate risk. Similar to models with multiple sectors, movements in interest rate cannot simultaneously offset changes in growth rates for all assets. Likewise, the price-earnings ratio will not be constant in models where the interest rate does not track the natural rate.

economy's aggregate surplus claim, and the price variable is the LDF. The equilibrium of this demand and supply system yields the equilibrium LDF and, thus, pins down all asset prices and labor demand. This demand and supply of risk representation is a convenient transformation of the asset-market clearing conditions (18): multiply both sides of the condition for risky assets by $\sigma_s[R_r(X,s,s')]$, and use that $\sigma_s[R_{i,n}(X,s,s')] = \omega_i(X,s)\sigma_s[R_r(X,s,s')]$, to obtain:

$$\underbrace{\sum_{i=1}^{I}\eta_i\sigma_s[R_{i,n}(X,s,s')]}_{\text{demand for risk}} = \underbrace{\sigma_s[R_r(X,s,s')]}_{\text{supply of risk}}.$$

The left of the expression is the *demand for risk*. The demand for risk corresponds to the volatility of the households' total wealth needed to guarantee asset market clearing at given asset prices. The right-hand side represents the *supply of risk*. The supply of risk is the volatility of the economy's surplus claim induced by the firm's labor demand at given asset prices. The following proposition expresses the demand and supply of risk as a function of the LDF.

Proposition 3 (The demand and supply of risk). *Suppose $x_L > \alpha x_H$. Then,*

(i) The supply of risk is

$$\sigma_s[R_r(X,s,s')] = \frac{1}{\beta}\frac{x_s}{x_s - \frac{\alpha}{1+\nu}\mathcal{L}} \times \frac{\sigma_s[x_{s'}]\mathcal{L}'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{\mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}}}. \tag{21}$$

(ii) The demand for risk is

$$\sum_{i=1}^{I}\eta_i\sigma_s[R_{i,n}(X,s,s')] = \sigma_s[x_{s'}]R_b(X,s)\left[\frac{\overline{p}^m_{sH}(X)}{\mathcal{L}'(X,s)-x_L} - \frac{\overline{p}^m_{sL}(X)}{x_H-\mathcal{L}'(X,s)}\right], \tag{22}$$

where $\overline{p}^m_{ss'}(X) \equiv \sum_{i=1}^{I}\eta_i p^i_{ss'}$.

Proposition 3 implies that the supply of risk increases while the demand for risk decreases with the LDF, as shown in Figure 1. As with any demand system, the equilibrium falls at the intersection of both curves. The supply of risk increases with the LDF. Recall that with log utility, the price-dividend ratio of the surplus claim is constant. Therefore, its return volatility is proportional to the volatility of the surplus growth. Notice that the volatility of the surplus increases with the LDF due to the operating leverage channel. As firms hire more labor, the surplus, like profits, is more exposed to productivity shocks, leading to higher volatility.

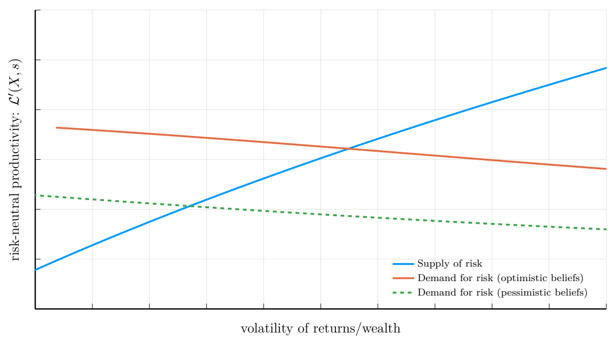

Figure 1: Market for risky assets

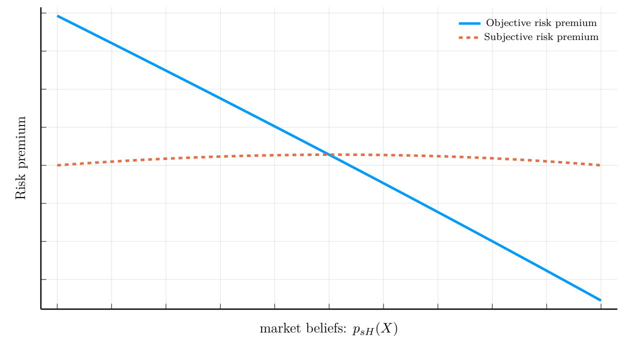

Figure 2: Beliefs and risk premia

Whereas the supply of risk increases with the LDF, the demand for risk is decreasing in the LDF. As shown in Proposition 2, the risk premium is inversely related to $\mathcal{L}'(X, S)$, so investors are more willing to hold risky assets when the risk premium is high. The demand for risk is itself a function of *market beliefs*, a weighted average of investors' beliefs, $\overline{p}^m_{ss'}(X)$. As market beliefs become more optimistic, investors are willing to hold more risky assets for a given risk premium.

Figure 1 also illustrates how we can exploit this demand system representation to explain the effects of changes in market beliefs: When market beliefs are pessimistic, there is a decline in the demand for risk, which leads to a decrease in the equilibrium LDF and, ultimately, a drop in hours. The following corollary formally demonstrates this by presenting the solution to the equilibrium LDF as a function of market beliefs:

Corollary 1 (Labor demand factor)**.** *The risk-neutral expectation of productivity growth $\mathcal{L}'(X, s)$ corresponds to the smallest real root of the quadratic equation:*

$$\mathcal{L}'(X, s) = \Gamma(X) - \sqrt{\Gamma(X)^2 - \frac{1+\nu}{\alpha} x_L x_H}.$$

where $\Gamma(X)$ is a function of market beliefs:

$$\Gamma(X) \equiv \frac{\overline{p}^m_{sH}(X)}{2} \left(x_L \frac{1+\nu}{\alpha} + x_H \right) + \frac{\overline{p}^m_{sL}(X)}{2} \left(x_L + x_H \frac{1+\nu}{\alpha} \right).$$

The corollary shows that equilibrium LDF is a function of market beliefs, which are solely a function of the wealth distribution. Thus, wealth distribution affects asset prices and hiring only to the extent it affects market beliefs.

Beliefs and Risk Premia. We concluded that fluctuations in market beliefs generate movements in risk premia and labor demand. But why does optimism lead to changes in risk *premia*? Consider first the response of the *subjective risk premium*. With a representative investor, the subjective risk premium depends solely on risk perceptions:

$$\mathbb{E}_{i,s}[R^e_r(X,s,s')] \approx Var_{i,s}[R^e_r(X,s,s')].$$

Hence, only changes in second moments affect the subjective risk premium. In contrast, the objective risk premium also responds to differences in first moments:

$$\mathbb{E}_s[R^e_r(X,s,s')] = \mathbb{E}_{i,s}[R^e_r(X,s,s')] - \frac{\mathbb{E}_{i,s}[x_{s'}] - \mathbb{E}_s[x_{s'}]}{(1-\frac{\alpha}{1+\nu})\mathcal{L}'}.$$

In particular, when investors become optimistic, they expect higher future dividends, driving up prices. From the point of view of a rational investor, higher prices imply lower returns going forward. Figure 2 shows how objective and subjective risk premia respond to market beliefs, $\overline{p}^m_{sH}(X)$. While an increase in optimism compresses the objective risk premium, it has a muted effect on the subjective risk premium, consistent with the evidence in, e.g., De La O and Myers (2021) and Nagel and Xu (2023).

From the LDF to the Evolution of Market Beliefs. Corollary 1 shows that given market beliefs, we can compute the LDF and asset prices. Another convenient property of log preferences is that the law of motion of market beliefs and wealth also has an analytic representation:

Proposition 4 (Dynamics of wealth and market beliefs)**.** *Let $\psi = \gamma = 1$. Then,*

(i) the wealth share of investor $i \in \mathcal{I}$ evolves as:

$$\eta'_i(X,s,s') = \eta_i \frac{p^i_{ss'}}{\overline{p}^m_{ss'}(X)}, \tag{23}$$

(ii) market beliefs are:

$$\overline{p}^m_{s'H}(X') = \sum_{i=1}^{I} \eta_i \frac{p^i_{ss'}}{\overline{p}^m_{ss'}(X)} p^i_{s'H}. \tag{24}$$

Proposition 4 shows that the wealth of household i increases when it assigns a greater likelihood to the realized state than market beliefs.

Next, we present a taxonomy of belief types. We exploit the demand-supply representation to uncover how different belief structures classified according to our taxonomy impact business cycles.

5.1 Belief Taxonomy and Business Cycle Properties

Although an ample literature documents departures from full-information rational expectations models (Greenwood and Shleifer, 2014; Coibion and Gorodnichenko, 2015), there is no consensus on how belief dynamics deviate from rational expectations.[20] This section aims to classify belief structures and show how each has different business cycle implications. We provide the following taxonomy:

Definition 3 (Taxonomy of beliefs)**.** *Household i is optimistic (pessimistic) relative to a benchmark belief o at state s if $p^i_{sH} > p^o_{sH}$ ($p^i_{sH} > p^o_{sH}$). We further classify belief structures:*

*(i) Beliefs are **rank preserving** if i is optimistic or pessimistic relative to o, $\forall s$.*

*(ii) Beliefs are **rank alternating** if i is optimistic relative to o in one state, but i is pessimistic relative to o in the other state.*

Under this definition, the index o represents a belief benchmark: it can be another household's belief or rational beliefs. We exploit this taxonomy to show that different belief structures induce different business cycle amplification properties, both in terms of the amplitude and the phase of the business cycle.

Homogeneous Beliefs. With homogeneous beliefs, we have a representative investor. Hence, the model lacks internal propagation: the LDF only depends on the current state, x_s. Furthermore, if investors believe that productivity growth is iid, $p_{LH}(X) = p_{HH}(X)$, then the LDF and, thus, hours are constant.

Away from iid beliefs about growth, beliefs may amplify or dampen cycles relative to a rational expectations benchmark. Figure 3 illustrates this point. The figure simulates four periods of recession within a twenty-period interval for five types of homogeneous belief classifications: rational expectations ($p^i_{ss'} = p_{ss'}$ for all s, s'); two rank-preserving cases: always optimistic ($p^i_{sH} > p_{sH}$ for all s) or consistently pessimistic ($p^i_{sH} < p_{sH}$ for all s); and two rank-alternating cases: extrapolative ($p^i_{ss'} > p_{ss'}$ all $s = s'$) defined as optimistic during booms but pessimistic at busts, and intrapolative ($p^i_{ss'} < p_{ss'}$ all $s = s'$) defined as pessimistic at booms but pessimistic at busts.

Optimistic beliefs amplify expansions among rank-preserving beliefs but dampen recessions relative to rational expectations. The converse is true about pessimistic beliefs:

[20] Some studies emphasize over-extrapolation (Bordalo, Gennaioli, Ma and Shleifer, 2020b; Fuster, Laibson and Mendel, 2010), as in models with diagnostic expectations (Bordalo, Gennaioli and Shleifer, 2018b) or fading memory (Nagel and Xu, 2022), others capture under-reaction, such as sticky information (Mankiw and Reis, 2010), cognitive discounting (Gabaix, 2019), and level-k thinking (Farhi and Werning, 2019).

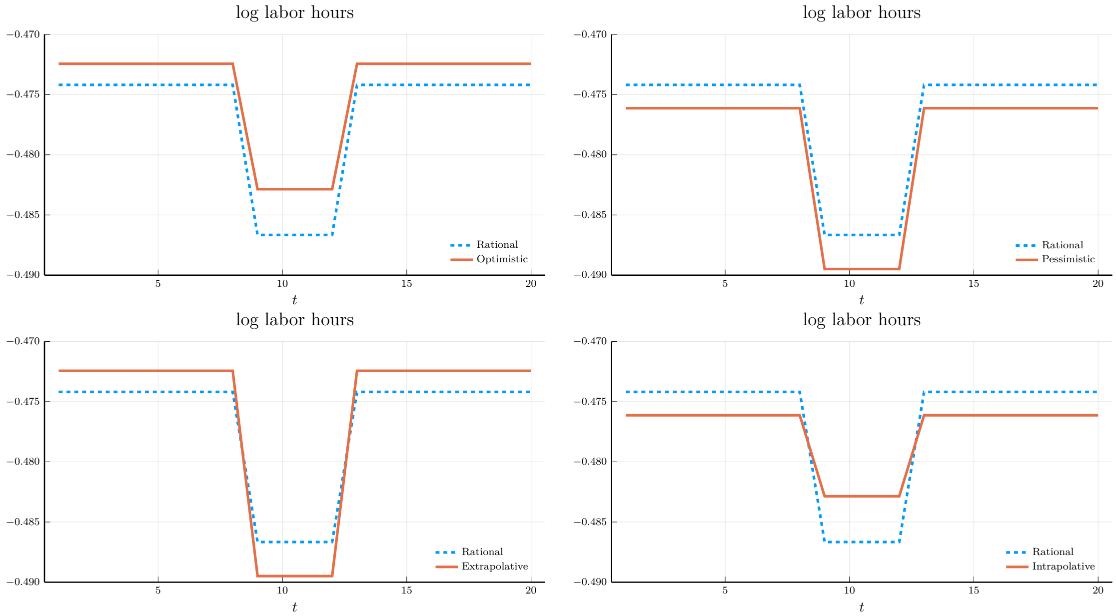

Figure 3: Homogeneous beliefs

Note: examples of cycles (four periods of bad shocks with sixteen periods of expansions) with homogeneous beliefs: rational, always optimistic (top-left panel), always pessimistic (top-right panel), extrapolative (bottom-left panel), and intrapolative (bottom-right panel) beliefs.

recessions are amplified and expansions dampened. Rank-preserving beliefs have state-dependent amplification properties. Rank-alternating beliefs operate differently. Recall that extrapolative beliefs are optimistic in good but pessimistic in bad states. Under extrapolative beliefs, the cycle's amplitude is magnified. Analogously, intrapolative beliefs produce the opposite: they reduce the cycle's amplitude. In conclusion, extrapolative beliefs are the belief structure that amplifies the cycle relative to rational expectations.

Heterogeneous Beliefs. Next, we investigate the internal propagation of the model. As we explained earlier, internal propagation is induced by belief heterogeneity through its effect on market beliefs. From Proposition 4, we deduce that $\eta_i'(X, s, H) > \eta_i$ if and only if investor i is optimistic at (X, s) relative to market beliefs. We know that optimists' wealth increases after good shocks and decreases after bad shocks. This observation is key to understanding the internal propagation of the economy:

Corollary 2. *If beliefs are heterogeneous, then:*

- *As the current state persists, the LDF increases (decreases) with time if the current state is high (low) growth.*

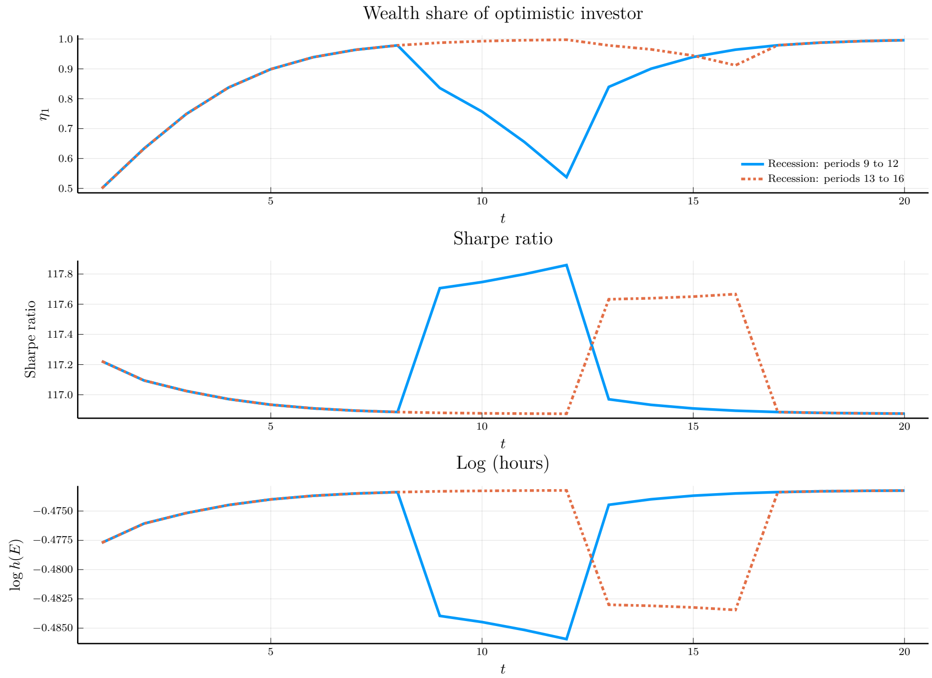

Figure 4: Heterogeneous beliefs: optimistic vs pessimistic

Note: examples of cycles (four periods of bad shocks with sixteen periods of expansions) with heterogeneous beliefs: $I = 2$ investors, always optimistic and always pessimistic. The top panel shows the wealth share of the optimistic investor, the middle panel shows the Sharpe ratio (under objective beliefs), and the bottom panel shows log labor hours.

- *Consider an initial state (X, H) and a first switch from $s = H$ to $s' = L$ at some future date. Then,*
  - *If beliefs are rank-preserving, the later the date of the switch in the state, the lower the reduction in output (the longer the boom, the lesser the bust).*
  - *If beliefs are rank-alternating, the later the date of the switch in the state (the longer the boom), the lower the reduction in output (the longer the boom, the greater the bust).*

Proof. See Appendix A.8. □

With belief heterogeneity, the economy evolves even without changes in TFP growth. The reason is the joint evolution of wealth and market beliefs. From Proposition 4, optimists' wealth share increases while the economy is in a boom. Market beliefs become more optimistic relative to rational expectations as optimists accumulate wealth. This force leads to increased labor demand throughout the boom. The converse is true during a low growth phase: pessimists accumulate wealth and market beliefs tilt toward greater pessimism.

The connection between the length of cycles and their amplitude (the drop in output after a change in state) crucially hinges on whether beliefs are rank-preserving or rank-alternating. Rank-preserving beliefs attenuate the subsequent decline in TFP growth in a recession that follows a more prolonged boom. The relationship between the high-growth phase's duration and the recession's severity is reversed with rank-alternating beliefs.

Figure 4 aids us in explaining this pattern. The figure shows simulations of business cycles in the case of rank-preserving beliefs—the figure uses $I = 2$, where investor 1 is optimistic, and investor 2 is pessimistic in both states. The figure shows the evolution of the wealth share of the optimist, the Sharpe ratio, and log hours. The lower panel shows that the drop in hours is smaller as the economy remains longer in the high state. As optimists accumulate wealth during the high-growth phase. Their accumulated wealth implies that when the economy switches states, optimists arrive at the bad state with more wealth, the longer the boom. Since optimists remain optimistic during downturns when beliefs are rank-preserving, their greater wealth makes market beliefs more optimistic during crashes. This rank-preservation attenuates the increase in risk premia and the decline in hours.

Figure 5 is the analogue figure for rank-alternating beliefs—investor 1 is rational and investor 2 is extrapolative. The top panel shows that the wealth share of the rational investor declines during the boom phase. This means that extrapolative investors are getting wealthier as the boom lasts longer. Consequently, extrapolative households have a larger wealth share during busts, precisely when they become pessimistic. As a result, market beliefs become more pessimistic after the economy transitions to a low state when beliefs are extrapolative. Belief extrapolation amplifies the increase in risk-premia and the decline in hours.

Belief Taxonomy and Trading Volume. Our taxonomy also has implications for trading dynamics. As is typical in models of belief heterogeneity, trading volume increases with the level of belief disagreement. More importantly, Appendix B shows that the relationship between trading volume and belief disagreement is amplified during economic downturns under *rank-alternating* beliefs, not under *rank-preserving* beliefs. The intuition is as follows: optimistic investors lose wealth and rebalance their portfolios by selling shares when the economy enters a recession. If their beliefs are rank-alternating, they also become pessimistic, which induces additional selling, further increasing trading volume. Appendix C provides empirical support for this prediction. We find that not only does stock market turnover rise with belief disagreement, but this relationship also strengthens during recessions, consistent with the predictions under rank-alternating beliefs.

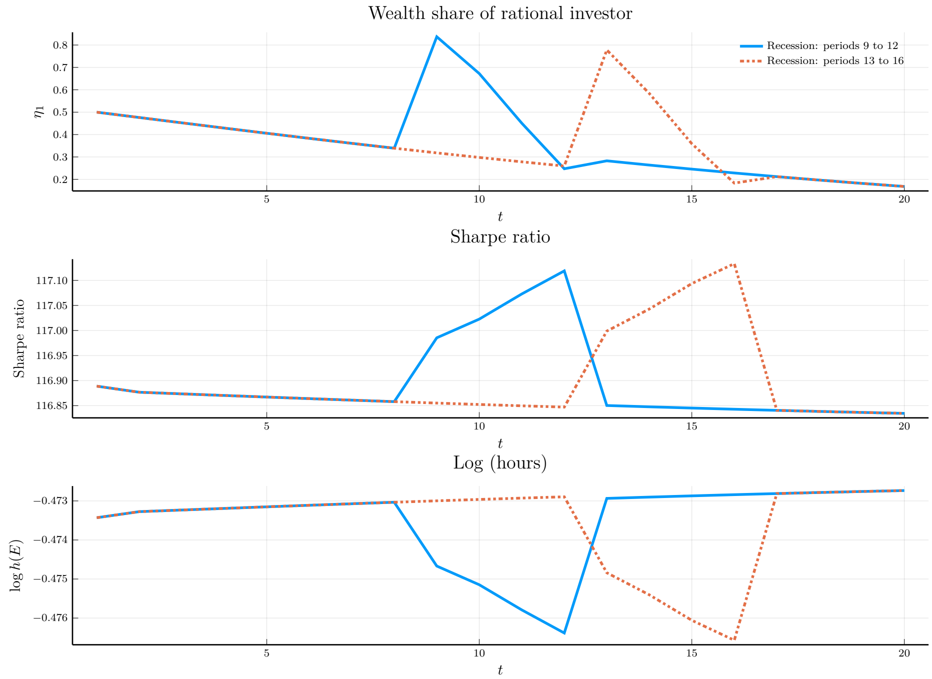

Figure 5: Heterogeneous beliefs: rational vs extrapolative

Note: examples of cycles (four periods of bad shocks with sixteen periods of expansions) with heterogeneous beliefs: $I = 2$ investors, rational and extrapolative. The top panel shows the wealth share of the optimistic investor, the middle panel shows the Sharpe ratio (under objective beliefs), and the bottom panel shows log labor hours.

Frothy markets and risk build-ups. An implication of Corollary 2 is that the risk premium declines with the length of economic expansions. To the extent that risk premia drive credit spreads, the model is consistent with the discussions in López-Salido et al. (2017) and Krishnamurthy and Muir (2017) that argue that economic booms are characterized by "froth" market conditions driven by credit-market *sentiments*. Our framework shows that with belief extrapolation, an increase in optimism leads to a reduction in risk premia and an increase in volatility, as shown in Figure 1. Under this interpretation, there is endogenous *risk build-up* during booms. As argued by Krishnamurthy and Li (2020), the combination of risk build-up and low spreads is challenging to generate for standard macro-finance models, such as He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014). This observation suggests that heterogeneous extrapolative beliefs explain credit and asset-market dynamics during boom-bust cycles.

It is worth clarifying that risk-build-up requires heterogeneity only when beliefs about the state are themselves Markovian. If beliefs are history-dependent, for example, if agents become more optimistic as boom states persist but more pessimistic once the economy switches states as boom states persist, the economy can also feature risk build-up. This risk build-up would not respond to the model's internal propagation through leverage, risk-exposure, and wealth accumulation, though.

Discussion: Pareto and Belief-Neutral Welfare. The environment's inherent risk build-up leads to subtle normative implications. It is well known that, under complete markets, the equilibrium is Pareto efficient from a subjective standpoint. The First Welfare Theorem does not distinguish whether trade is driven by differences in preferences over goods or by differences in beliefs across future states— treating both as equally valid sources of heterogeneity.

Classic Pareto efficiency is not the only sensible welfare criterion. Brunnermeier, Simsek and Xiong (2014) considers a paternalistic criterion, albeit one that does not presume superior knowledge by the planner. Instead, this belief-neutral criterion only requires the allocation to be efficient for all possible (convex combinations of) the agents' beliefs. Our environment is generically inefficient under this belief-neutral criterion: the simplest way to see this is to assume all agents believe TFP growth is iid but differ in the weights they assign states. In such a case, under all possible belief combinations, TFP growth is iid. Hence, labor fluctuations are inefficient under this criterion. This criterion can be used to justify leaning against the wind policies even though the economy is efficient in a standard sense.

Discussion: belief heterogeneity vs. risk aversion. As discussed above, rank-alternating beliefs amplify inherent economic fluctuations. Therefore, not only do differences in investors' risk appetite matter for business cycles, but also differences in their propensity to take risks. For instance, models of heterogeneous risk aversion can generate portfolio dispersion and time variation in expected returns. However, the ranking of agents in terms of risk-taking is always the same. This property is analogous to the case of rank-preserving beliefs shown in Figure 4. In contrast, there is no counterpart to the rank-alternating beliefs with heterogeneous risk aversion, as the agent taking less risk in booms is also taking more risk in the bust.

6 Quantitative analysis

We now consider the quantitative implications. We extend the baseline formulation in Section 3: First, productivity growth x_t now follows a more general process. This is necessary to capture the empirical patterns of dividend growth. Second, we introduce a force that induces a non-degenerate stationary wealth distribution. A force like this is needed so that belief heterogeneity survives in the ergodic distribution of the model. We do so through Uzawa endogenous discounting for convenience. This specific approach is not indispensable; we can obtain non-degeneracy through birth-death processes for example.

Belief process with a continuum of states. We start by specifying a continuous Markov process for x_t under both objective and subjective beliefs. Let $\hat{x}_t \equiv \log x_t$ denote log aggregate productivity growth. Under the objective measure, $\hat{x}_t$ follows:

$$\hat{x}_{t+1} = \mu_x + \sigma_x \epsilon_{x,t+1}. \tag{25}$$

where $\epsilon_{x,t+1} \overset{\text{iid}}{\sim} N(0,1)$. Thus, productivity growth is iid under the objective measure.

In turn, subjective beliefs about $\hat{x}_t$ for household $i \in \mathcal{I}$ are given by:

$$\hat{x}_{t+1} = \mu_{x,i} + \rho_{x,i}(\hat{x}_t - \mu_{x,i}) + v_{i,t} + \sigma_{x,i}\epsilon_{x,i,t+1}, \tag{26}$$

where $\epsilon_{x,i,t+1} \overset{\text{iid}}{\sim} N(0,1)$. Beliefs deviate from the process (25) in two critical ways: First, the subjective conditional expectation responds to current productivity growth $\hat{x}_t$. This dimension is important to allow households to over- or under-react to past information, consistent with Fact 2 in Section 2. Overreaction is controlled by $\rho_{x,i}$. For instance, a positive $\rho_{x,i}$ implies the household overreacts to productivity news, given our assumption that productivity growth is iid under the objective measure.

Second, beliefs are also exposed to a persistent *sentiment shock* v_t, which evolve according to

$$v_{i,t+1} = \rho_{v,i} v_{i,t} + \sigma_{v,i}\epsilon_{v,i,t+1} \tag{27}$$

where $\epsilon_{v,i,t+1} \overset{\text{iid}}{\sim} N(0,1)$, and $\epsilon_{v,i,t}$ and $\epsilon_{x,i,t}$ are uncorrelated.

The sentiment is independent of TFP growth. Thus, this shock leads to fluctuations in the degree of optimism unrelated to fundamentals. Introducing this shock is necessary to quantitatively match the volatility of subjective TFP growth expectations relative to the objective one—Fact 1 in Section 2. While extrapolation, through higher values of $\rho_{x,i}$, increases the volatility of subjective expectations, it is insufficient, so the sentiment is needed.

We note that the belief system in this section is a hybrid of the one in our taxonomy. It encompasses both extrapolation and time-dependent optimism through sentiment shock. We let the survey data discipline the strength of each force.

Endogenous discounting. To ensure the wealth distribution is stationary, we assume that households' subjective discount rate responds to their consumption share:

$$\beta_{i,t} = \beta e^{\kappa\left(1-\frac{C_{i,t}}{Y_t}\right)}. \tag{28}$$

This assumption, a form of Uzawa (2017) preferences, implies that households' marginal propensity to consume increases with their share of consumption, ensuring that no investor type concentrates all the wealth asymptotically. Following Schmitt-Grohé and Uribe (2003), we assume that $\beta_{i,t}$ depends on the average consumption share of type-i households, so households take the process for $\beta_{i,t}$ as given.[21]

Model solution. We describe the model characterization with a continuum of states and Uzawa preferences in the supplementary appendix. Previous results are essentially unchanged. As in the binary case, an exact closed-form solution is unavailable away from log preferences. We compute the solution using a third-order perturbation around the non-stochastic steady state. A third-order perturbation (or higher) is indispensable to capture time variation in expected returns.

Calibration. We use the following calibration, where parameters are expressed in quarterly terms. Preferences and technology parameters are standard. *Preferences*: We set $\beta = 0.978$ to match an unconditional annualized risk-free rate of 1%. The risk aversion is set to $\gamma = 10.0$ and the EIS to $\psi = 2.0$, typical values in the macro-finance literature. We choose the labor disutility parameter ξ to normalize the average hours to 1 and set the Frish elasticity to one, $\nu = 1$, a standard value in the literature. *Technology:* We set $\alpha = 0.66$ and choose μ_x and σ_x to match the average and standard deviation of annual consumption growth of 2% and 3.3%, respectively, consistent with Campbell and Cochrane (1999).

The calibration of the belief processes is novel. *Beliefs:* We focus on the case $J = 2$, and assume that households of type $i = 2$ have rational beliefs, which allows for some rational agents in the environment. We set $\mu_{x,1} = \mu_x$ and $\sigma_{x,1} = \sigma_x$, so households agree on the mean and conditional volatility of productivity growth. We set $\mu_2 = 0.1$, so the fraction of rational agents in the population is 10%, consistent with the fact that average beliefs in surveys of households and analysts show substantial deviations from rational expectations. We set $\rho_{v,1}$ and $\sigma_{v,1}$ to match the persistence of subjective dividend growth and the share of variance explained by movements in expectations, based on estimations of

[21] This mechanism is often used in small open-economy models to ensure a stationary distribution of external debt.

	Rational Beliefs (1)		(2)		Subjective Beliefs (3)		(4)		(5)		Data (6)	
Variables	**Mean**	**Std**	**Mean**	**Std**	**Mean**	**Std**	**Mean**	**Std**	**Mean**	**Std**	**Mean**	**Std**
Interest rate	10.0	0.0	9.7	0.7	9.7	1.4	1.0	6.4	0.8	6.5	0.8	5.7
Excess Returns (equity)	0.9	3.0	1.4	7.9	1.5	9.3	5.8	17.3	6.0	17.1	6.2	16.5
Consumption growth	1.8	3.0	1.8	3.0	1.8	3.1	1.8	3.1	1.8	3.3	1.8	3.0
Dividend growth	1.8	3.0	1.8	10.3	1.8	11.2	1.8	11.1	1.8	10.9	-	11.5
Log hours	0.0	0.0	-0.1	0.0	-0.1	1.5	-0.1	1.3	0.0	2.1	-	2.4

Table 3: Unconditional moments

Note: The table reports annualized unconditional moments in percentage points. Column 1 shows results for the endowment economy limit with rational beliefs. Column 2 introduces the labor friction. Column 3 shows the case of homogeneous subjective beliefs. Column 4 presents the case of heterogeneous beliefs with our calibration. Column 5 presents results for an alternative calibration with a higher labor supply elasticity. The last set of columns shows the corresponding values in the data. Mean and standard deviation of the interest rate and excess equity returns are from Mehra and Prescott (1985). Average consumption growth and the volatility of consumption and dividend growth are from Bansal and Yaron (2004). The standard deviation of total hours is obtained from Table 5 in Ohanian and Raffo (2012): total hours include changes in employment and hours worked for 1985Q1:2007Q4. Standard deviations in that paper are reported as deviations from an HP-filter. We convert their quarterly figure to annual rates.

De La O and Myers (2021). In turn, we set $\rho_{x,1}$ to match the correlation between subjective expectations and current productivity growth observed in the data. This calibration is agnostic about the relevant belief system: the subjective belief moments pin down these parameters, determining whether non-rational beliefs are extrapolation or rank-preserving. Finally, we set $\kappa = 0.5\%$, consistent with the mean-reversion in consumption shares in Gârleanu and Panageas (2015).

6.1 Unconditional moments

We first study the model's ability to match unconditional moments. To isolate the role of each ingredient, we start from a stripped-down version of the model and progressively add features until we reach the complete model. This progression allows us to discuss the role of each ingredient in the model. Table 3 presents the results under each formulation.

Column 1 presents the stripped-down version of the model with a single representative rational investor, for which hiring is done after TFP is known and labor supply is inelastic. This economy behaves as an Epstein-Zin version of the endowment economy studied by Mehra and Prescott (1985). The model fails to generate a sizeable equity premium and excess stock volatility. Moreover, without the hiring friction, the volatility of consumption and dividends coincide. Given that rational beliefs are iid, the price-dividend ratio is constant, so return volatility equals dividend volatility.

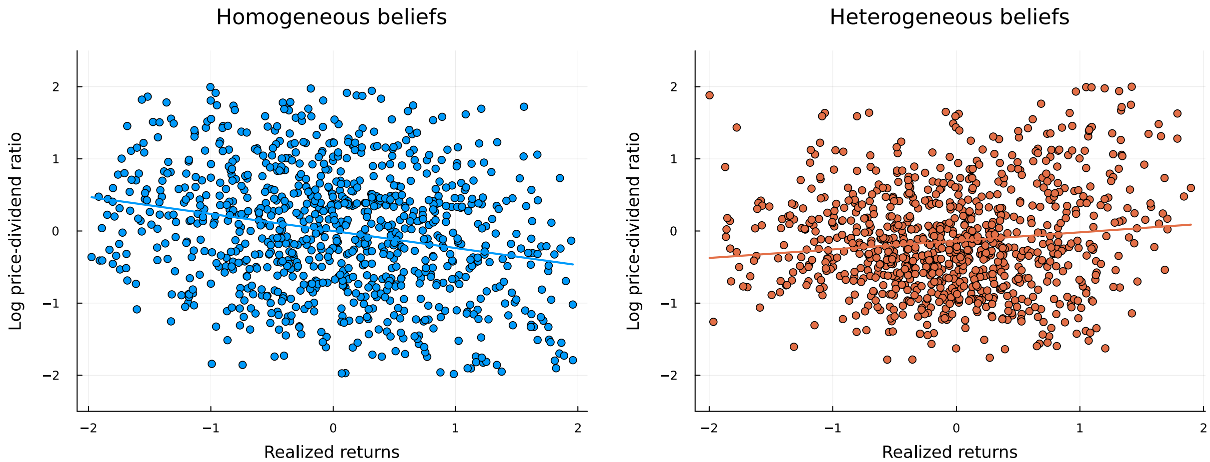

Figure 6: Price amplification: homogeneous vs. heterogeneous beliefs

Note: Figure shows a scatterplot of the log price-dividends and realized log returns in a 1,000-period simulation of the model. The left panel shows the simulation for the model with homogeneous beliefs, and the right panel shows the simulation for the model with heterogeneous beliefs. We standardized both variables and removed outliers to improve the visualization.

Column 2 introduces the timing of hires assumption and the elastic labor supply. The operating leverage channel endogenously generates differences in volatility between consumption and dividends in line with the data, even though this is not a target moment. With iid beliefs, there is no time variation in the price of risk, so the model fails to generate fluctuations in hours. Moreover, the hiring friction implies that dividend growth follows a mean-reverting process.[22] Without time-varying risk premia, mean-reversion in dividends causes the price-dividend ratio to negatively correlate with realized returns. This counterfactual behavior dampens return volatility relative to dividend volatility.

Column 3 considers the role of non-rational beliefs calibrated to match the survey data, while abstracting from differences in beliefs. In this case, the model does generate movements in hours, given that subjective beliefs are volatile, consistent with the discussion in Section 4. However, deviations from rational expectations are insufficient to generate an equity premium and stock return volatility in line with the data. Movements in subjective beliefs only attenuate the negative correlation between returns and the price-dividend ratio, so return volatility is still dampened relative to dividend volatility.

Column 4 introduces heterogeneity in beliefs. It brings along rational investors and investors with distorted beliefs. The model now generates the level and volatility of asset prices consistent with the evidence, with significant volatility in hours (more than half of that of the data). To obtain a high equity premium, it is crucial to generate excess volatility of returns. This requires the price-dividend ratio to have the right comovement

[22] For models with mean-reverting dividend process, see e.g. Menzly, Santos and Veronesi (2004) and Santos and Veronesi (2006).

with returns. Figure 6 shows that this is the case for the model with heterogeneous beliefs, but not for the model with homogeneous beliefs. This success is explained by the time-varying movements in the relative wealth of investors.

Column 5 considers an alternative calibration with a higher labor supply elasticity, increasing it from 1.0 to 2.0, the upper limit of the estimates of the macro labor supply elasticity in Keane and Rogerson (2012). Increasing the labor supply elasticity is motivated by the presence of unmodeled frictions: it is akin to increasing the real wage rigidities. While asset-price figures do not change substantially, the volatility of hours almost doubles, aligning the volatility of hours with that of the data.

6.2 Excess volatility and return predictability

The model's version with heterogeneous beliefs successfully generates empirically relevant levels of excess volatility, i.e., returns that are more volatile than cash flows. Following Cochrane (1992), we can use a Campbell-Shiller approximation to decompose the variance of the price-dividend ratio into a cash-flow and a discount-rate component:

$$Var[pd_t] = Cov\left[\sum_{k=1}^{\infty} \rho^k \Delta d_{t+k}, pd_t\right] - Cov\left[\sum_{k=1}^{\infty} \rho^k r_{t+k}, pd_t\right], \tag{29}$$

where pd_t is the log price-dividend ratio, Δd_{t+1} denotes log dividend growth, and r_{t+1} denotes realized log returns. Expression (29) connects volatility and predictability. It shows that movements in the price-dividend ratio either predict changes in dividend growth or future returns. To quantify the relative importance of movements in discount rates, consider the share of variance explained by movements in expected returns, defined as $\beta_r \equiv -\frac{Cov\left[\sum_{k=1}^{\infty} \rho^k r_{t+k}, pd_t\right]}{Var[pd_t]}$. As shown, e.g., in Cochrane (2011), empirical evidence suggests that $\beta_r \approx 1$, so discount rates explain most of the price-dividend ratio variance.

Given that β_r corresponds to the slope of a regression of cumulative returns on the price-dividend ratio, the importance of discount rates to explain return volatility is tightly connected to the ability of the price-dividend ratio in predicting future long-run returns. Figure 7 shows that the model with heterogeneous beliefs can capture the predictability patterns observed in the data. The figure shows a strong negative association between the price-dividend ratio and cumulative returns in model-simulated data. Moreover, the coefficient β_r in models (4) and (5), as defined in Table 3, is given by $\beta_r = 0.93$ and $\beta_r = 0.97$, respectively. In contrast, we have $\beta_r = -0.16$ for model (2), so the price-dividend ratio predict returns with the wrong sign, and $\beta_r = 0.40$ for model (3), which has the correct sign but movements in cash flows drive the majority of the fluctuations

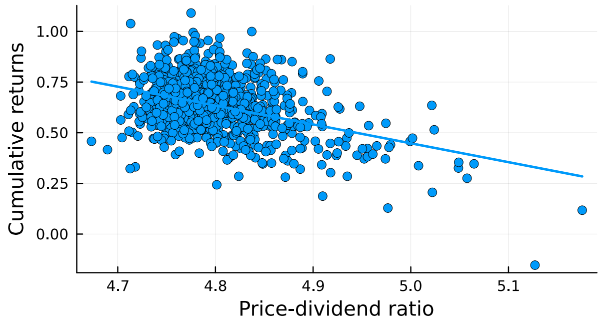

Figure 7: Return predictability

Dependent Variable:	cum_ex_returns			
Model:	(1)	(2)	(3)	(4)
Variables				
labor_income	-2.89		0.87	
	(0.53)		(0.68)	
p/d		-0.20		-0.02
		(0.10)		(0.12)
Observations	4,000	4,000	4,000	4,000

Table 4: Labor and p/d ratio regressions

Note: The left panel shows a scatter plot based on model simulations of the price-dividend ratio and a measure of future cumulative returns, $\sum_{k=1}^{T} \rho^{k-1} r_{t+k}$, for $T = 60$ quarters. The right panel shows the regression coefficients of future cumulative excess returns for $T = 60$ on the price-dividend ratio and labor income. The data for the first two columns was simulated under the objective measure, while the data for the last two columns was simulated under subjective beliefs. Standard errors are in parentheses.

in the price-dividend ratio. Hence, only versions of the model with belief heterogeneity generate a degree of return predictability that aligns with the data.

Labor income and return predictability. Time variation in risk premia generates fluctuations in hours worked in our setting. As a result, in addition to the price-dividend ratio, movements in labor income should also predict excess returns. Table 4 reports the results of a regression of cumulative excess returns on the price-dividend ratio and labor income in our simulated data. Column 1 shows that periods of high labor income are associated with lower future returns. This finding is consistent with the evidence from Santos and Veronesi (2006), who show that a high labor income share predicts lower aggregate stock returns. Similarly, Belo, Donangelo, Lin and Luo (2023) documents that the aggregate hiring rate of publicly traded firms negatively predicts stock market excess returns.[23] Our model links this evidence to fluctuations in subjective beliefs. As discussed in Section 4, waves of optimism lead to both an increase in labor income and a compression of risk premia, jointly generating predictability from labor income and the price-dividend ratio.

Subjective vs. objective predictability. The predictability results above are from the perspective of an econometrician. Recent work by Nagel and Xu (2023) shows that return predictability looks very different under subjective beliefs: the standard predictors of returns are only weakly associated with survey-based measures of subjective risk premia. While expected excess returns are countercyclical under the objective measure, they

[23] Belo et al. (2014) and Favilukis and Lin (2016) show similar patterns for the cross-section of stock returns and U.S. states. For recent evidence using administrative data, see Meeuwis, Papanikolaou, Rothbaum and Schmidt (2024).

appear acyclical under subjective beliefs.

Table 4 shows that the model replicates these empirical patterns. Columns 1 and 2 report predictability regressions using labor income and the price-dividend ratio as predictors under the objective measure. In contrast, Columns 3 and 4 present the same regressions when the model is simulated under subjective beliefs. Consistent with Nagel and Xu (2023), standard predictors are only weakly associated with future returns under subjective beliefs. The coefficient on labor income has the wrong sign and is statistically insignificant, while the coefficient on the price-dividend ratio is near zero. Thus, our model generates subjective risk premia that are essentially acyclical, which aligns with survey evidence. As discussed in Section 4, this acyclicity arises because subjective expectations of future cash flows closely track asset price movements, dampening fluctuations in perceived risk premia.

7 Conclusion

When asked about the nature of business cycles, Thomas Sargent[24], a rational expectations pioneer, answered:[25]

> "[...] economists have been working hard to refine rational expectations theory. [...] An influential example of such work is the 1978 QJE paper by Harrison and Kreps. [...], for policymakers to know whether and how they can moderate bubbles, we need to have well-confirmed quantitative versions of such models up and running."

This response embraces the idea that "belief heterogeneity" matters but also calls for quantitative models that link beliefs with the real economy.

This paper responds to that call and adapts a standard real business cycle and asset pricing model to fit the narrative that waves of optimism and pessimism drive the business cycle. We argue that a combination of heterogeneity in beliefs, with substantial extrapolation, coupled with an operational leverage channel, can succesfully reproduce business cycle, asset pricing and survey data patterns. Extrapolative beliefs and heterogeneity deliver the amplification that is key for this success. Extrapolation adds to risk build up.

[24]Hyman Minsky was Thomas Sargent's undergraduate advisor. Whereas Sargent departs methodologically and calls for a quantitative approach to economic research, there is an agreement in their views regarding nature of business cycles.

[25]Interview with Thomas Sargent, *The Region*, August 26, 2010. Available at https://www.minneapolisfed.org/article/2010/interview-with-thomas-sargent.

We foresee that our framework can be extended to study the interaction with other forms of amplification. Namely, nominal rigidities open the door to amplification through aggregate demand externalities. Additional financial frictions, such as margin calls or short-selleing constraints, can amplify asset prices thourhg fire-sale and pecuniary externalities that impact wealth and risk-taking capacity. We expect our conclusions to remain relevant in contexts with these additional frictions.

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Online Appendix

A Proofs

A.1 Derivation of the investor's modified problem

We start by providing a derivation of the investor's modified problem.

Proof. First, we adopt a change of variables and write the investor's problem as follows

$$V_{i,t} = \max_{\{\tilde{C}_{i,t}, h_{i,t}, B_{i,t}, S_{i,t}\}} (1-\beta) U\left(\tilde{C}_{i,t}\right) + \beta U\left(\Psi^{-1}\left(\mathbb{E}_{i,t}\left[\Psi\left(U^{-1}(V_{i,t+1})\right)\right]\right)\right), \tag{A.1}$$

subject to

$$\tilde{C}_{i,t} + Q_t S_{i,t} + B_{i,t} = R_{e,t} Q_{t-1} S_{i,t-1} + R_{b,t} B_{i,t-1} + W_t h_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu}, \tag{A.2}$$

and the natural borrowing

$$(Q_t + \pi_t) S_{i,t-1} + R_{b,t} B_{i,t-1} + W_t h_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu} \geq -\mathcal{H}_{i,t} \tag{A.3}$$

It is immediate that the optimal value of $h_{i,t}$ satisfies

$$W_t = \xi_t h_{i,t}^{\nu}. \tag{A.4}$$

We show next that, given $h_{i,t}$ satisfying (A.4), if the sequence $(\tilde{C}_{i,t}, B_{i,t}, S_{i,t})$ satisfies (A.2) and (A.3), then there exists $(N_{i,t}, \omega_{i,t})$ such that $(\tilde{C}_{i,t}, N_{i,t}, \omega_{i,t})$ satisfies (10) and $N_{i,t} \geq 0$. Conversely, if $(\tilde{C}_{i,t}, N_{i,t}, \omega_{i,t})$ satisfies (10) and $N_{i,t} \geq 0$, there exists $(B_{i,t}, S_{i,t})$ such that $(\tilde{C}_{i,t}, B_{i,t}, S_{i,t})$ satisfies (A.2) and (A.3).

From the definition of the return on human wealth, we have that $W_t h_{i,t} - \xi_t \frac{h_{i,t}^{1+\nu}}{1+\nu} = R_{h,t-1} \mathcal{H}_{i,t-1} - \mathcal{H}_{i,t}$, which allow us to write (A.2) and (A.3) as follows:

$$\tilde{C}_{i,t} + Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t} = N_{i,t}, \qquad N_{i,t} \geq 0. \tag{A.5}$$

We consider next the law of motion of total wealth:

$$N_{i,t+1} = \left[R_{e,t+1} \frac{Q_t S_{i,t}}{Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t}} + R_{b,t+1} \frac{B_{i,t}}{Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t}} + R_{h_i,t+1} \frac{\mathcal{H}_{i,t}}{Q_t S_{i,t} + B_{i,t} + \mathcal{H}_{i,t}}\right] \left(N_{i,t} - \tilde{C}_{i,t}\right). \tag{A.6}$$

As markets are dynamically complete, there exists replicating portfolios $(\omega_{h_i,t}, \omega_{e,t})$ such that

$$R_{k,t+1} = \omega_{k,t} R_{r,t+1} + (1 - \omega_{k,t}) R_{b,t+1}, \tag{A.7}$$

for $k \in \{h_i, e\}$.

Combining the previous two conditions, we obtain

$$N_{i,t+1} = \left[R_{r,t+1} \frac{\omega_{e,t} Q_t S_{i,t} + \omega_{h_i,t} \mathcal{H}_{i,t}}{N_{i,t} - \tilde{C}_{i,t}} + R_{b,t+1} \frac{B_{i,t} + (1 - \omega_{e,t}) Q_t S_{i,t} + (1 - \omega_{h_i,t}) \mathcal{H}_{i,t}}{N_{i,t} - \tilde{C}_{i,t}} \right] \left(N_{i,t} - \tilde{C}_{i,t} \right). \tag{A.8}$$

Using the first condition in (A.5) to solve for $B_{i,t}$, we obtain

$$N_{i,t+1} = \left[\left(R_{r,t+1} - R_{b,t+1} \right) \omega_{i,t} + R_{b,t+1} \right] \left(N_{i,t} - \tilde{C}_{i,t} \right), \tag{A.9}$$

where $\omega_{i,t} \equiv \frac{\omega_{e,t} Q_t S_{i,t} + \omega_{h_i,t} \mathcal{H}_{i,t}}{N_{i,t} - \tilde{C}_{i,t}}$.

□

A.2 Proof of Lemma 1

The next lemma characterizes the value function, the consumption function, and the Euler equations of each investor.

Lemma 1 (Consumption and Euler equations)**.** *The household's value function takes the form: $V_i(N, X, s) = U\left(v_i(X, s) N\right)$, where $v_i(X, s)$ denotes the wealth multiplier. The consumption-wealth ratio $c_i(X, s) = \frac{\tilde{C}_i(N, X, s)}{N}$ and Euler equations for investor $i \in \mathcal{I}$ are given by*

(i) Consumption-wealth ratio:

$$c_i(X, s) = \frac{(\beta^{-1} - 1)^{\psi} \mathcal{R}_i(X, s)^{1-\psi}}{1 + (\beta^{-1} - 1)^{\psi} \mathcal{R}_i(X, s)^{1-\psi}}, \tag{A.10}$$

where $\mathcal{R}_i(X, s) \equiv \Psi^{-1}\left(\mathbb{E}_i\left[\Psi\left(v(X', s') R_{i,n}(X, s, s')\right) | X\right]\right)$.

(ii) Euler equation for an asset $j \in \{r, b\}$:

$$1 = \mathbb{E}_i \left[\Lambda_i(X, s, s') R_j(X, s, s') \right], \tag{A.11}$$

where, for $\theta \equiv \frac{1-\gamma}{1-\psi^{-1}}$, the investor's SDF is given by

$$\Lambda_i(X, s, s') = \beta^{\theta} \left(\frac{c_i(\chi(X, s, s'), s') N'}{c_i(X, s) N} \right)^{-\frac{\theta}{\psi}} R_{i,n}(X, s, s')^{-(1-\theta)}. \tag{A.12}$$

(iii) The wealth multipliers satisfy:

$$v_i(X,s) = U^{-1}\left[U\left(c_i(X,s)\right) + \beta U\left(\mathcal{R}_i(X,s)\left(1 - c_i(X,s)\right)\right)\right]. \tag{A.13}$$

Proof. First, we verify that the value function takes the form (**??**). Given the conjecture about the value function, the Bellman equation for investor i can be written as

$$\frac{(v_i(X,s)N)^{1-\psi^{-1}} - 1}{1-\psi^{-1}} = \max_{\tilde{c}_i,\omega_i}(1-\beta)\frac{(\tilde{c}_i N)^{1-\psi^{-1}} - 1}{1-\psi^{-1}} + \beta\frac{\mathbb{E}_i\left[(v_i(X',s')N')^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}} - 1}{1-\psi^{-1}}, \tag{A.14}$$

subject to $N' = R_{i,n}(X,s,s')(1-\tilde{c}_i)N$ and $N' \geq 0$.

The first-order conditions for the consumption-wealth ratio and the portfolio share are given by

$$(1-\beta)\tilde{c}_i^{-\psi^{-1}} = \beta\mathcal{R}_i(X,s)^{1-\psi^{-1}}(1-\tilde{c}_i)^{-\psi^{-1}} \tag{A.15}$$

$$0 = \mathbb{E}_i\left[(v_i(X',s')R_{i,n}(X,s,s'))^{-\gamma}v_i(X')(R_r(X,s,s') - R_b(X,s))\right] \tag{A.16}$$

where $\mathcal{R}_i(X,s) = \mathbb{E}_i\left[(v_i(X',s')R_{i,n}(X,s,s'))^{1-\gamma}|X,s\right]^{\frac{1}{1-\gamma}}$.

Given $\mathcal{R}_i(X,s)$, we can solve for the consumption-wealth ratio:

$$\tilde{c}_i(X,s) = \frac{(\beta^{-1}-1)^{\psi}\mathcal{R}_i(X,s)^{1-\psi}}{1+(\beta^{-1}-1)^{\psi}\mathcal{R}_i(X,s)^{1-\psi}}. \tag{A.17}$$

The envelope condition with respect to N is given by

$$v_i(X)^{1-1/\psi} = \beta\mathcal{R}_i(X)^{1-1/\psi}(1-\tilde{c}_i(X))^{-1/\psi} \Rightarrow \tilde{c}_i(X) = (1-\beta)^{\psi}v_i(X)^{1-\psi}. \tag{A.18}$$

From the optimality condition for the risky asset, we obtain

$$\mathbb{E}_i\left[(v_i(X',s')R_{i,n}(X,s,s'))^{1-\gamma}\right] = \mathbb{E}_i\left[v_i(X',s')^{1-\gamma}R_{i,n}(X,s,s')^{-\gamma}R_j(X,s,s')\right], \tag{A.19}$$

for $j \in \{r,b\}$.

Raising the envelope condition (A.18) to the power $\theta \equiv \frac{1-\gamma}{1-\psi^{-1}}$, using the definition of $\mathcal{R}_i(X)$ and condition (A.19), we obtain

$$1 = \mathbb{E}_i\left[\beta^{\theta}\left(\frac{v_i(X',s')}{v_i(X,s')}\right)^{1-\gamma}R_{i,n}(X,s,s')^{-\gamma}R_j(X,s,s')(1-\tilde{c}_i(X,s))^{-\theta/\psi}\right]. \tag{A.20}$$

Using the condition $v_i(X) = (1-\beta)^{\frac{1}{1-\psi^{-1}}} \tilde{c}_i(X)^{-\frac{\psi^{-1}}{1-\psi^{-1}}}$, we obtain the Euler equations

$$1 = \mathbb{E}_i \left[\beta^\theta \left(\frac{\tilde{c}_i(X', s')N'}{\tilde{c}_i(X, s)N} \right)^{-\frac{\theta}{\psi}} R_{i,n}(X, s, s')^{-(1-\theta)} R_j(X, s, s') \right]. \tag{A.21}$$

This concludes the derivation of the consumption-wealth ratio and the Euler equations for the two assets. It remains to check that the value function takes the form (**??**), which amounts to show that $v_i(X)$ indeed does not depend on N. Notice that $\tilde{c}_i(X, s)$ and $\omega_i(X, s)$ do not depend on N. We can then write the Bellman equation as follows:

$$v_i(X, s)^{1-\psi^{-1}} = (1-\beta)\tilde{c}_i(X, s)^{1-\psi^{-1}} + \beta \mathbb{E}_i \left[(v_i(X', s')R_{i,n}(X, s, s')(1 - \tilde{c}_i(X)))^{1-\gamma} \right]^{\frac{1-\psi^{-1}}{1-\gamma}}, \tag{A.22}$$

for $\psi \neq 1$ and

$$\log v_i(X, s) = (1-\beta) \log \tilde{c}_i(X, s) + \beta \log \mathbb{E}_i \left[(v_i(X', s')R_{i,n}(X, s, s')(1 - \tilde{c}_i(X, s)))^{1-\gamma} \right]^{\frac{1}{1-\gamma}}. \tag{A.23}$$

which is independent of N, which confirms our conjecture for the value function (**??**).

□

A.3 Proof of Lemma 2

The following lemma characterizes households' portfolio weight in the surplus claim in terms of the economy-wide SDF, the market prices, and their beliefs.

Lemma 2 (Portfolio share)**.** *The shares of total wealth invested in the risky asset are*

$$\omega_i(X, s) = \frac{1}{\Delta R_r(X, s)} \left[\frac{\tilde{p}_i(X, s, H)}{p_{s,H} \Lambda(X, s, H)} - \frac{\tilde{p}_i(X, s, L)}{p_{s,L} \Lambda(X, s, L)} \right], \tag{A.24}$$

where $\tilde{p}_i(X, s, s')$ *is*

$$\tilde{p}_i(X, s, s') = \frac{(p^i_{ss'})^{\frac{1}{\gamma}} \left[v_i(\chi(X, s, s'), s') | R^e_r(X, s, s') | \right]^{\frac{1}{\gamma}-1}}{\sum_{\tilde{s}' \in \{L,H\}} (p^i_{s\tilde{s}'})^{\frac{1}{\gamma}} \left[v_i(\chi(X, s, \tilde{s}'), \tilde{s}') | R^e_r(X, s, \tilde{s}') | \right]^{\frac{1}{\gamma}-1}}.$$

Lemma 2, describes how portfolio shares depend on distorted probabilities, $\tilde{p}_i(X, s, s')$, and $p_{ss'} \times \Lambda(X, s, s')$. The portfolio share $\omega_i(X, s)$ in (A.24) is increasing in p^i_{sH}. This means that relatively optimistic investors hold more of the risky surplus claim.

Proof. The optimal portfolio share satisfies the condition

$$\frac{p^i_{sL}}{p^i_{sH}} \frac{v_i(\chi(X,s,L),L)^{1-\gamma}}{v_i(\chi(X,s,H),H)^{1-\gamma}} \left(\frac{\omega_i(X,s)R^e_r(X,s,L)+1}{\omega_i(X,s)R^e_r(X,s,H)+1}\right)^{-\gamma} \frac{|R^e_r(X,s,L)|}{R^e_r(X,s,H)} = 1 \tag{A.25}$$

Raising both sides to $-\frac{1}{\gamma}$, we obtain

$$\left(\frac{p^i_{sL}}{p^i_{sH}}\right)^{-\frac{1}{\gamma}} \frac{v_i(\chi(X,s,L),L)^{1-\frac{1}{\gamma}}}{v_i(\chi(X,s,H),H)^{1-\frac{1}{\gamma}}} \frac{\omega_i(X,s)R^e_r(X,s,L)+1}{\omega_i(X,s)R^e_r(X,s,H)+1} \frac{|R^e_r(X,s,L)|^{-\frac{1}{\gamma}}}{R^e_r(X,s,H)^{-\frac{1}{\gamma}}} = 1 \tag{A.26}$$

Rearranging the expression above, we obtain

$$\omega_i(X,s) = \frac{\tilde{p}_i(X,s,H)}{|R^e_r(X,s,L)|} - \frac{\tilde{p}_i(X,s,L)}{R^e_r(X,s,H)}, \tag{A.27}$$

where

$$\tilde{p}_i(X,s,s') = \frac{(p^i_{ss'})^{\frac{1}{\gamma}} \left[v_i(\chi(X,s,s'),s')|R^e_r(X,s,s')|\right]^{\frac{1}{\gamma}-1}}{\sum_{s'\in\{L,H\}} (p^i_{ss'})^{\frac{1}{\gamma}} \left[v_i(\chi(X,s,s'),s')|R^e_r(X,s,s')|\right]^{\frac{1}{\gamma}-1}}. \tag{A.28}$$

The SDF in this economy is given by

$$\Lambda(X,s,s') = \frac{1}{p_{ss'}} \frac{1}{R_b(X,s)} \frac{|R_r(X,s,-s') - R_b(X,s)|}{\Delta R_r(X,s)}, \tag{A.29}$$

where $\Delta R_r(X,s) = R_r(X,s,H) - R_r(X,s,L)$.

We can then write $\omega_i(X,s)$ as follows

$$\omega_i(X,s) = \frac{1}{\Delta R_r(X,s)} \left[\frac{\tilde{p}_i(X,s,H)}{p_{s,H}\Lambda(X,s,H)} - \frac{\tilde{p}_i(X,s,L)}{p_{s,L}\Lambda(X,s,L)}\right]. \tag{A.30}$$

Diffusion-like approximation. To better interpret the expression for the portfolio share, it is useful to consider an approximation analogous to the continuous-time limit for diffusion processes. Given $R_r(X,s,s')$, probabilities $p^i_{ss'}$ for household i, and a small parameter $\epsilon > 0$, we can find $\mu_{i,r}(X,s)$ and $\sigma_{i,r}(X,s)$ that satisfies the conditions

$$R^e_r(X,s,H) = \mu_{i,r}(X,s)\epsilon + \sqrt{\frac{p_{sL}}{p_{sH}}}\sigma_{i,r}(X,s)\sqrt{\epsilon}, \qquad R^e_r(X,s,L) = \mu_{i,r}(X,s)\epsilon - \sqrt{\frac{p_{sH}}{p_{sL}}}\sigma_{i,r}(X,s)\sqrt{\epsilon}, \tag{A.31}$$

which gives us the expected value and variance for household i:

$$\mathbb{E}_i[R^e_r(X,s,s')|X,s] = \mu_{i,r}(X,s)\epsilon, \qquad Var_i[R^e_r(X,s,s')|X,s] = \sigma^2_{i,r}(X,s)\epsilon. \tag{A.32}$$

Similarly, we can write $R_b(X,s) = 1 + r_b(X,s)\epsilon$.

From Equation (B.6), and assuming $\gamma = 1$, we obtain

$$\begin{aligned} \omega_i(X,s) &= R_b(X,s)\frac{p^i_{s,H}R^e_r(X,s,H) + p^i_{s,L}R^e_r(X,s,L)}{|R^e_r(X,s,L)|R^e_r(X,s,H)} \\ &= (1 + r_b(X,s)\epsilon)\frac{\mu_{i,r}(X,s)\epsilon}{\left(\sqrt{\frac{p_{sH}}{p_{sL}}}\sigma_{i,r}(X,s)\sqrt{\epsilon} - \mu_{i,r}(X,s)\epsilon\right)\left(\mu_{i,r}(X,s)\epsilon + \sqrt{\frac{p_{sL}}{p_{sH}}}\sigma_{i,r}(X,s)\sqrt{\epsilon}\right)}, \end{aligned} \tag{A.33}$$

where we used the fact that $R^e_r(X,s,L) < 0$ by no-arbitrage.

In general, $(\mu_{i,r}(X,s), \sigma_{i,r}(X,s))$ and $p^i_{ss'}$ are functions of ϵ. Assuming that $\mu_{i,r}(X,s) = \mathcal{O}(1)$, $\sigma_{i,r}(X,s)) = \mathcal{O}(1)$, and $p^i_{ss'} = \mathcal{O}(1)$, we can write the expression $\omega_i(X,s)$ as follows:[26]

$$\omega_i(X,s) = \frac{\mu_{i,r}(X,s)}{\sigma^2_{i,r}(X,s)} + \mathcal{O}(\epsilon). \tag{A.34}$$

□

A.4 Proof of Proposition 1

Proof. First, we compute the Sharpe ratio on the risky asset. We will compute expectations using the objective measure, but a similar calculation gives the Sharpe ratio using the investors' subjective beliefs. The expected excess return is given by

$$\mathbb{E}\left[R^e_r(X,s,s')\right] = p_{sL}R^e_r(X,s,L) + p_{sH}R^e_r(X,s,H). \tag{A.35}$$

The variance of excess returns is given by

$$Var[R^e_r(X,s,s')] = p_{sL}p_{sH}\Delta R^e_r(X,s)^2. \tag{A.36}$$

The Sharpe ratio in the risky asset is then given by

$$\frac{\mathbb{E}[R^e_r(X,s,s')]}{\sqrt{Var[R^e_r(X,s,s')]}} = \sqrt{\frac{p_{sL}}{p_{sH}}}\frac{R^e_r(X,s,L)}{\Delta R^e_r(X,s)} + \sqrt{\frac{p_{sH}}{p_{sL}}}\frac{R^e_r(X,s,H)}{\Delta R^e_r(X,s)}. \tag{A.37}$$

We can write the expression above in terms of the economy's SDF. The SDF under the

[26]These assumptions are analogous to the ones used by e.g. Merton (1992) to derive the continuous-time limit with diffusion processes. Allowing for rare events, $p^i_{ss'} = \mathcal{O}(\epsilon)$ for some s', would lead to a jump-diffusion process.

objective measure can be written as

$$\Lambda(X,s,L) = \frac{\mathbb{E}[\Lambda(X,s,s')]}{p_{sL}} \frac{R_r^e(X,s,H)}{\Delta R_r^e(X,s)}, \qquad \Lambda(X,s,H) = -\frac{\mathbb{E}[\Lambda(X,s,s')]}{p_{sH}} \frac{R_r^e(X,s,L)}{\Delta R_r^e(X,s)}. \tag{A.38}$$

Combining the expressions above, we obtain

$$\frac{\mathbb{E}[R_r^e(X,s,s')]}{\sqrt{Var[R_r^e(X,s,s')]}} = \sqrt{p_{sL}p_{sH}} \frac{\Lambda(X,s,L) - \Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]}. \tag{A.39}$$

We consider next how the Sharpe ratio affects the risk-neutral expectation of future productivity growth. The risk-neutral expectation of productivity is given by

$$\mathbb{E}^Q[x_{t+1}] = p_{sL} \frac{\Lambda(X,s,L)}{\mathbb{E}[\Lambda(X,s,s')]} x_L + p_{sH} \frac{\Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]} x_H. \tag{A.40}$$

The difference between the expected value of productivity under the physical measure and the risk-neutral measure is given by

$$\mathbb{E}[x_{t+1}] - \mathbb{E}^Q[x_{t+1}] = p_{sL} \frac{\mathbb{E}[\Lambda(X,s,s')] - \Lambda(X,s,L)}{\mathbb{E}[\Lambda(X,s,s')]} x_L + p_{sH} \frac{\mathbb{E}[\Lambda(X,s,s')] - \Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]} x_H. \tag{A.41}$$

Rearranging the expression above, we obtain

$$\mathbb{E}[x_{t+1}] - \mathbb{E}^Q[x_{t+1}] = p_{sL}p_{sH} \frac{\Lambda(X,s,L) - \Lambda(X,s,H)}{\mathbb{E}[\Lambda(X,s,s')]} \Delta x, \tag{A.42}$$

where $\Delta x = x_H - x_L$.

Using the expression for the Sharpe ratio, we obtain

$$\mathbb{E}^Q[x_{t+1}] = \mathbb{E}[x_{t+1}] - \sqrt{p_{sL}p_{sH}} \frac{\mathbb{E}[R_r^e(X,s,s')]}{\sqrt{Var[R_r^e(X,s,s')]}} \Delta x. \tag{A.43}$$

□

A.5 Proof of Proposition 2

Proof. We start by deriving the process for returns. From the market clearing condition for goods, we obtain

$$\frac{x_s h(\mathcal{L})^\alpha - \xi \frac{h(\mathcal{L})^{1+\nu}}{1+\nu}}{P(X,s)} = 1 - \beta \tag{A.44}$$

The return on the surplus claim is given by

$$R_p(X,s,s') = \frac{x_s P(\chi(X,s,s'),s')}{P(X,s) - \left(x_s h(\mathcal{L})^\alpha - \xi \frac{h(\mathcal{L})^{1+\nu}}{1+\nu}\right)} = \frac{x_s}{\beta} \frac{x_{s'} h(\mathcal{L}'(X,s))^\alpha - \xi \frac{h(\mathcal{L}'(X,s))^{1+\nu}}{1+\nu}}{x_s h(\mathcal{L})^\alpha - \xi \frac{h(\mathcal{L})^{1+\nu}}{1+\nu}}. \tag{A.45}$$

Using the conditions in (14), we can rewrite the expression as follows

$$R_p(X,s,s') = \frac{x_s}{\beta} \frac{x_{s'} \mathcal{L}'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} \mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} \mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{A.46}$$

Note that the denominator in the expression above is positive if and only if $\mathcal{L} < \frac{1+\nu}{\alpha} x_s$. A sufficient condition is given by $\alpha x_H < x_L$, as shown below

$$\mathcal{L} \leq x_H < \frac{x_L}{\alpha} < \frac{1+\nu}{\alpha} x_s, \tag{A.47}$$

and, similarly, this condition guarantees that the numerator is also positive.

Interest rate. The interest rate satisfies the condition $R_b(X,s) = \mathbb{E}\left[\frac{\Lambda(X,s,s')}{\mathbb{E}[\Lambda(X,s,s')]} R_p(X,s,s')\right]$, so $R_b(X,s)$ is given by

$$R_b(X,s) = \left(1 - \frac{\alpha}{1+\nu}\right) \frac{x_s}{\beta} \frac{\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} \mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{A.48}$$

using the fact that $\mathbb{E}\left[\frac{\Lambda(X,s,s')}{\mathbb{E}[\Lambda(X,s,s')]} x_{s'}\right] = \mathcal{L}'(X,s)$.

The expression above is increasing in $\mathcal{L}'(X,s)$, decreasing in x_s, and it is increasing in $\mathcal{L}$ for $s = L$.

Risk premium. The risk asset's excess return is given by

$$\frac{R_p(X,s,s')}{R_b(X,s)} = \frac{1}{1 - \frac{\alpha}{1+\nu}} \frac{x_{s'} - \frac{\alpha}{1+\nu} \mathcal{L}'(X,s)}{\mathcal{L}'(X,s)}. \tag{A.49}$$

The conditional risk premium is then given by

$$\mathbb{E}_s[R_p^e(X,s,s')] = \frac{1}{1 - \frac{\alpha}{1+\nu}} \frac{\mathbb{E}_s[x_{s'}] - \mathcal{L}'(X,s)}{\mathcal{L}'(X,s)}, \tag{A.50}$$

given the definition $R_p^e(X,s,s') \equiv \frac{R_p(X,s,s') - R_b(X,s)}{R_b(X,s)}$.

$\square$

A.6 Proof of Proposition 3 and Corollary 1

Proof. We start by deriving the expression for the SDF. Note that we can express the SDF in terms of $R_e(X,s,s')$ and $R_b(X,s)$ instead of $R_r(X,s,s')$ and $R_b(X,s)$, as we can always obtain the SDF in terms of any two (linearly independent) assets. The SDF is then given by

$$\Lambda(X,s,s') = \frac{1}{p_{ss'}}\frac{1}{R_b(X,s)}\frac{|R_p^e(X,s,-s')|}{\Delta R_p^e(X,s)}. \tag{A.51}$$

The excess return on the surplus claim is given by

$$R_p^e(X,s,s') = \frac{1}{1-\frac{\alpha}{1+\nu}}\frac{x_{s'}-\mathcal{L}'(X,s)}{\mathcal{L}'(X,s)}. \tag{A.52}$$

Combining the previous two expressions, we obtain

$$\Lambda(X,s,s') = \frac{1}{p_{ss'}}\frac{1}{R_b(X,s)}\frac{|x_{-s'}-\mathcal{L}'(X,s)|}{\Delta x} \tag{A.53}$$

using the fact that $\frac{R_p^e(X,s,s')}{\Delta R_p^e(X,s)} = \frac{1}{R_b(X,s)}\frac{x_{s'}-\mathcal{L}'(X,s)}{\Delta x}$.

Demand for risk. The demand for risk in this economy is given by

$$\sum_{i=1}^{I}\eta_i\sigma_s[R_{i,n}(X,s,s')] = \sqrt{p_{sL}p_{sH}}\left[\frac{p_{sH}(X,s)}{p_{sH}\Lambda(X,s,H)} - \frac{p_{sL}(X,s)}{p_{sL}\Lambda(X,s,L)}\right], \tag{A.54}$$

where $p_{ss'}(X,s) = \sum_{i=1}^{I}\eta_{i,t}p_{ss'}^i$, using the fact that $\sigma_s[R_r(X,s,s')] = \sqrt{p_{sH}p_{sL}}\Delta R_r(X,s)$ and the results in Lemma 2.

Using the expression for the SDF, the demand for risk can be written as

$$\sum_{i=1}^{I}\eta_i\sigma_s[R_{i,n}(X,s,s')] = \sigma_s[x_{s'}]\frac{\frac{1+\nu-\alpha}{1+\nu}\frac{x_s}{\beta}}{x_s\mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}\mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}}\left[\frac{p_{sH}(X,s)\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{\mathcal{L}'(X,s)-x_L} - \frac{p_{sL}(X,s)\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_H-\mathcal{L}'(X,s)}\right], \tag{A.55}$$

given $\sigma_s[x_{s'}] = \sqrt{p_{sL}p_{sH}}\Delta x$.

The first term inside brackets in the expression above is decreasing in $\mathcal{L}'(X,s)$ if and only if the following condition holds

$$\frac{1+\nu}{1+\nu-\alpha}\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}-1}(\mathcal{L}'(X,s)-x_L) - \mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}} < 0 \iff \mathcal{L}'(X,s) < \frac{x_L}{\alpha}, \tag{A.56}$$

which holds, given that $\mathcal{L}'(X,s) \leq x_H < \frac{x_L}{\alpha}$.

Therefore, the demand for risk is decreasing in $\mathcal{L}'(X,s)$. As $\mathcal{L}'(X,s)$ is decreasing in the Sharpe ratio of the risky asset, then the demand for risk is increasing in the Sharpe ratio.

Supply of risk. The volatility of returns is given by

$$\sigma_s[R_p(X,s,s')] = \frac{x_s}{\beta} \frac{\sigma_s[x_{s'}]\mathcal{L}'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}\mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{A.57}$$

which is increasing in $\mathcal{L}'(X,s)$ and decreasing in x_s.

Equilibrium. Combining supply and demand for risk, we obtain

$$\frac{x_s}{\beta} \frac{\sigma_s[x_{s'}]\mathcal{L}'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}\mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}} = \frac{1+\nu-\alpha}{1+\nu} \frac{x_s}{\beta} \frac{\sigma_s[x_{s'}]\mathcal{L}'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s \mathcal{L}^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}\mathcal{L}^{\frac{1+\nu}{1+\nu-\alpha}}} \left[\frac{p_{sH}(X,s)}{\mathcal{L}'(X,s) - x_L} - \frac{p_{sL}(X,s)}{x_H - \mathcal{L}'(X,s)}\right]. \tag{A.58}$$

The left-hand side is strictly increasing in $\mathcal{L}'(X,s)$, while the right-hand side is strictly decreasing in $\mathcal{L}'(X,s)$ in the interval $x_L < \mathcal{L}'(X,s) < x_H$. The right-hand side converges to $+\infty$ as $\mathcal{L}'(X',s)$ approaches x_L from above, and it converges to $-\infty$ as $\mathcal{L}'(X,s)$ approaches x_H from below. Therefore, there exists a unique value of $\mathcal{L}'(X,s)$ solving the equation above in this interval. Note that the two curves intersect again for $\mathcal{L}'(X,s) > x_H$, which can be seen by noticing that the right-hand is decreasing in $\mathcal{L}'(X,s)$ for $\mathcal{L}'(X,s) > x_H$ and converges to $+\infty$ as $\mathcal{L}'(X,s)$ approaches x_H from above. Therefore, the economically relevant solution corresponds to the smallest of the two points of intersection.

Rearranging the expression above, we obtain

$$1 = \frac{1+\nu-\alpha}{1+\nu}\mathcal{L}'(X,s)\frac{p_{sH}(X,s)(x_H - \mathcal{L}'(X,s)) - p_{sL}(X,s)(\mathcal{L}'(X,s) - x_L)}{(\mathcal{L}'(X,s) - x_L)(x_H - \mathcal{L}'(X,s))}. \tag{A.59}$$

We then obtain a quadratic equation for $\mathcal{L}'(X,s)$:

$$\frac{\alpha}{1+\nu}\mathcal{L}'(X,s)^2 - \left[\left(1 - \frac{1+\nu-\alpha}{1+v}p_{sH}(X,s)\right)x_H + \left(1 - \frac{1+\nu-\alpha}{1+v}p_{sL}(X,s)\right)x_L\right]\mathcal{L}'(X,s) + x_L x_H = 0 \tag{A.60}$$

The equilibrium value is given by the smallest root of the equation above.

□

A.7 Proof of Proposition 4

Proof. We start by deriving the return on the investor's portfolio. Given that markets are complete, there exists a replicating portfolio $\omega^r(X,s)$ such that

$$R_r(X,s,s') = \omega^r(X,s)R_p(X,s,s') + (1-\omega^r(X,s))R_b(X,s) \Rightarrow R_r^e(X,s,s') = \omega^r(X,s)R_p^e(X,s,s'), \tag{A.61}$$

where $\omega^r(X,s) = \frac{\sigma_s[R_r(X,s,s')]}{\sigma_s[R_p(X,s,s')]} = \frac{\Delta R_r(X,s)}{\Delta R_p(X,s)}$ We can then write the return on the portfolio of investor i as follows:

$$R_{i,n}(X,s,s') = \omega_i(X,s)R_{r,t}^e(X,s,s') + R_b(X,s) = \omega_i(X,s)\frac{\Delta R_r(X,s)}{\Delta R_p(X,s)}R_{p,t}^e(X,s,s') + R_b(X,s). \tag{A.62}$$

Using condition (A.24) and the expression for the SDF, we obtain

$$R_{i,n}^e(X,s,s') = R_b(X,s)\left[\frac{p_{sH}^i}{\mathcal{L}'(X,s)-x_L} - \frac{p_{sL}^i}{x_H-\mathcal{L}'(X,s)}\right]\frac{\Delta x R_p^e(X,s,s')}{\Delta R_p(X,s)} \tag{A.63}$$

$$= R_b(X,s)\left[\frac{p_{sH}^i}{\mathcal{L}'(X,s)-x_L} - \frac{p_{sL}^i}{x_H-\mathcal{L}'(X,s)}\right](x_{s'}-\mathcal{L}'(X,s)). \tag{A.64}$$

The return on the portfolio is then given by

$$R_{i,n}(X,s,L) = \frac{\Delta x R_b(X,s)}{x_H-\mathcal{L}'(X,s)}p_{sL}^i, \qquad R_{i,n}(X,s,H) = \frac{\Delta x R_b(X,s)}{\mathcal{L}'(X,s)-x_L}p_{sH}^i. \tag{A.65}$$

Wealth share dynamics. The share of wealth of investor i is given by

$$\eta_i'(X,s,s') = \frac{\eta_i R_{i,n}(X,s,s')}{\sum_{j=1}^I \eta_j R_{j,n}(X,s,s')} = \frac{\eta_i p_{ss'}^i}{\sum_{j=1}^I \eta_j p_{ss'}^j} = \eta_i\frac{p_{ss'}^i}{p_{ss'}(X)}. \tag{A.66}$$

Long-run wealth dynamics. Note that the wealth share is a (bounded) martingale under market beliefs:

$$p_{sH}(X)\eta_i'(X,s,H) + p_{sL}(X)\eta_i'(X,s,L) = \eta_i(p_{sH}^i + p_{sL}^i) = \eta_i. \tag{A.67}$$

Therefore, from the martingale convergence theorem, the wealth share of investor i

converges. This implies that, for every ϵ, there exits T such that

$$|\eta_{i,T+1} - \eta_{i,T}| < \epsilon \iff \eta_i \left| \frac{p^i_{ss'} - p_{ss'}(X)}{p_{ss'}(X)} \right| < \epsilon, \tag{A.68}$$

almost surely, where the economy is at the state (X, s) at period T.

This implies that either η_i converges to zero or $p^i_{ss'}$ converges to $p_{ss}(X)$. If $p^i_{ss'} \neq p^j_{ss'}$ for any $i, j \in \mathcal{I}$, $i \neq j$, then the wealth share of a single investor converges to one.

By definition, $p^i_{sH} > p_{sH}(X)$ for an optimistic investor in state (X, s), then the wealth share of optimists increase in the good state and decline in the bad state. This implies that market beliefs evolve according to

$$p_{s'H}(X') = \sum_{i=1}^{I} \eta'_i p^i_{s'H} = \sum_{i=1}^{I} \eta_i \frac{p^i_{ss'}}{p_{ss'}(X)} p^i_{s'H}, \tag{A.69}$$

where

$$p_{HH}(X') = \sum_{i=1}^{I} \eta_i \frac{p^i_{sH}}{p_{sH}(X)} p^i_{HH} \geq p_{sH}(X), \qquad p_{LH}(X') = \sum_{i=1}^{I} \eta_i \frac{p^i_{sL}}{p_{sL}(X)} p^i_{LH} \leq p_{sH}(X). \tag{A.70}$$

This implies that the relative wealth share for investors i and j is given by

$$\frac{\eta'_i(X, s, s')}{\eta'_j(X, s, s')} = \frac{\eta_i}{\eta_j} \frac{p^i_{ss'}}{p^j_{ss'}}. \tag{A.71}$$

Suppose investor j beliefs coincides with the objective measure. Then, the ratio above is a martingale:

$$\mathbb{E}_s \left[\frac{\eta'_i}{\eta'_j} \right] = p_{sL} \frac{\eta_i}{\eta_j} \frac{p^i_{sL}}{p_{sL}} + p_{sH} \frac{\eta_i}{\eta_j} \frac{p^i_{sH}}{p_{sH}} = \frac{\eta_i}{\eta_j}. \tag{A.72}$$

If the wealth of investor j is bounded away from zero, then the above martingale is bounded and, from the martingale convergence theorem, it converges almost surely. □

A.8 Proof of Corollary 2

Proof. Consider an economy that starts at $s = H$ with wealth distribution $\{\eta_i\}_{i=1}^{I}$ which switches to the low state after either one period (early transition) or two periods (late transition). Market beliefs on the low state in the case of an early transition are given by

$$p_{LH}(X') = \sum_{i=1}^{I} \eta_i \frac{p^i_{HL}}{p_{HL}(X)} p^i_{LH}, \tag{A.73}$$

and market beliefs on the low state in the case of a late transition are given by

$$p_{LH}(X'') = \sum_{i=1}^{I} \eta'_i \frac{p^i_{HL}}{p_{HL}(X')} p^i_{LH}, \tag{A.74}$$

where $\eta'_i = \eta_i \frac{p^i_{HH}}{p_{HH}(X)}$.

Note that if investor i is optimistic, $p^i_{HH} > p_{HH}(X)$, then $\eta'_i > \eta_i$ and $p^{-i}_{HL}(X') \leq p^{-i}_{HL}(X)$, where $p^{-i}_{HL}(X) \equiv \frac{1}{1-\eta_i} \sum_{j \neq i} \eta_j p^j_{HL}$. This implies that the following inequality holds:

$$\eta'_i \frac{p^i_{HL}}{p_{HL}(X')} = \frac{\eta'_i p^i_{HL}}{\eta'_i p^i_{HL} + (1-\eta'_i) p^{-i}_{HL}(X')} > \frac{\eta_i p^i_{HL}}{\eta_i p^i_{HL} + (1-\eta_i) p^{-i}_{HL}(X)} = \eta_i \frac{p^i_{HL}}{p_{HL}(X)}. \tag{A.75}$$

Therefore, there is more weight on the beliefs of investors who were optimistic in the original state in the case of a late transition. In the case of rank-preserving beliefs, these agents are also optimistic in the low state, so the market is more optimistic under a late transition:

$$p_{LH}(X'') > p_{LH}(X'). \tag{A.76}$$

Alternatively, the market is now more pessimistic after a late transition in the case of rank-alternating beliefs:

$$p_{LH}(X'') < p_{LH}(X'). \tag{A.77}$$

A similar argument shows that, under rank-preserving beliefs, the market is more pessimistic after a late transition when the economy starts at state $s = L$:

$$p_{HH}(X'') = \sum_{i=1}^{I} \eta'_i \frac{p^i_{LH}}{p_{LH}(X')} p^i_{HH} < \sum_{i=1}^{I} \eta_i \frac{p^i_{LH}}{p_{LH}(X)} p^i_{HH} = p_{HH}(X), \tag{A.78}$$

where $\eta'_i = \eta_i \frac{p^i_{LL}}{p_{LL}(X)}$. Alternatively, the market is more optimistic under a late transition in the case of rank-alternating beliefs.

□

B Trading volume

B.1 Belief Taxanomy and Trading Volume

We consider next the implications of heterogeneous beliefs regarding stock turnover, a measure of trading volume. To compute the stock turnover, we first map the portfolio holdings of the surplus claim, ω_i, into the effective number of shares on firm equity in the primitive economy. This mapping is straightforward in the case of linear labor disutility, $\nu = 0$ because human wealth is zero in this case.[27] To simplify the exhibition, we adopt this assumption for the rest of the section. The traded volume in this case is:

$$\tau_t = \frac{1}{2}\sum_{i=1}^{I} |\omega_{i,t}\eta_{i,t} - \omega_{i,t-1}\eta_{i,t-1}|.$$

This formula shows that the volume traded depends on the level of disagreement.

We consider a small deviation from homogeneous beliefs to study the effect of belief dispersion on volume. We express investor i's beliefs as follows

$$p^i_{ss'} = p^*_{ss'} + \delta^i_{ss'}\epsilon,$$

where $\delta^i_{sH} + \delta^i_{sL} = 0$ and ϵ captures belief heterogeneity as discussed earlier. Also, for parsimony, set $p^*_{ss'} = \frac{1}{2}$, such that, in the absence of heterogeneity, beliefs are iid and symmetric—the proofs hold for general common belief case.

The portfolio share of investor i is:

$$\omega_i(X, s; \epsilon) = 1 + \kappa_\omega \left[p^i_{sH} - \overline{p}^m_{sH}(X)\right] + \mathcal{O}(\epsilon^2),$$

where κ_ω is a positive constant. This expression showcases how optimistic investors, for whom $p^i_{sH} > \overline{p}^m_{sH}(X)$, are levered up in stocks.

The following lemma characterizes the trading behavior of a given investor.

Lemma 3. *Consider current and future states s and s'. The effect of a perturbation in ϵ on the trades of investor i is:*

$$\Delta S_i(X, s, s'; \epsilon) = \underbrace{\Delta\eta_i(X, s, s')}_{\textit{rebalancing effect}} + \underbrace{\Delta\omega_i(X, s, s')\eta_i}_{\textit{change-in-beliefs effect}} + \mathcal{O}(\epsilon^2), \tag{B.1}$$

[27] The share of wealth invested in stocks is ω_i, given that human wealth is equal to zero, $\mathcal{H}_i = 0$, and $R_e(X, s, s') = R_r(X, s, s')$ under this assumption.

as the economy switches from state (X,s) to (X',s'), where

$$\Delta\eta_i(X,s,s') \equiv \eta_i \frac{p^i_{ss'} - \overline{p}^m_{ss'}(X)}{p^*_{ss'}}, \qquad \Delta\omega_i(X,s,s') \equiv \kappa_\omega \left[p^i_{s'H} - \overline{p}^m_{s'H}(X) - (p^i_{sH} - \overline{p}^m_{sH}(X)) \right].$$

Expression (B.1) reveals two effects. The *rebalancing effect* captures the extent to which investors trade after a change in the state to keep portfolio shares constant: investors who put more likelihood on the realized state relative to the market belief, increased (decreased) their wealth share. Thus, they must buy (sell) the risky asset when that state is realized, to keep the portfolio share constant. Of course, as the economy evolves from s to s', portfolio shares themselves change as beliefs are modified. The *change-in-beliefs effect* captures the trade that follows the change in portfolio shares as the state changes. The change-in-beliefs effect equals zero if $s = s'$, as individual beliefs are constant.

In tandem, the effects of rebalancing and change-in-beliefs determine equilibrium turnover.

Proposition 5 (Turnover)**.** *The economy's turnover, as it switches from state (X,s) to state (X',s'), is given by*

$$\tau(X,s,s';\epsilon) = \frac{1}{2}\sum_{i=1}^{I} \eta_i \left| \frac{p^i_{ss'} - \overline{p}^m_{ss'}(X)}{p^*_{ss'}} + \kappa_\omega \left[p^i_{s'H} - \overline{p}^m_{s'H}(X) - (p^i_{sH} - \overline{p}^m_{sH}(X)) \right] \right| + \mathcal{O}(\epsilon^2). \quad \text{(B.2)}$$

Proposition 5 provides a characterization of turnover. When $s = s'$, the change-in-beliefs effect vanishes; turnover is driven solely by the rebalancing effect:

$$\tau(X,s,s';\epsilon) = \frac{1}{2}\sum_{i=1}^{I} \eta_i \frac{|p^i_{ss'} - \overline{p}^m_{ss'}(X)|}{p^*_{ss'}} + \mathcal{O}(\epsilon^2).$$

Thus, when there is no change in the state of the economy, turnover is proportional to the average absolute deviation of beliefs. The formula is consistent with the evidence in Section C.4, which shows that dispersion on subjective beliefs about cash flows is correlated with stock market turnover.

The change-in-beliefs effect emerges when the economy switches states, that is, when $s \neq s'$. This effect may either amplify or dampen the rebalancing effect, depending on the type of belief and the direction of change in the economy. For instance, suppose that $s = H$ and $s' = L$ and beliefs are rank-alternating. Optimistic investors lose wealth as the economy switches to a bad state. The rebalancing effect implies that they must sell some risky assets to maintain their portfolio shares once stocks lose value. These investors also become pessimists in downturns, leading them to sell even more stocks. Thus, the two effects go in the same direction, amplifying the impact on the turnover when the economy switches from high to low states. The two effects are opposite when $s = L$ and $s' = H$. Pessimists become optimistic as the economy switches to the high state, which induces them to increase their stock portfolio share. At the same time, the rebalancing effect dictates that they sell stocks once stocks appreciate to keep the portfolio balanced.

Connecting with the Turnover Evidence. It is convenient to express heterogeneity in beliefs, p^i_{ss}, in terms of heterogeneity in the perceived *persistence* of fundamentals. Assuming investors agree on the unconditional mean of x_t, $\overline{x}$, we can write $\mathbb{E}_{i,t}[x_{t+1}] - \overline{x} = \theta_i(x_t - \overline{x})$, where θ_i is a function of $p^i_{ss'}$. The following corollary shows heterogeneous beliefs lead to larger turnover rates as the economy switches from booms to recessions.

Corollary 3. *Suppose investors agree on the unconditional mean of x_t, i.e. $p^i_{LH}/p^i_{HL} = \overline{p}_H/\overline{p}_L$ and that the following condition is satisfied: $p^*_{ss'} = \overline{p}_H = \frac{1}{2}$. Turnover as the economy switches from s to s' is given by*

$$\tau(X, H, L; \epsilon) = \frac{\zeta(s, s')}{2} \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| + \mathcal{O}(\epsilon^2), \tag{B.3}$$

where

$$\zeta(s, s') \equiv \begin{cases} \kappa_\omega + 1, & \text{if } s = H \text{ and } s' = L \\ |\kappa_\omega - 1|, & \text{if } s = L \text{ and } s' = H \\ 1, & \text{if } s = s' \end{cases}$$

The key message from Corollary 3 is that turnover increases in belief dispersion and, furthermore, that the effect is more pronounced during busts. Both predictions are in line with the evidence discussed in Section C.4. The assumption of rank-alternating beliefs is important to obtain this asymmetric effect. If investors have rank-preserving beliefs, where they are equally optimistic or pessimistic in both states, so $\tilde{\delta}^i_{s'H} = \tilde{\delta}^i_{sH}$ even for $s' \neq s$, then the change-in-beliefs effect will be equal to zero and we would not obtain a stronger response of turnover to disagreement during bad times. Therefore, rank-alternating beliefs are key to capturing the dynamics of stock market turnover.

B.2 Proof of Lemma 3

Proof. The portfolio share of stocks for a type-i investor is defined as $\omega_{i,t} \equiv \frac{Q_t S_{i,t}}{N_{i,t}(1-c_{i,t})}$, so $S_{i,t} = \frac{\omega_{i,t} N_{i,t}(1-c_{i,t})}{Q_t}$. Given that $1 - c_{i,t} = \beta$ and $Q_t = \beta P_t$, we obtain $\mu_{i,t} S_{i,t} = \frac{\omega_{i,t}\mu_i N_{i,t}}{P_t} = \omega_{i,t}\eta_{i,t}$. Shares traded by type-$i$ investors are given by $\mu_{i,t}|S_{i,t} - S_{i,t-1}| =$

$|\omega_{i,t}\eta_{i,t} - \omega_{i,t-1}\eta_{i,t-1}|$. Trading volume is then given by

$$\tau_t = \frac{1}{2}\sum_{i=1}^{I} |\omega_{i,t}\eta_{i,t} - \omega_{i,t-1}\eta_{i,t-1}|. \tag{B.4}$$

In recursive notation, we can write

$$\tau(X,s,s') = \frac{1}{2}\sum_{i=1}^{I} |\omega_i(X',s')\eta_i'(X,s,s') - \omega_i(X,s)\eta_i|, \tag{B.5}$$

where $X' = \chi(X,s,s')$ and $\eta_i'(X,s,s') = \eta_i \frac{p^i_{ss'}}{p_{ss'}(X)}$.

Solving for the portfolio share. Using the expression for the economy-wide SDF and Equation (A.24), we can write the portfolio share as follows

$$\omega_i(X,s) = p^i_{sH}\frac{R_b(X,s)}{|R^e_r(X,s,L)|} - p^i_{sL}\frac{R_b(X,s)}{R^e_r(X,s,H)}. \tag{B.6}$$

The return on the risky and riskless assets can be written as follows:

$$R^e_r(X,s,s') = \frac{x_s}{\beta}\frac{(x_{s'} - \mathcal{L}'(X,s))\mathcal{L}'(X,s)^{\frac{\alpha}{1-\alpha}}}{x_s\mathcal{L}^{\frac{\alpha}{1-\alpha}} - \alpha\mathcal{L}^{\frac{1}{1-\alpha}}}, \qquad R_b(X,s) = (1-\alpha)\,\frac{x_s}{\beta}\frac{\mathcal{L}'(X,s)^{\frac{1}{1-\alpha}}}{x_s\mathcal{L}^{\frac{\alpha}{1-\alpha}} - \alpha\mathcal{L}^{\frac{1}{1-\alpha}}}, \tag{B.7}$$

Combining the previous expressions, we obtain

$$\omega_i(X,s) = (1-\alpha)\left[p^i_{sH}\frac{\mathcal{L}'(X,s)}{\mathcal{L}'(X,s) - x_L} - p^i_{sL}\frac{\mathcal{L}'(X,s)}{x_H - \mathcal{L}'(X,s)}\right], \tag{B.8}$$

which is strictly decreasing in $\mathcal{L}'(X,s)$ and $\omega_i(X,s) > 1$ if and only if $p^i_{sH} > p_{sH}(X)$.

Turnover is then given by

$$\tau(X,s,s') = (1-\alpha)\sum_{i=1}^{I}\eta_i\left|\left(\frac{p^i_{s'H}\mathcal{L}'(X',s')}{\mathcal{L}'(X',s') - x_L} - \frac{p^i_{s'L}\mathcal{L}'(X',s')}{x_H - \mathcal{L}'(X',s')}\right)\frac{p^i_{ss'}}{p_{ss'}(X)} - \left(\frac{p^i_{sH}\mathcal{L}'(X,s)}{\mathcal{L}'(X,s) - x_L} - \frac{p^i_{sL}\mathcal{L}'(X,s)}{x_H - \mathcal{L}'(X,s)}\right)\right| \tag{B.9}$$

Perturbation. It is useful to parameterize the dispersion in beliefs as follows:

$$p^i_{ss'} = p^*_{ss'} + \epsilon\delta^i_{ss'}, \tag{B.10}$$

where $\delta^i_{sH} + \delta^i_{sL} = 0$. If $\epsilon = 0$, then there is no belief heterogeneity and $\tau(X,s,s') = 0$. We consider next how turnover depends on belief heterogeneity for small deviations of this benchmark, that is, for ϵ close to zero.

Notice that all equilibrium variables now depend on ϵ. For instance, the average probability of the high state can be written as

$$p_{sH}(X;\epsilon) = p^*_{sH} + \delta_{sH}(X)\epsilon + \mathcal{O}(\epsilon^2), \tag{B.11}$$

where $\delta_{sH}(X) \equiv \sum_{i=1} \eta_i \delta^i_{sH}$. Risk-neutral expectation of productivity growth is a function of $\mathcal{L}'(X,s;\epsilon) = f_s(p_{sH}(X))$, where $f_s(p)$ satisfies the condition

$$1 = (1-\alpha)\left[p\frac{f_s(p)}{f_s(p)-x_L} - (1-p)\frac{f_s(p)}{x_H - f_s(p)}\right] \Rightarrow f'_s(p) = \frac{\frac{f_s(p)}{f_s(p)-x_L} + \frac{f_s(p)}{x_H-f_s(p)}}{p\frac{x_L}{(f(p)-x_L)^2} + (1-p)\frac{x_H}{(x_H-f_s(p))^2}}. \tag{B.12}$$

Let $\mathcal{L}^*(X,s) \equiv \mathcal{L}'(X,s;0)$ denote the value of $\mathcal{L}'(X,s)$ when $\epsilon = 0$. In this case, we can drop the dependence on X and simply write $\mathcal{L}^*(s)$, as $\mathcal{L}'(X,s)$ would only depend on the state s. We can then expand $\mathcal{L}'(X,s;\epsilon)$ in ϵ to obtain:

$$\mathcal{L}'(X,s;\epsilon) = \mathcal{L}^*(s) + \tilde{\mathcal{L}}(X,s)\epsilon + \mathcal{O}(\epsilon^2), \tag{B.13}$$

where $\tilde{\mathcal{L}}(X,s) = f'(p^*_{sH})\sum_{i=1}^I \eta_i \delta^i_{sH}$, where $f'(\cdot) > 0$.

We can then write the portfolio share of investor i as follows

$$\omega_i(X,s;\epsilon) = 1 + \left[\theta_{\omega,1}(s)\delta^i_{sH} - \theta_{\omega,2}(s)\delta_{sH}(X)\right]\epsilon + \mathcal{O}(\epsilon^2), \tag{B.14}$$

where $\theta_{\omega,1}(s) > 0$ and $\theta_{\omega,2}(s) > 0$

$$\theta_{\omega,1}(s) \equiv (1-\alpha)\left(\frac{\mathcal{L}^*(s)}{\mathcal{L}^*(s)-x_L} + \frac{\mathcal{L}^*(s)}{x_H - \mathcal{L}^*(s)}\right) \tag{B.15}$$

$$\theta_{\omega,2}(s) \equiv (1-\alpha)\left[\frac{p^*_{sH}x_L}{(\mathcal{L}^*(s)-x_L)^2} + \frac{p^*_{sL}x_H}{(x_H-\mathcal{L}^*(s))^2}\right] f'(p^*_{sH}). \tag{B.16}$$

Using the expression for $f'(\cdot)$, we obtain that $\theta_{\omega,1} = \theta_{\omega,2}$. We can then write $\omega_i(X,s;\epsilon)$ as follows:

$$\omega_i(X,s;\epsilon) = 1 + \theta_{\omega,1}(s)\left[\delta^i_{sH} - \delta_{sH}(X)\right]\epsilon + \mathcal{O}(\epsilon^2), \tag{B.17}$$

The evolution of wealth is given by

$$\eta'_i(X,s,s';\epsilon) = \eta_i + \eta_i\frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}}\epsilon + \mathcal{O}(\epsilon^2) \tag{B.18}$$

Let $p_H(X,s,s';\epsilon) = \sum_{i=1}^I \eta'_i(x,s,s';\epsilon)p^i_{s'H}$ denote the market-implied probability of the high state after a transition to state s', then

$$p_H(X,s,s';\epsilon) = p^*_{s'H} + \delta_{s'H}(X)\epsilon + \mathcal{O}(\epsilon^2), \tag{B.19}$$

where $\delta_{s'H}(X) \equiv \sum_{i=1}^{I} \eta_i \frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}} p^*_{s'H} + \sum_{i=1}^{I} \eta_i \delta^i_{s'H} = \sum_{i=1}^{I} \eta_i \delta^i_{s'H}$.

The portfolio share next period is given by

$$\omega_i'(X,s,s';\epsilon) = 1 + \theta_{\omega,1}(s')\left[\delta^i_{s'H} - \delta_{s'H}(X)\right]\epsilon + \mathcal{O}(\epsilon^2). \tag{B.20}$$

Investor i's net purchases of shares is given by

$$\begin{aligned}\Delta S_i(X,s,s';\epsilon) = \eta_i &\left[\frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}} + \theta_{\omega,1}(s')\left(\delta^i_{s'H} - \delta_{s'H}(X)\right)\right]\epsilon \\ &- \theta_{\omega,1}(s)\eta_i\left[\delta^i_{sH} - \delta_{sH}(X)\right]\epsilon + \mathcal{O}(\epsilon^2)\end{aligned} \tag{B.21}$$

For simplicity, suppose that investors believe productivity growth to be iid in the reference economy, that is, $p^*_{Ls'} = p^*_{Hs'}$. We can then write the

$$\Delta S_i(X,s,s';\epsilon) = \left[\underbrace{\Delta\tilde{\omega}_i(X,s,s')\eta_i}_{\text{change-in-beliefs effect}} + \underbrace{\Delta\tilde{\eta}_i(X,s,s')}_{\text{rebalancing effect}}\right]\epsilon + \mathcal{O}(\epsilon^2), \tag{B.22}$$

where

$$\Delta\tilde{\omega}_i(X,s,s') \equiv \theta_{\omega,1}\left[\left(\delta^i_{s'H} - \delta_{s'H}(X,s)\right) - \left(\delta^i_{sH} - \delta_{sH}(X)\right)\right] \tag{B.23}$$

$$\Delta\tilde{\eta}_i(X,s,s') \equiv \eta_i \frac{\delta^i_{ss'} - \delta_{ss'}(X)}{p^*_{ss'}}. \tag{B.24}$$

□

B.3 Proof of Proposition 5 and Corollary 3

Proof. Turnover is given by

$$\tau(X,s,s';\epsilon) = \frac{1}{2}\sum_{i=1}^{I} \eta_i \left|\frac{\tilde{\delta}^i_{ss'}}{p^*_{ss'}} + \kappa_\omega\left(\tilde{\delta}^i_{s'H} - \tilde{\delta}^i_{sH}\right)\right|\epsilon + \mathcal{O}(\epsilon^2), \tag{B.25}$$

where $\tilde{\delta}^i_{ss'} = \delta^i_{ss'} - \delta_{ss'}(X)$.

Suppose $s = s'$, then

$$\begin{aligned}
\tau(X, s, s'; \epsilon) &= \frac{1}{2} \sum_{i=1}^{I} \eta_i \frac{\left|\tilde{\delta}^i_{ss'}(X)\right|}{p^*_{ss'}} \epsilon + \mathcal{O}(\epsilon^2) && \text{(B.26)} \\
&= \frac{1}{2} \left[\sum_{i=1}^{I} \eta_i \frac{\tilde{\delta}^i_{ss'}(X)}{p^*_{ss'}} \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) \geq 0} - \sum_{i=1}^{I} \eta_i \frac{\tilde{\delta}^i_{ss'}(X)}{p^*_{ss'}} \mathbf{1}_{\tilde{\delta}^i_{ss'} < 0} \right] \epsilon + \mathcal{O}(\epsilon^2) && \text{(B.27)} \\
&= \frac{1}{2} \left[\eta_B \frac{\tilde{\delta}^B_{ss'}(X)}{p^*_{ss'}} + \eta_S \frac{|\tilde{\delta}^S_{ss'}(X)|}{p^*_{ss'}} \right] \epsilon + \mathcal{O}(\epsilon^2). && \text{(B.28)}
\end{aligned}$$

where

$$\eta_B \equiv \sum_{i=1}^{I} \eta_i \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) \geq 0}, \qquad \tilde{\delta}^B_{ss'}(X) \equiv \frac{1}{\eta_B} \sum_{i=1}^{I} \eta_i \tilde{\delta}^i_{ss'}(X) \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) \geq 0}, \tag{B.29}$$

$$\eta_S \equiv \sum_{i=1}^{I} \eta_i \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) < 0}, \qquad \tilde{\delta}^S_{ss'}(X) \equiv \frac{1}{\eta_S} \sum_{i=1}^{I} \eta_i \tilde{\delta}^i_{ss'}(X) \mathbf{1}_{\tilde{\delta}^i_{ss'}(X) < 0}. \tag{B.30}$$

We can write turnover in this case as follows

$$\tau(X, s, s'; \epsilon) = \eta_B \eta_S \frac{\delta^B_{ss'}(X) + |\delta^S_{ss'}(X)|}{p^*_{ss'}} \epsilon + \mathcal{O}(\epsilon^2), \tag{B.31}$$

using the fact that $\delta_{ss'}(X) = \eta_B \delta^B_{ss'}(X) + \eta_S \delta^S_{ss'}(X)$.

Heterogeneous persistence. We consider next the special case where investors agree about the unconditional mean of x, but they disagree about the persistence of the aggregate productivity growth.

The stationary distribution of beliefs for investor i is given by

$$p^i_L = \frac{p^i_{HL}}{p^i_{LH} + p^i_{HL}}. \tag{B.32}$$

We assume that p^i_L is equalized across investors, so all investors agree about the unconditional mean of x_t. Note this implies that the likelihood ratio p^i_{LH}/p^i_{HL} is equalized across investors. The unconditional mean is given by

$$\overline{x} = \frac{p^i_{HL}}{p^i_{LH} + p^i_{HL}} x_L + \frac{p^i_{LH}}{p^i_{LH} + p^i_{HL}} x_H. \tag{B.33}$$

The expected value of x_{t+1} relative to the mean $\overline{x}$ conditional on $x_t = x_L$ is given by

$$\mathbb{E}_i[x_{t+1} - \overline{x} | x_t = x_L] = p^i_{LL}(x_L - \overline{x}) + p^i_{LH}(x_H - \overline{x}) \tag{B.34}$$
$$= \left[1 + p^i_{LH}\frac{x_H - x_L}{x_L - \overline{x}}\right](x_L - \overline{x}) \tag{B.35}$$
$$= \left[1 - (p^i_{LH} + p^i_{HL})\right](x_L - \overline{x}), \tag{B.36}$$

using the fact that $\overline{x} - x_L = \frac{p^i_{LH}}{p^i_{LH} + p^i_{HL}}(x_H - x_L)$

We obtain a similar expression conditioning on $x_t = x_H$ instead:

$$\mathbb{E}_i[x_{t+1} - \overline{x} | x_t = x_H] = p^i_{HL}(x_L - \overline{x}) + p^i_{HH}(x_H - \overline{x}) \tag{B.37}$$
$$= \left[1 - p^i_{HL}\frac{x_H - x_L}{x_H - \overline{x}}\right](x_H - \overline{x}) \tag{B.38}$$
$$= \left[1 - (p^i_{LH} + p^i_{HL})\right](x_H - \overline{x}), \tag{B.39}$$

using the fact that $x_H - \overline{x} = \frac{p^i_{HL}}{p^i_{LH} + p^i_{HL}}(x_H - x_L)$.

Let $\hat{x}_t = x_t - \overline{x}$, we can then write

$$\mathbb{E}_i[\hat{x}_{t+1} | \hat{x}_t] = \theta_i \hat{x}_t, \tag{B.40}$$

where $\theta_i \equiv 1 - (p^i_{LH} + p^i_{HL}) = p^i_{HH} - p^i_{LH}$.

Given that investors agree about the unconditional mean of x, we are able to pin down beliefs as a function of θ_i:

$$p^i_{LH} = \overline{p}_H(1 - \theta_i), \qquad p^i_{HH} = \overline{p}_H + \overline{p}_L\theta_i. \tag{B.41}$$

Corollary. Under the assumption investors agree about the unconditional mean of x_t, we have that

$$p^i_{LH} - p_{LH}(X) = -\overline{p}_H(\theta_i - \theta(X)), \qquad p^i_{HH} - p_{HH}(X) = \overline{p}_L(\theta_i - \theta(X)), \tag{B.42}$$

where $\overline{\theta}(X) \equiv \sum_{i=1}^{I} \eta_i \theta_i$.

Notice that we have that $\tilde{\delta}^i_{ss'}(X)\epsilon = p^i_{ss'} - p_{ss'}(X)$, which gives us

$$\tilde{\delta}^i_{LH}(X)\epsilon = -\overline{p}_H(\theta_i - \theta(X)), \qquad \tilde{\delta}^i_{HH}(X)\epsilon = (1 - \overline{p}_H)(\theta_i - \theta(X)). \tag{B.43}$$

We can then write turnover in the case $s = L$ and $s' = H$ as follows:

$$\tau(X, L, H; \epsilon) = \frac{1}{2}\left|\kappa_\omega - \frac{\overline{p}_H}{p_H^*}\right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| + \mathcal{O}(\epsilon^2). \tag{B.44}$$

Consider now the case $s = H$ and $s' = L$:

$$\tau(X, H, L; \epsilon) = \frac{1}{2}\left|\kappa_\omega + \frac{\overline{p}_L}{p_L^*}\right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| + \mathcal{O}(\epsilon^2), \tag{B.45}$$

Suppose now that $s = s' = L$, then

$$\tau(X, H, H; \epsilon) = \frac{1}{2}\left|\frac{\overline{p}_L}{p_H^*}\right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| \epsilon + \mathcal{O}(\epsilon^2) \tag{B.46}$$

$$\tau(X, L, L; \epsilon) = \frac{1}{2}\left|\frac{\overline{p}_H}{p_L^*}\right| \sum_{i=1}^{I} \eta_i |\theta_i - \theta(X)| \epsilon + \mathcal{O}(\epsilon^2). \tag{B.47}$$

□

C Estimating the Heterogeneity in Beliefs

C.1 The process for realized and expected earnings

Let $i \in \mathcal{I}$ denote a firm-analyst pair. We index both firm-level outcomes and the expectations of the analyst covering this firm by i. We denote (realized) earnings for firm i at period t by $e_{i,t}$ and the first-difference of realized earnings by $\Delta e_{i,t} = e_{i,t} - e_{i,t-1}$.[28] We denote aggregate earnings by e_t and the first-difference of aggregate earnings by Δe_t. Realized earnings follows the process:

$$\Delta e_{i,t} = \beta_i \Delta e_t + u_{i,t}, \tag{C.1}$$

where $u_{i,t} = \rho_i u_{i,t-1} + \epsilon_{i,t}$ and $\epsilon_{i,t} \sim \mathcal{N}(0, \sigma_\epsilon^2)$. The error term $\epsilon_{i,t}$ is assumed to be i.i.d. and independent of Δe_t. We assume that $\Delta e_{i,t}$ and Δe_t have already been de-meaned, so we can omit the intercept. We also assume that $\Delta e_{i,t}$ and Δe_t have been normalized to have unit variance.

[28] As $e_{i,t}$ can potentially be negative, we work with first differences instead of proportional differences, $\frac{\Delta e_{i,t}}{e_{i,t}}$, or log-differences, $\Delta \log(e_{i,t})$. By focusing on first differences, we do not have to drop firms which experience negative earnings, which is a significant fraction of our sample.

Given the formulation above, individual earnings depend on aggregate shocks, i.e. shocks that affect aggregate earnings, as well as idiosyncratic shocks, as captured by $u_{i,t}$. The parameters ρ_i controls the persistence of idiosyncratic shocks. Hence, firms are allowed to be heterogeneous on their exposure to the aggregate shock as well as the persistence of idiosyncratic shocks.

We assume that analysts understand that individual earnings follows the process (C.1), but they potentially disagree on the process followed by aggregated earnings. In particular, we assume that analyst i believe (in a dogmatic fashion) that Δe_t follows the following process:

$$\Delta e_t = \theta_i \Delta e_{t-1} + \nu_{i,t}, \tag{C.2}$$

where $\nu_{i,t}$ is an i.i.d. process given by $\nu_{i,t} \sim \mathcal{N}(0, \sigma_\nu^2)$. We assume that analysts agree on the unconditional mean for Δe_t, which we normalize to zero. This allow us to focus only on disagreement about the persistence of shocks to aggregate earnings.

The expected change in aggregate earnings using the subjective beliefs of analyst i is given by

$$\mathbb{E}_{i,t}[\Delta e_{t+1}] = \theta_i \Delta e_t, \tag{C.3}$$

where $\mathbb{E}_{i,t}[\cdot]$ denote the conditional expectation at t according to the subjective beliefs of analyst i.

We assume that Δe_t is perfectly observed by investors at time t, so differences in beliefs are controlled by θ_i. A relatively high value for θ_i implies that analyst i is more optimistic about aggregate earnings after a positive shock and more pessimistic after a negative shock, capturing a form of belief extrapolation.

Notice that expectations of changes in *individual* earnings depend on the degree of persistence of shocks to *aggregate* earnings θ_i:

$$\mathbb{E}_{i,t}[\Delta e_{i,t+1}] = \beta_i \theta_i \Delta e_t + \rho_i u_{i,t}. \tag{C.4}$$

Equation (C.4) shows that we can infer properties of the process for subjective beliefs on *aggregate* earnings using information on subjective beliefs about *individual* earnings. This is important as beliefs on aggregate earnings are not directly available.

C.2 Estimation procedure

We show next how to estimate $(\beta_i, \rho_i, \theta_i)$ in two stages. First, we estimate the parameters in Equation (C.1). In a second stage, we obtain the distribution of θ_i, using Equation (C.4) and the parameters estimated in the first stage.

First stage. Consider first Equation (C.1). We can rewrite the process for $\Delta e_{i,t}$ as follows:

$$\Delta e_{i,t} = \beta_i \Delta e_t + \rho_i \left(\Delta e_{i,t-1} - \beta_i \Delta e_{t-1} \right) + \epsilon_{i,t}, \tag{C.5}$$

where we used the fact that $u_{i,t} = \Delta e_{i,t} - \beta_i \Delta e_t$.

To ensure that $-1 < \rho_i < 1$, we consider the following change of variables. Assume that ρ_i is given by the a non-linear transformation of the parameter $\tilde{\rho}_i \in \mathbb{R}$: $\rho_i = -1 + 2\frac{\exp(\tilde{\rho}_i)}{1+\exp(\tilde{\rho}_i)} \in (-1,1)$. The parameters $(\beta_i, \tilde{\rho}_i)$ can in principle be estimated using, for instance, non-linear least squares for each company i. We proceed instead by estimating the parameters simultaneously for all i using Bayesian methods. The Bayesian approach is useful as it allow us to regularize the individual estimates and avoid overfitting, which can be a concern in settings where the length of the time series is not particularly long.[29]

Formally, we consider the following multi-level priors:

$$\beta_i \sim \mathcal{N}(\overline{\beta}, \sigma^2_\beta), \qquad \tilde{\rho}_i \sim \mathcal{N}(\overline{\rho}, \sigma^2_\rho), \tag{C.6}$$

The coefficients $(\overline{\beta}, \overline{\rho})$ and $(\sigma_\beta, \sigma_\rho)$ are referred to as *hyperparameters* and they have their own priors, which are given by

$$\overline{\beta} \sim \mathcal{N}(0, 1.50^2), \qquad \overline{\rho} \sim \mathcal{N}(0, 0.50^2), \tag{C.7}$$

and the standard-deviation for each parameter is assumed to follow a Half Student-t distribution with 3 degrees of freedom, a standard value for this class of models. These priors are set to be wide enough to capture the range of plausible values for the parameters.

The multi-level structure allow us to obtain a form of adaptive regularization. If (say) σ_β is very large, then the prior on β_i is not very informative, and this would be analogous to estimate β_i independently for each i. If $\sigma_\beta \approx 0$, then we have effectively a pooling estimator, where β_i will be the same for all i. For intermediate values of σ_β, the parameters are allowed to vary across units, but they are partially shrunk towards the population mean. The shrinkage of the parameters limits the effect of noise or measurement error, as the model is essentially skeptical of extreme values. Because σ_β is also an estimated parameter, the extent to which estimates are regularized is directly informed by the data.[30]

[29] This procedure is analogous to a ridge regression, where the estimates are regularized using a L2 penalty (see e.g. Hastie, Tibshirani, Friedman and Friedman, 2009). For a discussion of how regularized regressions can be reinterpreted as a Bayesian procedure, see e.g. Nagel (2021).

[30] For more details on how multi-level models provide a form of adaptive regularization, see e.g. the discussion in McElreath (2020).

Second stage. Consider next Equation (C.4), which relates subjective beliefs about individual earnings to realized aggregate and individual earnings. To capture the fact that (subjective) expectations are potentially measured with error, we assume that only a noisy version of the analyst's expectation is observed, which is given by $\hat{\mathbb{E}}_{i,t}[\Delta e_{i,t+1}] = \mathbb{E}_{i,t}[\Delta e_{i,t+1}] + \tilde{w}_{i,t}$. The measurement error $\tilde{w}_{i,t}$ is assumed to be a mean-zero normally distributed i.i.d. process with variance given by σ_w^2. Combining this measurement equation with Equation (C.4) and isolating the terms estimated in the first stage, we obtain the following estimating equation:

$$z_{i,t} = \alpha_i + \theta_i x_{i,t} + w_{i,t}, \tag{C.8}$$

where $z_{i,t} \equiv \hat{\mathbb{E}}_{i,t}[\Delta e_{i,t+1}] - \rho_i u_{i,t}$ and $x_{i,t} \equiv \beta_i \Delta e_t$. Notice that $z_{i,t}$ and $x_{i,t}$ are known at this stage, so it only remains to estimate $\theta_{i,t}$.

As before, we use a Bayesian multi-level model to adaptively regularize our estimates. We also consider the transformation $\theta_i = -1 + 2\frac{\exp(\tilde{\theta}_i)}{1+\exp(\tilde{\theta}_i)}$, where $\tilde{\theta}_i \in \mathbb{R}$, such that we can ensure that $\theta_i \in (-1, 1)$. We assume the following prior for $\tilde{\theta}_{i,t}$:

$$\theta_{i,t} \sim \mathcal{N}(\overline{\theta}, \sigma_\theta^2), \tag{C.9}$$

where $\overline{\theta} \sim \mathcal{N}(0, 0.5^2)$ and σ_θ follows a half Student-t distribution with 3 degrees of freedom.

C.3 Data and estimation results

Data. We use data from I/B/E/S on analysts expectations about firms' future earnings. For firms with coverage of more than one analyst, we use the consensus expectation for that firm. We drop firms with missing values for realized or expected earnings in more than 20% of the sample. We ended up with 579 firms covering the time period from March 1977 until December 2020, with a total of 44, 267 company-quarter pairs.

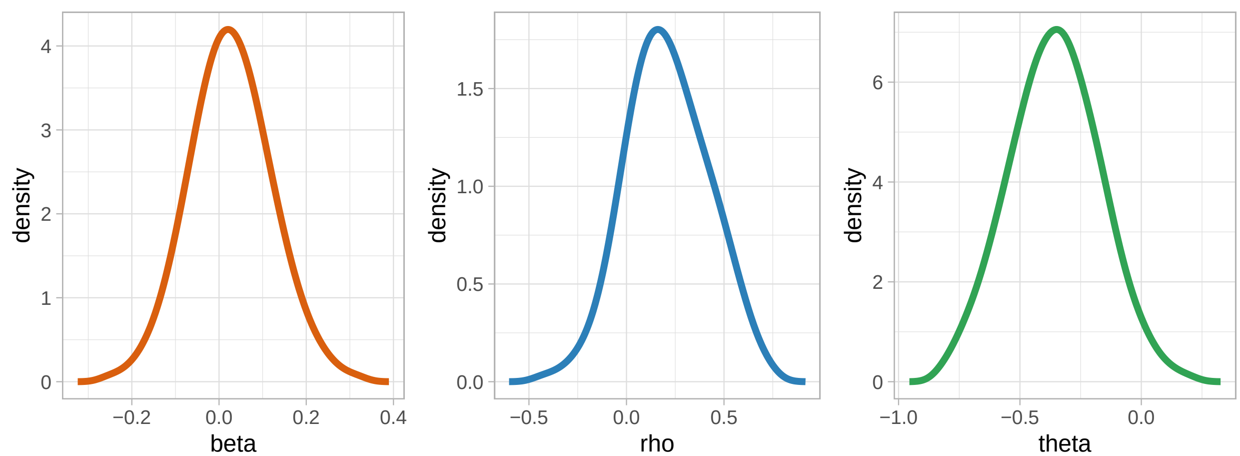
Model fitting and results. We sample the model using an extension of Hamiltonian Monte Carlo, the no-U-turn sampler (NUTS) by Hoffman, Gelman et al. (2014), as implemented in R Stan. Table 5 reports the posterior mean and 95% credible intervals for the cross-sectional mean and dispersion of parameters $(\beta_i, \rho_i, \theta_i)$. Because we have standardized all the variables, the parameter β_i captures the correlation between individual and aggregate earnings. The correlation is close to zero reflecting the fact that typically most of the variation in a company's earnings reflect idiosyncratic shocks. However, there

Table 5: Cross-sectional mean and dispersion of parameters

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat
$\overline{\mathbb{E}}[\beta_i]$	0.03	0.01	0.01	0.04	1.00
$\overline{\mathbb{E}}[\rho_i]$	0.45	0.02	0.41	0.50	1.00
$\overline{\mathbb{E}}[\theta_i]$	-0.48	0.12	-0.72	-0.24	1.00
$\overline{\sigma}[\beta_i]$	0.09	0.01	0.08	0.10	1.00
$\overline{\sigma}[\rho_i]$	0.47	0.02	0.43	0.51	1.00
$\overline{\sigma}[\theta_i]$	0.19	0.13	0.01	0.49	1.00

Note: Posterior mean and credible intervals (CI) for the cross-sectional mean, $\overline{\mathbb{E}}[x_i]$, and cross-sectional standard-deviation, $\overline{\sigma}[x_i]$, for parameters $x \in \{\beta, \rho, \theta\}$. Rhat is an indicator of the convergence of the chains during sampling. Rhat = 1 indicates convergence.

Figure 8: Kernel estimate of cross-sectional distribution of the different parameters



Note: Posterior mean of the kernel density for the cross-section of θ_i (left panel), ρ_i (middle panel), and θ_i (right panel).

is substantial heterogeneity in this parameter, with the cross-sectional dispersion being three times the average β_i. This can be seen in the left panel of Figure 8, which shows the posterior mean of the kernel density for β_i, where β_i ranges from -0.3 to 0.4. The average autocorrelation coefficient ρ_i is positive, but it is also very dispersed across firms, as shown in the middle panel of Figure 8. Finally, we have that θ_i is on average negative, which is consistent with the fact that Δe_t has a negative autocorrelation. However, the average subjective coefficient of autocorrelation is more negative than its objective counterpart, as $\mathbb{E}[\theta_i] = -0.48$ and we obtain a coefficient of autocorrelation of -0.28 for Δe_t using aggregate data. As before, we observe substantial heterogeneity in θ_i, as shown in the right panel of Figure 8.

C.4 Belief disagreement

We consider next a measure of belief disagreement. Notice that the expectation of analyst of aggregate earnings growth is given by $\mathbb{E}_i[\Delta e_{t+1}] = \theta_i \Delta e_t$. This motivates our definition of a *disagreement index DI_t*, which corresponds to the cross-sectional dispersion in beliefs about aggregate earnings growth:

$$DI_t = \underbrace{\overline{\sigma}[\theta_i] \times |\Delta e_t|}_{\overline{\sigma}[\mathbb{E}_i[\Delta e_{t+1}]]}. \tag{C.10}$$

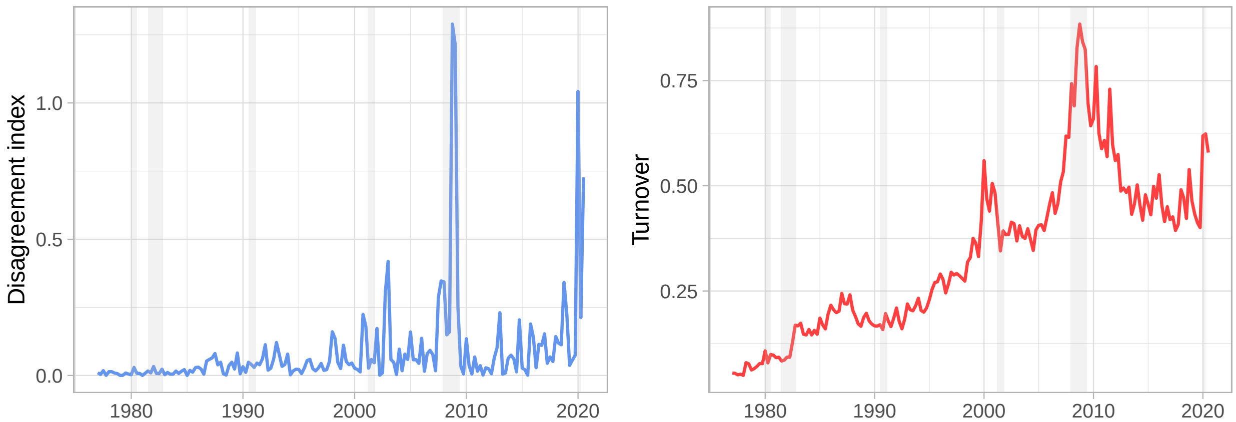
The disagreement index has two components. First, the cross-sectional dispersion in the parameter θ_i. If all analysts agree on the persistence of aggregate earnings growth, such that $\overline{\sigma}[\theta_i] = 0$, then the disagreement index would be equal to zero. Second, the absolute value of aggregate earnings growth, $|\Delta e_t|$. Given that Δe_t has been already demeaned, $|\Delta e_t|$ captures the distance of aggregate earnings growth to its mean. If aggregate earnings growth is already at its average value, $|\Delta e_t| = 0$, then disagreement on how Δe_t reverts to its plays no role in determining expectations. Therefore, the level of disagreement in the economy depends on the interaction between dispersion in beliefs and deviations of aggregate earnings growth from its mean.

The left panel of Figure 9 shows the time series of the disagreement index. The disagreement index is typically low during normal times, and it significantly spikes in periods of crises, where aggregate earnings growth deviates substantially from its average value.

Turnover. One important implication of theories with heterogeneous beliefs is that the level of disagreement is related to the amount of trading in the economy. To test this implication, we consider next a measure of trading activity, the (value-weighted) stock market turnover.[31] We measure the stock turnover—shares traded divided by shares outstanding—for individual securities on the New York and American Stock Exchanges from January 1977 to December 2021. We measure turnover at the quarterly frequency and compute an aggregate turnover measure using a value-weighted average (similar results are obtained by using an equal-weight measure). The right panel of Figure 9 shows the time series of turnover. We can observe that the turnover level changed significantly over time and that turnover has an important cyclical component.

[31] For a discussion of turnover as a measure of trading volume and its connection with standard portfolio theory, see Lo and Wang (2010).

Figure 9: Time series of the disagreement index and stock market turnover



Note: Left panel shows the time series of the disagreement index and the right panel shows the time-series of stock market turnover. The smooth line in the right panel is the HP-filter trend of turnover. The vertical bars represent NBER recessions.

Belief disagreement and turnover. We consider next the relationship between belief disagreement and turnover. Table 6 shows the result of a time-series regression of turnover on the disagreement index. As shown in Figure 9, the disagreement index series has a few outliers, in particular, during crisis periods. To ensure that the relationship between turnover and disagreement is not driven only by these extreme periods, we consider a sample where we exclude observations where the disagreement index is below the 2.5% percentile or above the 97.5% percentile. Column (1) shows that there is a strong statistically significant association between DI and turnover, where we compute Newey-West standard-errors with four lags. If the disagreement index goes from its 25% percentile to its 75% percentile, turnover increases by 8.0 percentage points, an increase of almost 30%. Column (2) tests whether this relationship is nonlinear by introducing a quadratic term, again in the example where we exclude outliers. We find that the quadratic term is not significant, consistent with a linear relationship. This can be verified visually in Figure 10, which shows the scatterplot of turnover and the disagreement index for this sample. Column (3) shows the regression of turnover on DI and DI^2 for the full sample. We find that the quadratic term is now statistically significant, indicating the necessity of considering a nonlinear relationship to capture the effect of the extreme crisis-level disagreement. The magnitude of the marginal effect of changing DI is similar to the linear case for large of values for the disagreement index. Therefore, we conclude that belief disagreement is strongly associated with stock market turnover.

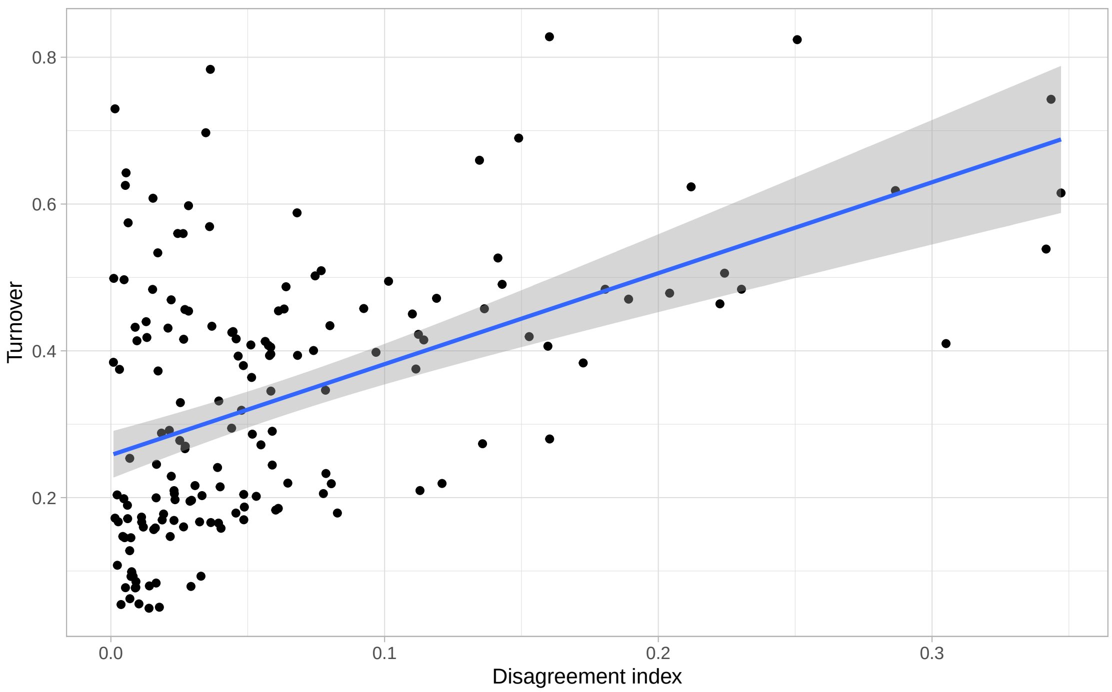
Table 6: Regression of turnover on disagreement index

Dependent Variable:		*turnover*	
Model:	(1)	(2)	(3)
Variables			
(Intercept)	0.2580***	0.2420***	0.2549***
	(0.03373)	(0.04375)	(0.0369)
DI	1.239***	1.798**	1.260***
	(0.2277)	(0.6277)	(0.2898)
DI^2		-2.068	-0.6879**
		(1.6920)	(0.2094)
Fit statistics			
Observations	165	165	175
R^2	0.24084	0.24786	0.30386
Adjusted R^2	0.23618	0.23857	0.29576

Newey-West standard-errors in parentheses (4 lags)
*Signif. Codes: ***: 0.01, **: 0.05, *: 0.1*

Note: Columns (1) and (2) .

Figure 10: Scatterplot of the disagreement index and stock market turnover



Note: Scatterplot of disagreement index and turnover for a sample without outliers.

Online Appendix (not for publication)

O1 The case with an arbitrary number of states

O1.1 Environment

We consider an extension of the model in Section 3 where aggregate productivity growth x_t takes N possible values, i.e. $x_t \in \{x^1, x^2, \ldots, x^N\} \equiv \mathcal{X}$, where $x^1 < x^2 < \ldots < x^N$. The objective probability of switching from state $s \in \{1, 2, \ldots, N\} \equiv \mathcal{S}$ to state $s' \in \mathcal{S}$ is denoted by $p_{ss'}$ and the corresponding subjective probability for household $i \in \mathcal{I}$ is denoted by $p^i_{ss'}$. Households can trade Arrow securities that pay off conditional on every possible state. We also assume that households die with probability κ and leave their financial wealth to their child, which will have type j with probability μ_j. This assumption ensures that a non-degenerate stationary distribution of wealth exists. The next proposition provides a characterization of the equilibrium in this N-state economy. The main conclusion is that the results of Section 3 are essentially unchanged in this more general setting.

Proposition O.1 (N-state economy). *Suppose that $x_t \in \mathcal{X}$, where x_t takes N possible values.*

i. The (scaled) household's problem can be written as follows

$$\frac{v_i(X,s)^{1-\psi^{-1}} - 1}{1-\psi^{-1}} = \max_{c_i, R'_{i,n}} (1-\beta)\frac{c_i^{1-\psi^{-1}} - 1}{1-\psi^{-1}} + \beta \frac{\mathbb{E}_i\left[(v_i(X', s')n')^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}} - 1}{1-\psi^{-1}}, \tag{O1.1}$$

subject to the flow budget constraint $n' = R'_{i,n}(1-c_i)$, the natural borrowing limit $n' \geq 0$, and the portfolio-return constraint

$$\mathbb{E}_s[\Lambda' R'_{i,n}] = 1. \tag{O1.2}$$

ii. Consumption-wealth ratio and the investor's SDF are given by

$$c_i(X,s) = \frac{(\beta^{-1}-1)^{\psi}\mathcal{R}_i(X,s)^{1-\psi}}{1 + (\beta^{-1}-1)^{\psi}\mathcal{R}_i(X,s)^{1-\psi}}, \tag{O1.3}$$

$$\Lambda_i(X,s,s') = \beta^{\theta}\left(\frac{c_i(\chi(X,s,s'),s')N'}{c_i(X,s)N}\right)^{-\frac{\theta}{\psi}} R_{i,n}(X,s,s')^{-(1-\theta)}, \tag{O1.4}$$

and the change-of-measure condition is given by

$$\Lambda_i(X,s,s') = \frac{p_{ss'}}{p^i_{ss'}}\Lambda(X,s,s'). \tag{O1.5}$$

iii. *Wages, hours, and profits are given by*

$$h(E) = \left(\frac{\alpha E}{\xi}\right)^{\frac{1}{1+\nu-\alpha}}, \qquad w(E) = \xi\left(\frac{\alpha E}{\xi}\right)^{\frac{\nu}{1+\nu-\alpha}}, \qquad \pi(E,s) = \left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\left[x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \alpha E^{\frac{1+\nu}{1+\nu-\alpha}}\right]. \tag{O1.6}$$

iv. *The law of motion of the endogenous aggregate state variables is given by*

$$E'(X,s) = \sum_{s'\in\mathcal{S}} \frac{p_{ss'}\Lambda(X,s,s')}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}\Lambda(X,s,\tilde{s})} x_{s'}, \tag{O1.7}$$

$$\eta_i'(X,s,s') = \frac{(1-\kappa)\eta_i R_{i,n}(X,s,s')(1-c_i(X,s))}{\sum_{j=1}^I \eta_j R_{j,n}(X,s,s')(1-c_j(X,s))} + \kappa\mu_i. \tag{O1.8}$$

v. *The market clearing conditions for consumption and the Arrow security for state $s\in\mathcal{S}$ are given by*

$$\sum_{i=1}^I \eta_i c_i(X,s) = \frac{x_s h(E)^\alpha - \xi\frac{h(E)^{1+\nu}}{1+\nu}}{P(X,s)}, \qquad \sum_{i=1}^I \tilde{\eta}_i R_{n,i}(X,s,s') = R_p(X,s,s'), \tag{O1.9}$$

where $\tilde{\eta}_i \equiv \frac{\eta_i(1-c_i(X,s)}{\sum_{j=1}^I \eta_j(1-c_j(X,s)}$.

Proof. See Online Appendix O3.1. □

An implication of the result above is that the LDF corresponds to the risk-neutral expectation of productivity growth. The following corollary shows that $E'(X,s)$ can be expressed as the expected productivity growth (under the objective probability measure) discounted by a risk premium.

Corollary 4. *Let $R_g(X,s,s')$ denote the return on a claim on productivity growth, then*

$$\log E'(X,s) = \log \mathbb{E}_s[x_{s'}] - \log \overline{R}^e_g(X,s), \tag{O1.10}$$

where $\overline{R}^e_g(X,s) \equiv \sum_{s'\in\mathcal{S}} p_{ss'}\frac{R_g(X,s,s')}{R_b(X,s)}$ is the risk premium on a claim on productivity growth.

Proof. The price of a claim on productivity growth is given by

$$P_g(X,s) = \mathbb{E}_s[\Lambda(X,s,s')x_{s'}], \tag{O1.11}$$

and the return on this claim is given by $R_g(X,s,s') = \frac{x_{s'}}{P_g(X,s)}$.

Expressing the pricing condition above in terms of covariances, we obtain

$$\mathbb{E}_s[R_g(X,s,s')] - R_b(X,s) = -Cov_s\left(\frac{\Lambda(X,s,s')}{\mathbb{E}_s[\Lambda(X,s,s')]}, \frac{x_{s'}}{P_g(X,s)}\right). \tag{O1.12}$$

Similarly, we can write $E'(X,s)$ in terms of covariances:

$$E'(X,s) = \mathbb{E}_s[x_{s'}] + Cov_s\left(\frac{\Lambda(X,s,s')}{\mathbb{E}_s[\Lambda(X,s,s')]}, x_{s'}\right). \tag{O1.13}$$

Using the fact that $P_g(X,s) = E'(X,s)/R_b(X,s)$, we can combine the expressions above to obtain

$$E'(X,s) = \mathbb{E}_s[x_{s'}] - \left(\frac{\mathbb{E}_s[R_g(X,s,s')]}{R_b(X,s)} - 1\right)E'(X,s) \Rightarrow E'(X,s) = \frac{\mathbb{E}_s[x_{s'}]}{\overline{R}^e_g(X,s)}. \tag{O1.14}$$

□

O1.2 Special Case I: Log utility

We consider next the special case where $\psi = \gamma = 1$ for the economy with an arbitrary number of states. Proposition O.2 below shows that the main implications from Section 5 extends to this more general economy.

Proposition O.2 (Log-utility)**.** *Suppose $\psi = \gamma = 1$ and that the following condition is satisfied $x^N < \frac{x^1}{\alpha}$.*

i. Consumption and portfolio decisions are given by

$$c_i(X,s) = 1-\beta, \qquad R_{i,n}(X,s,s') = \frac{p^i_{ss'}}{p_{ss'}(X)}R_p(X,s,s'). \tag{O1.15}$$

ii. The economy's SDF is given by

$$\Lambda(X,s,s') = \frac{p_{ss'(X)}}{p_{ss'}}R'_p(X,s,s')^{-1}. \tag{O1.16}$$

iii. The price and return on the surplus claim are given by

$$P(X,s) = \frac{x_s h(E)^\alpha - \xi \frac{h(E)^{1+\nu}}{1+\nu}}{1-\beta}, \tag{O1.17}$$

$$R_p(X,s,s') = \frac{x_s}{\beta} \frac{x_{s'} E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{O1.18}$$

iv. The risk premium on the surplus claim and the interest rate are given by

$$R_b(X,s) = \left(1 - \frac{\alpha}{1+\nu}\right) \frac{x_s}{\beta} \frac{E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{O1.19}$$

$$\mathbb{E}_s[R_p^e(X,s,s')] = \frac{x_s}{\beta} \frac{[\mathbb{E}_s[x_{s'}] - E'(X,s)] E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{O1.20}$$

v. The law of motion of the aggregate state variables are given by

$$E'(X,s) = \sum_{s' \in \mathcal{S}} x_{s'} \frac{p_{ss'}(X) \left[x_{s'} - \frac{\alpha}{1+\nu} E'(X,s)\right]^{-1}}{\sum_{\tilde{s} \in \mathcal{S}} p_{s\tilde{s}}(X) \left[x_{\tilde{s}} - \frac{\alpha}{1+\nu} E'(X,s)\right]^{-1}} \tag{O1.21}$$

$$\eta_i'(X,s,s') = (1-\kappa)\eta_i \frac{p_{ss'}}{p_{ss'}(X)} + \kappa \mu_i, \tag{O1.22}$$

and there exists a unique value of $E'(X,s) \in (x_1, x_N)$ satisfying the law of motion of $\mathcal{L}$.

Proof. See Online Appendix O3.2. □

O1.3 Special Case II: Representative Agent with IID Returns

We consider next a different special case which is also particularly tractable: investors have common iid beliefs, $p^i_{ss'} = p^*_{s'}$, and the supply and demand of labor converge to zero. Formally, we assume $\alpha = \hat{\alpha}\epsilon$ and $\xi = \hat{\xi}\epsilon$ and take the limit as ϵ goes to zero. For simplicity, we focus on the case $\kappa = 0$. Because labor is chosen in advance, returns on financial assets would not be iid even if the process for aggregate productivity is iid. By taking the limit as supply and demand goes to zero, we ensure that all equilibrium objects are well-defined in the limit and the economy behaves essentially as an endowment economy, analogous to an iid version of the Mehra and Prescott (1985) economy.

Proposition O.3 provides a characterization of this limit economy. To highlight these results apply to this particular limit, we denote the equilibrium objects in the limiting economy with an $*$, e.g. $v^*(X,s)$ and $c^*(X,s)$.

Proposition O.3 (IID Returns). *Suppose $p^i_{ss'} = p^*_{s'}$, $\alpha = \hat{\alpha}\epsilon$, and $\xi = \hat{\xi}\epsilon$. Suppose also the following condition is satisfied: $\beta^* \equiv \beta \mathbb{E}^*[x_{s'}^{1-\gamma}]^{\frac{1-\psi^{-1}}{1-\gamma}} < 1$. Then, the economy in the limit as $\epsilon \to 0$ satisfies the following conditions:*

i. Consumption and portfolio decisions:

$$c^*_i(X,s) = 1 - \beta^*, \qquad R^*_{i,n}(X,s,s') = R^*_p(X,s,s'). \tag{O1.23}$$

ii. The net-worth multiplier $v^*_i(X,s)$ is given

$$v^*_i(X,s) = (1-\beta)^{\frac{1}{1-\psi^{-1}}}(1-\beta^*)^{-\frac{\psi^{-1}}{1-\psi^{-1}}}. \tag{O1.24}$$

iii. Wages, hours, and profits are given by

$$h^*(E) = \left(\frac{\hat{\alpha}E}{\hat{\xi}}\right)^{\frac{1}{1+\nu}}, \qquad w^*(E) = 0, \qquad \pi^*(E,s) = x_s. \tag{O1.25}$$

iv. The economy's SDF is given by

$$\Lambda^*(X,s,s') = \beta \mathbb{E}^*[x_{s'}^{1-\gamma}]^{\frac{\gamma-\psi^{-1}}{1-\gamma}} x_{s'}^{-\gamma}. \tag{O1.26}$$

v. The price and return on the surplus claim are given by

$$P^*(X,s) = \frac{x_s}{1-\beta^*}, \qquad R^*_p(X,s,s') = \frac{x'_s}{\beta^*}. \tag{O1.27}$$

vi. The risk-free rate and the expected return on the surplus claim are given by

$$R^*_b(X,s) = \frac{1}{\beta^*}\frac{\mathbb{E}^*[x_{s'}^{1-\gamma}]}{\mathbb{E}^*[x_{s'}^{-\gamma}]}, \tag{O1.28}$$

$$\mathbb{E}^*[R_p(X,s,s')] = R_b(X,s)\frac{\mathbb{E}^*[x_{s'}]\mathbb{E}^*[x_{s'}^{-\gamma}]}{\mathbb{E}^*[x_{s'}^{1-\gamma}]}. \tag{O1.29}$$

vii. The law of motion of the state variables are given by

$$E'(X,s) = E^*, \qquad \eta'_i(X,s,s') = \eta_i. \tag{O1.30}$$

where $E^* \equiv \frac{\mathbb{E}^*[x_{s'}^{1-\gamma}]}{\mathbb{E}^*[x_{s'}^{-\gamma}]}$.

Proof. See Online Appendix O3.3. □

The following corollary shows that we recover the standard asset pricing formulae for iid economies in continuous time if we assume that x_s is approximately log-normal.

Corollary 5. *Suppose* $\log x_s$ *can be approximated by a normal distribution with mean* μ *and variance* σ^2*. Then, under the assumption of Proposition O.3, we obtain*

i. Interest rate:

$$\log R_b^*(X,s) \approx \rho + \psi^{-1}\left(\mu + \frac{\sigma^2}{2}\right) - \frac{\gamma(1+\psi^{-1})}{2}\sigma^2, \tag{O1.31}$$

where $\rho \equiv -\log\beta$.

ii. Risk-premium:

$$\log \mathbb{E}\left[\frac{R_p^*(X,s,s')}{R_b^*(X,s)}\right] \approx \gamma\sigma^2. \tag{O1.32}$$

iii. Risk-neutral expectation of productivity growth:

$$\log \frac{E'(X,s)}{\mathbb{E}[x_{s'}]} \approx -\gamma\sigma^2. \tag{O1.33}$$

The corollary above shows how $E'(X,s)$ depends on $x_{s'}$ and the equity risk premium.

O2 Approximate Solution of the General Economy

In the previous section, we derived exact analytical solutions for two special cases: i) log-utility; ii) homogeneous beliefs and iid returns. In this section, we derive asymptotic closed-form solutions for a general economy with an arbitrary number of states, an arbitrary number of households with heterogeneous beliefs, and Epstein-Zin preferences with unrestricted EIS and risk aversion. The derivations for the benchmark case with homogeneous beliefs and iid returns will be useful in deriving the approximate solution.

O2.1 Perturbation

Consider a family of economies indexed by ϵ. The parameter ϵ controls three dimensions through which these economies differ from each other. First, it determines the degree of belief heterogeneity:

$$p^i_{ss'} = p^*_{s'} + \delta^i_{ss'}\epsilon, \tag{O2.1}$$

where $\sum_{s'\in\mathcal{S}} \delta^i_{ss'} = 0$. We also assume that the objective measure coincides with beliefs in the reference economy, i.e. $p_{ss'} = p^*_{s'}$. Second, ϵ scales both supply and demand for labor:

$$\xi = \hat{\xi}\epsilon, \qquad \alpha = \hat{\alpha}\epsilon. \tag{O2.2}$$

The economy satisfying $\epsilon = 0$ is essentially an endowment economy with iid common beliefs, a special case of the Mehra and Prescott (1985) economy, as described above. Third, we assume that $\kappa = \hat{\kappa}\epsilon$, such that there is no mortality risk in the benchmark economy.

All equilibrium objects are now indexed by ϵ. For instance, the net worth multiplier is now given by $v_i(X, s; \epsilon)$. We are interested in the expansion of $v_i(X, s; \epsilon)$ on ϵ, for ϵ small:

$$v_i(X, s; \epsilon) = v^*_i(X, s) + \hat{v}_i(X, s)\epsilon + \mathcal{O}(\epsilon^2), \tag{O2.3}$$

where $v^*_i(X, s) \equiv v_i(X, s; 0)$ and $\hat{v}_i(X, s)$ represents the first-order correction of $v_i(X, s; \epsilon)$ in ϵ.

Similarly, we can write the consumption-wealth ratio $c_i(X, s; \epsilon)$ as follows:

$$c_i(X, s; \epsilon) = c^*_i(X, s) + \hat{c}_i(X, s)\epsilon + \mathcal{O}(\epsilon^2), \tag{O2.4}$$

and analogously for the remaining equilibrium variables.

The functions $v^*_i(X, s)$ and $c^*_i(X, s)$ are already known, as they correspond to the solution of the case with homogeneous beliefs and iid returns, which we characterized above. It remains to solve for $\hat{v}_i(X, s)$, $\hat{c}_i(X, s)$, and the first-order correction for the other variables.

We start by providing a characterization of the households' problem in this general economy.

Proposition O.4. *Suppose that $p^i_{ss'} = p^*_{s'} + \delta^i_{ss'}\epsilon$, $\alpha = \hat{\alpha}\epsilon$, and $\xi = \hat{\xi}\epsilon$. Suppose also that $\beta^* < 1$. Then,*

i. Net-worth multiplier:

$$\frac{\hat{v}_i(X, s)}{v^*_i(X, s)} = \beta^* \sum_{s'\in\mathcal{S}} \omega^*_{s'} \left[\frac{1}{1-\gamma}\frac{\delta^i_{ss'}}{p^*_{s'}} - \frac{\hat{\Lambda}(X, s, s')}{\Lambda^*(X, s, s')}\right] + \beta^*\overline{v}, \tag{O2.5}$$

where $\omega^*_{s'} \equiv \frac{p^*_{s'} x_{s'}^{1-\gamma}}{\sum_{\tilde{s}\in\mathcal{S}} p^*_{\tilde{s}} x_{\tilde{s}}^{1-\gamma}}$, $X^* = (E^*, \{\eta_i\}_{i=1}^I)$, *and*

$$\overline{v} \equiv \frac{\beta^*}{1-\beta^*} \sum_{\tilde{s}\in\mathcal{S}} \omega^*_{\tilde{s}} \sum_{\tilde{s}'\in\mathcal{S}} \omega^*_{\tilde{s}'} \left[\frac{1}{1-\gamma} \frac{\delta^i_{\tilde{s}\tilde{s}'}}{p^*_{\tilde{s}'}} - \frac{\hat{\Lambda}(X^*, \tilde{s}, \tilde{s}')}{\Lambda^*(X^*, \tilde{s}, \tilde{s}')} \right]. \tag{O2.6}$$

ii. Consumption-wealth ratio:

$$\frac{\hat{c}_i(X,s)}{c^*(X,s)} = (1-\psi) \frac{\hat{v}_i(X,s)}{\hat{v}^*(X,s)}. \tag{O2.7}$$

iii. Portfolio return:

$$\frac{\hat{R}_{n,i}(X,s,s')}{R^*_p(X,s,s')} = \frac{1}{\gamma} myopic + \frac{1-\gamma}{\gamma} hedging, \tag{O2.8}$$

where

$$myopic = \left[\frac{\delta^i_{ss'}}{p^*_{s'}} - \sum_{\tilde{s}'\in\mathcal{S}} \omega^*_{\tilde{s}'} \frac{\delta^i_{s\tilde{s}'}}{p^*_{\tilde{s}'}} \right] - \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} \tag{O2.9}$$

$$hedging = \left[\frac{\hat{v}_i(X^*,s')}{v^*(X,s)} - \sum_{\tilde{s}\in\mathcal{S}} \omega^*_{\tilde{s}} \frac{\hat{v}_i(X^*,\tilde{s})}{v^*(X,s)} \right] + \sum_{\tilde{s}\in\mathcal{S}} \omega^*_{\tilde{s}} \frac{\hat{\Lambda}(X,s,\tilde{s})}{\Lambda^*(X,s,\tilde{s})}. \tag{O2.10}$$

Proof. See Online Appendix O3.4. □

Proposition O.4 provides asymptotic closed-form solutions to the value function and policy functions. The net-worth multiplier $\hat{v}_i(X,s)$ is high when investor i is relatively optimistic and state-prices are relatively low. The effect of beliefs can be seen by writing the term involving $\delta^i_{ss'}$ as follows:

$$\sum_{s'\in\mathcal{S}} \omega^*_{s'} \frac{1}{1-\gamma} \frac{\delta^i_{ss'}}{p^*_{s'}} = \sum_{s'\in\mathcal{S}} p^*_{s'} \frac{1}{1-\gamma} \frac{x_{s'}^{1-\gamma}}{\mathbb{E}^*[x_{s'}^{1-\gamma}]} \frac{\delta^i_{ss'}}{p^*_{s'}} = Cov^* \left(\frac{1}{1-\gamma} \frac{x_{s'}^{1-\gamma}}{\mathbb{E}^*[x_{s'}^{1-\gamma}]}, \frac{\delta^i_{ss'}}{p^*_{s'}} \right), \tag{O2.11}$$

using the fact that $\sum_{s'\in\mathcal{S}} p^*_{s'} \frac{\delta^i_{ss'}}{p^*_{s'}} = 0$. The covariance above will be positive when $\delta^i_{ss'}$ is on average positive when $x_{s'}$ is high, i.e. the covariance is increasing in how optimistic investor i is.

The term involving $\hat{\Lambda}(X,s,s')$ captures the effect of changes in the SDF on the portfolio return that can be achieved by the household:

$$1 = \mathbb{E}_s \left[\Lambda(X,s,s';\epsilon) R_{i,n}(X,s,s';\epsilon) \right] \Rightarrow \sum_{s'\in\mathcal{S}} \omega^*_{s'} \frac{\hat{R}_{i,n}(X,s,s')}{R^*_{i,n}(X,s,s')} = -\sum_{s'\in\mathcal{S}} \omega^*_{s'} \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O2.12}$$

Hence, if $\sum_{s' \in \mathcal{S}} \omega^*_{s'} \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}$ is negative, then the household is able to achieve higher weighted portfolio returns in the $\epsilon > 0$ economy.

Given $\hat{v}_i(X,s)$, we can characterize the policy functions. The consumption-wealth ratio $\hat{c}_i(X,s)$ is proportional to $\hat{v}_i(X,s)$. If $\psi > 1$, such that the substitution effect on savings dominates, households save a larger fraction of their wealth when average portfolio returns are high.

As in the continuous-time model of Merton (1992), portfolio returns have two components: the *myopic demand* and the *hedging demand*. The myopic demand depends on current market conditions, while the hedging demand depends on future expected returns as captured by $\hat{v}_i(X^*, s')$.

We consider next the labor market outcomes and firms' profits.

Proposition O.5 (Hours, wages, and profits)**.** *Suppose that* $p^i_{ss'} = p^*_{s'} + \delta^i_{ss'}\epsilon$*,* $\alpha = \hat{\alpha}\epsilon$*, and* $\xi = \hat{\xi}\epsilon$*. Suppose also that* $\beta^* < 1$*. Then,*

i. Wages:

$$\hat{w}(E) = \hat{\xi}\left(\frac{\hat{\alpha}E}{\hat{\xi}}\right)^{\frac{\nu}{1+\nu}}. \tag{O2.13}$$

ii. Hours:

$$\hat{h}(E) = \left(\frac{\hat{\alpha}E}{\hat{\xi}}\right)^{\frac{1}{1+\nu}} \frac{\log\left(\frac{\hat{\alpha}E}{\hat{\xi}}\right)}{(1+\nu)^2}\hat{\alpha}. \tag{O2.14}$$

iii. Profits:

$$\hat{\pi}(X,s) = \left[x_s \frac{\log\left(\hat{\alpha}E/\hat{\xi}\right)}{1+\nu} - E\right]\hat{\alpha}. \tag{O2.15}$$

Proof. See Online Appendix O3.5 □

We consider next the behavior of the price of the surplus claim and the riskless asset.

Proposition O.6 (Asset Prices)**.** *Suppose that* $p^i_{ss'} = p^*_{s'} + \delta^i_{ss'}\epsilon$*,* $\alpha = \hat{\alpha}\epsilon$*, and* $\xi = \hat{\xi}\epsilon$*. Suppose also that* $\beta^* < 1$*. Then,*

i. Price of surplus claim:

$$\frac{\hat{P}(X,s)}{P^*(X,s)} = \left[\log(\hat{\alpha}E/\hat{\xi}) - \frac{E}{x_s}\right]\frac{\hat{\alpha}}{1+\nu} + (\psi - 1)\sum_{i=1}^{I} \eta_i \frac{\hat{v}_i(X,s)}{v^*(X,s)}. \tag{O2.16}$$

ii. Return on the surplus claim:

$$\frac{\hat{R}_p(X,s,s')}{R^*_p(X,s,s')} = \left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu} + (\psi-1)\sum_{i=1}^{I}\eta_i\left[\frac{\hat{v}_i(X^*,s')}{v^*(X^*,s')} - \frac{1}{\beta^*}\frac{\hat{v}_i(X,s)}{v^*(X,s)}\right]. \tag{O2.17}$$

iii. Risk-free rate:

$$\frac{\hat{R}_b(X,s)}{R^*_b(X,s)} = -\sum_{s'\in\mathcal{S}} \frac{p_{s'} x_{s'}^{-\gamma}}{\mathbb{E}^*[x_{s'}^{-\gamma}]}\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O2.18}$$

iv. Conditional risk premium:

$$\frac{\widehat{\overline{R}}_E(X,s)}{\overline{R}_E^*(X,s)} = \sum_{s'\in\mathcal{S}} \frac{p_{s'}^* x_{s'}}{\mathbb{E}^*[x_{s'}]} \frac{\hat{R}_E(X,s,s')}{R_E^*(X,s,s')} - \frac{\hat{R}_b(X,s)}{R_b^*(X,s)}, \tag{O2.19}$$

where $\overline{R}_E(X,s;\epsilon) = \mathbb{E}^*\left[\frac{R_E(X,s,s';\epsilon)}{R_b(X,s;\epsilon)}\right]$.

Proof. See Online Appendix O3.6 □

The next proposition provides the law of motion of the aggregate state variables.

Proposition O.7 (Aggregate state variables.). *Suppose that* $p_{ss'}^i = p_{s'}^* + \delta_{ss'}^i \epsilon$, $\alpha = \hat{\alpha}\epsilon$, *and* $\xi = \hat{\xi}\epsilon$. *Suppose also that* $\beta^* < 1$. *Then,*

i. Wealth distribution:

$$\frac{\hat{\eta}_i'(X,s,s')}{\eta_i} = \frac{\hat{R}_{i,n}(X,s,s')}{R_{i,n}^*(X,s,s')} - \sum_{j=1}^{I} \eta_i \frac{\hat{R}_{j,n}(X,s,s')}{R_{j,n}^*(X,s,s')} - ((\beta^*)^{-1} - 1)\left(\frac{\hat{c}_i(X,s)}{c_i^*(X,s)} - \sum_{j=1}^{I} \eta_j \frac{\hat{c}_j(X,s)}{c_j^*(X,s)}\right) + \kappa \frac{\mu_i - \eta_i}{\eta_i}. \tag{O2.20}$$

ii. Risk-neutral expectation of productivity growth:

$$\frac{\hat{E}'(X,s)}{E^*} = \frac{\hat{R}_b(X,s)}{R_b^*(X,s)} + \sum_{s'\in\mathcal{S}} \omega_{s'}^* \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O2.21}$$

Proof. See Online Appendix O3.7. □

Propositions O.4 to O.7 characterize the behavior of all equilibrium objects given the economy's SDF $\hat{\Lambda}(X,s,s')$. The next proposition provides an expression for $\Lambda(X,s,s')$ in terms of the primitives of the economy.

Proposition O.8 (The economy's SDF). *Suppose that* $p_{ss'}^i = p_{s'}^* + \delta_{ss'}^i \epsilon$, $\alpha = \hat{\alpha}\epsilon$, *and* $\xi = \hat{\xi}\epsilon$. *Suppose also that* $\beta^* < 1$. *Then, the economy's SDF is given by*

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \gamma b^{\Lambda}(X,s,s') - (\gamma - \psi^{-1})\left[\omega^* b^{\Lambda}(X,s) - \beta^* \omega^* \cdot b^{\Lambda}(X^*,s') + \beta^* \sum_{\tilde{s}\in\mathcal{S}} \omega_{\tilde{s}}(\omega^* \cdot b^{\Lambda}(X^*,\tilde{s}))\right], \tag{O2.22}$$

where

$$b^{\Lambda}(X,s,s') = \frac{1}{\gamma}\frac{\delta_{ss'}(X)}{p_{s'}^*} - \frac{\psi - \gamma^{-1}}{\gamma - 1}\left[\omega^* \cdot \delta_s(X) - \beta^* \omega^* \cdot \delta_{s'}(X) + \beta^* \sum_{\tilde{s}} \omega_{\tilde{s}}^* (\omega^* \cdot \delta_{\tilde{s}}(X))\right] - \left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu}. \tag{O2.23}$$

Proof. See Online Appendix O3.8. □

A particularly simple special case is given by the case of CRRA preferences, i.e. $\gamma = \psi^{-1}$.

Corollary 6. *Suppose $\gamma = \psi^{-1}$. Then,*

i. SDF:

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \frac{\delta_{ss'}(X)}{p^*_{s'}} - \gamma \left[\log \frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right] \frac{\hat{\alpha}}{1+\nu}. \tag{O2.24}$$

O2.2 Special cases

Consider the special case where $\delta^i_{ss'} = 0$ for all $i \in \mathcal{I}$ and $s, s' \in \mathcal{S}$. In this case, investors still have common iid beliefs, but returns will not be iid due to the fact that labor is chosen one period in advance.

In this case, the economy's is given by

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \psi^{-1} \left[\log \frac{E}{E^*} + \frac{E^*}{x_{s'}} - \frac{E}{x_s}\right] \frac{\hat{\alpha}}{1+\nu} - (\gamma - \psi^{-1})(1-\beta^*) \sum_{\tilde{s}'} \omega^*_{\tilde{s}'} \left(\frac{E^*}{x_{\tilde{s}'}} - \frac{E^*}{x_{s'}}\right) \frac{\hat{\alpha}}{1+\nu}. \tag{O2.25}$$

The interest rate is given by

$$\frac{\hat{R}_b(X,s)}{R^*_b(X,s)} = -\sum_{s' \in \mathcal{S}} \frac{p_{s'} x_{s'}^{-\gamma}}{\mathbb{E}^*[x_{s'}^{-\gamma}]} \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O2.26}$$

O2.3 Conditional moments

Consider the conditional risk premium

$$R^e_p(X,s;\epsilon) \equiv \sum_{s' \in \mathcal{S}} p^*_{s'} \frac{R_p(X,s,s';\epsilon)}{R_b(X,s;\epsilon)}. \tag{O2.27}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\hat{R}^e_p(X,s)}{R^{e,*}_p} = \sum_{s' \in \mathcal{S}} \frac{p^*_{s'} R^*_p(X,s,s')}{\sum_{\tilde{s}' \in \mathcal{S}} p^*_{\tilde{s}'} R^*_p(X,s,\tilde{s}')} \left[\frac{\hat{R}_p(X,s,s')}{R^*_p(X,s,s')} - \frac{\hat{R}_b(X,s)}{R^*_b(X,s)}\right] = \sum_{s' \in \mathcal{S}} \frac{p_{s'} x_{s'}}{\mathbb{E}^*[x_{s'}]} \frac{\hat{R}_p(X,s,s')}{R^*_p(X,s)} - \frac{\hat{R}_b(X,s,s')}{R^*_b(X,s)}. \tag{O2.28}$$

The conditional volatility of excess returns is given by

$$\sigma_p(X,s;\epsilon) \equiv \left[\sum_{s' \in \mathcal{S}} p_{s'} \left(R^e_p(X,s,s';\epsilon) - R^e_p(X,s;\epsilon)\right)^2\right]^{\frac{1}{2}}. \tag{O2.29}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\hat{\sigma}_p(X,s)}{\sigma^*_p(X,s)} = \frac{1}{\sigma^*_p(X,s)^2} \sum_{s' \in \mathcal{S}} p_{s'} \left(R^{e,*}_p(X,s,s') - R^{e,*}_p(X,s)\right) \left(\hat{R}^e_p(X,s,s') - \hat{R}^e_p(X,s)\right), \tag{O2.30}$$

where $\hat{R}^e_p(X,s,s') = \hat{R}^{e,*}_p(X,s,s') \left(\frac{\hat{R}_p(X,s,s')}{R^*_p(X,s,s')} - \frac{\hat{R}_b(X,s)}{R^*_b(X,s)}\right)$.

O2.4 Stock prices in the log economy

Suppose $\psi = \gamma = 1$ and that investors have homogeneous iid beliefs, $\delta^i_{ss'} = 0$. The stock price satisfies the following recursion:

$$Q(X,s) = \sum_{s' \in \mathcal{S}} p_{ss'} \Lambda(X,s,s') \left[\pi(E'(X,s),s') + x_{s'} Q(X',s') \right], \tag{O2.31}$$

where $\pi(E,s) = \left(\frac{\alpha E}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}} x_s \left[1 - \frac{\alpha}{x_s} E\right]$ and $p_{ss'}\Lambda(X,s,s') = p_{ss'}(X) \frac{\beta}{x_{s'}} \left(\frac{E}{E'(X,s)}\right)^{\frac{\alpha}{1+\nu-\alpha}} \frac{1+\nu-\alpha\frac{E}{x_s}}{1+\nu-\alpha\frac{E'(X,s)}{x_{s'}}}$.

Define the price-dividend ratio $q(X,s) = x_s \frac{Q(X,s)}{\pi(X,s)}$. The price-dividend ratio satisfies the recursion:

$$q(X,s) = \sum_{s' \in \mathcal{S}} p_{ss'} \Lambda(X,s,s') \left[\pi(E'(X,s),s') + x_{s'} Q(X',s') \right], \tag{O2.32}$$

We can then write the expression above as follows:

$$q(X,s) = \beta \sum_{s' \in \mathcal{S}} p_{ss'}(X) \frac{1+\nu-\alpha\frac{E}{x_s}}{1+\nu-\alpha\frac{E'(X,s)}{x_{s'}}} \frac{1-\alpha\frac{E'(X,s)}{x_{s'}}}{1-\alpha\frac{E}{x_s}} \left[1 + q(X',s')\right]. \tag{O2.33}$$

Let's assume that $\nu = \bar{\nu}\epsilon$. We can then write $q(X,s;\epsilon)$ as follows

$$q(X,s;\epsilon) = \sum_{k=0}^{\infty} q_k(X,s)\epsilon^k. \tag{O2.34}$$

Define $g(X,s,s';\epsilon) \equiv \frac{1+\nu-\alpha\frac{E}{x_s}}{1+\nu-\alpha\frac{E'(X,s)}{x_{s'}}} \frac{1-\alpha\frac{E'(X,s)}{x_{s'}}}{1-\alpha\frac{E}{x_s}}$. We can expand $g(X,s,s')$ as follows

$$g(X,s,s';\epsilon) = \sum_{k=0}^{\infty} g_k(X,s,s')\epsilon^k, \tag{O2.35}$$

where $g_0(X,s,s') = 1$ and, for $k > 0$, we obtain

$$g_k(X,s,s') = \frac{\alpha\bar{\nu}^k \left(\frac{E'(X,s)}{x_{s'}} - \frac{E}{x_s}\right)}{\left(1-\alpha\frac{E}{x_s}\right)\left(\alpha\frac{E'(X,s')}{x_{s'}} - 1\right)^k} \tag{O2.36}$$

This gives the following recursion for $q_k(X,s)$:

$$q_k(X,s) = \beta \sum_{s' \in \mathcal{S}} p_{ss'}(X) \left[g_k(X,s) + \sum_{j=0}^{k} g_j(X,s) q_{k-j}(X',s') \right]. \tag{O2.37}$$

Under our assumptions, the risk-neutral expectation of productivity growth is constant

$E'(X,s) = \overline{E}$. In this case, we can write the recursion

$$Q(\overline{X},s) = \beta\left(1 - \frac{\alpha}{1+\nu}\frac{\overline{E}}{x_s}\right)\sum_{s'\in\mathcal{S}}\frac{p^*_{s'}x_{s'}}{x_{s'} - \frac{\alpha}{1+\nu}\overline{E}}\left[Q(\overline{X},s') + \left(\frac{\alpha\overline{E}}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\left(1 - \alpha\frac{\overline{E}}{x_{s'}}\right)\right]$$

Let $\tilde{Q}(X,s) \equiv \frac{Q(x,s)}{1-\frac{\alpha}{1+\nu}\frac{E'(X,s)}{x_s}}$, $\tilde{Q}(X) \equiv [\tilde{Q}(X,1),\ldots,\tilde{Q}(X,N)]'$, and $b^Q \equiv \beta\left(\frac{\alpha\overline{E}}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}$. Then, we can write

$$\left[I - \beta\mathbb{1}_N(p^*)'\right]\tilde{Q}(\overline{X}) = b^Q\mathbb{1}_N, \tag{O2.38}$$

Inverting the matrix above, we obtain

$$Q(\overline{X},s) = \left[\frac{\beta}{1-\beta}\left(\frac{\alpha\overline{E}}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}\right]\left(1 - \frac{\alpha}{1+\nu}\frac{\overline{E}}{x_s}\right). \tag{O2.39}$$

The price-dividend ratio is given by

$$\frac{x_s Q(\overline{X},s)}{\pi(\overline{X},s)} = \frac{\beta}{1-\beta}\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}\frac{x_s - \frac{\alpha}{1+\nu}\overline{E}}{x_s - \alpha\overline{E}}. \tag{O2.40}$$

Equity returns are given by

$$R_E(X,s,s') = \frac{\pi(\overline{E},s') + x'Q(\overline{X},s')}{Q(\overline{X},s)} = \frac{x_{s'}}{\beta}\left[\frac{1-\beta}{\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}}\frac{1-\alpha\frac{\overline{E}}{x_{s'}}}{1-\frac{\alpha}{1+\nu}\frac{\overline{E}}{x_s}} + \beta\frac{1-\frac{\alpha}{1+\nu}\frac{\overline{E}}{x_{s'}}}{1-\frac{\alpha}{1+\nu}\frac{\overline{E}}{x_s}}\right]. \tag{O2.41}$$

Excess returns are given by

$$R^e_E(X,s,s') = a_E x_{s'} + b_E, \tag{O2.42}$$

where

$$a_E \equiv \frac{1}{\left(1-\frac{\alpha}{1+\nu}\right)\overline{E}}\left(\frac{1-\beta}{\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}} + \beta\right) \tag{O2.43}$$

$$b_E \equiv \tag{O2.44}$$

The conditional risk premium is given by

$$R_E(\overline{X},s) = \frac{1}{\left(1-\frac{\alpha}{1+\nu}\right)\overline{E}}\left[\left(\frac{1-\beta}{\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}} + \beta\right)\mathbb{E}[x_{s'}] - \left(\frac{(1-\beta)(1+\nu)}{\sum_{s'\in\mathcal{S}}p^*_{s'}\frac{x_{s'}-\alpha\overline{E}}{x_{s'}-\frac{\alpha}{1+\nu}\overline{E}}} + \beta\right)\frac{\alpha}{1+\nu}\overline{E}\right] - 1. \tag{O2.45}$$

We can write the expression above as follows:

$$R_E(\overline{X}, s) = \frac{1}{\left(1 - \frac{\alpha}{1+\nu}\right)\overline{E}} \left[\left(\frac{1-\beta}{1 - \nu \sum_{s' \in \mathcal{S}} \frac{p^*_{s'} \frac{\alpha}{1+\nu}\overline{E}}{x_{s'} - \frac{\alpha}{1+\nu}\overline{E}}} + \beta \right) \mathbb{E}[x_{s'}] - \left(\frac{(1-\beta)(1+\nu)}{1 - \nu \sum_{s' \in \mathcal{S}} \frac{p^*_{s'} \frac{\alpha}{1+\nu}\overline{E}}{x_{s'} - \frac{\alpha}{1+\nu}\overline{E}}} + \beta \right) \frac{\alpha}{1+\nu}\overline{E} \right] - 1. \tag{O2.46}$$

O2.5 Quantitative implications

Let z_t denote demeaned log productivity growth, which we assume follows an AR(1) process:

$$z_{t+1} = \rho z_t + \sigma \sqrt{1-\rho^2} \epsilon_{t+1}, \tag{O2.47}$$

where ϵ_{t+1} follows a standard normal distribution and it is serially uncorrelated. In levels, the (gross) productivity growth is given by $x_t = e^{\mu + z_t}$, where μ denotes average productivity growth.

We discretize the process above following the method of Rouwenhurst (1995). Let $\hat{z}_t$ denote the discretized variable taking values in the equally-spaced grid $\{z_1, \ldots, z_N\}$, where $z_i = -\overline{\psi} + \frac{2\overline{\psi}}{N-1}(i-1)$, so $z_1 = -\overline{\psi}$ and $z_N = \overline{\psi}$. We set $\overline{\psi} \equiv \sigma\sqrt{N-1}$, so we match the unconditional variance.

O2.6 A more general process for productivity growth

Discretization. The evolution of $\hat{x}_t$, under subjective beliefs, can be written in a convenient matrix form:

$$\begin{bmatrix} \hat{x}_{t+1} \\ \hat{z}_{i,t+1} \end{bmatrix} = \begin{bmatrix} w_{t+1} \\ 0 \end{bmatrix} + \begin{bmatrix} \theta_i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{z}_{i,t} \end{bmatrix} + \begin{bmatrix} \sigma_{i,u} & 0 \\ 0 & \sigma_v \end{bmatrix} \begin{bmatrix} u_{i,t+1} \\ v_{t+1} \end{bmatrix}, \tag{O2.48}$$

where $\hat{z}_{i,t} \equiv \mathbb{E}_{i,t}[\hat{x}_{t+1}] - \theta_i \hat{x}_t$. We recover objective beliefs in the special case $\theta_i = \sigma_v = 0$. As w_{t+1} follows a Markov chain, the process above corresponds to a Markov-switching vector autoregression (MS-VAR), with state-dependent conditional means. To discretize the system above, we adapt the methods of Gospodinov and Lkhagvasuren (2014), who extended the Rouwenhurst (1995) method to VARs, and Liu (2017), who proposed a discretization of univariate Markov-Switching models. The discretization provides a state space with dimension S for x_t, so $x_t \in \mathcal{X} = \{x^1, x^2, \ldots, x^S\}$, and transition probabilities $\{p^i_{ss'}\}$, for $s, s' \in \mathcal{S} = \{1, 2, \ldots, S\}$, that approximate the MS-VAR (O2.48). Notice that our discretization implies that the grid $\mathcal{X}$ is the same for all investors, so they agree on the state s, but they disagree on the transition probabilities $p^i_{ss'}$.

Let $\hat{x}_t \equiv \log x_t - \mu$ denote the demeaned log productivity growth. We assume that investor i

believes the process for $\hat{x}_t$ is given by

$$\hat{x}_{t+1} = \mathbb{E}_{i,t}[\hat{x}_{t+1}] + \sigma_{i,u} u_{i,t+1} \tag{O2.49}$$

$$\mathbb{E}_{i,t}[\hat{x}_{t+1}] = \theta_i \hat{x}_t + \sigma_v v_{i,t}, \tag{O2.50}$$

$v_{i,t} = \overline{v}_t + \tilde{v}_{i,t}$, where $u_{i,t}$ and v_t are mutually independent, serially uncorrelated, standard normal random variables. Notice that $u_{i,t+1}$ represents the period $t+1$ innovation according to investor i and v_t represents an *expectation shock*. We assume that this expectation shock is common across investors, so heterogeneity comes only from θ_i.

The presence of this expectation shock is important to quantitatively match the volatility of expectations in the data. To see the role of v_t, notice that the unconditional variance of $\hat{x}_{t+1}$ and $\mathbb{E}_t[\hat{x}_{t+1}]$ are given by

$$Var[\hat{x}_{t+1}] = \frac{\sigma_{i,u}^2 + \sigma_v^2}{1-\theta_i^2}, \qquad Var[\mathbb{E}_t[\hat{x}_{t+1}]] = \frac{\theta_i^2 \sigma_{i,u}^2 + \sigma_v^2}{1-\theta_i^2}. \tag{O2.51}$$

The fraction of total variance explained by movements in expectations is given by

$$\frac{Var[\mathbb{E}_t[\hat{x}_{t+1}]]}{Var[\hat{x}_{t+1}]} = \theta_i^2 + (1-\theta_i^2)\frac{\sigma_v^2}{\sigma_{i,u}^2 + \sigma_v^2}. \tag{O2.52}$$

Hence, by adjusting σ_v, it is possible to obtain any value in the interval $[\theta_i^2, 1)$ for the fraction of variance explained by movements in expectations. In the special case $\sigma_v = 0$, we obtain an AR(1) process for $\hat{x}_{t+1}$, which achieves the lower bound of this interval.

Discretization. We discretize the process above using the generalization of the method of Rouwenhurst (1995) proposed by Gospodinov and Lkhagvasuren (2014). The method consists of mixing the distribution for independent AR(1) processes to approximate the distribution of a VAR(1) with uncorrelated shocks. Define $\hat{z}_t \equiv \mathbb{E}_{i,t}[\hat{x}_{i,t}] - \theta_i \hat{x}_t$, so we can write the system above in matrix form:

$$\begin{bmatrix} \hat{x}_{t+1} \\ \hat{z}_{t+1} \end{bmatrix} = \begin{bmatrix} \theta & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{z}_t \end{bmatrix} + \begin{bmatrix} \sigma_u & 0 \\ 0 & \sigma_v \end{bmatrix} \begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix}, \tag{O2.53}$$

where we dropped the dependence on the investor i to ease notation. Given this representation, we can construct the discrete approximation following the three steps described below.

Step 1: grid construction. We construct the grids for $\hat{x}$ and $\hat{z}$ as in Rouwenhurst (1995). Let $\hat{x}(N_x, \sigma_x) = \{\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^{N_x}\}$ denote the grid for $\hat{x}$, where

$$\hat{x}^i = -\psi_x(N_x, \sigma_x) + 2\psi_x(N_x, \sigma_x)\frac{i-1}{N_x - 1}, \tag{O2.54}$$

$\psi_x(N_x, \sigma_x) \equiv \sigma_x \sqrt{N_x - 1}$, and σ_x denotes the unconditional standard-deviation for $\hat{x}_t$. Notice that the grid is equally spaced, $\bar{x}^1 = -\psi_x(N_x, \sigma_x)$, and $\bar{x}^{N_x} = \psi_x(N_x, \sigma_x)$. The grid for $\hat{z}$ is constructed analogously.

Step 2: transition matrix for independent AR(1). Let $\Pi(N, \rho, \sigma)$ denote the $N \times N$ transition matrix for the Rouwenhurst (1995) approximation of an AR(1) process with autocorrelation ρ and unconditional variance σ^2. We denote the k-th row of this matrix by

$$\pi_k(N, \rho, \sigma) = \{\pi_{k,1}(N, \rho, \sigma), \pi_{k,2}(N, \rho, \sigma), \ldots, \pi_{k,N}(N, \rho, \sigma)\}, \tag{O2.55}$$

where $\pi_{k,l}(N, \rho, \sigma)$ is the probability of transitioning from state k to state l. In the special case where $\rho = 0$, the transition probability is independent of the current state, so we can write $\pi_{k,j}(N, 0, \sigma) = \bar{\pi}_j(N, \sigma)$.

Step 3: Markov chain construction. Given N_x points in the grid for $\hat{x}$ and N_z points in the grid for $\hat{z}_t$, we have a total of $S = N_x \times N_z$ states. Denote the state space by $\mathcal{S} = \{1, 2, \ldots, S\}$. Let $s = i + (k-1) \times N_x$ and $s' = j + (l-1) \times N_x$, where $i, j \in \{1, \ldots, N_x\}$ and $k, l \in \{1, \ldots, N_z\}$. Denote the probability of $\hat{x}_{t+1} = \bar{x}^j$ given state s by $p_s^x(j)$ and the probability of $\hat{z}_{t+1} = \bar{z}^l$ given state s by $p_s^z(l)$. As $\hat{x}_{t+1}$ and $\hat{z}_{t+1}$ are conditionally independent, the probability of switching from state s to state s' is given by

$$p_{ss'} = p_s^x(j) \times p_s^z(l). \tag{O2.56}$$

As z is serially uncorrelated, we have that $p_s^z(l) = \bar{\pi}_l(N_z, \sigma_z)$. The transition probability for $\hat{x}_t$ will be obtained by appropriately mixing the distribution of an AR(1) process with autocorrelation $\rho_x \equiv \sqrt{1 - \frac{\sigma_u^2}{\sigma_x^2}}$ and unconditional variance σ_x^2.

Let $\mu_x(s) \equiv \theta \hat{x}^i + \hat{z}^k$ denote the conditional expectation of $\hat{x}$ at state s. Suppose first that $\mu_x(s) \in [\rho_x \bar{x}^1, \rho_x \bar{x}^{N_x}]$. Define the probability of $\hat{x}_{t+1} = \bar{x}^j$ given state s as follows:

$$p_s^x(j) = \lambda(\rho_x)\pi_{\iota,j}(N_x, \rho_x, \sigma_x) + (1 - \lambda(\rho_x))\pi_{\iota+1,j}(N_x, \rho_x, \sigma_x), \tag{O2.57}$$

where ι is such that $\rho_x \bar{x}^\iota \leq \mu_x(s) \leq \rho_x \bar{x}^{\iota+1}$ and $\lambda(\rho_x) \equiv \frac{\rho_x \bar{x}^{\iota+1} - \mu_x(s)}{\rho_x \bar{x}^{\iota+1} - \rho_x \bar{x}^\iota}$.

This choice of $\lambda(\rho_x)$ implies that we match the conditional moments:

$$\sum_{j=1}^{N_x} p_s^x(j)\bar{x}^j = \lambda(\rho_x)\rho_x \bar{x}^\iota + (1 - \lambda(\rho_x))\rho_x \bar{x}^{\iota+1} = \mu_x(s). \tag{O2.58}$$

The conditional second moment is given by

$$\sum_{j=1}^{N_x} p_s^x(j)(\bar{x}^j)^2 = \sigma_x^2(1 - \rho_x^2) + \rho_x^2 \left[\lambda(\rho_x)(\bar{y}^\iota)^2 + (1 - \lambda(\rho_x))(\bar{y}^{\iota+1})^2\right]. \tag{O2.59}$$

Denote the conditional variance of the discrete process by $\tilde{\sigma}_u^2$, which is given by

$$\begin{aligned}
\tilde{\sigma}_u^2 &= \sigma_x^2(1-\rho_x^2) + \rho_x^2\left[\lambda(\rho_x)(\overline{y}^\imath)^2 + (1-\lambda(\rho_x))(\overline{y}^{\imath+1})^2\right] - \rho_x^2\left[\lambda(\rho_x)\overline{y}^\imath + (1-\lambda(\rho_x))\overline{y}^{\imath+1}\right]^2 \\
&= \sigma_x^2(1-\rho_x^2) + \rho_x^2\lambda(\rho_x)(1-\lambda(\rho_x))(\overline{x}^{\imath+1} - \overline{x}^\imath)^2 \\
&= \sigma_x^2(1-\rho_x^2) + \sigma_x^2\rho_x^2\frac{4\lambda(\rho_x)(1-\lambda(\rho_x))}{N_x - 1}, \qquad \text{(O2.60)}
\end{aligned}$$

using the fact that $(\overline{x}^{i+1} - \overline{x}^i)^2 = \frac{4\sigma_x^2}{N-1}$. As $N_x \to \infty$, the second term on the right converges to zero and $\tilde{\sigma}_u = \sigma_x^2(1-\rho_x^2) = \sigma_u^2$, given our choice of ρ_x. If $\mu_x(s)/\rho_x$ does not belongs to the grid of $\hat{x}$, then the discretization matches the conditional mean of $\hat{x}$, but it overstates the conditional variance.

Suppose now that $\mu_x(s) \notin [\rho_x\overline{x}^1, \rho_x\overline{x}^{N_x}]$. In this case, we set $p_s^x(j) = \pi_{1,j}(N_x,\rho_x,\sigma_x)$ if $\mu_x(s) < \rho_x\overline{x}^1$ and $p_s^x(j) = \pi_{N_x,j}(N_x,\rho_x,\sigma_x)$ if $\mu_x(s) > \rho_x\overline{x}^{N_x}$. In both cases, the conditional variance is matched exactly and the conditional mean achieves the value closest to $\mu_x(s)$ given the grid points.

A different representation. An equivalent representation of the system is given by

$$\hat{x}_{t+1} = z_t + \sigma_u u_{t+1} \qquad \text{(O2.61)}$$

$$z_{t+1} = \theta_i z_t + \theta_i \sigma_u u_{t+1} + \sigma_v v_{t+1}, \qquad \text{(O2.62)}$$

where $z_t \equiv \mathbb{E}_{i,t}[\hat{x}_{t+1}]$. Hence, expected growth follows an AR(1) process and it is exposed to both expectation shocks, v_{t+1}, and shocks to realized growth rates, u_{t+1}. Notice that we cannot independently choose the persistence of expectations and the correlation between z_{t+1} and $\hat{x}_{t+1}$.

The impact of v_t in expected future growth is

$$\frac{\partial \mathbb{E}_t[\hat{x}_{t+k}]}{\partial v_t} = \sigma_v \theta_i^{k-1}, \qquad \text{(O2.63)}$$

for $k \geq 1$.

O2.7 A process with richer heterogeneity

Under the objective measure, log productivity follows a Markov-Switching process:

$$\log(x_{t+1}) = \mu_{t+1} + \theta(\log(x_t) - \mu_t) + u_{t+1}, \qquad \text{(O2.64)}$$

where $u_{t+1} \sim \mathcal{N}(0,\sigma_u^2)$ and μ_{t+1} follows a two-state Markov chain, that is, $\mu_{t+1} \in \{\mu^1,\mu^2\}$ and $Pr(\mu_{t+1} = \mu^j | \mu_t = \mu^i) = p_{ij}^\mu$ for $i,j \in \{1,2\}$. The different regimes enable us to capture the fact that productivity is subject to small fluctuations most of the time with occasional rare large shocks.

Under subjective beliefs, productivity follows the process

$$\log(x_{t+1}) = \mu_{t+1} + \theta_i(\log(x_t) - \mu_t) + v_{i,t} + u_{i,t+1}, \tag{O2.65}$$

where $u_{i,t+1} \sim \mathcal{N}(0, \sigma_{i,u}^2)$, $v_{i,t} = \rho\sigma_{i,v}\overline{v}_t + \sqrt{1-\rho^2}\sigma_{i,v}\hat{v}_{i,t}$, and $(\overline{v}_t, v_{i,t})$ are iid standard normal random variables. We assume that $(u_{i,t}, \hat{v}_{i,t}, \overline{v}_t)$ are mutually independent.

Subjective beliefs differ from the objective one in two important dimensions. First, the persistent parameter θ_i may differ from the objective one θ. Second, subjective beliefs are exposed to expectation shocks $v_{i,t}$. These expectations shocks are exposed to a common component $\overline{v}_t$ and an investor-specific component $v_{i,t}$. Differences in θ_i capture the fact that investors differ on how they react to news, with some investors extrapolating and some investors under-reacting. The expectation shocks $v_{i,t}$ are important to capture the volatility of subjective expectations observed in the data.

Define $\hat{x}_t \equiv \log(x_t) - \mu_t$ and the vector $\hat{v}_t = [\hat{v}_{1,t}, \ldots, \hat{v}_{I,t}]'$. Investor i believes that $[\hat{x}_t, \overline{v}_t, \hat{v}_t]$ follows the process:

$$\begin{bmatrix} \hat{x}_{t+1} \\ \overline{v}_{t+1} \\ \hat{v}_{t+1} \end{bmatrix} = \begin{bmatrix} \theta_i & \rho\sigma_{i,v} & \sqrt{1-\rho^2}\sigma_{i,v}\mathbf{e}_i' \\ 0 & 0 & \mathbf{0}_{1\times I} \\ \mathbf{0}_{I\times 1} & \mathbf{0}_{I\times 1} & \mathbf{0}_{I\times I} \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \overline{v}_t \\ \hat{v}_t \end{bmatrix} + \begin{bmatrix} u_{i,t+1} \\ \overline{v}_{t+1} \\ \hat{v}_{t+1} \end{bmatrix} \tag{O2.66}$$

Notice that the total variance and the variance of the conditional expectation are given by

$$Var[\hat{x}_{t+1}] = \frac{\sigma_{i,u}^2 + \sigma_{i,v}^2}{1-\theta_i^2}, \qquad Var[\mathbb{E}_t[\hat{x}_{t+1}]] = \frac{\theta_i^2\sigma_{i,u}^2 + \sigma_{i,v}^2}{1-\theta_i^2}. \tag{O2.67}$$

The fraction of total variance explained by movements in expectations is given by

$$\frac{Var[\mathbb{E}_t[\hat{x}_{t+1}]]}{Var[\hat{x}_{t+1}]} = \theta_i^2 + (1-\theta_i^2)\frac{\sigma_{i,v}^2}{\sigma_{i,u}^2 + \sigma_{i,v}^2}. \tag{O2.68}$$

O2.8 A more general process for productivity growth

Let $\hat{x}_t \equiv \log x_t - \mu$ denote the demeaned log productivity growth. We assume that $\hat{x}_t$ follows the process:

$$\hat{x}_{t+1} = z_t + \sigma_x\left[\sqrt{1-\rho_{xz}^2}u_{t+1} + \rho_{xz}v_{t+1}\right] \tag{O2.69}$$

$$z_{t+1} = \theta_z z_t + \sigma_z v_{t+1}, \tag{O2.70}$$

where u_t and v_t are standard normal random variables, serially uncorrelated, and uncorrelated with each other. Notice that $\mathbb{E}_t[x_{t+1}] = z_t$, so z_t corresponds to expected productivity growth. The disturbance v_{t+1} can then be interpreted as expectations shocks. These expectations shocks are potentially correlated with cash-flow shocks, with correlation coefficient ρ_{xz}. In the long-run risk literature, v_{t+1} is referred to as a long-run risk shock, while u_{t+1} corresponds to a short-run risk shock.

The ARMA(1,1) case. Suppose $\rho_{xz} = 1$. This implies that the process for $\hat{x}_t$ specializes to

$$\hat{x}_{t+1} = \theta_z \hat{x}_t + \sigma_x v_{t+1} - (\theta_z \sigma_x - \sigma_z) v_t, \tag{O2.71}$$

which is an ARMA(1,1) process. If we further assume that $\sigma_z = \theta_z \sigma_x$, then we obtain an AR(1) process.

Notice that we can write the conditional expectation of x_{t+1} as follows

$$\mathbb{E}_t[x_{t+1}] = \theta_z \hat{x}_t - b \frac{\hat{x}_t - \mathbb{E}_{t-1}[x_t]}{\sigma_x} \Rightarrow \mathbb{E}_t[x_{t+1}] = \frac{\theta_z - b/\sigma_x}{1 - bL} \hat{x}_t. \tag{O2.72}$$

where $b \equiv \theta_z \sigma_x - \sigma_z$ and L is the lag operator.

Define $\hat{w}_t$ as follows

$$\hat{w}_t \equiv \frac{\hat{x}_t}{1 - bL} = \sum_{j=1}^{\infty} b^j \hat{x}_{t-j}. \tag{O2.73}$$

Unconditional moments. The unconditional variance of z_{t+1} is given by

$$Var[z_{t+1}] = \frac{\sigma_z^2}{1 - \theta_z^2}, \tag{O2.74}$$

and the unconditional variance of $\hat{x}_{t+1}$ is given by

$$Var[\hat{x}_{t+1}] = \mathbb{E}\left[Var_t[\hat{x}_{t+1}]\right] + Var[\mathbb{E}_t[\hat{x}_{t+1}]] = \sigma_x^2 + \frac{\sigma_z^2}{1 - \theta_z^2}. \tag{O2.75}$$

In this general case, we can choose θ_z and σ_z to match the persistence and variance of expectations and choose σ_x^2 to match the unconditional variance of productivity growth. The parameter ρ_{xz} controls the correlation between expected and realized productivity growth.

In the special case $\rho_{xz} = 1$ and $\sigma_z = \theta_z \sigma_x$. This allows us to match the persistence of expectations and either the unconditional variance of expected productivity growth or unconditional variance of realized productivity growth.

If $\sigma_z = \theta_z \sigma_x$, then

$$Var[z_{t+1}] = \theta_z^2 \frac{\sigma_x^2}{1 - \theta_z^2} = \theta_z^2 Var[x_{t+1}]. \tag{O2.76}$$

Discretization of the productivity growth process. Using the process for z_{t+1} to eliminate v_{t+1} from the expression for $\hat{x}_t$, we obtain

$$\hat{x}_{t+1} = \theta_z^{-1} z_{t+1} + \left(\frac{\rho_{xz}\sigma_x}{\sigma_z} - \frac{1}{\theta_z} \right) (z_{t+1} - \theta_z z_t) + \sigma_x \sqrt{1-\rho_{xz}^2} u_{t+1}. \tag{O2.77}$$

Given z_t, z_{t+1}, and u_{t+1}, this allow us to solve for $\hat{x}_{t+1}$. Suppose z_t takes on N_z discrete values and u_t takes on N_u values. This implies that $\hat{x}_t$ can take on $N \equiv N_z^2 \times N_u$ values. If we impose the constraint $\rho_{xz} = 1$, then $\hat{x}_{t+1}$ is independent of u_{t+1}, so $\hat{x}_t$ takes on N_z^2 possible values. If we further assume that $\sigma_z = \theta_z \sigma_x$, then $\hat{x}_t$ can take only N_z values.

The current value of $\hat{x}$ is determined by $(z_{t-1}, z_t, u_t) = (z^i, z^j, u^k)$, where $i, j \in \{1, \ldots, N_z\}$ and $k \in \{1, \ldots, N_u\}$. We can define the current state as a function of (i, j, k): $s = i + (j-1)N_z + (k-1)N_z^2$. The transition matrix is then given by

$$Pr(s' = i' + (j'-1)N_z + (k'-1)N_z^2 | s) = \begin{cases} Pr(z' = z^{j'} | z = z^j) Pr(u' = u^k), & \text{if } i' = j \\ 0, & \text{if } i' \neq j \end{cases}, \tag{O2.78}$$

where $s = i + (j-1)N_z + (k-1)N_z^2$.

We can write the system above in matrix form:

$$\begin{bmatrix} \hat{x}_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \theta_z \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ z_t \end{bmatrix} + \begin{bmatrix} \sigma_x & 0 \\ \rho_{xz}\sigma_z & \sqrt{1-\rho_{xz}^2}\sigma_z \end{bmatrix} \begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix}. \tag{O2.79}$$

Notice that the spectral decomposition of the matrix of coefficients is given by

$$\begin{bmatrix} 0 & 1 \\ 0 & \theta_z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \theta_z & 0 \end{bmatrix} \begin{bmatrix} \theta_z & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \theta_z^{-1} \\ 1 & -\theta_z^{-1} \end{bmatrix}. \tag{O2.80}$$

Define the following transformed variables:

$$\begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} \equiv \begin{bmatrix} 0 & \theta_z^{-1} \\ 1 & -\theta_z^{-1} \end{bmatrix} \begin{bmatrix} x_t - \mu \\ z_t \end{bmatrix}. \tag{O2.81}$$

The difference equation for w_t is given by

$$\begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix} = \begin{bmatrix} \theta_z & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} + \begin{bmatrix} \rho_{xz}\frac{\sigma_z}{\theta_z} & \sqrt{1-\rho_{xz}^2}\frac{\sigma_z}{\theta_z} \\ \sigma_x - \rho_{xz}\frac{\sigma_z}{\theta_z} & -\sqrt{1-\rho_{xz}^2}\frac{\sigma_z}{\theta_z} \end{bmatrix} \begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix}. \tag{O2.82}$$

We can write the original variables in terms of $w_{1,t}$ and $w_{2,t}$:

$$\log x_t = \mu + w_{1,t} + w_{2,t}, \qquad z_t = \theta_z w_{1,t} \tag{O2.83}$$

We can then discretize $w_{1,t}$ and $w_{2,t}$.

$$w_{2,t+1} = \sqrt{1-\rho_{xz}^2}\sigma_x u_{t+1} + (\rho_{xz}\sigma_x - \sigma_z\theta_z^{-1})v_{t+1} \tag{O2.84}$$

$$w_{1,t+1} = \theta_z w_{1,t} + \sigma_z\theta_z^{-1}u_{t+1} \tag{O2.85}$$

$$x_{t+1} = \mu + w_{1,t+1} + \sqrt{1-\rho_{xz}^2}\sigma_x(w_{1,t+1} - \theta_z w_{1,t})\frac{\theta_z}{\sigma_z} + (\rho_{xz}\sigma_x - \sigma_z\theta_z^{-1})v_{t+1}. \tag{O2.86}$$

O3 Proofs

O3.1 Proof of Proposition O.1

Proof. We provide the characterization of the economy with N-possible states in steps, proceeding from the households' problem to the market clearing conditions.

Step 1: households' problem. Household i chooses consumption C_i, hours h_i, and arrow securities $B_i(X,s,s')$ to maximize (3) subject to the budget constraint:

$$C_i + \mathbb{E}_s[\Lambda(X,s,s')B_i(X,s,s')] = B_i + Wh_i, \tag{O3.1}$$

and an appropriate No-Ponzi condition.

As in the two-state case, it is useful to transform this budget constraint in terms of net consumption and total wealth:

$$\tilde{C}_i + \mathbb{E}_s\left[\Lambda'\left(B_i' + W'h_i' - \xi'\frac{(h_i')^{1+\nu}}{1+\nu} + \mathcal{H}_i'\right)\right] = B_i + Wh_i - \xi\frac{h_i^{1+\nu}}{1+\nu} + \mathcal{H}_i \equiv N_i, \tag{O3.2}$$

where we used the fact that $\mathcal{H}_i = \mathbb{E}_s\left[\Lambda'\left(W'h_i' - \xi'\frac{(h_i')^{1+\nu}}{1+\nu} + \mathcal{H}_i'\right)\right]$ and $\tilde{C}_i = C_i - \xi\frac{h_i^{1+\nu}}{1+\nu}$.

We can then write the budget constraint above as follows

$$\tilde{C}_i + \mathbb{E}_s\left[\Lambda' N_i'\right] = N_i. \tag{O3.3}$$

The household's problem can then be equivalently expressed as choosing $(\tilde{C}_i(N,X,s), N_i'(N,X,s,s'))$ to maximize (3) subject to the constraint above and the natural borrowing limit $N_i'(N,X,s,s') \geq 0$. The solution takes the form in Equation (**??**). It will be useful to define the consumption-wealth ratio $c_i \equiv \frac{\tilde{C}_i}{N}$ and the normalized net worth $n_i' \equiv \frac{N_i'}{N_i}$. Define the portfolio return as $R_{i,n}(X,s,s') \equiv \frac{n_i'(X,s,s')}{1-c_i(X,s)}$, which gives the budget constraint $n' = R_{i,n}'(1-c)$. The function $v_i(X,s)$ must then satisfy the condition

$$\frac{(v_i(X,s)N)^{1-\psi^{-1}}-1}{1-\psi^{-1}} = \max_{c_i,n_i'}(1-\beta)\frac{(c_iN)^{1-\psi^{-1}}-1}{1-\psi^{-1}} + \beta\frac{\mathbb{E}_i\left[(v_i(X',s')n'N)^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}}-1}{1-\psi^{-1}}, \quad \text{(O3.4)}$$

subject to $n' = R_n'(1-c_i)$, $\mathbb{E}_s[\Lambda' R_n'] = 1$, and $n' \geq 0$.

Step 2: optimality conditions. The first-order conditions for the consumption-wealth ratio and the portfolio share are given by

$$(1-\beta)c_i^{-\psi^{-1}} = \beta\mathcal{R}_i(X,s)^{1-\psi^{-1}}(1-c_i)^{-\psi^{-1}} \quad \text{(O3.5)}$$

$$p^i_{ss'}v_i(X',s')^{1-\gamma}R'_{i,n}(X,s,s')^{-\gamma} = p_{ss'}\Lambda(X,s,s')\mu(X,s), \quad \text{(O3.6)}$$

where $\mathcal{R}_i(X,s) = \mathbb{E}_i\left[(v_i(X',s')R_{i,n}(X,s,s'))^{1-\gamma}|X,s\right]^{\frac{1}{1-\gamma}}$ and $\mu(X,s)$ is the (normalized) multiplier on the constraint on returns. From the first-order condition for consumption, we obtain Equation (A.10). The envelope condition is given by

$$v_i(X)^{1-\frac{1}{\psi}} = (1-\beta)c_i^{-\frac{1}{\psi}} \Rightarrow v_i(X)^{1-\gamma} = (1-\beta)^{\theta}c_i^{-\frac{\theta}{\psi}}. \quad \text{(O3.7)}$$

Notice that the multiplier is given by

$$\mu(X,s) = \mathbb{E}_i[(v_i(X,s')R'_{i,n}(X,s,s'))^{1-\gamma}] = \mathcal{R}_i(X,s)^{1-\gamma} = \left(\frac{1-\beta}{\beta}\right)^{\theta}\left[\frac{c_i}{1-c_i}\right]^{-\frac{\theta}{\psi}}. \quad \text{(O3.8)}$$

Combining the previous two expressions above with the first-order condition for $R'_{i,n}$, we obtain

$$p_{ss'}\Lambda(X,s,s') = p^i_{ss'}\frac{(1-\beta)^{\theta}(c_i')^{-\frac{\theta}{\psi}}R'_{i,n}(X,s,s')^{-\gamma}}{\left(\frac{1-\beta}{\beta}\right)^{\theta}\left[\frac{c_i}{1-c_i}\right]^{-\frac{\theta}{\psi}}} \quad \text{(O3.9)}$$

$$= p^i_{ss'}\beta^{\theta}\left(\frac{c_i'N'}{c_iN}\right)^{-\frac{\theta}{\psi}}(R'_{i,n})^{-(1-\theta)}, \quad \text{(O3.10)}$$

$$\equiv p^i_{ss'}\Lambda_i(X,s,s'), \quad \text{(O3.11)}$$

using the fact that $\frac{\theta}{\psi}+1-\theta=\gamma$.

Hence, expressions (A.10) and (11) hold unchanged with multiple states. Moreover, the change-of-measure equation $\Lambda_i(X,s,s') = \frac{p_{ss'}}{p^i_{ss'}}\Lambda(X,s,s')$ also holds.

Step 3: firms' problem and labor market outcomes. The firm's problem is essentially the same and the first-order condition (13) holds without change. The equations for hours and wages

(14) are also unchanged.

Step 4: law of motion of aggregate state variables. The aggregate state variables are the same as before. The law of motion of $\mathcal{L}$ is given by

$$E'(X,s) = \sum_{s' \in \mathcal{S}} \frac{p_{ss'}\Lambda(X,s,s')}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}\Lambda(X,s,\tilde{s})} x_{s'}, \tag{O3.12}$$

and the law of motion of η_i is unchanged.

Step 5: market clearing conditions. Notice that $\sum_{i=1}^{I} \mu_i B_i$ must coincide with the cum-dividend value of the firm. Hence, $\sum_{i=1}^{I} \mu_i N_i$ coincides with the cum-dividend value of the surplus claim, $A_{-}P(X,s)$, where A_{-} denotes lagged productivity.

The market clearing condition for net consumption is then given by

$$\sum_{i=1}^{I} \mu_i N_i c_i(X,s) = A_{-}\left(x_s h(E)^{\alpha} - \xi \frac{h(E)^{1+\nu}}{1+\nu}\right). \tag{O3.13}$$

Using the fact that $\sum_{i=1}^{I} \mu_i N_i = A_{-}P(X,s)$, we obtain the market clearing for consumption in Equation (18). The market clearing for Arrow securities is given by

$$\sum_{i=1}^{I} \mu_i N_i n_i(X,s,s') = x_s A_{-}P(X',s'). \tag{O3.14}$$

We can write the expression above in terms of portfolio returns:

$$\sum_{i=1}^{I} \frac{\mu_i N_i(1-c_i(X,s))}{\sum_{j=1}^{I} \mu_j N_j(1-c_j(X,s))} R_{n,i}(X,s,s') = \frac{x_s A_{-}P(X',s')}{A_{-}\left[P(X,s) - \left(x_s h(E)^{\alpha} - \xi\frac{h(E)^{1+\nu}}{1+\nu}\right)\right]}, \tag{O3.15}$$

using the fact that $\sum_{j=1}^{I} \mu_j N_j(1-c_j(X,s)) = A_{-}\left[P(X,s) - \left(x_s h(E)^{\alpha} - \xi\frac{h(E)^{1+\nu}}{1+\nu}\right)\right]$.

We can write the expression above as follows

$$\sum_{i=1}^{I} \tilde{\eta}_i R_{n,i}(X,s,s') = R_p(X,s,s'), \tag{O3.16}$$

for each $s' \in \mathcal{S}$. □

O3.2 Proof of Proposition O.2

Proof. We provide next a characterization of the economy with log-utility and an arbitrary number of states.

Step 1: consumption and portfolio decisions. Suppose $\psi = \gamma = 1$. This implies that $c_i(X,s) = 1-\beta$ and that $\Lambda_i(X,s,s') = R_{i,n}^{-1}(X,s,s')$. From the change-of-measure equation, we obtain

$$R_{i,n}^{-1}(X,s,s') = \frac{p_{ss'}\Lambda(X,s,s')}{p_{ss'}^i} \Rightarrow R_{i,n}(X,s,s') = \frac{p_{ss'}^i}{p_{ss'}\Lambda(X,s,s')}. \tag{O3.17}$$

Step 2: the economy's SDF. Plugging the expression for $R_{i,n}(X,s,s')$ into the market clearing condition for Arrow securities paying off in state s', we obtain

$$\frac{\sum_{i=1}^I \eta_i p_{ss'}^i}{p_{ss'}\Lambda(X,s,s')} = R_p(X,s,s') \Rightarrow \Lambda(X,s,s') = \frac{p_{ss'}(X)}{p_{ss'}} R_p^{-1}(X,s,s'). \tag{O3.18}$$

Notice that the portfolio return for household i is given by

$$R_{i,n}(X,s,s') = \frac{p_{ss'}^i}{p_{ss'}(X)} R_p(X,s,s'). \tag{O3.19}$$

Hence, optimistic investors, i.e. investors satisfying $p_{ss'}^i > p_{ss'}(X)$, hold a levered position on the surplus claim.

Step 3: the surplus claim. From the market clearing condition for goods, we obtain

$$P(X,s) = \frac{x_s h(E)^\alpha - \xi \frac{h(E)^{1+\nu}}{1+\nu}}{1-\beta}. \tag{O3.20}$$

This implies that the return on the surplus claim is given by

$$R_p(X,s,s') = \frac{x_s P(X',s')}{P(X,s) - \left(x_s h(E)^\alpha - \xi \frac{h(E)^{1+\nu}}{1+\nu}\right)} = \frac{x_s}{\beta} \frac{x_{s'} h(E'(X,s))^\alpha - \xi \frac{h(E'(X,s))^{1+\nu}}{1+\nu}}{x_s h(E)^\alpha - \xi \frac{h(E)^{1+\nu}}{1+\nu}}. \tag{O3.21}$$

Using the expression for $h(E)$, we can simplify the expression above

$$R_p(X,s,s') = \frac{x_s}{\beta} \frac{x_{s'} E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{O3.22}$$

which coincides with the expression for the two-type case.

Step 4: interest rate and risk premium. Using the fact that $R_b(X,s)$ is the risk-neutral expectation of $R_p(X,s,s')$ and $E'(X,s)$ is the risk-neutral expectation of $x_{s'}$, we obtain

$$R_b(X,s) = \left(1 - \frac{\alpha}{1+\nu}\right) \frac{x_s}{\beta} \frac{E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu} E^{\frac{1+\nu}{1+\nu-\alpha}}}, \tag{O3.23}$$

which coincides with the expression for the two-type case.

The expected return on the surplus claim is given by

$$\mathbb{E}_s[R_p(X,s,s')] = \frac{x_s}{\beta} \frac{\mathbb{E}_s[x_{s'}]E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{O3.24}$$

Taking the difference of the previous two equations, we obtain the risk premium on the surplus claim:

$$\mathbb{E}_s[R_p^e(X,s,s')] = \frac{x_s}{\beta} \frac{[\mathbb{E}_s[x_{s'}] - E'(X,s)]E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}}. \tag{O3.25}$$

Step 5: law of motion of aggregate state variables. The risk-neutral probability is given by

$$\frac{p_{ss'}\Lambda(X,s,s')}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}\Lambda(X,s,\tilde{s})} = \frac{p_{ss'}(X)R_p^{-1}(X,s,s')}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}(X)R_p^{-1}(X,s,\tilde{s})} = \frac{p_{ss'}(X)\left[x_{s'} - \frac{\alpha}{1+\nu}E'(X,s)\right]^{-1}}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}(X)\left[x_{\tilde{s}} - \frac{\alpha}{1+\nu}E'(X,s)\right]^{-1}}. \tag{O3.26}$$

From the law of motion of $\mathcal{L}$, we obtain

$$E'(X,s) = \sum_{s'\in\mathcal{S}} x_{s'} \frac{p_{ss'}(X)\left[x_{s'} - \frac{\alpha}{1+\nu}E'(X,s)\right]^{-1}}{\sum_{\tilde{s}\in\mathcal{S}} p_{s\tilde{s}}(X)\left[x_{\tilde{s}} - \frac{\alpha}{1+\nu}E'(X,s)\right]^{-1}}. \tag{O3.27}$$

Rearranging the expression above, we obtain

$$\sum_{s'\in\mathcal{S}} \frac{p_{ss'}(X)(x_{s'} - E'(X,s))}{x_{s'} - \frac{\alpha}{1+\nu}E'(X,s)} = 0 \tag{O3.28}$$

The left-hand side is positive for $E'(X,s) = x^1$, it is negative for $E'(X,s) = x^N$, and it is strictly decreasing in $E'(X,s)$, assuming the condition $x^N < \frac{x^1}{\alpha}$ such that the denominator is positive in the range $x^1 < E'(X,s) < x^N$. Therefore, a solution exists and it is unique.

The law of motion of the wealth share is given by

$$\eta_i'(X,s,s') = \frac{\eta_i R_{i,n}(X,s,s')}{\sum_{j=1}^{I} \eta_j R_{j,n}(X,s,s')} = \eta_i \frac{R_{i,n}(X,s,s')}{R_p(X,s,s')} = \eta_i \frac{p_{ss'}}{p_{ss'}(X)}. \tag{O3.29}$$

□

O3.3 Proof of Proposition O.3

Proof. We will construct an equilibrium that has iid returns for any financial asset. We guess-and-verify that the consumption-wealth ratio and the net-worth multiplier are constant.

Step 1: consumption and portfolio decisions. Let the consumption-wealth ratio be given by $c_i^*(X,s) = 1-\beta^*$, given a constant β^* that we need to determine. Given that there is no heterogeneity in beliefs, we obtain from the market clearing condition for Arrow securities $R_{i,n}^*(X,s,s') = R_p^*(X,s,s')$. Plugging $c_i^*(X,s)$ and $R_{i,n}^*(X,s,s')$ into the expression for $\Lambda_i^*(X,s,s')$, we obtain

$$\Lambda_i^*(X,s,s') = \beta^\theta(\beta^*)^{-\frac{\theta}{\psi}}[R_p^*(X,s,s')]^{-\gamma}. \tag{O3.30}$$

Step 2: net-worth multiplier. From the envelope condition, we obtain

$$v_i^*(X,s)^{1-\psi^{-1}} = (1-\beta)c_i^*(X,s)^{-\psi^{-1}} \Rightarrow v_i^*(X,s) = (1-\beta)^{\frac{1}{1-\psi^{-1}}}(1-\beta^*)^{-\frac{\psi^{-1}}{1-\psi^{-1}}}. \tag{O3.31}$$

Step 3: wages, hours, and profits. Using $\alpha = \hat{\alpha}\epsilon$, $\xi = \hat{\xi}\epsilon$, and taking the limit of the expressions for wages, hours, and profits as $\epsilon \to 0$, we obtain the expressions provided in the proposition.

Step 4: The price and return on the surplus claim. For an arbitrary α and ξ, the market clearing condition for goods implies that

$$P^*(X,s) = \left.\frac{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}}{1-\beta^*}\right|_{\epsilon=0} = \frac{x_s}{1-\beta^*}. \tag{O3.32}$$

The return on the surplus claim is given by

$$R_p^*(X,s,s') = \left.\frac{x_s}{\beta^*}\frac{x_{s'}E'(X,s)^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E'(X,s)^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}}\right|_{\epsilon=0} = \frac{x_{s'}}{\beta^*}. \tag{O3.33}$$

Step 4: The economy's SDF. From the pricing equation, we obtain

$$\mathbb{E}_i[\Lambda_i^*(X,s,s')R_p^*(X,s,s')] = 1 \Rightarrow \beta^\theta(\beta^*)^{-\theta}\mathbb{E}_i[(x_s')^{1-\gamma}] = 1 \tag{O3.34}$$

Rearranging the expression above, we obtain

$$\beta^* = \beta\mathbb{E}_i[(x_s')^{1-\gamma}]^{\frac{1-\psi^{-1}}{1-\gamma}}. \tag{O3.35}$$

Notice that the condition $\beta^* < 1$ is required to ensure that the consumption-wealth ratio is positive.

The SDF is then given by

$$\Lambda^*(X,s,s') = \beta\mathbb{E}^*[x_{s'}^{1-\gamma}]^{\frac{\gamma-\psi^{-1}}{1-\gamma}}(x_{s'})^{-\gamma}, \tag{O3.36}$$

using the fact that $\Lambda^*(X,s,s') = \Lambda_i^*(X,s,s')$.

Step 6: Law of motion of aggregate state variables. The risk-neutral probability is given by

$$\frac{p_{s'}^* \Lambda^*(X,s,s')}{\sum_{\tilde{s}\in\mathcal{S}} p_{\tilde{s}}^* \Lambda^*(X,s,\tilde{s})} = \frac{p_{s'}^* x_{s'}^{-\gamma}}{\mathbb{E}^*[x_{s'}^{-\gamma}]} \tag{O3.37}$$

Hence, $E'(X,s)$ is given by

$$E'(X,s) = \frac{\mathbb{E}^*[x_{s'}^{1-\gamma}]}{\mathbb{E}^*[x_{s'}^{-\gamma}]}. \tag{O3.38}$$

Step 5: The interest rate and risk premium on surplus claim. The interest rate and risk premium are given by

$$R_b^*(X,s) = \frac{E'(X,s)}{\beta^*}, \qquad R_p^e(X,s) = \frac{\mathbb{E}^*[x_{s'}] - E'(X,s)}{\beta^*}. \tag{O3.39}$$

Using the expression for β^* and $E'(X,s)$, we can write the interest rate as follows

$$R_b^*(X,s) = \beta^{-1}\mathbb{E}^*[(x_s')^{1-\gamma}]^{\frac{\psi^{-1}-\gamma}{1-\gamma}}\mathbb{E}^*[(x_{s'})^{-\gamma}]^{-1}, \tag{O3.40}$$

The expected return on the surplus claim is given by

$$\mathbb{E}^*\left[R_p^*(X,s,s')\right] = \beta^{-1}\mathbb{E}^*[(x_s')^{1-\gamma}]^{\frac{\psi^{-1}-1}{1-\gamma}}\mathbb{E}^*[x_{s'}] \tag{O3.41}$$

and the risk premium on the surplus claim is given by

$$\mathbb{E}^*\left[\frac{R_p^*(X,s,s')}{R_b^*(X,s)}\right] = \frac{\mathbb{E}^*[x_{s'}]\mathbb{E}^*[x_{s'}^{-\gamma}]}{\mathbb{E}^*[x_{s'}^{1-\gamma}]} \tag{O3.42}$$

□

O3.4 Proof of Proposition O.4

Proof. We provide a characterization of the first-order correction for the value function, summarized by the net-worth multiplier $v_i(X,s;\epsilon)$, and the policy functions, namely the consumption-wealth ratio $c_i(X,s;\epsilon)$ and the portfolio return $R_{i,n}(X,s,s';\epsilon)$, given the expansion for the economy's SDF

$$\Lambda(X,s,s';\epsilon) = \Lambda^*(X,s,s') + \hat{\Lambda}(X,s,s')\epsilon + \mathcal{O}(\epsilon), \tag{O3.43}$$

where $\hat{\Lambda}(X,s,s')$ is the first-order correction for the SDF. We take $\hat{\Lambda}(X,s,s')$ as given for now and we will solve for it in a later stage.

Step 1: value function. The Bellman equation for household i can be written as follows:

$$\frac{v_i(X,s;\epsilon)^{1-\psi^{-1}}}{1-\psi^{-1}} = (1-\beta)\frac{c_i^{1-\psi^{-1}}}{1-\psi^{-1}} + \beta\frac{\left[\sum_{s'\in\mathcal{S}} p^i_{ss'}(v_i(X',s';\epsilon)R'_n(1-c_i))^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}}}{1-\psi^{-1}} \tag{O3.44}$$
$$+ \mu(X,s;\epsilon)\left[1-\sum_{s'\in\mathcal{S}} p^*_{s'}\Lambda(X,s,s';\epsilon)R'_n\right],$$

where $X' = \chi(X,s,s';\epsilon)$.

Taking the derivative of the expression above with respect to ϵ and evaluating at $\epsilon = 0$, we obtain a condition involving the first-order correction for v_i:

$$(v_i^*)^{-\psi^{-1}}\hat{v}_i(X,s) = \beta\left[\sum_{s'\in\mathcal{S}} p^*_{s'}(v_i^* R^*_p(s')\beta^*)^{1-\gamma}\right]^{\frac{\gamma-\psi^{-1}}{1-\gamma}}\left[\sum_{s'\in\mathcal{S}}\delta^i_{ss'}\frac{(v_i^*R^*_p(s')\beta^*)^{1-\gamma}}{1-\gamma}+\right.$$
$$\left.\sum_{s'\in\mathcal{S}} p^*_{s'}(v_i^*)^{-\gamma}(R^*_p(s')\beta^*)^{1-\gamma}\hat{v}_i(X^*,s')\right] - \mu^*(X,s)\sum_{s'\in\mathcal{S}} p^*_{s'}\hat{\Lambda}_1(X,s,s')R^*_p(s'), \tag{O3.45}$$

where we used the fact that $R'_n(X,s,s';0) = R^*_p(X,s,s')$, $c_i(X,s;0) = 1-\beta^*$. We also used the fact that $\chi(X,s,s';0) = X^*$, where $X^* = (E^*, \{\eta_i\}_{i=1}^I)$ as $E'(x,s) = E^*$ and the wealth distribution is constant in the benchmark economy.

Using the results for the benchmark economy, we can simplify the expression above:

$$\hat{v}_i(X,s) = \beta\left[\sum_{s'\in\mathcal{S}} p^*_{s'}x_{s'}^{1-\gamma}\right]^{\frac{\gamma-\psi^{-1}}{1-\gamma}}\left[v_i^*\sum_{s'\in\mathcal{S}}\delta^i_{ss'}\frac{x_{s'}^{1-\gamma}}{1-\gamma} + \sum_{s'\in\mathcal{S}} p^*_{s'}x_{s'}^{1-\gamma}\hat{v}_i(X^*,s')\right]$$
$$-\mu^*(X,s)(v_i^*)^{\psi^{-1}}\sum_{s'\in\mathcal{S}} p^*_{s'}\hat{\Lambda}_1(X,s,s')R_p(s'), \tag{O3.46}$$

where we used the fact that $v^*(X,s)$ is constant and $R^*_p(s')\beta^* = x_{s'}$.

Let's solve for $\mu^*(X,s)$ next. The first-order condition for $R_n(X,s,s';\epsilon)$ is given by

$$\beta\left[\sum_{s'\in\mathcal{S}} p^i_{ss'}(v_i(X',s';\epsilon)R'_n(1-c_i))^{1-\gamma}\right]^{\frac{\gamma-\psi^{-1}}{1-\gamma}} p_{ss'}(v_i(X',s')(1-c_i))^{1-\gamma}R_n(X,s,s')^{-\gamma} = \mu(X,s;\epsilon)p^*_{s'}\Lambda(X,s,s';\epsilon) \tag{O3.47}$$

Multiplying by $R_n(X,s,s')$ both sides and adding across states, we obtain

$$\mu(X,s;\epsilon) = \beta\left[\sum_{s'\in\mathcal{S}} p^i_{ss'}(v_i(X',s';\epsilon)R'_n(1-c_i))^{1-\gamma}\right]^{\frac{1-\psi^{-1}}{1-\gamma}}. \tag{O3.48}$$

Evaluating the expression above at $\epsilon = 0$, we obtain

$$\mu^*(X,s) = \beta v^*(X,s)^{1-\psi^{-1}} \left[\sum_{s' \in \mathcal{S}} p^*_{s'} x^{1-\gamma}_{s'} \right]^{\frac{1-\psi^{-1}}{1-\gamma}}. \tag{O3.49}$$

Given $\mu^*(X,s)$, we obtain a system of equations for $\hat{v}_i(X,s')$:

$$\frac{\hat{v}_i(X,s)}{v^*_i(X,s)} - \beta \mathbb{E}^*[x^{1-\gamma}_{s'}]^{\frac{1-\psi^{-1}}{1-\gamma}} \sum_{s' \in \mathcal{S}} \omega^*_{s'} \frac{\hat{v}_i(X^*,s')}{v^*(X,s)} = \beta \mathbb{E}^*[x^{1-\gamma}_{s'}]^{\frac{1-\psi^{-1}}{1-\gamma}} \left[\sum_{s' \in \mathcal{S}} \frac{\omega^*_{s'}}{1-\gamma} \frac{\delta^i_{ss'}}{p^*_{s'}} - \sum_{s' \in \mathcal{S}} \omega^*_{s'} \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} \right] \tag{O3.50}$$

using the fact that $R_p(s')\Lambda^*(X,s,s') = \frac{x^{1-\gamma}_{s'}}{\mathbb{E}^*[x^{1-\gamma}_{s'}]}$ and the definition $\omega^*_s \equiv \frac{p^*_s x^{1-\gamma}_s}{\mathbb{E}^*[x^{1-\gamma}_{s'}]}$.

We will solve first for the case $X = X^*$. We can write the system above in matrix form:

$$\begin{bmatrix} 1-\chi_v\omega^*_1 & -\chi_v\omega^*_2 & \dots & -\chi_v\omega^*_N \\ -\chi_v\omega^*_1 & 1-\chi_v\omega^*_2 & \dots & -\chi_v\omega^*_N \\ \vdots & \vdots & \dots & \vdots \\ -\chi_v\omega^*_1 & -\chi_v\omega^*_2 & \dots & 1-\chi_v\omega^*_N \end{bmatrix} \begin{bmatrix} \frac{\hat{v}_i(X^*,1)}{v^*(X,s)} \\ \frac{\hat{v}_i(X^*,2)}{v^*(X,s)} \\ \vdots \\ \frac{\hat{v}_i(X^*,N)}{v^*(X,s)} \end{bmatrix} = \begin{bmatrix} b^v_{i,1}(X^*) \\ b^v_{i,2}(X^*) \\ \vdots \\ b^v_{i,N}(X^*) \end{bmatrix}, \tag{O3.51}$$

where $\chi_v \equiv \beta \mathbb{E}^*[x^{1-\gamma}_{s'}]^{\frac{1-\psi^{-1}}{1-\gamma}}$ and

$$b^v_{i,s}(X^*) \equiv \chi_v \left[\sum_{s' \in \mathcal{S}} \frac{\omega^*_{s'}}{1-\gamma} \frac{\delta^i_{ss'}}{p^*_{s'}} - \sum_{s' \in \mathcal{S}} \omega^*_{s'} \frac{\hat{\Lambda}(X^*,s,s')}{\Lambda^*(X,s,s')} \right]. \tag{O3.52}$$

Let $\omega^* = [\omega^*_1, \omega^*_2, \dots, \omega^*_N]$ denote a row vector, $\hat{v}_i(X) = [\hat{v}_i(X,1), \dots, \hat{v}_i(X,N)]'$ denote a column vector, $b^v_i(X) = \left[b^v_{i,1}(X), \dots, b^v_{i,N}(X) \right]'$ denote a column-vector, and $\mathbf{1}_N$ denote a N-dimensional column vector filled with ones. We can then write the expression above as follows:

$$[I - \chi_v \mathbb{1}_N \omega^*] \frac{\hat{v}_i(X^*)}{v^*_i} = b^v_i(X^*). \tag{O3.53}$$

The matrix on the left-hand side corresponds to the sum of an invertible matrix and rank-one matrix. An application of the Sherman-Morrison formula gives the inverse of this matrix, which gives the solution

$$\hat{v}_i(X^*) = v^*_i(X,s) \left[I + \frac{\chi_v}{1-\chi_v} \mathbb{1}_N \omega^* \right] b^v_i(X^*) \tag{O3.54}$$

The net-worth multiplier at state (X, s) is then given by

$$\frac{\hat{v}_i(X^*, s)}{v_i^*(X, s)} = b_{i,s}^v + \frac{\chi_v}{1-\chi_v} \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}^* b_{i,\tilde{s}}^v(X^*). \tag{O3.55}$$

Using the expression for $b_{i,s}^v$, we can write the expression above as follows

$$\frac{\hat{v}_i(X^*, s)}{v_i^*(X, s)} = \chi_v \sum_{\tilde{s} \in \mathcal{S}} \left(\mathbf{1}_{\tilde{s}=s} + \frac{\chi_v}{1-\chi_v} \omega_{\tilde{s}}^* \right) \left[\sum_{s' \in \mathcal{S}} \frac{\omega_{s'}^*}{1-\gamma} \frac{\delta_{\tilde{s}s'}^i}{p_{s'}^*} - \sum_{s' \in \mathcal{S}} \omega_{s'}^* \frac{\hat{\Lambda}(X^*, \tilde{s}, s')}{\Lambda^*(X, s, s')} \right]. \tag{O3.56}$$

Taking the average of the expression above using the weights ω_s^*, we obtain

$$\sum_{s \in \mathcal{S}} \omega_s^* \frac{\hat{v}_i(X^*, s)}{v_i^*(X, s)} = \frac{\chi_v}{1-\chi_v} \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}^* \left[\sum_{s' \in \mathcal{S}} \frac{\omega_{s'}^*}{1-\gamma} \frac{\delta_{\tilde{s}s'}^i}{p_{s'}^*} - \sum_{s' \in \mathcal{S}} \omega_{s'}^* \frac{\hat{\Lambda}(X^*, \tilde{s}, s')}{\Lambda^*(X, s, s')} \right]. \tag{O3.57}$$

The net-worth multiplier at (X, s) is then given by

$$\frac{\hat{v}_i(X, s)}{v_i^*(X, s)} = \chi_v \left[\sum_{s' \in \mathcal{S}} \frac{\omega_{s'}^*}{1-\gamma} \frac{\delta_{ss'}^i}{p_{s'}^*} - \sum_{s' \in \mathcal{S}} \omega_{s'}^* \frac{\hat{\Lambda}(X, s, s')}{\Lambda^*(X, s, s')} + \sum_{s' \in \mathcal{S}} \omega_{s'}^* \frac{\hat{v}_i(X^*, s')}{v^*(X, s)} \right] \tag{O3.58}$$

We can then write the expression above as follows:

$$\frac{\hat{v}_i(X, s)}{v_i^*(X, s)} = \chi_v \sum_{s' \in \mathcal{S}} \omega_{s'}^* \left[\frac{1}{1-\gamma} \frac{\delta_{ss'}^i}{p_{s'}^*} - \frac{\hat{\Lambda}(X, s, s')}{\Lambda^*(X, s, s')} \right] + \chi_v \overline{v}, \tag{O3.59}$$

where

$$\overline{v} \equiv \frac{\chi_v}{1-\chi_v} \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}^* \sum_{s' \in \mathcal{S}} \omega_{s'}^* \left[\frac{1}{1-\gamma} \frac{\delta_{\tilde{s}s'}^i}{p_{s'}^*} - \frac{\hat{\Lambda}(X^*, \tilde{s}, s')}{\Lambda^*(X^*, \tilde{s}, s')} \right]. \tag{O3.60}$$

Step 2: consumption-wealth ratio. From the envelope condition, the consumption-wealth ratio is given by

$$c_i(X, s; \epsilon) = (1-\beta)^{\psi} v_i(X, s; \epsilon)^{1-\psi}. \tag{O3.61}$$

The first-order correction for consumption is then given by

$$\hat{c}_1(X, s) = (1-\beta)^{\psi} (v_i^*(X, s))^{1-\psi} (1-\psi) \frac{\hat{v}_i(X, s)}{v_i^*(X, s)}. \tag{O3.62}$$

Step 3: portfolio return. Using the expression for the Lagrange multiplier, we can write the first-order condition for the portfolio return as follows

$$\frac{p^i_{ss'}}{p^*_{s'}} v_i(X', s')^{1-\gamma} R_n(X, s, s'; \epsilon)^{-\gamma} = \Lambda(X, s, s'; \epsilon) \sum_{s' \in \mathcal{S}} p^i_{ss'} (v_i(X', s'; \epsilon) R_n(X, s, s'; \epsilon))^{1-\gamma} \tag{O3.63}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\delta^i_{ss'}}{p^*_{s'}} + (1-\gamma)\frac{\hat{v}_i(X^*, s')}{v^*(X, s)} - \gamma \frac{\hat{R}_{n,i}(X, s, s')}{R^*_p(X, s, s')} = \frac{\hat{\Lambda}(X, s, s')}{\Lambda^*(X, s, s')} \tag{O3.64}$$

$$+ \sum_{s' \in \mathcal{S}'} \frac{p^*_{s'} (v^*(X, s) R^*_p(s'))^{1-\gamma}}{\sum_{\tilde{s} \in \mathcal{S}} p^*_{\tilde{s}} (v^*(X, s) R^*_p(\tilde{s}))^{1-\gamma}} \left[\frac{\delta^i_{ss'}}{p^*_{s'}} + (1-\gamma) \left(\frac{\hat{v}_i(X^*, s')}{v^*(X, s)} + \frac{\hat{R}_{n,i}(X, s, s')}{R^*_p(X, s, s')} \right) \right] \tag{O3.65}$$

Rearranging the expression above, we obtain

$$\gamma \frac{\hat{R}_{n,i}(X, s, s')}{R^*_p(X, s, s')} + (1-\gamma) \sum_{\tilde{s} \in \mathcal{S}'} \omega^*_{\tilde{s}} \frac{\hat{R}_{n,i}(X, s, \tilde{s})}{R^*_p(X, s, \tilde{s})} = b^R_i(X, s, s'), \tag{O3.66}$$

where

$$b^R_i(X, s, s') \equiv \frac{\delta^i_{ss'}}{p^*_{s'}} + (1-\gamma)\frac{\hat{v}_i(X^*, s')}{v^*(X, s)} - \sum_{\tilde{s} \in \mathcal{S}} \omega^*_{\tilde{s}} \left[\frac{\delta^i_{s\tilde{s}}}{p^*_{\tilde{s}}} + (1-\gamma)\frac{\hat{v}_i(X^*, \tilde{s})}{v^*(X, s)} \right] - \frac{\hat{\Lambda}_1(X, s, s')}{\Lambda^*(X, s, s')} \tag{O3.67}$$

We can write the system above in matrix form:

$$\begin{bmatrix} \gamma + (1-\gamma)\omega^*_1 & (1-\gamma)\omega^*_2 & \dots & (1-\gamma)\omega^* N \\ (1-\gamma)\omega^*_1 & \gamma + (1-\gamma)\omega^*_2 & \dots & (1-\gamma)\omega^*_N \\ \vdots & \vdots & \dots & \vdots \\ (1-\gamma)\omega^*_1 & (1-\gamma)\omega^*_2 & \dots & \gamma + (1-\omega)\omega^*_N \end{bmatrix} \begin{bmatrix} \frac{\hat{R}_{n,i}(X,s,1)}{R^*_p(X,s,1)} \\ \frac{\hat{R}_{n,i}(X,s,2)}{R^*_p(X,s,2)} \\ \vdots \\ \frac{\hat{R}_{n,i}(X,s,N)}{R^*_p(X,s,N)} \end{bmatrix} = \begin{bmatrix} b^R_i(X, s, 1) \\ b^R_i(X, s, 2) \\ \vdots \\ b^R_i(X, s, N) \end{bmatrix} \tag{O3.68}$$

Denote the matrix above by A^* and define the row vector $\omega^* \equiv [\omega^*(s_1), \omega^*(s_2), \dots, \omega^*(s_N)]$ and the column-vector $\mathbb{1}$ with 1 in every entry. We can then write A^* as follows:

$$A^* = \gamma I + (1-\gamma)\mathbb{1}\omega^*. \tag{O3.69}$$

The inverse of A^* is given by

$$(A^*)^{-1} = \frac{1}{\gamma} I - \frac{1-\gamma}{\gamma} \mathbb{1} \omega^*. \tag{O3.70}$$

The portfolio return is then given by

$$\frac{\hat{R}_{n,i}(X,s,s')}{R_p^*(X,s,s')} = \frac{1}{\gamma} b_i^R(X,s,s') + \frac{1-\gamma}{\gamma} \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}^* \frac{\hat{\Lambda}_1(X,s,\tilde{s})}{\Lambda^*(X,s,\tilde{s})}. \tag{O3.71}$$

We can write the expression above as follows:

$$\frac{\hat{R}_{n,i}(X,s,s')}{R_p^*(X,s,s')} = \frac{1}{\gamma} \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}^* \left[\left(\frac{\delta_{ss'}^i}{p_{s'}^*} - \frac{\delta_{s\tilde{s}}^i}{p_{\tilde{s}}^*} \right) - \frac{\hat{\Lambda}_1(X,s,s')}{\Lambda^*(X,s,s')} \right] + \frac{1-\gamma}{\gamma} \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}^* \left[\left(\frac{\hat{v}_i(X^*,s')}{v^*(X,s)} - \frac{\hat{v}_i(X^*,\tilde{s})}{v^*(X,s)} \right) + \frac{\hat{\Lambda}_1(X,s,\tilde{s})}{\Lambda^*(X,s,\tilde{s})} \right]. \tag{O3.72}$$

Notice that we can write the term involving $\hat{v}_i(X,s)$ as follows

$$\frac{\hat{v}_i(X^*,s')}{v^*(X^*,s')} - \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}^* \frac{\hat{v}_i(X^*,\tilde{s})}{v^*(X^*,\tilde{s})} = \chi_v \sum_{\tilde{s} \in \mathcal{S}} \omega_{\tilde{s}}^* \sum_{\tilde{s}' \in \mathcal{S}} \omega_{\tilde{s}'}^* \left[\frac{1}{1-\gamma} \left(\frac{\delta_{s'\tilde{s}'}^i}{p_{\tilde{s}'}^*} - \frac{\delta_{\tilde{s}\tilde{s}'}^i}{p_{\tilde{s}'}^*} \right) - \left(\frac{\hat{\Lambda}(X^*,s',\tilde{s}')}{\Lambda^*(X^*,s',\tilde{s}')} - \frac{\hat{\Lambda}(X^*,\tilde{s},\tilde{s}')}{\Lambda^*(X^*,\tilde{s},\tilde{s}')} \right) \right] \tag{O3.73}$$

□

O3.5 Proof of Proposition O.5

Proof. From the expression for wages, we obtain:

$$w(E;\epsilon) = \xi \left(\frac{\hat{\alpha} E}{\hat{\xi}} \right)^{\frac{\nu}{1+\nu-\alpha}} = \hat{\xi} \left(\frac{\hat{\alpha} E}{\hat{\xi}} \right)^{\frac{\nu}{1+\nu}} \epsilon + \mathcal{O}(\epsilon^2). \tag{O3.74}$$

Hours are given by

$$h(E;\epsilon) = \exp\left[\frac{1}{1+\nu-\alpha} \log\left(\frac{\hat{\alpha} E}{\hat{\xi}} \right) \right] = \left(\frac{\hat{\alpha} E}{\hat{\xi}} \right)^{\frac{1}{1+\nu}} + \left(\frac{\hat{\alpha} E}{\hat{\xi}} \right)^{\frac{1}{1+\nu}} \frac{\log\left(\frac{\hat{\alpha} E}{\hat{\xi}} \right)}{(1+\nu)^2} \hat{\alpha} \epsilon + \mathcal{O}(\epsilon^2). \tag{O3.75}$$

Profits are given by

$$\pi(X,s;\epsilon) = \left(\frac{\hat{\alpha}}{\hat{\xi}} \right)^{\frac{\alpha}{1+\nu-\alpha}} \left[x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \alpha E^{\frac{1+\nu}{1+\nu-\alpha}} \right] = x_s + \left[x_s \frac{\log(\hat{\alpha} E / \hat{\xi})}{1+\nu} - E \right] \hat{\alpha} \epsilon + \mathcal{O}(\epsilon^2), \tag{O3.76}$$

where we used the following Taylor expansion:

$$E^{\frac{\alpha}{1+\nu-\alpha}} = 1 + \frac{\log E}{1+\nu} \hat{\alpha} \epsilon + \mathcal{O}(\epsilon^2). \tag{O3.77}$$

$\square$

O3.6 Proof of Proposition O.6

Proof. We derive next the expression for the price and return for the surplus claim and riskless asset.

Step 1: price of surplus claim. The market clearing for consumption can be written as

$$P(X,s;\epsilon)\sum_{i=1}^{I}\eta_i c_i(X,s;\epsilon) = \left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}}\left[x_s E^{\frac{\alpha}{1+\nu-\alpha}} - \frac{\alpha}{1+\nu}E^{\frac{1+\nu}{1+\nu-\alpha}}\right]. \tag{O3.78}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\hat{P}(X,s)}{P^*(X,s)} + \sum_{i=1}^{I}\eta_i\frac{\hat{c}_i(X,s)}{c^*(X,s)} = \left[\frac{\log(\hat{\alpha}E/\hat{\xi})}{1+\nu} - \frac{1}{1+\nu}\frac{E}{x_s}\right]\hat{\alpha}. \tag{O3.79}$$

Rearranging the expression above, and using the expression for $\hat{c}_i(X,s)$, we obtain

$$\frac{\hat{P}(X,s)}{P^*(X,s)} = \left[\log(\hat{\alpha}E/\hat{\xi}) - \frac{E}{x_s}\right]\frac{\hat{\alpha}}{1+\nu} - (1-\psi)\sum_{i=1}^{I}\eta_i\frac{\hat{v}_i(X,s)}{v^*(X,s)}. \tag{O3.80}$$

Step 2: return on surplus claim. The return on the surplus claim is defined as follows

$$R_p(X,s,s';\epsilon) = \frac{x_s P(\chi(X,s,s';\epsilon),s';\epsilon)}{P(X,s;\epsilon) - \left(x_s h(E)^{\alpha} - \xi\frac{h(E;\epsilon)^{1+\nu}}{1+\nu}\right)}. \tag{O3.81}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\hat{R}_p(X,s,s')}{R_p^*(X,s,s')} = \frac{\hat{P}(X^*,s')}{P^*(X^*,s')} - \left[\frac{P^*(X,s)}{P^*(X,s)-x_s}\frac{\hat{P}(X,s)}{P^*(X,s)} - \frac{1}{P^*(X,s)-x_s}\left(x_s\log\left(\frac{\hat{\alpha}}{\hat{\xi}}E\right) - E\right)\frac{\hat{\alpha}}{1+\nu}\right]. \tag{O3.82}$$

We can write the expression above as follows:

$$\frac{\hat{R}_p(X,s,s')}{R_p^*(X,s,s')} = \frac{\hat{P}(X^*,s')}{P^*(X^*,s')} - \left[(\beta^*)^{-1}\frac{\hat{P}(X,s)}{P^*(X,s)} + (1-(\beta^*)^{-1})\left(\log\left(\frac{\hat{\alpha}}{\hat{\xi}}E\right) - \frac{E}{x_s}\right)\frac{\hat{\alpha}}{1+\nu}\right], \tag{O3.83}$$

using the fact that $P^*(X,s) = x_s/(1-\beta^*)$.

Using the expression for the price of the surplus claim, we obtain

$$\frac{\hat{R}_p(X,s,s')}{R_p^*(X,s,s')} = \left[\log(E^*/E) - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu} - (1-\psi)\sum_{i=1}^{I}\eta_i\left[\frac{\hat{v}_i(X^*,s')}{v^*(X^*,s')} - \frac{1}{\beta^*}\frac{\hat{v}_i(X,s)}{v^*(X,s)}\right], \tag{O3.84}$$

Step 3: interest rate. The interest rate is given by

$$R_b(X,s,s';\epsilon) = \left[\sum_{s'\in\mathcal{S}} p^*_{s'}\Lambda(X,s,s';\epsilon)\right]^{-1} \Rightarrow \frac{\hat{R}_b(X,s)}{R^*_b(X,s)} = -\sum_{s'\in\mathcal{S}} \frac{p_{s'}x_{s'}^{-\gamma}}{\mathbb{E}^*[x_{s'}^{-\gamma}]}\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O3.85}$$

Step 4: stock prices. Stock prices, normalized by current productivity, satisfy the functional equation:

$$Q(X,s;\epsilon) = \sum_{s'\in\mathcal{S}} p^*_{s'}\Lambda(X,s,s';\epsilon)\left[\left(\frac{\hat{\alpha}}{\hat{\xi}}\right)^{\frac{\alpha}{1+\nu-\alpha}}\left(x_{s'}E'(X,s;\epsilon)^{\frac{\alpha}{1+\nu-\alpha}} - \alpha E'(X,s;\epsilon)^{\frac{1+\nu}{1+\nu-\alpha}}\right) + x_{s'}Q(\chi(X,s,s';\epsilon),s';\epsilon)\right] \tag{O3.86}$$

For $\epsilon = 0$, we obtain

$$Q^*(X,s) = \sum_{s'\in\mathcal{S}} p^*_{s'}\Lambda^*(X,s,s')x_{s'}\left[1+Q^*(X^*,s')\right]. \tag{O3.87}$$

We can write the expression above as follows:

$$Q^*(X,s) = \beta^*\sum_{s'\in\mathcal{S}}\frac{p^*_{s'}x_{s'}^{1-\gamma}}{\mathbb{E}^*[x_{s'}^{1-\gamma}]}\left[1+Q^*(X^*,s')\right] \Rightarrow Q^*(X,s) = \frac{\beta^*}{1-\beta^*}. \tag{O3.88}$$

Expanding the expression for $Q(X,s)$, we obtain

$$\frac{\hat{Q}(X,s)}{Q^*(X,s)} = \sum_{s'\in\mathcal{S}}\omega^*_{s'}\left[\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} + (1-\beta^*)\left(\frac{\log\left(\frac{\hat{\alpha}E^*}{\hat{\xi}}\right)}{1+\nu} - \frac{E^*}{x_{s'}}\right)\hat{\alpha} + \beta^*\frac{\hat{Q}(X^*,s')}{Q(X^*,s)}\right]. \tag{O3.89}$$

Evaluating the expression above at $X = X^*$, we obtain

$$\left[I - \beta^*\mathbb{1}_N\omega^*\right]\hat{Q}(X^*) = b^Q(X^*), \tag{O3.90}$$

where $\hat{Q}(X) \equiv [\hat{Q}(X,1),\ldots,\hat{Q}(X,N)]'$, $b^Q(X) \equiv \left[b^Q(X,1),\ldots,b^Q(X,N)\right]'$, and $b^Q(X,s) \equiv Q^*(X,s)\sum_{s'\in\mathcal{S}}\omega^*_{s'}\left[\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} + (1-\beta^*)\left(\frac{\log\left(\frac{\hat{\alpha}E^*}{\hat{\xi}}\right)}{1+\nu} - \frac{E^*}{x_{s'}}\right)\hat{\alpha}\right]$.

Solving the system above, we obtain

$$\hat{Q}(X^*) = \left[I + \frac{\beta^*}{1-\beta^*}\mathbb{1}_N\omega^*\right]b^Q(X^*). \tag{O3.91}$$

We can then write $\hat{Q}(X,s)$ as follows:

$$\frac{\hat{Q}(X,s)}{Q^*(X,s)} = \sum_{s'\in\mathcal{S}}\omega^*_{s'}\left[\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} + \beta^*\frac{\hat{\Lambda}(X^*,s,s')}{\Lambda^*(X^*,s,s')} + \frac{(\beta^*)^2}{1-\beta^*}\sum_{\tilde{s}\in\mathcal{S}}\omega^*_{\tilde{s}}\frac{\hat{\Lambda}(X^*,\tilde{s},s')}{\Lambda^*(X^*,s,s')}\right] + \left(\frac{\log\left(\frac{\hat{\alpha}E^*}{\hat{\xi}}\right)}{1+\nu} - \sum_{s'\in\mathcal{S}}\omega^*_{s'}\frac{E^*}{x_{s'}}\right)\hat{\alpha}. \tag{O3.92}$$

Step 5: equity returns. Equity returns are given by

$$R_E(X,s,s';\epsilon) = \frac{x_{s'}Q(\chi(X,s,s';\epsilon),s') + \pi(E'(X,s;\epsilon;\epsilon),s')}{Q(X,s;\epsilon)}. \tag{O3.93}$$

Evaluating the expression above at $\epsilon = 0$, we obtain

$$R_E^*(X,s,s') = \frac{x_{s'}Q^*(X^*,s') + \pi^*(E^*,s')}{Q^*(X,s;\epsilon)} = \frac{x_{s'}}{\beta^*}. \tag{O3.94}$$

The first-order correction is given by

$$\frac{\hat{R}_E(X,s,s')}{R_E^*(X,s,s')} = \beta^* \frac{\hat{Q}(X^*,s')}{Q^*(X,s)} + (1-\beta^*)\frac{\hat{\pi}(E^*,s')}{x_{s'}} - \frac{\hat{Q}(X,s)}{Q^*(X,s)}. \tag{O3.95}$$

Step 6: conditional risk premium. The conditional risk premium is defined as follows:

$$\overline{R}_E(X,s;\epsilon) = \sum_{s'\in\mathcal{S}} p_{s'}^* \left[\frac{R_E(X,s,s';\epsilon)}{R_b(X,s;\epsilon)}\right]. \tag{O3.96}$$

The first-order correction is given by

$$\frac{\widehat{\overline{R}}_E(X,s)}{\overline{R}^*(X,s)} = \sum_{s'\in\mathcal{S}} \frac{p_{s'}^* x_{s'}}{\mathbb{E}^*[x_{s'}]} \frac{\hat{R}_E(X,s,s')}{R_E^*(X,s,s')} - \frac{\hat{R}_b(X,s)}{R_b^*(X,s)}. \tag{O3.97}$$

□

O3.7 Proof of Proposition O.7

Proof. We consider next the law of motion of η_i and $\mathcal{L}$.

Step 1: wealth distribution. The law of motion of η_i can be written as

$$\eta_i'(X,s,s';\epsilon)\sum_{j=1}^{I}\eta_j R_{j,n}(X,s,s';\epsilon)(1-c_j(X,s;\epsilon)) = \eta_i R_{i,n}(X,s,s';\epsilon)(1-c_i(X,s;\epsilon)). \tag{O3.98}$$

Expanding the expression above in ϵ, we obtain

$$\hat{\eta}_i'(X,s,s') = \eta_i\left[\frac{\hat{R}_{i,n}(X,s,s')}{R_{i,n}^*(X,s,s')} - \sum_{j=1}^{I}\eta_i\frac{\hat{R}_{j,n}(X,s,s')}{R_{j,n}^*(X,s,s')} - \frac{c_i^*(X,s)}{1-c_i^*(X,s)}\left(\frac{\hat{c}_i(X,s)}{c_i^*(X,s)} - \sum_{j=1}^{I}\eta_j\frac{\hat{c}_j(X,s)}{c_j^*(X,s)}\right)\right]. \tag{O3.99}$$

Using $c_i^*(X,s) = 1-\beta^*$ gives the expression in the proposition.

Step 2: risk-neutral probability of productivity growth. The law of motion of $\mathcal{L}$ can be written as

$$E'(X,s;\epsilon) = R_b(X,s;\epsilon) \sum_{s' \in \mathcal{S}} p^*_{s'} \Lambda(X,s,s';\epsilon) x_{s'}. \tag{O3.100}$$

Expanding the expression above in ϵ, we obtain

$$\frac{\hat{E}'(X,s)}{E^*(X,s)} = \frac{\hat{R}_b(X,s)}{R^*_b(X,s)} + \sum_{s' \in \mathcal{S}} \frac{p^*_{s'} \Lambda^*(X,s,s') x_{s'}}{\mathbb{E}^*[\Lambda^*(X,s,s') x_{s'}]} \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} \tag{O3.101}$$

We can write the expression above as follows:

$$\frac{\hat{E}'(X,s)}{E^*} = \frac{\hat{R}_b(X,s)}{R^*_b(X,s)} + \sum_{s' \in \mathcal{S}} \frac{p^*_{s'} x^{1-\gamma}_{s'}}{\mathbb{E}^*[x^{1-\gamma}_{s'}]} \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}. \tag{O3.102}$$

Using the definition of E^* and ω^*_s, we obtain the expression given in the proposition. □

O3.8 Proof of Proposition O.8

Proof. We consider the derivation of the economy's SDF $\hat{\Lambda}(X,s,s')$.

Step 1: the system of equations. The market clearing for the Arrow security paying off in state s' is given by

$$\sum_{i=1}^{I} \eta_i (1 - c_i(X,s;\epsilon)) R_{n,i}(X,s,s') = R_p(X,s,s') \sum_{i=1}^{I} \eta_i (1 - c_i(X,s;\epsilon)). \tag{O3.103}$$

Expanding the expression above, we obtain

$$\sum_{i=1}^{I} \eta_i \hat{R}_{n,i}(X,s,s') = \hat{R}_p(X,s,s'). \tag{O3.104}$$

Using the expression for $\hat{R}_{n,i}(X,s,s')$ and $\hat{R}_p(X,s,s')$, we obtain

$$\frac{1}{\gamma}\left[\frac{\delta_{ss'}(X)}{p^*_{s'}} - \sum_{\tilde{s}' \in \mathcal{S}} \omega^*_{\tilde{s}'} \frac{\delta_{s\tilde{s}'}(X)}{p^*_{\tilde{s}'}} - \frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')}\right] + \frac{1-\gamma}{\gamma}\left[\sum_{i=1}^{I} \eta_i \left[\frac{\hat{v}_i(X^*,s')}{v^*(X,s)} - \sum_{\tilde{s} \in \mathcal{S}} \omega^*_{\tilde{s}} \frac{\hat{v}_i(X^*,\tilde{s})}{v^*(X,s)}\right] + \sum_{\tilde{s} \in \mathcal{S}} \omega^*_{\tilde{s}} \frac{\hat{\Lambda}(X,s,\tilde{s})}{\Lambda^*(X,s,\tilde{s})}\right] =$$
$$\left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right] \frac{\hat{\alpha}}{1+\nu} + (\psi - 1) \sum_{i=1}^{I} \eta_i \left[\frac{\hat{v}_i(X^*,s')}{v^*(X^*,s')} - \frac{1}{\beta^*} \frac{\hat{v}_i(X,s)}{v^*(X,s)}\right]. \tag{O3.105}$$

Using the expression for $\hat{v}_i(X,s)$, we obtain

$$\frac{1}{\gamma}\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} + \frac{1-\gamma}{\gamma}\left[\beta^*\sum_{\tilde{s}\in\mathcal{S}}\omega^*_{\tilde{s}}\sum_{\tilde{s}'\in\mathcal{S}}\omega^*_{\tilde{s}'}\left(\frac{\hat{\Lambda}(X^*,s',\tilde{s}')}{\Lambda^*(X^*,s',\tilde{s}')} - \frac{\hat{\Lambda}(X^*,\tilde{s},\tilde{s}')}{\Lambda^*(X^*,\tilde{s},\tilde{s}')}\right) - \sum_{\tilde{s}'\in\mathcal{S}}\omega^*_{\tilde{s}'}\frac{\hat{\Lambda}(X,s,\tilde{s}')}{\Lambda^*(X,s,\tilde{s}')}\right] \tag{O3.106}$$

$$(\psi-1)\beta^*\left[-\sum_{\tilde{s}'\in\mathcal{S}}\omega^*_{\tilde{s}'}\left(\frac{\hat{\Lambda}(X^*,s',\tilde{s}')}{\Lambda^*(X^*,s',\tilde{s}')} - \frac{1}{\beta^*}\frac{\hat{\Lambda}(X,s,\tilde{s}')}{\Lambda^*(X,s,\tilde{s}')}\right) + \sum_{\tilde{s}\in\mathcal{S}}\omega^*_{\tilde{s}}\sum_{\tilde{s}'\in\mathcal{S}}\omega^*_{\tilde{s}'}\frac{\hat{\Lambda}(X^*,\tilde{s},\tilde{s}')}{\Lambda^*(X^*,\tilde{s},\tilde{s}')}\right] = b^{\Lambda}(X,s,s'), \tag{O3.107}$$

where

$$b^{\Lambda}(X,s,s') \equiv \frac{1}{\gamma}\left[\frac{\delta_{ss'}(X)}{p^*_{s'}} - \sum_{\tilde{s}'\in\mathcal{S}}\omega^*_{\tilde{s}'}\frac{\delta_{s\tilde{s}'}(X)}{p^*_{\tilde{s}'}}\right] + \frac{\beta^*}{\gamma}\left[\sum_{\tilde{s}\in\mathcal{S}}\omega^*_{\tilde{s}}\sum_{\tilde{s}'\in\mathcal{S}}\omega^*_{\tilde{s}'}\left[\frac{\delta_{s'\tilde{s}'}(X)}{p^*_{\tilde{s}'}} - \frac{\delta_{\tilde{s}\tilde{s}'}(X)}{p^*_{\tilde{s}'}}\right]\right] +$$
$$-\frac{\psi-1}{1-\gamma}\sum_{\tilde{s}\in\mathcal{S}}\omega^*_{\tilde{s}}\sum_{\tilde{s}'\in\mathcal{S}}\omega^*_{\tilde{s}'}\left[\frac{\delta_{s\tilde{s}'}(X)}{p^*_{\tilde{s}'}} - \beta^*\frac{\delta_{s'\tilde{s}'}(X^*)}{p^*_{\tilde{s}'}} + \beta^*\frac{\delta_{\tilde{s}\tilde{s}'}(X)}{p^*_{\tilde{s}'}}\right] - \left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu}. \tag{O3.108}$$

We can simplify the expression above as follows:

$$\frac{1}{\gamma}\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} + (\psi-\gamma^{-1})\left[\omega^*\cdot\hat{\Lambda}(X,s) - \beta^*\omega^*\cdot\hat{\Lambda}(X^*,s') + \beta^*\sum_{\tilde{s}\in\mathcal{S}}\omega_{\tilde{s}}(\omega^*\cdot\hat{\Lambda}(X^*,\tilde{s}))\right] = b^{\Lambda}(X,s,s'), \tag{O3.109}$$

where $\hat{\Lambda}(X,s) = \left[\frac{\hat{\Lambda}(X^*,s,1)}{\Lambda^*(X^*,s,1)}, \frac{\hat{\Lambda}(X^*,s,2)}{\Lambda^*(X^*,s,2)}, \ldots, \frac{\hat{\Lambda}(X^*,s,N)}{\Lambda^*(X^*,s,N)}\right]'$ and $\omega^*\cdot\hat{\Lambda}(X,s) = \sum_{\tilde{s}'}\omega^*_{\tilde{s}'}\frac{\hat{\Lambda}(X,s,\tilde{s}')}{\Lambda^*(X,s,\tilde{s}')}$.

Step 2: solving the system. We can write the system above in matrix form as follows:

$$\left[\gamma^{-1}I + (\psi-\gamma^{-1})\mathbb{1}_N\omega^*\right]\hat{\Lambda}(X,s) = \tilde{b}^{\Lambda}(X,s), \tag{O3.110}$$

where $\tilde{b}^{\Lambda}(X,s) = [\tilde{b}^{\Lambda}(X,s,1), \tilde{b}^{\Lambda}(X,s,2), \ldots, \tilde{b}^{\Lambda}(X,s,N)]'$ and

$$\tilde{b}^{\Lambda}(X,s,s') = b^{\Lambda}(X,s,s') + (\psi-\gamma^{-1})\beta^*\left[\omega^*\cdot\hat{\Lambda}(X^*,s') - \sum_{\tilde{s}\in\mathcal{S}}\omega_{\tilde{s}}(\omega^*\cdot\hat{\Lambda}(X^*,\tilde{s}))\right] \tag{O3.111}$$

Applying the Sherman-Morrison formula, we can invert the system above

$$\hat{\Lambda}(X,s) = \left[\gamma I - (\gamma-\psi^{-1})\mathbb{1}_N\omega^*\right]\tilde{b}^{\Lambda}(X,s). \tag{O3.112}$$

We can then write the expression above as follows:

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \gamma b^{\Lambda}(X,s,s') + (\gamma\psi - 1)\beta^* \left[\omega^* \cdot \hat{\Lambda}(X^*,s') - \sum_{\tilde{s}\in\mathcal{S}} \omega_{\tilde{s}}(\omega^* \cdot \hat{\Lambda}(X^*,\tilde{s}))\right] - (\gamma - \psi^{-1})\omega^* b^{\Lambda}(X,s). \tag{O3.113}$$

Step 3: solving for the average $\hat{\Lambda}(X,s,s')$. Assuming $X = X^*$, multiplying by $\omega^*_{s'}$, and adding across states, we obtain

$$\omega^* \hat{\Lambda}(X^*,s) = \psi^{-1}\omega^* b^{\Lambda}(X^*,s). \tag{O3.114}$$

Averaging across s, we obtain

$$\sum_{\tilde{s}\in\mathcal{S}} \omega^*_{\tilde{s}}[\omega^* \hat{\Lambda}(X^*,\tilde{s})] = \psi^{-1} \sum_{\tilde{s}\in\mathcal{S}} \omega^*_{\tilde{s}}[\omega^* b^{\Lambda}(X^*,\tilde{s})]. \tag{O3.115}$$

We can then write $\hat{\Lambda}(X,s,s')$ as follows

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \gamma b^{\Lambda}(X,s,s') - (\gamma - \psi^{-1})\omega^* b^{\Lambda}(X,s) + (\gamma - \psi^{-1})\beta^* \left[\omega^* \cdot b^{\Lambda}(X^*,s') - \sum_{\tilde{s}\in\mathcal{S}} \omega_{\tilde{s}}(\omega^* \cdot b^{\Lambda}(X^*,\tilde{s}))\right]. \tag{O3.116}$$

Step 4: simplifying the expression for $b^{\Lambda}(X,s,s')$. We can write $b^{\Lambda}(X,s,s')$ as follows:

$$\begin{aligned} b^{\Lambda}(X,s,s') = {}& \frac{1}{\gamma}\left[\frac{\delta_{ss'}(X)}{p^*_{s'}} - \omega^* \cdot \delta_s(X)\right] + \frac{\beta^*}{\gamma}\left[\omega^* \delta_{s'}(X) - \sum_{\tilde{s}} \omega^*_{\tilde{s}} \omega^* \cdot \delta_{\tilde{s}}(X)\right] \\ & + \frac{\psi - 1}{1-\gamma}\left[\omega^* \cdot \delta_s(X) - \beta^* \omega^* \cdot \delta_{s'}(X) + \beta^* \sum_{\tilde{s}} \omega^*_{\tilde{s}}(\omega^* \cdot \delta_{\tilde{s}}(X))\right] - \left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu}. \end{aligned} \tag{O3.117}$$

Combining terms, we obtain

$$b^{\Lambda}(X,s,s') = \frac{1}{\gamma}\frac{\delta_{ss'}(X)}{p^*_{s'}} - \frac{\psi - \gamma^{-1}}{\gamma - 1}\left[\omega^* \cdot \delta_s(X) - \beta^* \omega^* \cdot \delta_{s'}(X) + \beta^* \sum_{\tilde{s}} \omega^*_{\tilde{s}}(\omega^* \cdot \delta_{\tilde{s}}(X))\right] - \left[\log\frac{E^*}{E} - \left(\frac{E^*}{x_{s'}} - \frac{E}{x_s}\right)\right]\frac{\hat{\alpha}}{1+\nu}. \tag{O3.118}$$

Notice that we can $\hat{\Lambda}(X,s,s')$ as follows:

$$\frac{\hat{\Lambda}(X,s,s')}{\Lambda^*(X,s,s')} = \frac{\hat{\Lambda}(X^*,s,s')}{\Lambda^*(X,s,s')} + \psi^{-1}\left[\log\frac{E}{E^*} - \frac{E - E^*}{x_s}\right]\frac{\hat{\alpha}}{1+\nu}. \tag{O3.119}$$

$\square$

O3.9 The economy with no labor frictions and iid returns

Suppose that labor can be chosen conditional on the current productivity level. In this case, the problem of the firm can be written as

$$\max_{h_t} x_t h_t^\alpha - w_t h_t, \tag{O3.120}$$

where $w_t \equiv \frac{W_t}{A_{t-1}}$. Labor demand takes the familiar form:

$$\alpha x_t h_t^{\alpha-1} = w_t \Rightarrow h_t = \left(\frac{\alpha x_t}{w_t}\right)^{\frac{1}{1-\alpha}}. \tag{O3.121}$$

The labor supply from households is given by

$$h_t = \left(\frac{w_t}{\xi}\right)^{\frac{1}{\nu}}. \tag{O3.122}$$

Combining labor supply and labor demand, we obtain the equilibrium hours and wages:

$$h_t = \left(\frac{\alpha x_t}{\xi}\right)^{\frac{1}{1+\nu-\alpha}}, \qquad w_t = \xi^{\frac{1-\alpha}{1+\nu-\alpha}} (\alpha x_t)^{\frac{\nu}{1+\nu-\alpha}}. \tag{O3.123}$$

Firm's profits are given by

$$\pi_t = A_{t-1}(1-\alpha)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}} x_t^{\frac{1+\nu}{1+\nu-\alpha}}. \tag{O3.124}$$

Total surplus is given by

$$\tilde{C}_t = A_{t-1}\left[x_t h_t^\alpha - \xi \frac{h_t^{1+\nu}}{1+\nu}\right] = A_{t-1}\left(1 - \frac{\alpha}{1+\nu}\right)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}} x_t^{\frac{1+\nu}{1+\nu-\alpha}}. \tag{O3.125}$$

Let $P(X,s)$ denote the price of the surplus claim normalized by lagged productivity. Market clearing condition implies that

$$1-\beta^* = \left(1 - \frac{\alpha}{1+\nu}\right)\left(\frac{\alpha}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}} \frac{x_s^{\frac{1+\nu}{1+\nu-\alpha}}}{P(X,s)}, \tag{O3.126}$$

where $1-\beta^*$ is the consumption-wealth ratio, which we assume to be constant.

The return on the surplus claim is given by

$$R_p(X, s, s') = \frac{x_s}{\beta^*} \frac{x_{s'}^{\frac{1+\nu}{1+\nu-\alpha}}}{x_s^{\frac{1+\nu}{1+\nu-\alpha}}} = \frac{x_{s'}}{\beta^*} \left(\frac{x_{s'}}{x_s} \right)^{\frac{\alpha}{1+\nu-\alpha}}. \tag{O3.127}$$