Hedge Fund Investment: Optimal Portfolios with Regime-Switching

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Abstract

We investigate the benefits for a constant relative risk aversion investor of including different hedge fund styles into an optimal portfolio of stocks, bonds and commodities. We use a multivariate canonical vine copula regime-switching model which allows for non-linearity, asymmetry and time variation in hedge fund returns. We find that (i) investors with a five year horizon are willing to pay about 4 cents per dollar to gain access to hedge funds; (ii) risk-averse investors are willing to pay more to gain access to hedge fund investment, even though they end up holding less, compared to more risk-tolerant investors; (iii) for risk-averse investors, hedge funds play the role of the risky asset, whereas for more risk-tolerant investors, they play the role of the safe asset; (iv) optimally investing in hedge funds increases historical returns only until the subprime crisis, but produces diversification gains in the form of lower volatility even after.

JEL Classification codes: C32, C53, C58, G110.

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1 Introduction

Hedge funds are lightly regulated, sophisticated investment vehicles which pursue dynamic strategies and invest in complex and illiquid assets. As a result, unlike mutual funds, hedge funds typically display non-linear, option-like relations with risk factors, such as stock and bond market returns (see, e.g., Fung & Hsieh 2001, Mitchell & Pulvino 2001, Agarwal & Naik 2004, Diez del los Ríos & Garcia 2011, Mencía 2012, Jurek & Stafford 2015). Their returns also tend to be more dependent on markets in bad times than in good times, which is a source of non-diversifiable risk (see Brown & Spitzer 2006) and exhibit asymmetric dependence (see Kang, In, Kim & Kim 2010). Their great flexibility allows hedge funds to adjust their strategies and take advantage of new opportunities, which makes their exposures to risk factors vary over time (see, e.g., Fung & Hsieh 1997, Bollen & Whaley 2009, Billio, Getmansky & Pelizzon 2012, Patton & Ramadorai 2013). Moreover hedge fund returns are dependent in the tails and subject to contagion (see, e.g., Boyson, Stahel & Stulz 2010, Dudley & Nimalendran 2011, Patton 2009), which is particularly undesirable for investors and leads to a risk premium (see Kelly & Jiang 2014, Karagiannis & Tolikas 2019). These stylized facts of hedge fund returns call for specific models and they have to be taken into account by hedge fund investors.

We account for potential non-linearity, tail-dependence and time variation in hedge fund index returns and, following Ang & Bekaert (2002*a*), we compute the gains for an active investor with a constant relative risk aversion (CRRA) utility of adding any one of a number of hedge fund strategies to his optimal portfolio composed of three asset classes: U.S. stocks, a global bond index, and commodities.[1] We model time variation with regime-

[1]Choosing which specific hedge fund to invest in is outside the scope of our paper. In an industry characterized by its lack of transparency (see, e.g., Brown, Goetzmann, Liang & Schwarz 2008), identifying hedge fund managers that are able to consistently deliver attractive returns, provided they exist (for a recent analysis of performance persistence of hedge funds, see, e.g., Sun, Wang & Zheng 2018), is a very challenging task, which requires a commitment of time and resources to carry out the necessary research and monitoring. Moreover, hedge fund portfolio returns are increasingly becoming available to investors with the rapid development of hedge fund exchange traded funds (ETFs). For instance, since 2002, the Credit Suisse/Tremont (CSFB) Hedge Index Tracker tracks the CSFB Hedge Fund Index, which we use as our global hedge fund index.

switching and allow for non-linear dependence and tail dependence with the use of copulas. More specifically, we estimate skew-student t ARMA-GARCH models for the marginals and a multivariate regime-switching copula model of Chollete, Heinen & Valdesogo (2009), with one symmetric Gaussian dependence regime and a possibly asymmetric canonical vine regime. Canonical vines allow us to depart from the traditional bivariate setting and consider asymmetry and tail dependence between the hedge fund and the different asset classes in a consistent multivariate model. While our model is capable of fitting dynamics and asymmetry, the marginal also accommodates a plain normal with constant mean and variance, and the canonical vine regime can reduce to a symmetric Gaussian. Using the Ang & Chen (2002) H statistic, we show that the model can capture the asymmetric dependence in the data.

Our findings, based on optimal portfolios, are that, for a five year horizon, investors are willing to pay around 4 cents per dollar to gain access to hedge funds. This fee is similar to the cost of no international diversification found by Ang & Bekaert (2002*a*), when adding to US stocks the opportunity to invest in the UK. There is, however, a high level of heterogeneity in the benefits of hedge funds both across hedge fund styles and across levels of investor risk aversion. Surprisingly, we find that more risk-averse investors are willing to pay more to gain access to hedge funds than their less risk-averse counterparts, even though they end up being less invested in hedge funds than more risk-tolerant investors. More specifically, for most hedge fund styles, there is an inverse U-shape in hedge fund holdings across levels of risk aversion. A closer examination of the optimal portfolio weights reveals that hedge funds play the role of the safer asset in the portfolio of risk-tolerant investors, which are mostly composed of stocks and hedge funds, and for more risk-averse agents, they play the role of the risky asset in portfolios that are mostly composed of hedge funds and bonds, but no longer include stocks. For investors with intermediate levels of risk aversion hedge funds play both roles at the same time, which explains the inverse U-shape in hedge fund weights across levels of risk aversion. This suggests that while hedge funds play a different role in the portfolio depending on the investor's level of risk aversion, they are useful additions to a well-diversified portfolio, regardless of the risk aversion of the investor.

We use the optimal portfolio weights of our model to analyze the historical returns that an investor with a five year horizon following the strategy could have earned. Like Bali, Brown & Demirtas (2013) and Sullivan (2021), we find that the average returns of the portfolios including hedge funds are higher than those of the portfolios without hedge funds until the subprime crisis, and mostly lower since. On the other hand, the volatility of the hedge fund portfolio is systematically lower, which suggests that, even if they no longer deliver spectacular returns, hedge funds still produce significant gains in terms of diversification.

We further allow investors to leverage their portfolios by taking short positions in all asset classes, except hedge funds. We find that only the least risk-averse investors make use of significant amounts of leverage. Finally, we examine the effect on hedge fund investment of the sorts of lockup periods that hedge funds often impose on their investors. We find little evidence that restrictions in the withdrawal of funds have an effect on the benefits of hedge fund investment.

Our regime-switching copula model is able to capture the stylized facts of hedge fund returns, such as time-variation in the dependence, which is a well-known feature of financial returns (see Longin & Solnik 1995), dependence between extreme events, also known as tail dependence, as well as asymmetric dependence (see Longin & Solnik 2001). Our use of regime-switching is in line with Ang & Chen (2002), who find, using their H statistic, that among a number of models, regime-switching is the best at replicating empirical exceedance correlations. The multivariate canonical vine copula we use allows us to fit patterns of asymmetric dependence between a set of more than two asset returns in a flexible way. They extend the results of Patton (2004) who uses time-varying bivariate copulas and finds significant asymmetry in the dependence structure of financial returns and shows that taking asymmetric dependence into account leads to significant gains for an investor.[2]

[2]See also Patton (2006*b*) and Patton (2006*a*) for a theory of conditional copula models and their estimation.

Our analysis is related to the literature that assesses the value added of hedge funds according to whether they are able to consistently generate excess returns, in the form of a positive alpha from a regression of returns on risk factors (see, e.g., Fung & Hsieh 2001, Hasanhodzic & Lo 2007). Another strand of the literature identifies whether hedge funds produce superior returns using stochastic dominance (see, e.g., Olmo & Sanso-Navarro 2012) or utility-based measures, such as almost stochastic dominance and manipulation-proof performance measures (see Bali et al. 2013). Using a payoff distribution pricing model, Amin & Kat (2003) found that even hedge funds that do not show superior performance as a stand-alone investment can improve performance when combined with the S&P500. More recently, based on an ad hoc comparison between an equally weighted portfolio of stocks and bonds, and another that substitutes 20% of stocks with hedge funds, Bollen, Joenväärä & Kauppila (2021) find a drop in hedge fund performance after 2007. Instead, using a flexible model of dependence, we examine whether hedge funds can add value to an optimal portfolio composed of some broad asset classes meant to represent investment opportunities available to a relatively unsophisticated investor, such as a high net worth (HNW) individual, or to a fairly conservative institutional investor.[3] This is a different way to assess hedge funds, since even if they do not necessarily generate excess returns or dominate all other assets, they might still be of value to investor as part of a well-diversified portfolio.

Our paper is in the long tradition of regime-switching models, that were introduced into economics by Hamilton (1989) and have been widely applied in finance. For instance Ang & Bekaert (2002*b*), Guidolin & Timmermann (2006*a*) and Guidolin & Timmermann (2006*b*) use regime-switching models for interest rates. Ang & Bekaert (2002*a*) estimate a Gaussian Markov switching model for international returns and identify two regimes, a bear regime with negative returns, high volatilities and correlations and a bull regime with positive mean, low volatilities and correlations. Guidolin & Timmermann (2008) also use a regime-

[3]This avoids the statistical issue associated with the search for correct risk factor and/or functional form (see Diez del los Ríos & Garcia 2011). Moreover, we deliberately exclude options, since they are not easily accessible to most investors.

switching model for international financial returns. Moreover, the importance of accounting for regime switching has been shown in the context of portfolio allocation to stocks by Tu (2010) using a Bayesian approach. Regime-switching models have also been used to jointly model the returns of hedge fund style indices with their specific risk factors. For instance, Billio et al. (2012) assume a regime-switching model with two independent Markov chain processes: one for the alpha and the idiosyncratic volatility of each hedge fund, and another one for the risk factors. A number of papers combine copulas and regime-switching models in the bivariate case.[4] In most of these papers, means, variances and correlations switch together. Instead, we follow Chollete et al. (2009) and Garcia & Tsafack (2011) and we model potentially asymmetric ARMA-GARCH marginals with regime-switching dependence. We rely on copulas and use the flexibility they provide in modeling the marginals separately from the dependence structure. Moreover we allow for an additional source of asymmetry with the use of canonical vines, that are flexible multivariate copulas.

The paper is structured as follows. In Section 2 we present the model. Section 3 discusses the data and the empirical results. Section 4 presents the results of the portfolio exercise. Section 5 concludes.

2 Model

In this section we describe the Chollete et al. (2009) model that we use, and we refer to that paper for more details. We first briefly introduce the concept of copulas and the Gaussian and canonical vine copulas that we use for the symmetric and asymmetric dependence regime, we then discuss the regime-switching copula, and finally the marginal models.

[4] Rodriguez (2007) and Okimoto (2008) estimate regime-switching copulas for pairs of international stock indices. Okimoto (2008) focuses on the US-UK pair, whereas Rodriguez (2007) works with pairs of Latin American and Asian countries.

2.1 Copulas

Copulas are a flexible tool to model the non-linear dependence between variables separately from their marginal distributions, which makes them ideally suited for financial returns.[5] Copula theory goes back to the work of Sklar (1959), who showed that, for continuous variables, a joint distribution can be decomposed into its n marginal distributions and a copula, which fully characterizes the dependence between the variables. Specifically, let $H(y_1, \ldots, y_n)$ be a continuous n-variate cumulative distribution function with univariate margins $F_i(y_i)$, $i \in \{1, \ldots, n\}$, where $F_i(y_i) = H(\infty, \ldots, y_i, \ldots, \infty)$. According to Sklar (1959), there exists a function C, called a copula, mapping $[0,1]^n$ into $[0,1]$, such that:

$$H(y_1, \ldots, y_n) = C(F_1(y_1), \ldots, F_n(y_n)) \; .$$

The joint density function is given by the product of the marginals and the copula density:

$$\frac{\partial H(y_1, \ldots, y_n)}{\partial y_1 \ldots \partial y_n} = \prod_{i=1}^{n} f_i(y_i) \frac{\partial C(F_1(y_1), \ldots, F_n(y_n))}{\partial F_1(y_1) \ldots \partial F_n(y_n)} \; .$$

While there are many bivariate copulas, the choice is much more limited for multivariate copulas. We use the Gaussian copula, a symmetric copula with no tail dependence, which means that in the limit there is no dependence between extreme events.[6] We also consider the canonical vine copula, which is a flexible multivariate copula obtained by a hierarchical construction (see Bedford & Cooke 2002, Aas, Czado, Frigessi & Bakken 2009). The density of an n-dimensional canonical vine copula is:

$$c(u_1, \ldots, u_n) = \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{j,j+i|1,\ldots,j-1}(F(y_j|y_1, \ldots, y_{j-1}), F(y_{j+i}|y_1, \ldots, y_{j-1})),$$

[5] A more detailed account of copulas can be found in Joe (1997), Nelsen (1999) and, for a more finance-oriented presentation, in Cherubini, Luciano & Vecchiato (2004) and McNeil, Frey & Embrechts (2005).

[6] Tail dependence refers to the dependence between arbitrarily extreme events, far out in the tails of the distribution. See Appendix A for a rigorous definition of quantile and tail dependence.

where $u_i = F_i(y_i)$, $i = \{1, \dots, n\}$ are the probability integral transformations (PIT) of the marginal models, $c_{j,j+i|1,\dots,j-1}$ is the bivariate copula of y_j and y_{j+i}, conditional on variables $y_1, \dots, y_{j-1}$, and conditional distribution functions are computed, following Joe (1996), as

$$F(y|v) = \frac{\partial C_{y,v_j|v_{-j}}(F(y|v_{-j}), F(v_j|v_{-j}))}{\partial F(v_j|v_{-j})},$$

where v_{-j} denotes the vector v excluding the component v_j.

[FIGURE 1]

The canonical vine leads to a tree structure, where at each branch one variable plays a pivotal role: in our context, we put the hedge fund, h, first, followed by stocks, s, bonds, b and commodities, c. Figure 1 represents the dependence structure of such a canonical vine copula graphically. In the first stage of the canonical vine we model the bivariate copulas, c_{hs}, c_{hb}, c_{hc}, of the hedge fund with all other asset classes. We then condition on the hedge fund, and we consider the bivariate conditional copulas, $c_{sb|h}$ and $c_{sc|h}$, of stocks with all the remaining asset classes, finally we consider $c_{bc|h,s}$, the copula between bonds and commodities conditioning on hedge fund and stocks.[7] This results in the following expression for the copula:

$$\begin{aligned} c_{hsbc}(u_h, u_s, u_b, u_s) &= c_{hs}(F(h), F(s))c_{hb}(F(h), F(b))c_{hc}(F(h), F(c)) \\ &\quad c_{sb|h}(F(s|h), F(b|h))c_{sc|h}(F(s|h), F(c|h)) \\ &\quad c_{bc|hs}(F(b|h, s), F(c|h, s)). \end{aligned}$$

The advantages of a canonical vine copula are immediately apparent: whereas there are only very few flexible multivariate copulas, there exists an almost unlimited number of bivariate copulas that can be used as building blocks for the canonical vine. Different patterns of

[7] Canonical vines can be viewed as a generalization to the non-Gaussian context of the idea consisting in expressing a correlation matrix, whose components are $\{\rho_{hs}, \rho_{hb}, \rho_{hc}, \rho_{sb}, \rho_{sc}, \rho_{bc}\}$, by the unconditional correlations in the first branch of the tree, $\rho_{hs}, \rho_{hb}, \rho_{hc}$ along with the partial correlations for the remaining branches $\rho_{sb|h}, \rho_{sc|h}$ and $\rho_{bc|hs}$.

dependence emerge according to the bivariate copulas that are used in the canonical vine.[8] In particular, with only bivariate Gaussian copulas one recovers the multivariate Gaussian copula. More generally, as a rule of thumb, whenever there is an asymmetric copula used at a lower tree, this will generate tail dependence in subsequent trees for the variables involved. For instance, in Figure 1, if c_{hs} is asymmetric, this will imply some degree of asymmetry in the unconditional distribution of (s, b) and (s, c), even if the copulas $c_{sb|h}$ and $c_{sc|h}$ are symmetric (see Joe, Li & Nikoloulopoulos 2008).

2.2 Regime-Switching

We set up a regime-switching model for the dependence structure of the excess returns of the hedge fund and the other asset classes. Like most of the literature, we assume two regimes. The copula in the first regime is a symmetric Gaussian, whereas in the other regime we use a possibly asymmetric canonical vine, where the bivariate copulas of the hedge fund with the other asset classes are allowed to be asymmetric. Like Garcia & Tsafack (2011), we assume that the regime-switching only affects the copula, not the marginals. As is standard in the literature, see e.g. Hamilton (1989), we assume that the unobserved state variable follows a Markov chain process with transition probability

$$P = \begin{pmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{pmatrix}.$$

The estimation is done in two steps: we first estimate the marginals, which do not depend on the regime, and conditionally on the parameters of the marginals, we use the EM algorithm to estimate the parameters of the regime-switching copula model.[9]

[8]We list all the bivariate copulas we use in this paper along with their coefficients of tail dependence in Appendix B. The Gaussian copula does not exhibit any tail dependence. The Clayton and the Gumbel, which we use exclusively in its rotated version have lower, but no upper tail dependence.

[9]For more details about the estimation procedure, see Section 2 of Chollete et al. (2009).

2.3 Marginal Model

We allow for possible dynamics in the mean of the asset returns $y_{i,t}$, with ARMA(p, q) models for $p, q \in \{0, 1, 2\}$, and the conditional variance $h_{i,t}$, with a GARCH$(1, 1)$. We further allow for departures from conditional normality with the Student t and the skewed Student t distribution of Hansen (1994), and we use the BIC criterion to choose the best model. The marginal model ranges in complexity from a normal with constant mean and variance, to the most general case of the skew Student t with mean and volatility

$$\begin{aligned} y_{i,t} &= \mu_i + \textstyle\sum_{j=1}^{2} \phi_{ij} y_{i,t-j} + \sum_{j=0}^{2} \theta_{ij} \eta_{i,t-j}, \\ h_{i,t} &= \omega_i + \alpha_i \eta_{i,t-1}^2 + \beta_i h_{i,t-1}, \end{aligned}$$

where, for $i \in \{1, \ldots, n\}$, the standardized residual,

$$\varepsilon_{i,t} = \frac{\eta_{i,t}}{\sqrt{h_{it}}} \sim skew - t(\nu_i, \lambda_i),$$

and we define the probability integral transformation (PIT) of the marginal as

$$u_{i,t} = F_{\text{skew-t}}(\varepsilon_{i,t}),$$

where the skewed-t density is given by

$$g(\varepsilon_{i,t}|\nu, \lambda) = \begin{cases} bc\left(1 + \frac{1}{\nu-2}\left(\frac{b\varepsilon_{i,t}+a}{1-\lambda}\right)^2\right)^{-(\nu+1)/2} & \varepsilon_{i,t} < -a/b, \\ bc\left(1 + \frac{1}{\nu-2}\left(\frac{b\varepsilon_{i,t}+a}{1+\lambda}\right)^2\right)^{-(\nu+1)/2} & \varepsilon_{i,t} \geq -a/b, \end{cases}$$

and the constants a, b and c are:

$$a = 4\lambda c\left(\frac{\nu-2}{\nu-1}\right), \quad b^2 = 1 + 3\lambda^2 - a^2, \quad c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma\left(\frac{\nu}{2}\right)}.$$

A negative λ corresponds to a left-skewed density, which means that there is more probability of observing large negative than large positive returns. This is what we expect, since it captures the large negative returns associated with market crashes that are responsible for the skewness. When $\lambda = 0$, we recover the Student t distribution, and in the limit when the degrees of freedom are large, $\nu \to \infty$, we recover the normal distribution.

3 Data and estimation results

3.1 Data

To measure performance in the hedge fund industry we rely on the Credit Suisse Hedge Fund Index returns (formerly known as CSFB/Tremont) from January 1994 to April 2018 (for a comparison of the two most commonly used hedge fund index families, see Fung & Hsieh 2002). These are asset-weighted indices of funds with the following requirements: (1) A minimum of US $50 million assets under management (AUM), (2) a minimum one-year track record, and (3) current audited financial statements. The hedge fund indices are calculated and rebalanced monthly, while funds are reselected on a quarterly basis. We use net-of-fee returns in excess of the risk-free rate, proxied by the three-month Treasury Bill rate. We work with the Credit Suisse global hedge fund index and with the following style-based indices: convertible arbitrage, emerging markets, equity market neutral, event-driven, event-driven distressed, event-driven multi-strategy, event-driven risk arbitrage, fixed-income arbitrage, global macro, long/short equity, managed futures, and multi-strategy. Besides hedge funds, our investors are allowed to invest in three additional asset classes: equity, bonds and commodities, proxied, respectively, by the S&P500 index, the Barclays Global Aggregate index and the S&P GSCI Commodity index.

[TABLE 1]

Table 1 presents descriptive statistics of the excess returns of the hedge fund indices and the other asset classes. The annualized means vary from a minimum of 1.80% for equity market neutral to a maximum of 6.76% for global macro, while the annualized standard deviations range from 3.81% for event-driven risk arbitrage to 13.42% for dedicated short. All hedge fund strategies have negative skewness, and they show clear signs of positive excess kurtosis, two indications of departure from normality. This phenomenon is present in most financial assets, but it is much more severe for hedge funds, with values of skewness much below that of the S&P500 stock index, and levels of kurtosis beyond 10 for a number of hedge fund styles. The different asset classes have annualized mean returns ranging from 1.93% for commodities to 6.63% for the S&P500 index, and annualized standard deviations between 5.38% for bonds and a 21.37% for commodities.

[TABLE 2]

Table 2 shows the correlation matrix between the excess returns of the hedge fund indices and the different asset classes. The different hedge fund styles are quite correlated among each other, particularly the group of event-driven hedge funds and the long/short equity style, whose correlations are all above 0.60. Except for managed futures, all hedge fund styles have the highest correlations with the S&P500. The correlations with these asset classes can be as high as 0.67 for stocks with long/short equity and 0.25 for stocks with market neutral equity, which should in principle be uncorrelated. This is reflected in our use of the S&P500 as the pivot in the second tree of the canonical vine copula, see Figure 1. There is only a moderate amount of dependence among stocks, bonds and commodities.

3.2 Asymmetric dependence and exceedance correlation

It has been widely documented that hedge funds have non-linear relations with other assets. For instance Fung & Hsieh (2001) and Agarwal & Naik (2004) find that hedge fund returns are characterized by non-linear, option-like exposures to risk factor, and Dudley & Nimalendran (2011) show that a Gaussian distribution is not adequate as it cannot reproduce the exceedance correlation between different hedge fund styles and the S&P500, especially in the lower tail. Using piecewise linear regressions, Diez del los Ríos & Garcia (2011) also find asymmetric relations between hedge fund returns and a set of risk factors including stocks, bonds and commodities. Moreover, Patton (2009) finds dependence between hedge fund and stock index returns also in the variance, in the Value-at-Risk (VaR) and in the tails, while Mencía (2012) finds evidence of non-linear dependence in the hedge fund industry mainly in the variance.

We test for the presence of non-linear dependence in hedge fund returns using the asymmetric dependence test of Hong, Tu & Zhou (2007) (HTZ, henceforth), based on exceedance correlations, and the Jiang, Maasoumi, Pan & Wu (2018) test based on entropy (see also Jiang, Wu & Zhou 2018). Exceedance correlations, introduced by Longin & Solnik (2001) are correlations for tail observations. At the mean, the upper and lower exceedance correlations, ρ^+ and ρ^-, between z_h and z_{ac}, the standardized excess returns of hedge fund h and asset class ac, where $ac \in AC = \{s, b, c\}$, are

$$\begin{aligned} \rho^+ &= \text{Corr}(z_h, z_{ac} | z_h \geq 0 \text{ and } z_{ac} \geq 0), \\ \rho^- &= \text{Corr}(z_h, z_{ac} | z_h \leq 0 \text{ and } z_{ac} \leq 0). \end{aligned}$$

HTZ compare the upper and lower tail exceedance correlation and introduce the J-statistic to test the null hypothesis $\rho^+ = \rho^-$ of symmetric exceedance correlation:

$$J = T(\hat{\rho}^+ - \hat{\rho}^-)\hat{\Omega}^{-1}(\hat{\rho}^+ - \hat{\rho}^-),$$

where $\hat{\rho}^+$ and $\hat{\rho}^-$ are sample-based exceedance correlations, and $\hat{\Omega}$ is a Newey & West (1994) variance. The statistic follows a $\chi^2_{[1]}$ distribution, provided it is not affected by the estimation of the exceedance correlations.[10] Instead, we rely on a nested bootstrap version of the HTZ statistic, discussed in Jiang, Maasoumi, Pan & Wu (2018).

[10] When the estimation noise from the exceedance correlations is taken into account, Chen (2016) shows that the distribution can be quite different in practice.

The second test we use is the asymmetric dependence test of Jiang, Maasoumi, Pan & Wu (2018), which relies on a tail version of mutual information, which is the Kullback-Leibler relative entropy between the joint density of the hedge fund and asset class pair and the product of their marginals. Define ρ_e^+ and ρ_e^-, respectively, as the upper and lower tail expectations of the Kullback-Leibler entropy relative to the mean:

$$\begin{aligned}\rho_e^+ &= \int_0^\infty \int_0^\infty f_{h,ac}(z_h, z_{ac}) \log \tfrac{f_{h,ac}(z_h,z_{ac})}{f_h(z_h) f_{ac}(z_{ac})} dz_h dz_{ac}, \\ \rho_e^- &= \int_{-\infty}^0 \int_{-\infty}^0 f_{h,ac}(z_h, z_{ac}) \log \tfrac{f_{h,ac}(z_h,z_{ac})}{f_h(z_h) f_{ac}(z_{ac})} dz_h dz_{ac}.\end{aligned}$$

Jiang, Maasoumi, Pan & Wu (2018) test the null hypothesis $\rho_e^+ = \rho_e^-$ of symmetric dependence. The test uses kernel-density estimates of the univariate and bivariate densities and the critical values are computed using a nested bootstrap. Monte Carlo simulations show that this test is more sensitive than the HTZ test and detects more asymmetry than the HTZ test.

[TABLE 3]

Panel A of Table 3 displays the results of these tests with standardized excess returns when dependence above and below the mean are compared. Both tests find significant departures from symmetry at the 10% level in the dependence between the hedge fund index and the S&P500. As expected, the Jiang, Maasoumi, Pan & Wu (2018) mutual information test is more sensitive, and detects asymmetric dependence in all but convertible arbitrage, equity market neutral, long/short equity and managed futures. Whereas the HTZ finds no evidence of asymmetry with bonds, the mutual information test finds asymmetry for four hedge fund styles (event-driven, event-driven distressed, event-driven multi-strategy and multi-strategy). Both tests find no evidence of asymmetric dependence between any of the hedge fund styles and commodities. Our rejection of symmetry in the family of event-driven strategies is in line with results of Mencía (2012), who finds non-linear dependence both in the conditional mean and variance of that hedge fund style, and also with Diez del los Ríos

& Garcia (2011).

3.3 Estimation results

We start by modeling the marginal distributions. This is crucial, since estimation of the copula is made conditionally on the marginal distribution, and a misspecified marginal would lead to departure from uniformity of the inputs into the copula, which could in turn cause a misspecification of the copula, see Fermanian & Scaillet (2005). We use an ARMA model with up to two AR and MA terms for each series, to make sure that we take into account any possible dynamics in the conditional mean. We choose the best specification according to the Bayesian information criterion (BIC) in order to ensure parsimony of the model. We select an ARMA(1,1) for about half the excess returns, and an MA(1) otherwise. We also take into account the possibility of conditional heteroscedasticity in the excess returns with a GARCH(1,1) process and we entertain different distributions for the innovations, such as the Gaussian, Student-t, or skewed Student t. Table 4 shows the results of the GARCH processes for each series. In line with the clear signs of non-normality in the descriptive statistics, we mostly select Student t or skewed Student t distributions with degrees of freedom below 10, for all but the S&P500 index, with a minimum of 2.51 for the equity market neutral style, which reflects the presence of significantly fatter tails than the Gaussian distribution. The skewness parameter is always negative, with a minimum of -0.36 for the S&P500 and a maximum of -0.16 for emerging markets. A negative skewness parameter implies that the left tail of the distribution is longer than the right one, which is undesirable from the point of view of an investor. Heteroscedasticity is significant in about half the hedge fund strategies, and therefore a GARCH specification is only needed for those cases.

[TABLE 4]

To ensure that the marginal models are well specified we apply different goodness of fit tests, which appear in the right panels of Table 4. We compute the Anderson-Darling and Kuiper tests, as well as three versions of the Kolmogorov-Smirnov test of uniformity of the PIT of the marginal models. We also test the uniformity and lack of correlation of the probability integral transformation using the Berkowitz (2001) test, and we compute the Ljung-Box test of autocorrelation for the squared residuals of the models. According to the test results the marginal models are well specified.

[TABLE 5]

After selecting the best marginal models, we estimate a regime-switching copula model for each individual hedge fund strategy with the different asset classes. We use the multivariate Gaussian copula for one of the regimes and a canonical vine copula for the other. We select the bivariate copulas between the hedge fund and each of the three asset classes, that make up the canonical vine, among the symmetric Gaussian, the rotated Gumbel and the Clayton. The best specification is chosen according to the BIC criterion and the results are displayed in Table 5. Consistent with the asymmetry test results of Section 3.2, in the canonical vine, most of the dependence structure between the different hedge fund strategies and the S&P500 index is captured by an asymmetric copula, with the exception of the equity market neutral, managed futures and multi-strategy styles. These results are in line with previous findings of Kang et al. (2010) and Distaso, Fernandes & Zikes (2008). Moreover we select asymmetric copulas with bonds for the convertible arbitrage, emerging markets, equity market neutral and event-driven distressed styles, while the dependence structure of commodities with the convertible arbitrage and multi-strategy subindices is modeled using a Clayton copula. These results imply that some of the hedge fund strategies have asymmetric dependence and lower tail dependence not only with the S&P500 but also with other asset classes. As can be seen from the transition probabilities, both the asymmetric and Gaussian regimes are in general very persistent. The regime-switching asymmetric copula model dominates a specification without asymmetry in the dependence, with two Gaussian copula regimes, for all hedge fund styles, according to the loglikelihood.

3.4 Fit of the model in the tails

In order to make sure that the canonical vine copula regime-switching models are able to capture the asymmetric dependence in hedge fund returns, we compare the exceedance correlation computed from our model with that of the data at a number of different thresholds. We adapt the approach of Ang & Bekaert (2002*a*) by computing exceedance correlations with respect to a quantile. This makes it easier to guarantee that there will be enough observations available to compute exceedance correlations in all pairs of hedge fund style and asset class, even in the presence of highly non-Gaussian distributions. From a bivariate vector of the excess returns of hedge fund h and asset class ac, (r_h, r_{ac}), we compute the exceedance correlation at quantile q_i as follows. When q_i is above (below) 0.5, we select the observations where both r_h and r_{ac} are above (below) their q_i-quantile. In other words, we select observations in the set $B^+(q_i) = \{(r_h, r_{ac}) | r_h \geq F_h^{-1}(q_i) \text{ and } r_{ac} \geq F_{ac}^{-1}(q_i)\}$ for $q_i \geq 0.5$ and $B^-(q_i) = \{(r_h, r_{ac}) | r_h \leq F_h^{-1}(q_i) \text{ and } r_{ac} \leq F_{ac}^{-1}(q_i)\}$ for $q_i \leq 0.5$, where, for $j \in \{h, ac\}$, F_j^{-1} is the empirical quantile function. The correlation of this subset of points is the exceedance correlation $\rho(q_i)$ at quantile q_i:

$$\rho(q_i) = \begin{cases} \operatorname{corr}(r_h, r_{ac} | B^+(q_i)) & \text{if } q_i \geq 0.5 \\ \operatorname{corr}(r_h, r_{ac} | B^-(q_i)) & \text{if } q_i \leq 0.5 \end{cases} .$$

We provide a formal Ang & Chen (2002) H statistic of adequacy of the regime-switching copula model to check whether the exceedance correlations of the model, $\rho^{(m)}(q)$, computed by simulation, can reproduce those of the data, $\bar{\rho}(q)$, for an N-dimensional vector of quantiles q:

$$H = (\rho^{(m)}(q) - \bar{\rho}(q))' W(q) (\rho^{(m)}(q) - \bar{\rho}(q)),$$

where $\bar{\rho}(q)$ and $\rho^{(m)}(q)$ denote N-dimensional vectors of empirical and model-based exceedance correlation, respectively, and $W(q)$ is a diagonal matrix of weights $\omega(q_i)$ that are proportional to $T(q_i)$, the number of observations available to compute exceedance correlations at each quantile:

$$\omega(q_i) = \frac{T(q_i)}{\sum_{j=1}^{N} T(q_j)}.$$

We compare this statistic to its distribution under the null, computed from $1,000$ simulated trajectories from the model, of the same length as the data. Given the length of our time series, we consider quantiles in the range $[0.3, 0.7]$ in order to guarantee that we have a sufficient number of observations to reliably compute correlations.[11] The p-values of the test for the canonical vine and the Gaussian copula regime-switching model appear in panel B of Table 3. The regime-switching model with a canonical vine and a Gaussian copula successfully reproduces the empirical exceedance correlations for nearly all pairs of hedge fund style and asset class at the 10% level.[12] On the other hand, the regime-switching model where both regimes are governed by a Gaussian copula has lower loglikelihood, and is also rejected for six styles with stocks, two styles with bonds and three styles with commodities. This indicates that the model with one canonical vine regime is adequate and that the asymmetric copulas are required to properly fit the exceedance correlations.[13]

[FIGURE 2]

In order to focus on the fit of the copulas, we compute correlations of the probability integral transformation (PIT), $u_{i,t}$, which is the quantile of the standardized residuals $\varepsilon_{i,t}$ of the excess returns, after mean and GARCH modeling. Figure 2 displays the exceedance correlation between the PITs of the hedge fund and the stock, bond and commodities indices, respectively in panels (a), (b) and (c). The exceedance correlation of the data is plotted with a 95% bootstrap confidence interval, calculated using $1,000$ replications, whereas the

[11] Under independence, for quantile q_i in the lower (upper) tail, on average we have a fraction q_i^2 $(1-q_i)^2$ of the initial observations available to compute correlations. So, for instance, when $q_i = 0.3$, this would on average still leave us with $292 \times 0.3^2 \simeq 26$ observations. The vector of thresholds q we use is built on a grid with an increment of 0.02 in the range $[0.3, 0.7]$, where 0.5 is repeated because we calculate exceedance correlations both below and above the median.

[12] The only exceptions are event-driven risk arbitrage, managed futures with stocks; fixed-income arbitrage with bonds and commodities; and multi-strategy with commodities.

[13] At 5% we only reject for two pairs of hedge fund style and asset class for our model, while there are still 7 rejections for the Gaussian copula regime-switching model. These results are robust to the use of equal weights.

exceedance correlation of the regime-switching copula model is computed on a simulated sample of $100,000$ observations. There are clear signs of asymmetry in the dependence between the hedge fund index and the S&P500 in Figure 2a, which is larger in the lower than in the upper tail. This illustrates the well-known fact that negative returns are more dependent than positive returns. The 95% confidence bands are fairly wide, which reflects the uncertainty in the estimation of exceedance correlations. The regime-switching model is able to capture this asymmetry with a correlation of 0.4 in the lower tail, but only 0.3 in the upper tail. While there is no such clear sign of asymmetry with bonds and commodities in Figures 2b and 2c, the model is always within the 95% confidence interval around the data.

[FIGURE 3]

Figure 3 displays the exceedance correlation between the stock index and the twelve hedge fund styles, most of which have asymmetric dependence, with more dependence in the left than in the right tail. This is particularly true for styles that trade in stocks, such as long/short equities, convertible arbitrage, which bear an exposure to the stock price of the companies whose bonds they trade, but also for emerging market hedge funds, which trade stocks in less mature markets, that offer fewer opportunities for hedging, and that might end up being quite correlated with US equities, especially in the face of large adverse shocks. The same holds for the different event-driven hedge funds, whose strategies involve taking advantage of stock mispricing around corporate events, which exposes them to equity risk. The regime-switching model is quite successful at capturing this type of asymmetry for all these styles. For instance, the correlation implied by the model jumps from about 0.2 in the right to 0.5 in the left tail for the event-driven and event-driven distressed styles in sub-figures (d) and (e). For convertible arbitrage hedge funds, the overall dependence with stocks is fairly low, but while the returns are virtually uncorrelated in the right tail with a correlation of 0.05, this correlation jumps to 0.2 in the left tail. However, there are also investment styles that display weaker or no asymmetric dependence with US stock markets and where the model does not pick up any asymmetric dependence. This is the case, for instance, for fixed-income arbitrage, global macro, managed futures and multi-strategy in sub-figures (h), (i), (k) and (l).

4 Hedge fund investment

4.1 Portfolio optimization

We perform a simulation exercise, similar to the one in Ang & Bekaert (2002a), where we assume that the data has been generated by the regime-switching model. We compare the outcomes for investors with and without access to hedge fund investment. We consider investors with constant relative risk aversion (CRRA) utility and coefficients of relative risk aversion of $\gamma = 3, 5, 7, 10, 15, 20$:

$$U(W_T) = \frac{W_T^{1-\gamma}}{1-\gamma},$$

where T is the investment horizon and final wealth W_T is

$$W_T = W_0 \exp\left(r_T\right), \qquad \text{with } r_T = \sum_{t=1}^{T} w_t' R_t,$$

where r_T is the log excess return of the portfolio over the investment horizon T, w_t and R_t are the vectors of weights and monthly log returns of the hedge fund and the three asset classes at time t, and we normalize initial wealth $W_0 = 1$. The investor maximizes utility under the constraint that the time t weights of the different asset classes, $w_{ac,t}$, and the weight of the hedge fund, $w_{h,t}$, which is positive since it is not possible to short hedge funds, add up to one at every period t:

$$w_{h,t,} + \sum_{ac \in AC} w_{ac,t} = 1, \qquad \text{and } w_{h,t} \geq 0, \quad \forall t, \tag{1}$$

and a no short selling constraint that further imposes positivity of the weights of all asset classes

$$w_{ac,t} \geq 0 \quad \forall ac \in \text{AC and } \forall t, \tag{2}$$

with $AC = \{s, b, c\}$ the set of asset classes we consider, where s, b and c stand for stocks, bonds and commodities, and we additionally impose $w_{h,t} = 0$, $\forall t$ when the investor is not allowed to invest in the hedge fund. We think of this as describing a fairly conservative investor who does not go short and is constrained to invest 100% of his wealth. This could be a high net worth (HNW) individual or an institutional investor, like a pension fund for example, who is subjected to regulation preventing him from going short.

Like Ang & Bekaert (2002a), we assume that investors know in which regime they are. Portfolios are computed using dynamic programming as in Ang & Bekaert (2002a), by recursive optimization backwards in time for horizons of one to sixty months, which corresponds to five year. Whereas Ang & Bekaert (2002a) use quadrature to compute expected utility of terminal wealth for various models of returns, we follow Patton (2004) in using simulation instead.[14]

We evaluate the gains as the "cents per dollar" compensation required by an investor who has access to hedge funds to forego such an investment opportunity and stick to more traditional asset classes. This is the method used by Ang & Bekaert (2002a) and Ang & Chen (2002) in a Gaussian regime-switching model. The fee is calculated as

$$100 \times \left\{ \left(\frac{E\left[U(W^h_{T,s_t})\right]}{E\left[U(W_{T,s_t})\right]} \right)^{1/(1-\gamma)} - 1 \right\},$$

where W^h_{T,s_t} and W_{T,s_t} are the values of the final wealth derived from an optimal portfolio, respectively with and without hedge funds, when the investment starts in regime s_t and for horizon T.

[14] We run $100,000$ simulations for each regime of the regime-switching model.

4.2 Benefits of investing in hedge funds

Table 6 shows the economic cost, measured in cents per dollar invested, of not including hedge funds in the optimal portfolio for an investor with a risk aversion equal to 5 or 10. As expected, the cost increases with horizon T for both regimes and for different levels of risk aversion γ. For the hedge fund index, for an investor with a relative risk aversion of 5, the economic cost in the asymmetric (Gaussian) regime goes from 0.05 (0.08) cents per dollar after one month to 3.64 (3.99) cents per dollar after five years. The values are similar to the cost of no international diversification found by Ang & Bekaert (2002a), when adding to US stocks, the opportunity to invest in the UK. We are mostly interested in longer horizons, as these are more realistic for a hedge fund investor. The average fee over all hedge fund styles and across both regimes for a five year horizon is 4.26, but these costs are quite variable across strategies and they range from 0 for the equity market neutral style to 11.73 for event-driven distressed, with a standard deviation of 3.88 across strategies. For instance, the equity market neutral strategy has a cost of zero for all horizons and regimes when we consider a risk aversion of 5. This is due to the lower performance of this strategy and its high volatility during the sample period. It is therefore not surprising that such a hedge fund is almost never included in the optimal portfolio, and therefore the costs of not being allowed to invest in it is close to zero. The event-driven distressed and global macro strategies are the hedge fund styles with the largest benefits, with more than ten cents per dollar after five years. On the other hand, the emerging markets, equity market neutral, fixed-income arbitrage and managed futures strategies seem to be the least interesting, as they lead to gains less or little more than one cent per dollar after five years. This difference in performance is similar for different levels of risk aversion and also across regimes. This implies that the gains of hedge fund investment depend very much on the type of strategy the hedge funds are following.

[TABLE 6]

4.3 Portfolio weights

[FIGURE 4]

Beyond the fees, we also examine the weights of the optimal portfolios. The first row of panel (a) in Figure 4 plots the proportion of the optimal portfolio invested in the hedge fund index, as well as the corresponding cents per dollar fee for a five year investment horizon (60 months) for different levels of risk aversion γ, when no short selling is allowed. The weights are averaged over the holding period by the forward probabilities from the Markov chain. Initially, as the risk aversion increases, the optimal weights in the hedge fund increase sharply from 46% up to a maximum of 77% at $\gamma = 7$ for both regimes. As risk aversion increases beyond this value, the weight of the hedge fund decreases. The fees, which correspond to the benefits of hedge fund investment, increase sharply when risk aversion increases from 3 to 5, and at a much lower pace for higher values of risk aversion, which is when the proportion of the wealth invested in hedge funds starts decreasing. This means that more risk-averse investors value the opportunity of investing in hedge funds more, even though their actual investments in them is less than that of more risk-tolerant investors.

A closer look at the portfolio weights reveals that, when investors do not have access to hedge funds, for low risk aversion, most of the portfolio is in stocks, and, as risk aversion increases, there is a gradual move away from stocks and into bonds. Instead, when given the opportunity, they invest mainly in hedge funds and in the stock market when their level of risk aversion is low and, at higher levels of risk aversion, they move out of stocks and into bonds, but they keep their hedge fund investment close to its low risk aversion level. Thus the hedge fund index is attractive for investors with low risk aversion because it allows them to diversify away from stocks, while offering higher returns than bonds, and for more risk-averse agents, it plays the role of stocks in the traditional trade-off between stocks and bonds.

It is also instructive to look at the moments of the optimal portfolios (for portfolio selection with higher moments, see, e.g., Jondeau & Rockinger 2006, Guidolin & Timmermann

2007, Campbell, Liechty, Liechty & Müller 2010). Our investor derives utility from the terminal wealth $W_T = \exp(r_T)$. A CRRA utility function, such as the one we use, places weight on moments higher than the mean and the variance. This can be seen by Taylor expanding the expected utility up to fourth order around zero (see, e.g., Skoulakis 2012):

$$E[U(W_T)] \simeq \frac{1}{1-\gamma} + E[r_T] + \frac{(1-\gamma)}{2}E[r_T^2] + \frac{(1-\gamma)^2}{6}E[r_T^3] + \frac{(1-\gamma)^3}{24}E[r_T^4].$$

At very low levels of risk aversion, the investor will care a lot about the mean and much less about higher moments (for instance when $\gamma = 3$, the weights on the first four moments are, respectively, 1, -1, 2/3 and -1/3), but this changes with higher levels of risk aversion, when the investor starts to fear negative third and large fourth moments (for instance when $\gamma = 10$, the weights on the first four moments are, respectively, 1, -4.5, 13.5 and -30.38).

[FIGURE 5]

Figure 5 illustrates how the mean, the second, the third and the fourth moments change in both regimes, with and without hedge fund investment, with the level of risk aversion for a time horizon of five years ($T = 60$ months). The moments are computed from simulated paths of returns, by applying the optimal portfolio weights that correspond to the maturity and the current regime. As γ increases, the investor takes less risk and the second moment in panel (b) decreases, but this is at the expense of the mean return, as can be seen in panel (a). The third moment in panel (c), which hurts the investor when it is negative, increases from negative to positive values, and the fourth moment in panel (d), which measures the fatness of the tails of the portfolio return distribution also decreases. The same picture holds for both regimes.

Overall, investing in the hedge fund allows to achieve a higher mean with fairly similar second, third and fourth moments compared with the no hedge fund allocation. For very low levels of risk aversion, the investor with the opportunity to invest in the hedge fund is willing to sacrifice a little bit of mean return in exchange for a stark reduction in the second and fourth moments. This shows that the less risk-averse investor uses the hedge fund mainly as a means of reaping the benefits of higher diversification by reducing the second moment, whereas the more risk-averse investor uses the hedge fund to enjoy the high returns while keeping a similar level of the higher order moments.[15]

[FIGURE 6]

Figure 6 displays the hedge fund weights and the fees, as a function of the level of risk aversion for all hedge fund style indices. There is a lot of heterogeneity in the weights of the different hedge fund styles. For example, optimal portfolios will only include very low proportions of the emerging markets or managed futures hedge fund styles, whereas the weights are above 90% for the event-driven and event-driven distressed styles. However, for most hedge fund styles, the portfolio weights have an inverse U-shape relation with the level of risk aversion of the investor, which is qualitatively similar to that of the hedge fund index. Also, like for the hedge fund index, for nearly all hedge fund styles the fees increase with the investor's risk aversion.

4.4 Historical analysis of the benefits of hedge fund investment

We now turn to an evaluation of how well investors using the regime-switching copula model would have done historically with a five year investment horizon ($T = 60$), had they used the optimal portfolio weights from the model along with the forward probabilities from the Markov chain. We assume that, every period, investors use the weights of the most likely of the two regimes according to the current state probability. We focus on an investor with risk aversion $\gamma = 5$.

[FIGURE 7]

[15] Although the portfolio including the hedge fund has a slightly higher second moment than the one without hedge fund, its variance is lower.

Figure 7, panel (a) plots the historical five year horizon portfolio returns with and without the hedge fund index in the top row, with the shaded area corresponding to the difference between them when the portfolio including the hedge fund index dominates and the difference between the two in the middle row. The dates correspond to the initial period of the five year investment window, which means, for instance, that the effect of a crisis will start to be reflected in the portfolio returns starting five years prior to the actual event. The returns of both portfolios in the first row of panel (a) vary dramatically over the sample period, from very high levels in the mid-nineties and before the subprime crisis down to the troughs during the bursting of the dot-com bubble in 2000 and the subprime crisis in 2008, and back up into positive territory when the markets rally after 2008. The portfolio including the hedge fund index, in panel (a), shows a very long downward trend relative to the one without hedge funds over almost the entire sample period. It beats it very consistently until the subprime crisis, and never again afterwards.

[FIGURE 8]

Figure 8 plots the difference between the historical portfolio returns with and without hedge funds, for all styles, when short selling is not allowed. The figure illustrates that the benefits of hedge fund investment vary dramatically over time, and with the exception of convertible arbitrage, fixed-income arbitrage and multi strategy in panels (a), (h) and (l), portfolios that include hedge funds underperform in the most recent period. The event-driven, event-driven distressed, event-driven multi strategy, global macro and long/short equity styles, in panels (d), (e), (f), (i) and (j), perform well in the first part of the sample, up to 2006, but underperform the portfolio without hedge funds after that. For risk aversion $\gamma = 5$, the holdings of the equity market neutral hedge fund index are nearly zero, which explains why there is practically no difference between the two portfolios in panel (c). The performance of the portfolio including the emerging markets hedge fund index, in panel (b), was poor during the emerging markets crisis in the second half of the nineties, it improved from 1998 to 2006, but it has not outperformed the portfolio without hedge fund ever since.

[FIGURE 9]

We now turn to the volatility of these portfolios to see how their risk evolved historically and to compare the volatility of the portfolios with and without hedge fund. The plots in the last row of Figure 7 reveal dramatic fluctuations between a minimum of around 10% in December 2012 and a maximum of 27% in August 2007 with two humps, one due to the dot-com bubble in 2001 and the other to the subprime crisis in 2008. Except for the very beginning of the sample period, the volatility of the portfolio including hedge funds is always lower. Figure 9 displays a similar pattern for all hedge fund styles with two humps and a lower volatility for the portfolios that include hedge funds. Overall, this shows that, while hedge funds were particularly valuable due to their high returns during the 90s, this is no longer the case. However, consistent with Bollen et al. (2021), we find that hedge funds still produce benefits in terms of diversification, as they make it possible for an investor to achieve significantly lower risk.

4.5 Relaxing the short selling constraint

So far, we have examined the results when investors are not allowed to take short positions. While maintaining the constraint of Equation (1), we relax the no short selling constraint of Equation (2) and instead impose a short selling constraint in the form:

$$w_{h,t} + \sum_{ac \in AC} |w_{ac,t}| \leq \delta, \quad \forall t, \tag{3}$$

where δ is gross leverage. The constraint implies that short positions aggregated over all assets other than the hedge fund are allowed up to $(\delta - 1)/2$. We are thinking of δ as a measure of the sophistication of our investor. When $\delta = 1$, we recover the no short selling constraint of Equation (2) and as δ increases, our investor is allowed to take increasingly larger short positions, of up to 50%, when $\delta = 2$, and 100%, when $\delta = 3$.

The first row of panels (b) and (c) in Figure 4 plots the holdings of the hedge fund index and the fees as a function of the risk aversion parameter γ, with a short selling constraint of, respectively, 50% and 100%. The second row of Figure 4 displays the gross leverage, which is the actual amount of δ, or equivalently the $\sum_i |w_i|$ used by the investors in their portfolios, averaged over the holding period, for different values of risk aversion γ. As expected, only the least risk-averse investors (with γ up to 5) use a significant amount of short positions, and depending on the regime, these investors use nearly as much leverage as they are allowed to when $\delta = 2$, but far less than the maximum allowed when $\delta = 3$. Interestingly, these investors will take moderately sized short positions of up to 8%, which implies a gross leverage of 1.16, even when they are not allowed to invest in hedge funds, but it is when they have access to the hedge fund index, that they will take very significant short positions reaching the limit of 50% when $\delta = 2$, and up to 60% or 70%, depending on the regime, when $\delta = 3$. The short selling constraint $\delta = 2$ ($\delta = 3$) is only binding for the least risk-averse hedge fund investors, with γ less than 7 ($\gamma = 3$). But even the most risk-averse investors (with $\gamma = 20$) will make use of up to 4.5% and 2.4% of leverage, respectively, with and without hedge funds, depending on the regime.

[TABLE 7]

Given the patterns in the investors' use of short selling, most of the difference with respect to the no short selling case should be expected for low risk aversion investors. This is confirmed by the results of the fees for 50% short selling that appear in Table 7. The fees are somewhat higher than the ones of Table 6, when no short selling is allowed, particularly when $\gamma = 5$ and only very slightly higher when $\gamma = 10$.[16]

The historical five year horizon portfolio returns with and without the hedge fund index in panels (b) and (c) of Figure 7 show the same long term downward trend as when short selling is not allowed, except for a very brief period after 2012. As can be expected, compared to their non-leveraged counterpart, when $\delta = 2$ and $\delta = 3$, the difference in performance

[16]The results with $\delta = 3$ are very similar and are omitted in the interest of space.

between the portfolios with and without the hedge fund index is more extreme, both on the up and on the down side.

4.6 Hedge fund investment with lockup periods

Hedge funds typically impose constraints on their investors, such as lockup periods, during which investors cannot redeem their investment in the hedge fund, or other liquidity limitations, such as restrictions on rebalancing frequency (see, e.g., Joenväärä, Kosowski & Tolonen 2019). The existence of such lockup periods will make monthly rebalancing of hedge fund holdings impossible. In order to check the effect of such lockup periods on the benefits of hedge fund investment, we redo our portfolio optimization exercise, with the additional constraint that the hedge fund holdings have to be chosen initially and held constant for the entire horizon T. We carry out a "brute force"computation, by setting up a grid of hedge fund holdings in 1% increments from 0 to the maximum allowed (100% when $\delta = 1$, 150%, when $\delta = 2$, and 200% when $\delta = 3$), and optimizing over the remaining portfolio weights as per Section 4.1 for each possible value of the hedge fund weight on the grid. We allow for lockup periods of up to 12 months, which is the most frequent lockup period in the sample of hedge funds analyzed by Aragon (2007) (followed by six, three, one and 24 months).

[TABLE 8]

Results in Table 8 reveal that the imposition of a lockup period makes almost no difference. The fees after six months ($T = 6$) or a year ($T = 12$ months) are nearly identical to those in Table 6. This can be explained by the fact that the regimes are in general highly persistent and, moreover, the optimal weights tend to vary very slowly with the investment horizon, a result that is also documented in Ang & Bekaert (2002*a*). Our estimates of the cost of lockups are lower than those of Derman (2007) and Derman, Park & Whitt (2009), which are computed under the assumption that hedge fund performance is persistent and that investors are able to forecast the returns of the hedge funds. We do not make such an assumption, which explains the difference in results. Both our exercise and the one of Derman and coauthors evaluate the cost of imposing a lockup *ceteris paribus*, under the assumption that lockup periods do not affect the strategies nor the returns of the hedge funds that impose them. This ignores the fact that the funds that impose lockups tend to have higher returns, because the lesser risk of withdrawals from investors reduces their exposure to funding risk (see, e.g., Aragon 2007) and because it allows those hedge funds to take advantage of equity-mispricing anomalies (see Aiken, Clifford, Ellis & Huang 2020). In the same way, these exercises do no apply to the discretionary liquidity restrictions (DLR) analyzed by Aiken et al. (2020), as these are more likely to be imposed when fund returns are low, and are thus more likely to be detrimental to hedge fund investors.

5 Conclusion

We account for non-linearity, asymmetry and time variation in hedge fund returns, and we investigate the benefits for an active investor with a constant relative risk aversion (CRRA) utility, of including different hedge fund styles into his portfolio composed of U.S. stocks, a global bond index, and commodities. We model time variation with regime-switching and allow for an additional source of non-linearity with the use of copulas. We estimate the multivariate regime-switching copula model of Chollete et al. (2009), with a symmetric Gaussian and a possibly asymmetric canonical vine copula regime, that allows for tail dependence. Using the Ang & Chen (2002) H statistic, we show that the regime-switching canonical vine model is able to capture the asymmetry present in the data.

We follow Ang & Bekaert (2002*a*) and compute the gains for an active investor with constant relative risk aversion (CRRA) utility, who considers different asset classes, from investing in any one of a number of hedge fund strategies. The fees we compute for the right of access to hedge fund investment increase with the investment horizon and reach about 4 cents per dollar for the hedge fund index at the five year horizon, which is in line with the cost of no diversification found by Ang & Bekaert (2002*b*). These gains vary a lot across hedge fund styles. The event-driven distressed and global macro strategies produce the largest benefits, whereas emerging markets, equity market neutral, fixed-income arbitrage and managed futures seem to be the least interesting for an investor.

We find that more risk-averse investors are willing to pay more to gain access to hedge funds even though their holdings of hedge funds are lower than those of more risk-tolerant investors. More specifically, the hedge fund holdings have a U-shape relation across levels of risk aversion, that can partly be explained by the fact that hedge funds play different roles for different investors. Risk-tolerant investors use hedge funds as the safe asset, as a replacement for bonds, whereas risk-averse agents use hedge funds as the risky asset, as a replacement for stocks. This is confirmed by the first four moments of the optimal portfolio. Indeed, we find that more risk-tolerant investors use hedge funds mainly for its diversification potential to reduce the second moment, whereas the more risk-averse investors use hedge funds to enjoy the high returns while keeping a similar level of the higher order moments.

The historical returns of a portfolio following the strategy based on the canonical vine copula regime-switching model show that including hedge funds delivers higher returns, but only up to the subprime crisis. However, even though they might no longer be expected to produce the stellar returns they used to deliver in the 90s, hedge funds continue to add value in terms of diversification after the subprime crisis, a period during which even some hedge funds with hitherto impeccable track records suffered significant losses and had to face withdrawals from investors (see, e.g., Aiken, Clifford & Ellis 2015). Our results are robust to the imposition of lockup periods and they only change very slightly when investors are allowed to take short positions in the traditional asset classes, mainly for the more risk-tolerant investors.

An interesting conclusion that emerges from our results is that, while hedge funds are typically reserved for high net worth individuals with some appetite for risk, we find that even more risk-averse agents would benefit from adding them to their portfolios, if given the opportunity.

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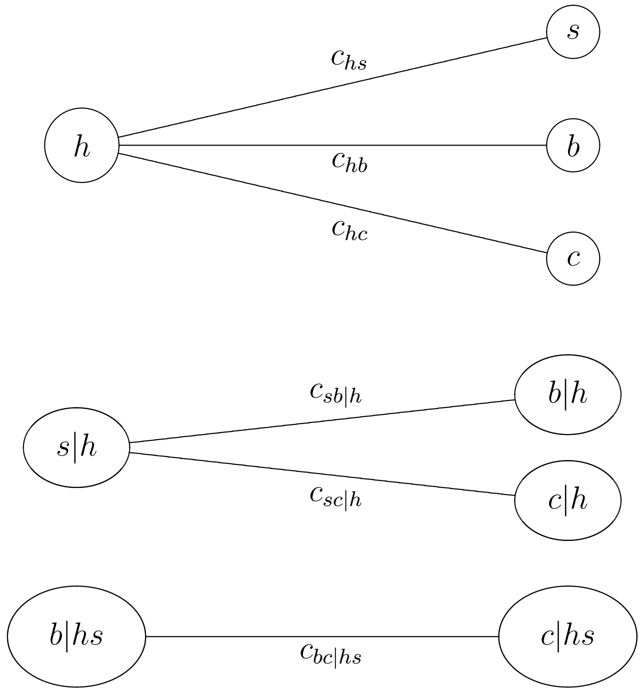
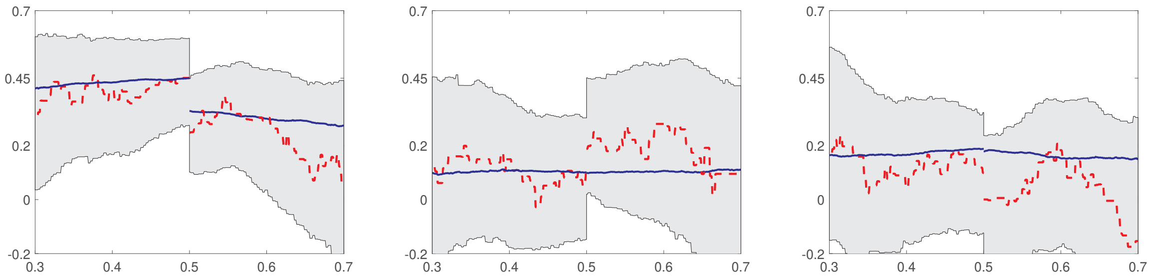
6 Figures

Figure 1: Dependence structure of a canonical vine



This figure shows the structure of a canonical vine copula with four variables. In the first tree, the dependence between the hedge fund returns h and stocks s, bonds b and commodities c is modeled with bivariate copulas c_{hs}, c_{hb} and c_{hc}. The second tree consists in the dependence of variables s with variables b and c, conditionally on hedge fund returns h. In the last tree, one uses a bivariate copula to model the dependence between bonds and commodities b and c, conditionally on hedge fund returns h and stocks s. In the case of this system with 4 variables, the dependence is modeled with 6 bivariate copulas

Figure 2: Exceedance correlation of hedge fund index with S&P500, bonds and commodities

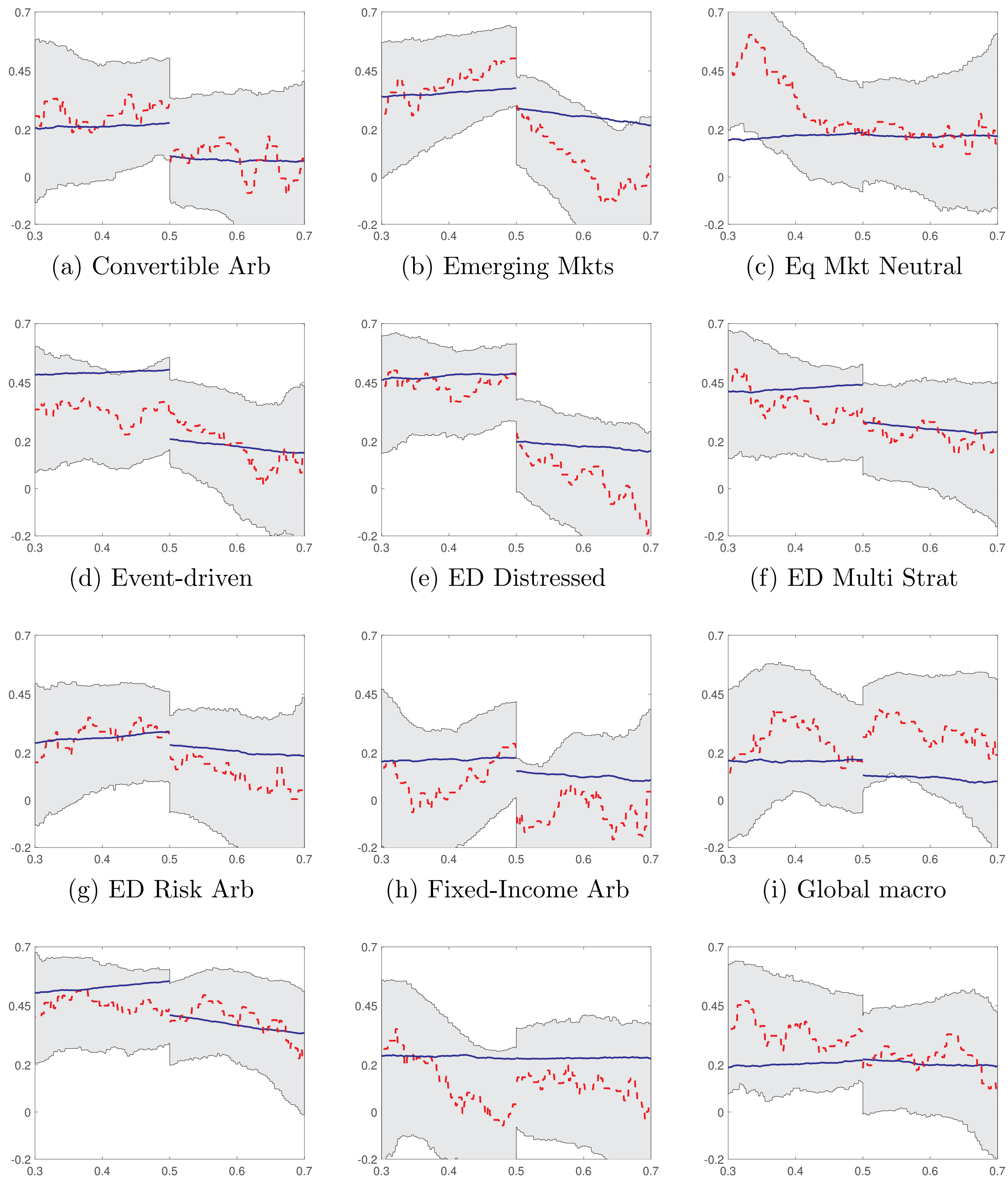


(a) S&P500 (b) Bonds (c) Commodities

This figure displays the exceedance correlation between the hedge fund and the stock, bond and commodities indices, respectively in panels (a), (b) and (c). The red dashed lines represent the exceedance correlation of the data, with a 95% bootstrap confidence interval, calculated using $1,000$ replications, reproduced by the grey area. The blue solid lines represent the exceedance correlation of the regime-switching copula model, computed on a simulated sample of $100,000$ observations. We compute the exceedance correlations $\rho(q)$ at quantile $q_i \in [0.3, 0.7]$, based on the probability integral transformation (PIT) of the standardized residuals, after mean and volatility modeling, of the hedge fund h with asset class ac, (u_h, u_{ac}), as follows:

$$\rho(q_i) = \begin{cases} \text{corr}(u_h, u_{ac} | u_h \geq q_i \text{ and } u_{ac} \geq q_i) & \text{if } q_i \geq 0.5 \\ \text{corr}(u_h, u_{ac} | u_h \leq q_i \text{ and } u_{ac} \leq q_i) & \text{if } q_i \leq 0.5. \end{cases}$$

Figure 3: Exceedance correlation of hedge fund styles with the S&P500 stock market index



(a) Convertible Arb (b) Emerging Mkts (c) Eq Mkt Neutral

(d) Event-driven (e) ED Distressed (f) ED Multi Strat

(g) ED Risk Arb (h) Fixed-Income Arb (i) Global macro

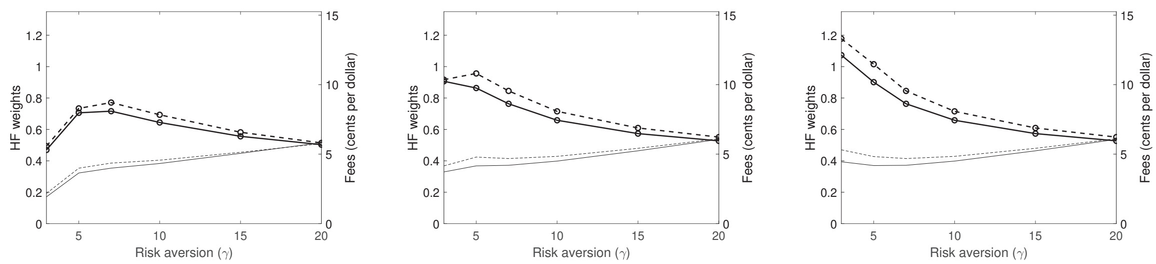
(j) Long/Short Eq (k) Managed futures (l) Multi Strat

This figure displays the exceedance correlation between each of the hedge fund styles and the S&P500 stock market index. The red dashed lines represent the exceedance correlation of the data, with a 95% bootstrap confidence interval, calculated using $1,000$ replications, reproduced by the grey area. The blue solid lines represent the exceedance correlation of the regime-switching copula model, computed on a simulated sample of $100,000$ observations. We compute the exceedance correlations $\rho(q)$ at quantile $q_i \in [0.3, 0.7]$, based on the probability integral transformation (PIT) of the standardized residuals, after mean and volatility modeling, of the hedge fund h with asset class ac, (u_h, u_{ac}), as follows:

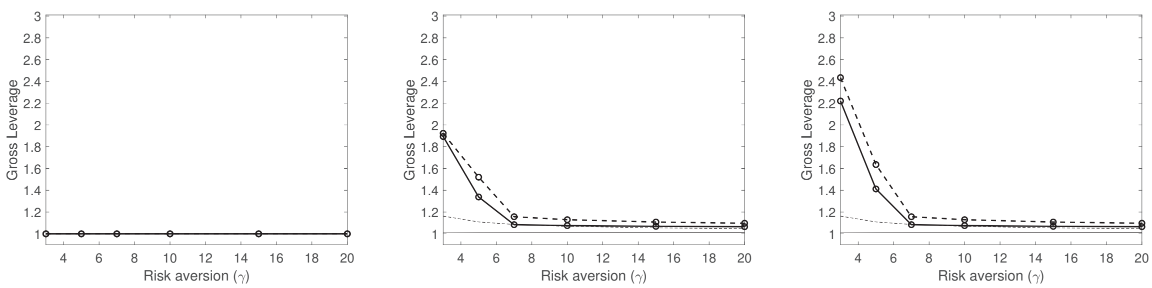
$$\rho(q_i) = \begin{cases} \text{corr}(u_h, u_{ac} | u_h \geq q_i \text{ and } u_{ac} \geq q_i) & \text{if } q_i \geq 0.5 \\ \text{corr}(u_h, u_{ac} | u_h \leq q_i \text{ and } u_{ac} \leq q_i) & \text{if } q_i \leq 0.5. \end{cases}$$

Figure 4: Fees and portfolio weights for hedge fund index, across levels of risk aversion for a five year investment horizon

Hedge fund weights & fees



Gross leverage



(a) No short selling ($\delta = 1$) (b) 50% short selling ($\delta = 2$) (c) 100% short selling ($\delta = 3$)

This figure shows the weights of the hedge fund index and the corresponding fees in the first row, and the gross leverage in the second row, for a five year ($T = 60$ months) investment horizon, as a function of the level of risk aversion γ, when there is no short selling allowed, in panel (a), when short selling of up to 50% is allowed, in panel (b) and when short selling of up to 100% is allowed, in panel (c).
The continuous line corresponds to starting the investment in the asymmetric regime, while the dashed line corresponds to starting the investment in the Gaussian regime.

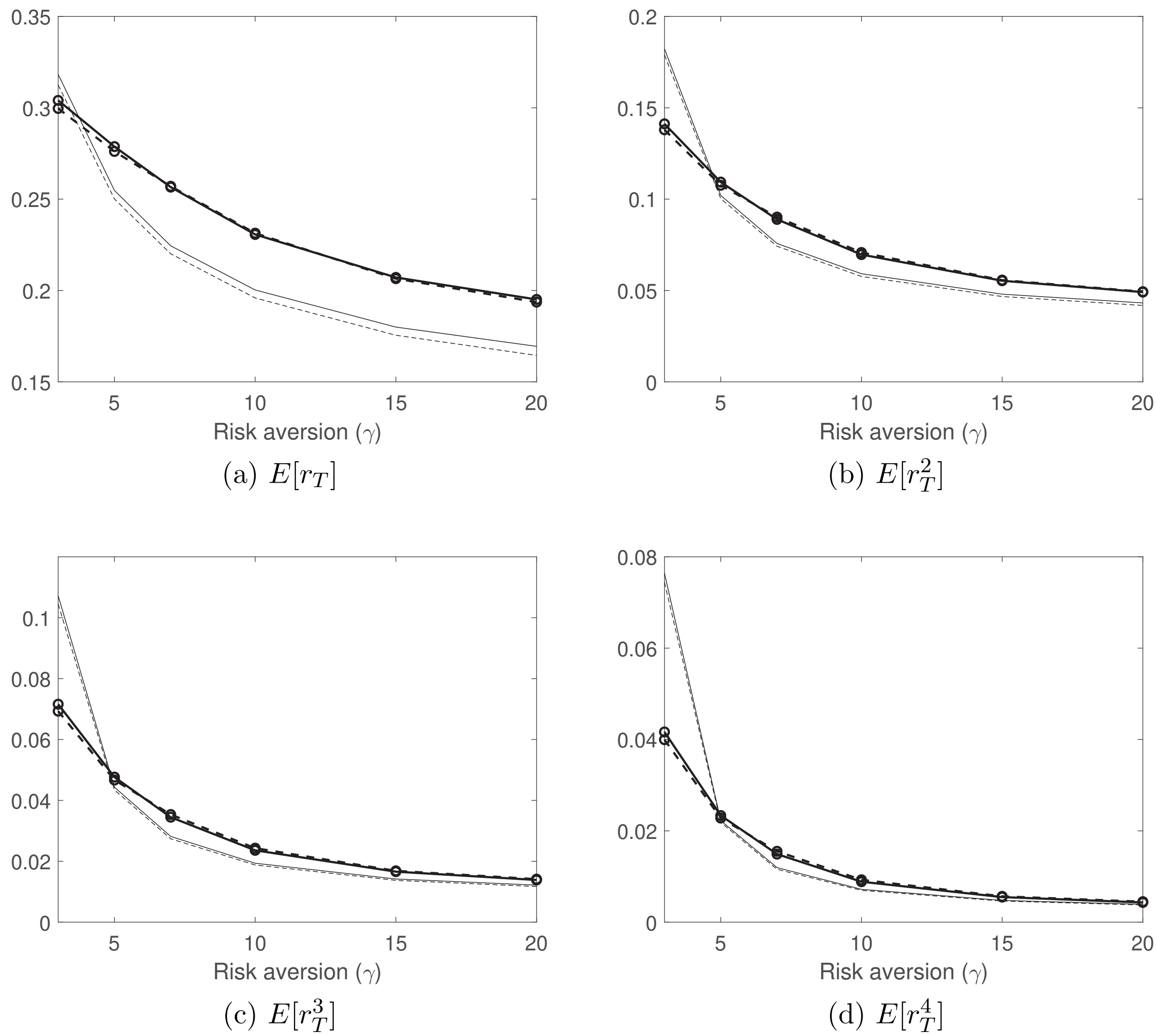
In the first row, the hedge fund portfolio weights are represented with circles, and the values should be read on the left axis, whereas the corresponding fees, in cents per dollar, are shown without the circles, and the values should be read on the right axis.
The portfolio weights $w(p_0)$ are averaged over the holding period by the forward probabilities from the Markov chain:

$$\hat{w}(\gamma, p_0) = \frac{1}{T} \sum_{t=0}^{T-1} \hat{w}_{T-t}(\gamma) P^t p_0,$$

where, for n assets, $\hat{w}_t(\gamma)$ is an $(n, 2)$-dimensional matrix containing the optimal portfolio weights for a horizon of t for both regimes for an investor with risk aversion γ, P is the transition matrix of the Markov chain, and p_0 is a bivariate column vector of initial state probabilities, which we choose to be equal to $(1, 0)$ and $(0, 1)$, respectively, when starting from the asymmetric and Gaussian regime.

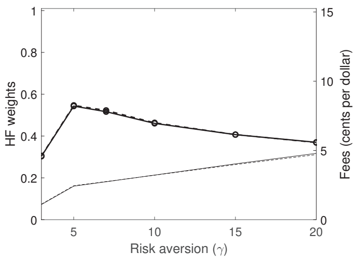
The second row of the figure shows the gross leverage, which is the actual amount of δ used by the investors in their portfolios, for different values of risk aversion γ. The gross leverage of the portfolios with the hedge fund are represented with circles, the gross leverage of the portfolios without the hedge fund are represented without circles.

Figure 5: Moments of the optimal portfolio with and without hedge fund, for a five year ($T = 60$ months) horizon, as a function of the level of risk aversion

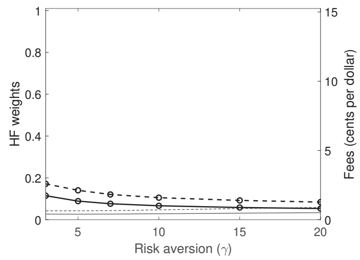


This figure shows the moments of the optimal portfolio with and without hedge fund, for both regimes, for a five year ($T = 60$ months) time horizon, as a function of the level of risk aversion. The moments are computed from the returns of $S = 100,000$ simulated paths of $T = 60$ periods, where at every period t the optimal weights were chosen according to the current regime and the remaining horizon $T - t$, and the next period's returns were chosen at random using the appropriate transition probabilities. The asymmetric regime is represented with the solid line, the Gaussian regime with the dashed lines, and circles are denote the portfolios that include hedge funds, while the lines without circles denote the portfolios without hedge funds.

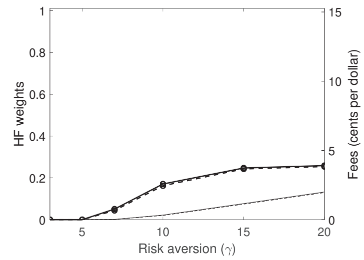
Figure 6: Fees and portfolio weights for all hedge fund styles, across levels of risk aversion (no short selling allowed), for a five year investment horizon



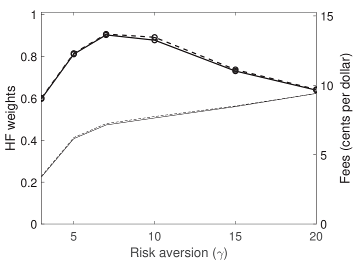
(a) Convertible Arb



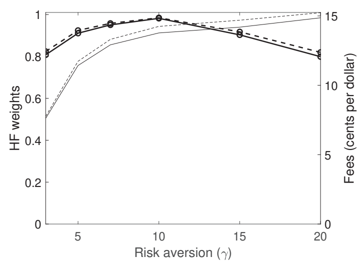
(b) Emerging Mkts



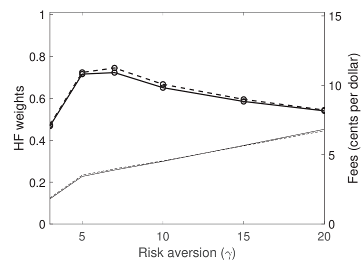
(c) Eq Mkt Neutral



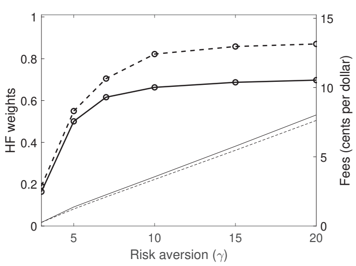
(d) Event-driven



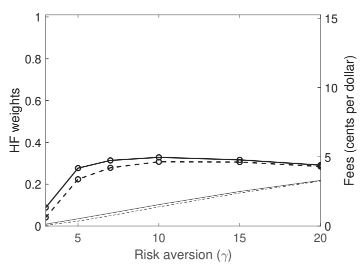
(e) ED Distressed



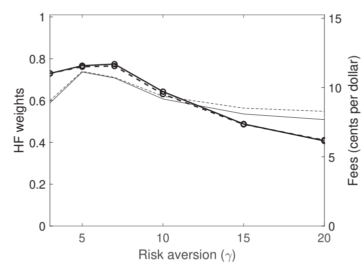
(f) ED Multi Strat



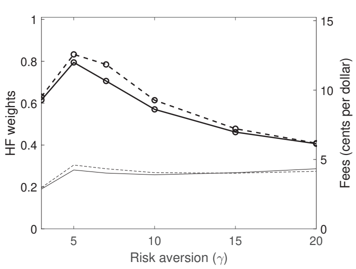
(g) ED Risk Arb



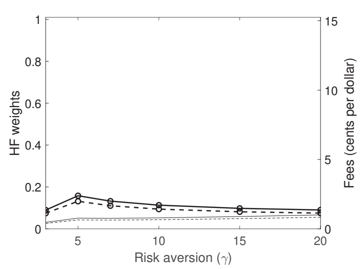
(h) Fixed-Income Arb



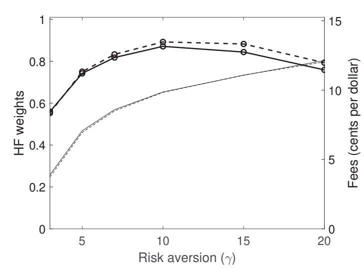
(i) Global macro



(j) Long/Short Eq



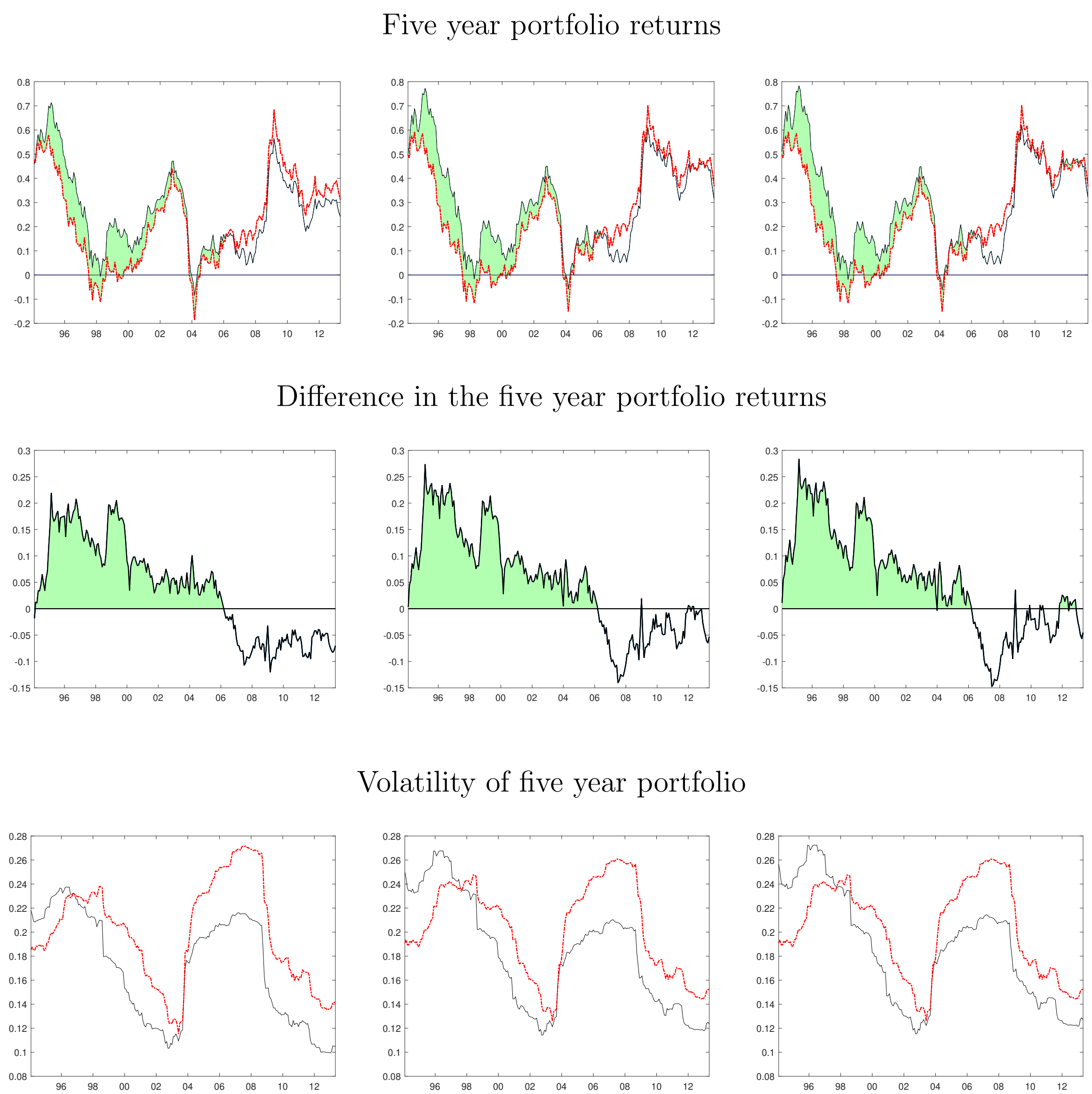
(k) Managed futures



(l) Multi Strat

This figure shows the weights of all hedge fund styles and the corresponding fees for a five year ($T = 60$ months) investment horizon, as a function of the level of risk aversion γ, when there is no short selling allowed. The continuous line corresponds to starting the investment in the asymmetric regime, while the dashed line corresponds to starting the investment in the Gaussian regime. The portfolio weights of the hedge fund are represented with circles, and the values should be read on the left axis, whereas the corresponding fees, in cents per dollar, are shown without the circles, and the values should be read on the right axis.

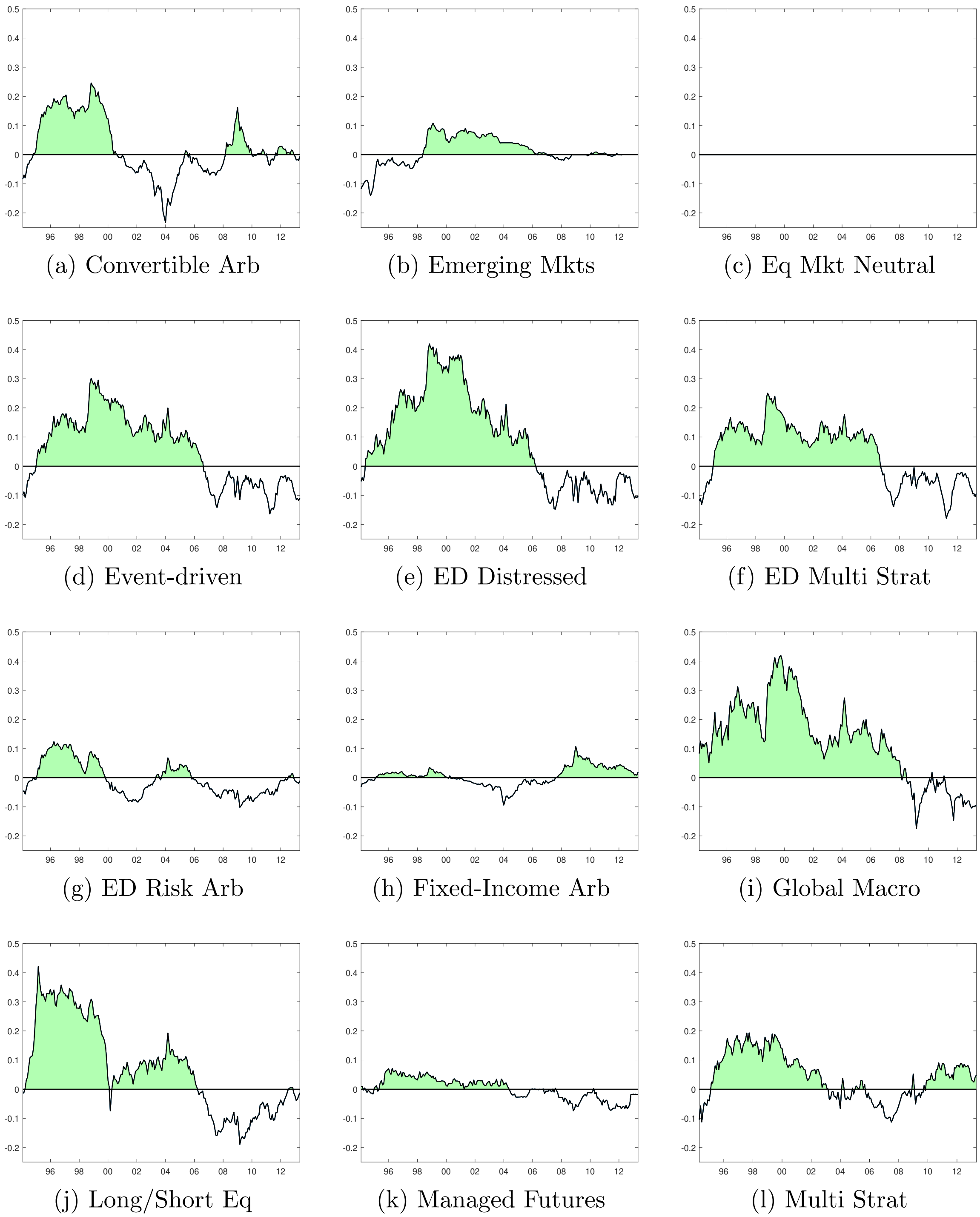
Figure 7: Historical returns from optimal investment with and without hedge fund index, their difference, and volatility, with five years investment horizon and a relative risk aversion of $\gamma = 5$



(a) No short selling ($\delta = 1$) (b) 50% short selling ($\delta = 2$) (c) 100% short selling ($\delta = 3$)

This figure shows the historical five year horizon portfolio returns with (solid black line) and without the hedge fund index (dashed red line), with the green shaded area corresponding to the difference between them when the portfolio including the hedge fund index dominates, as well as the corresponding excess returns of the hedge fund portfolio over the no hedge fund portfolio and the volatility of the portfolio with (solid black line) and without hedge fund (dashed red line). The dates correspond to the initial period of the five year investment window. Panel (a) shows the returns when no short selling is allowed, whereas panels (b) and (c) display the returns when short selling is allowed up to 50% and 100%, respectively.

Figure 8: Difference between the returns with and without hedge fund from investment in hedge fund style indices with five years investment horizon with a relative risk aversion of $\gamma = 5$



(a) Convertible Arb (b) Emerging Mkts (c) Eq Mkt Neutral

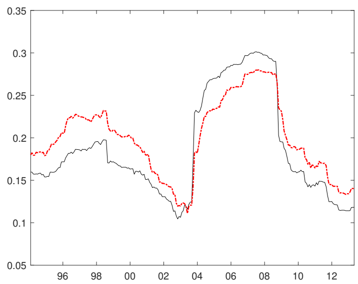
(d) Event-driven (e) ED Distressed (f) ED Multi Strat

(g) ED Risk Arb (h) Fixed-Income Arb (i) Global Macro

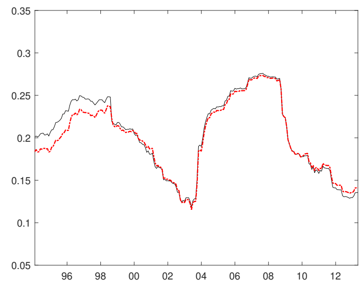
(j) Long/Short Eq (k) Managed Futures (l) Multi Strat

This figure shows the difference between the historical five year horizon portfolio returns with and without hedge fund for the twelve hedge fund styles. The green shaded area corresponds to the difference between them when the portfolio including the hedge fund index dominates. The dates correspond to the initial period of the five year investment window.

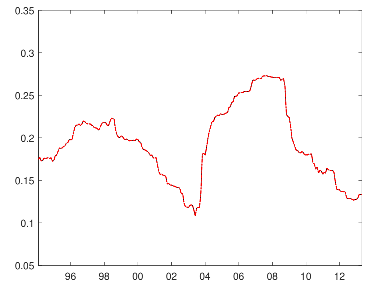
Figure 9: Volatility of portfolio returns with and without hedge fund from investment in hedge fund style indices with five years investment horizon with a relative risk aversion of $\gamma = 5$



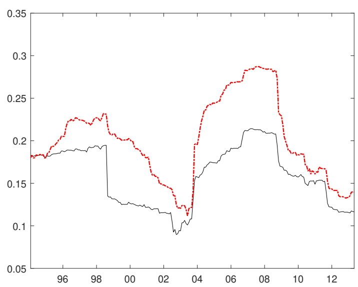
(a) Convertible Arb



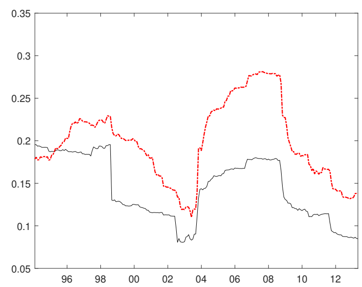
(b) Emerging Mkts



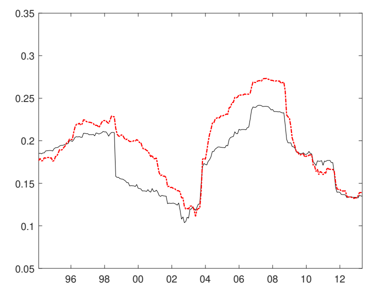
(c) Eq Mkt Neutral



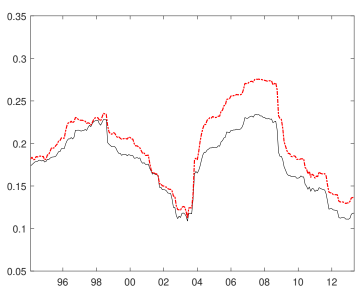
(d) Event-driven



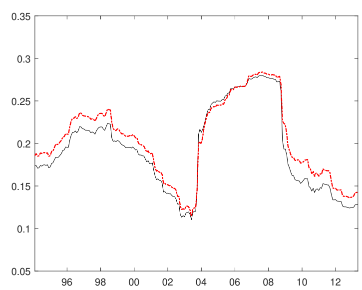
(e) ED Distressed



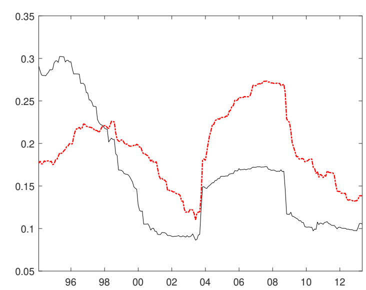
(f) ED Multi Strat



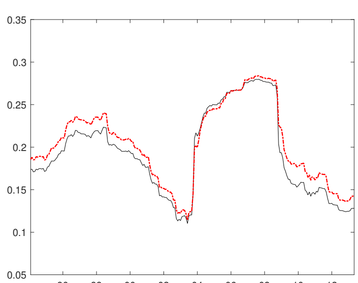
(g) ED Risk Arb



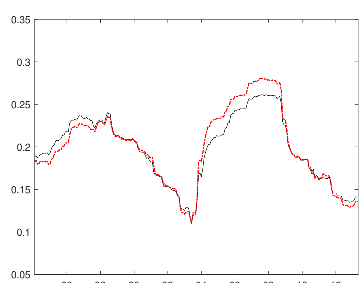
(h) Fixed-Income Arb



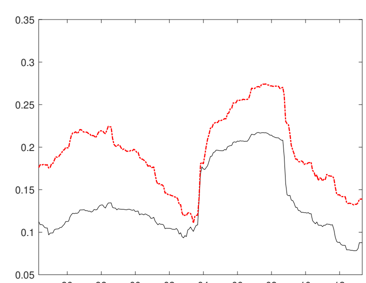
(i) Global Macro



(j) Long/Short Eq



(k) Managed Futures



(l) Multi Strat

This figure shows the historical five year horizon portfolio volatility of the portfolio with (solid black line) and without hedge fund (dashed red line) for the twelve hedge fund styles. The dates correspond to the initial period of the five year investment window.

7 Tables

	Mean	StdDev.	Skew.	Kurt.	Min	Max
Index	4.90	6.70	-0.42	6.42	-8.27	7.74
Convertible Arb	3.89	6.32	-3.03	24.53	-13.53	5.63
Emerging Mkts	4.42	13.42	-1.39	12.17	-26.59	14.83
Eq Mkt Neutral	1.80	11.25	-14.18	227.59	-51.88	3.58
Event-driven	5.22	5.98	-2.30	15.72	-12.95	4.12
ED Distressed	6.25	6.06	-2.44	18.17	-13.73	4.06
ED Multi Strat	4.73	6.51	-1.83	11.73	-12.67	4.66
ED Risk Arb	3.06	3.81	-1.17	8.92	-6.78	3.31
Fixed-Income Arb	2.61	5.25	-5.05	44.48	-15.20	4.23
Global macro	6.76	8.58	-0.27	8.29	-12.64	9.61
Long/Short Eq	5.83	8.86	-0.33	7.20	-12.57	11.79
Managed futures	1.95	11.45	-0.07	2.91	-10.27	9.06
Multi Strat	4.91	4.89	-1.83	10.48	-7.77	4.18
S&P500	6.63	14.53	-0.86	4.83	-18.46	10.37
Barclays Global	2.30	5.38	0.00	3.63	-4.08	6.03
S&P GSCI Commodity	1.93	21.37	-0.65	5.48	-32.60	19.14

Table 1: Summary statistics

Descriptive statistics of the monthly returns of the hedge fund indices and the other asset classes. All returns are expressed in US dollars from January 1994 to April, 2018, which corresponds to a sample of 292 observations.

	Index	Conv Arb	Emerg Mkts	Eq Mkt Neutral	Event Driven	ED Distr.	ED Mult. Strat	ED Risk Arb	Fix. I Arb	Global Macro	L/S Eq	Man. Futures	Multi Strat
Index	1.00												
Convertible Arb	0.55	1.00											
Emerging Mkts	0.72	0.46	1.00										
Eq Mkt Neutral	0.27	0.18	0.14	1.00									
Event-driven	0.76	0.64	0.69	0.27	1.00								
ED Distressed	0.69	0.60	0.64	0.31	0.93	1.00							
ED Multi Strat	0.76	0.62	0.68	0.22	0.96	0.81	1.00						
ED Risk Arb	0.51	0.47	0.53	0.14	0.67	0.60	0.66	1.00					
Fixed-Income Arb	0.53	0.78	0.41	0.29	0.50	0.49	0.49	0.30	1.00				
Global macro	0.80	0.33	0.45	0.04	0.39	0.33	0.42	0.21	0.38	1.00			
Long/Short Eq	0.83	0.45	0.67	0.19	0.74	0.67	0.73	0.60	0.36	0.43	1.00		
Managed futures	0.20	-0.09	-0.02	-0.02	-0.03	-0.05	-0.02	-0.06	-0.06	0.32	0.08	1.00	
Multi Strat	0.52	0.69	0.32	0.33	0.56	0.50	0.56	0.34	0.62	0.27	0.48	0.07	1.00
S&P500	0.57	0.38	0.55	0.25	0.63	0.61	0.59	0.51	0.34	0.23	0.67	-0.08	0.39
Barclays Global	0.11	0.12	0.05	-0.01	0.05	0.04	0.03	0.21	0.13	0.08	0.16	0.29	0.17
S&P GSCI Commodity	0.34	0.36	0.26	0.21	0.37	0.33	0.36	0.28	0.39	0.15	0.36	0.08	0.38

	S&P500	Barclays Global	Commodities
S&P500	1.00		
Barclays Global	0.15	1.00	
S&P GSCI Commodity	0.26	0.22	1.00

Table 2: Unconditional correlation of the monthly returns of the hedge fund indices and the other asset classes.

Unconditional Pearson correlation matrix between the returns of the monthly hedge fund indices and the other asset classes.

	Panel A: Asymmetry tests for excess returns					
	HTZ asymmetry test			Mutual information asymmetry test		
	S&P500	**Bonds**	**Commodities**	**S&P500**	**Bonds**	**Commodities**
Index	0.06	0.94	0.17	0.03	0.49	0.25
Convertible Arb	0.09	0.51	0.29	0.28	0.45	0.33
Emerging Mkts	0.03	0.87	0.77	0.03	0.19	0.22
Eq Mkt Neutral	0.39	0.11	0.44	0.44	0.51	0.64
Event-Driven	0.16	0.98	0.47	0.01	0.07	0.28
ED Distressed	0.01	0.44	0.29	0.00	0.05	0.36
ED Multi Strat	0.06	0.91	0.29	0.01	0.07	0.22
ED Risk Arb	0.02	0.73	0.40	0.08	0.21	0.23
Fixed-Income Arb	0.51	0.52	0.61	0.07	0.19	0.40
Global Macro	0.58	0.47	0.28	0.02	0.61	0.34
Long/Short Eq	0.20	0.35	0.31	0.28	0.16	0.24
Managed Futures	0.11	0.17	0.16	0.53	0.44	0.90
Multi Strat	0.17	0.69	0.29	0.02	0.09	0.22
	Panel B: Ang & Chen (2002) model adequacy test					
	vine-Gaussian RS copula			Gaussian-Gaussian RS copula		
	S&P500	**Bonds**	**Commodities**	**S&P500**	**Bonds**	**Commodities**
Index	0.77	0.49	0.30	0.30	0.52	0.28
Convertible Arb	0.15	0.32	0.14	0.02	0.28	0.08
Emerging Mkts	0.12	0.82	0.92	0.03	0.73	0.88
Eq Mkt Neutral	0.50	0.81	0.65	0.47	0.19	0.60
Event-Driven	0.28	0.12	0.61	0.12	0.05	0.45
ED Distressed	0.51	0.15	0.36	0.04	0.18	0.41
ED Multi Strat	0.83	0.22	0.69	0.16	0.24	0.72
ED Risk Arb	0.06	0.46	0.61	0.03	0.46	0.66
Fixed-Income Arb	0.25	0.01	0.06	0.12	0.02	0.07
Global Macro	0.22	0.71	0.34	0.06	0.69	0.38
Long/Short Eq	0.62	0.16	0.50	0.32	0.14	0.45
Managed Futures	0.03	0.42	0.44	0.04	0.41	0.42
Multi Strat	0.25	0.47	0.07	0.28	0.45	0.06

Table 3: Bivariate asymmetry, exceedance correlation and model adequacy tests of hedge fund styles with stock, bond and commodities indices (p-values)

This table shows p-values of the bootstrap versions of the Hong et al. (2007) and the Jiang, Maasoumi, Pan & Wu (2018) test of asymmetry for excess returns in Panel A. Panel B shows the Ang & Chen (2002) test for the adequacy between the exceedance correlations of the excess returns and of both a copula regime-switching model with a canonical vine and a Gaussian regime and a model with two Gaussian copula regimes. For each hedge fund style, we perform the tests pairwise with each of the S&P500 index, bonds and commodities.

	GARCH parameters					Distribution tests						Ljung Box (squared residuals)					
	ω/Var	α	β	ν	λ	**KS**	**KS+**	**KS-**	**Berk**	**AD**	**K**	**1**	**2**	**3**	**4**	**8**	**12**
Index	0.00	0.21	0.77	7.43		0.33	0.70	0.17	0.97	0.53	0.32	0.83	0.97	0.52	0.67	0.80	0.68
Convertible Arb	0.00	0.22	0.73	3.79	-0.19	0.55	0.28	0.59	0.88	0.88	0.38	0.70	0.92	0.94	0.97	0.40	0.72
Emerging Mkts	0.20			2.58	-0.16	0.63	0.33	0.39	0.92	0.69	0.24	0.82	0.98	0.14	0.19	0.13	0.12
Eq Mkt Neutral	0.03			2.51	-0.19	0.25	0.94	0.12	0.04	0.10	0.56	0.88	0.95	0.04	0.08	0.35	0.69
Event-driven	0.02			4.77	-0.26	0.91	0.83	0.54	0.72	0.98	0.95	0.93	0.85	0.91	0.96	1.00	1.00
ED Distressed	0.02			4.56		0.37	0.85	0.19	0.95	0.46	0.54	0.38	0.64	0.77	0.88	0.98	1.00
ED Multi Strat	0.03			4.69	-0.21	0.97	0.74	0.63	0.62	1.00	0.95	0.61	0.48	0.63	0.78	0.96	1.00
ED Risk Arb	0.01			5.09		0.30	0.64	0.15	0.97	0.52	0.23	0.62	0.79	0.88	0.91	0.10	0.30
Fixed-Income Arb	0.00	0.66	0.31	3.19	-0.20	0.61	0.66	0.31	0.23	0.80	0.52	0.58	0.76	0.80	0.86	0.00	0.00
Global macro	0.00	0.20	0.79	5.45		0.86	0.50	0.48	0.80	0.89	0.56	0.95	0.82	0.69	0.78	0.24	0.23
Long/Short Eq	0.00	0.20	0.75	9.73		0.44	0.78	0.22	0.96	0.70	0.51	0.92	0.99	0.96	0.99	0.99	1.00
Managed futures	0.11					0.92	0.65	0.54	1.00	0.98	0.82	0.67	0.82	0.83	0.92	0.98	0.82
Multi Strat	0.00	0.16	0.74	6.32	-0.25	0.97	0.75	0.63	0.95	0.95	0.96	0.83	0.96	0.88	0.95	0.91	0.97
S&P500	0.00	0.18	0.78	16.27	-0.36	0.94	0.81	0.57	0.48	0.99	0.96	0.79	0.93	0.98	0.99	1.00	1.00
Barclays Global	0.00	0.07	0.86			0.82	0.55	0.45	0.97	0.83	0.58	0.89	0.96	0.88	0.94	0.88	0.55
S&P GSCI Commodity	0.00	0.16	0.80			0.33	0.39	0.17	0.93	0.34	0.09	0.69	0.82	0.91	0.94	0.87	0.72

Table 4: GARCH estimates, goodness of fit statistics and Ljung-Box of squared residuals

Columns two to five are parameter estimates of univariate GARCH(1,1) models, with the mean omitted. Columns six to eleven report p-values of Goodness of Fit () statistics of the probability integral transform (PIT) of the marginal models. We present the p-values for the following tests. The Kolmogorov-Smirnov (KS) test evaluates the alternative hypothesis that the population cdf is different from a Uniform $[0,1]$. $KS+$, tests the alternative hypothesis that the population cdf is larger than a Uniform $[0,1]$, while $KS-$, tests the alternative hypothesis that the population cdf is smaller than a uniform $[0,1]$. $Berk$ reports the p-value of a test proposed by Berkowitz (2001). The test consists in transforming the PIT of the data into a normal variate with the inverse cdf of the normal, Φ^{-1}, and to test uniformity and lack of correlation, which corresponds to zero mean, variance one and zero correlation against the alternative of and an AR(1) model with unrestricted mean and variance. AD is the Anderson-Darling test for uniformity. K is Kuiper's test for uniformity, which puts more weight on the tails of the distribution than the other tests. Columns 12 to 17 report the p-values of the Ljung-Box statistics for tests of lack of correlation of squared residuals from the GARCH(1,1) models for numbers of lags $1, 2, 3, 4, 8, 12$.

	Asymmetric regime			Gaussian regime			Trans. prob.		Loglikelihood	
	S&P500	Bonds	Commod.	S&P500	Bonds	Commod.	P_{11}	P_{22}	Model	Gaussian
Index	r. Gumbel	Gaussian	Gaussian							
	0.38	-0.05	0.10	0.57	0.33	0.43	0.96	0.95	129.73	126.09
Convertible Arb	Clayton	Clayton	Clayton							
	0.17	0.00	0.06	0.32	0.36	0.31	0.94	0.85	54.43	52.48
Emerging Mkts	r. Gumbel	r. Gumbel	Gaussian							
	0.57	0.37	0.50	0.32	-0.04	0.07	0.96	0.98	109.63	104.77
Eq Mkt Neutral	Gaussian	r. Gumbel	Gaussian							
	0.20	0.10	0.09	0.67	0.23	0.55	0.99	0.88	59.16	59.06
Event Driven	Clayton	Gaussian	Gaussian							
	0.40	0.00	0.16	0.56	0.27	0.45	0.94	0.86	132.82	126.25
ED Distressed	Clayton	Clayton	Gaussian							
	0.33	0.00	0.15	0.68	0.37	0.50	0.98	0.89	129.75	114.88
ED Multi Strat	r. Gumbel	Gaussian	Gaussian							
	0.38	-0.06	0.12	0.55	0.27	0.47	0.95	0.88	119.19	114.81
ED Risk Arb	r. Gumbel	Gaussian	Gaussian							
	0.19	-0.04	-0.03	0.42	0.31	0.32	0.99	1.00	93.50	90.99
Fixed Income Arb	r. Gumbel	Gaussian	Gaussian							
	0.27	0.31	0.27	0.18	0.01	0.07	0.96	0.98	51.96	50.77
Global Macro	r. Gumbel	Gaussian	Gaussian							
	0.29	0.39	0.30	0.14	-0.01	0.07	0.92	0.97	51.80	50.5
Long/Short Eq	r. Gumbel	Gaussian	Gaussian							
	0.47	-0.01	0.12	0.66	0.30	0.44	0.98	0.96	169.43	163.7
Managed Futures	Gaussian	Gaussian	Gaussian							
	0.43	0.21	0.10	-0.29	0.16	0.09	0.88	0.91	52.82	52.82
Multi Strat	Gaussian	Gaussian	Clayton							
	0.15	0.03	0.05	0.52	0.36	0.40	0.96	0.94	73.31	72.92

Table 5: Estimation results for regime-switching copula

Columns two to four contain a description of the bivariate copulas of the hedge funds with the three asset classes, that make up the canonical vine, along with the implied Kendall's tau. Columns five to seven contain the parameters of the Gaussian copula regime. Columns eight and nine contain transition probabilities; columns ten and eleven contain respectively the loglikelihood of the model and the loglikelihood of a regime-switching model with two symmetric Gaussian regimes.

$\gamma = 5$	Asymmetric regime						Gaussian regime					
	T=1	**T=12**	**T=24**	**T=36**	**T=48**	**T=60**	**T=1**	**T=12**	**T=24**	**T=36**	**T=48**	**T=60**
Index	0.05	0.65	1.37	2.12	2.88	3.64	0.08	0.88	1.68	2.45	3.22	3.99
Convertible Arb	0.04	0.47	0.96	1.44	1.93	2.42	0.05	0.50	0.99	1.48	1.96	2.45
Emerging Mkts	0.00	0.03	0.10	0.19	0.29	0.40	0.01	0.15	0.29	0.41	0.53	0.64
Eq Mkt Neutral	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Event Driven	0.10	1.19	2.40	3.64	4.89	6.15	0.11	1.26	2.48	3.72	4.97	6.23
ED Distressed	0.17	2.15	4.39	6.68	9.03	11.42	0.21	2.38	4.67	6.98	9.33	11.73
ED Multi Strat	0.05	0.66	1.34	2.03	2.73	3.43	0.07	0.74	1.43	2.12	2.82	3.52
ED Risk Arb	0.02	0.28	0.56	0.83	1.10	1.38	0.02	0.25	0.49	0.74	0.98	1.23
Fixed Income Arb	0.01	0.14	0.26	0.35	0.44	0.53	0.00	0.05	0.12	0.20	0.27	0.35
Global Macro	0.17	2.11	4.29	6.52	8.79	11.12	0.18	2.15	4.33	6.56	8.84	11.17
Long/Short Eq	0.06	0.79	1.62	2.48	3.36	4.24	0.09	0.98	1.91	2.81	3.70	4.60
Managed Futures	0.03	0.21	0.35	0.49	0.63	0.77	0.00	0.09	0.22	0.36	0.50	0.64
Multi Strat	0.12	1.40	2.79	4.20	5.63	7.07	0.11	1.31	2.68	4.08	5.50	6.95

$\gamma = 10$	Asymmetric regime						Gaussian regime					
	T=1	**T=12**	**T=24**	**T=36**	**T=48**	**T=60**	**T=1**	**T=12**	**T=24**	**T=36**	**T=48**	**T=60**
Index	0.06	0.80	1.66	2.54	3.43	4.33	0.08	0.96	1.87	2.76	3.66	4.56
Convertible Arb	0.05	0.64	1.28	1.92	2.57	3.22	0.05	0.63	1.27	1.91	2.56	3.21
Emerging Mkts	0.00	0.03	0.11	0.21	0.32	0.43	0.02	0.17	0.32	0.46	0.58	0.71
Eq Mkt Neutral	0.01	0.07	0.14	0.20	0.27	0.33	0.00	0.04	0.11	0.17	0.24	0.30
Event Driven	0.12	1.46	2.97	4.51	6.06	7.64	0.14	1.55	3.07	4.61	6.17	7.75
ED Distressed	0.21	2.56	5.25	8.02	10.86	13.77	0.26	2.91	5.67	8.46	11.32	14.25
ED Multi Strat	0.07	0.88	1.78	2.68	3.60	4.52	0.08	0.91	1.81	2.72	3.63	4.55
ED Risk Arb	0.06	0.71	1.42	2.13	2.84	3.56	0.06	0.66	1.33	2.00	2.68	3.36
Fixed Income Arb	0.03	0.34	0.66	0.96	1.25	1.55	0.02	0.26	0.53	0.81	1.09	1.38
Global Macro	0.13	1.70	3.50	5.36	7.25	9.17	0.15	1.84	3.68	5.55	7.44	9.38
Long/Short Eq	0.06	0.75	1.52	2.30	3.09	3.90	0.07	0.83	1.64	2.44	3.24	4.05
Managed Futures	0.03	0.21	0.36	0.50	0.65	0.79	0.00	0.09	0.23	0.37	0.51	0.66
Multi Strat	0.16	1.90	3.84	5.81	7.82	9.86	0.16	1.89	3.81	5.78	7.79	9.83

Table 6: Cents per dollar fees for hedge fund investments.

$\gamma = 5$	Asymmetric regime						Gaussian regime					
	T=1	**T=12**	**T=24**	**T=36**	**T=48**	**T=60**	**T=1**	**T=12**	**T=24**	**T=36**	**T=48**	**T=60**
Index	0.05	0.70	1.52	2.38	3.26	4.16	0.10	1.12	2.07	2.98	3.89	4.79
Convertible Arb	0.04	0.47	0.96	1.45	1.94	2.43	0.05	0.51	0.99	1.48	1.97	2.46
Emerging Mkts	0.00	0.04	0.11	0.21	0.32	0.43	0.01	0.16	0.29	0.42	0.54	0.65
Eq Mkt Neutral	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Event Driven	0.11	1.42	2.90	4.40	5.93	7.48	0.14	1.57	3.06	4.56	6.09	7.64
ED Distressed	0.24	2.94	6.00	9.16	12.41	15.77	0.28	3.17	6.28	9.46	12.72	16.08
ED Multi Strat	0.05	0.69	1.42	2.16	2.91	3.66	0.08	0.84	1.59	2.33	3.08	3.83
ED Risk Arb	0.02	0.29	0.58	0.86	1.15	1.44	0.02	0.28	0.57	0.85	1.13	1.42
Fixed Income Arb	0.01	0.14	0.25	0.34	0.43	0.51	0.00	0.05	0.12	0.19	0.27	0.35
Global Macro	0.22	2.55	5.10	7.70	10.36	13.08	0.20	2.44	4.96	7.54	10.20	12.91
Long/Short Eq	0.06	0.82	1.72	2.66	3.62	4.60	0.11	1.20	2.28	3.31	4.31	5.32
Managed Futures	0.03	0.21	0.36	0.50	0.64	0.78	0.00	0.09	0.23	0.37	0.51	0.65
Multi Strat	0.16	1.93	3.88	5.86	7.88	9.93	0.15	1.86	3.79	5.76	7.77	9.83

$\gamma = 10$	Asymmetric regime						Gaussian regime					
	T=1	**T=12**	**T=24**	**T=36**	**T=48**	**T=60**	**T=1**	**T=12**	**T=24**	**T=36**	**T=48**	**T=60**
Index	0.06	0.82	1.71	2.62	3.56	4.50	0.09	1.04	2.00	2.95	3.89	4.84
Convertible Arb	0.05	0.64	1.27	1.92	2.56	3.21	0.05	0.62	1.26	1.90	2.55	3.20
Emerging Mkts	0.00	0.03	0.11	0.21	0.32	0.44	0.02	0.18	0.32	0.46	0.59	0.71
Eq Mkt Neutral	0.01	0.07	0.13	0.20	0.26	0.33	0.00	0.03	0.09	0.16	0.22	0.29
Event Driven	0.12	1.47	2.98	4.52	6.08	7.67	0.14	1.56	3.08	4.62	6.18	7.77
ED Distressed	0.21	2.64	5.42	8.29	11.24	14.27	0.29	3.13	6.02	8.92	11.89	14.93
ED Multi Strat	0.07	0.89	1.81	2.74	3.67	4.61	0.08	0.95	1.88	2.81	3.74	4.69
ED Risk Arb	0.06	0.71	1.43	2.15	2.87	3.60	0.06	0.70	1.41	2.12	2.83	3.55
Fixed Income Arb	0.03	0.34	0.65	0.95	1.25	1.54	0.02	0.26	0.53	0.81	1.09	1.38
Global Macro	0.14	1.72	3.53	5.40	7.30	9.24	0.15	1.84	3.69	5.56	7.47	9.41
Long/Short Eq	0.06	0.76	1.56	2.38	3.22	4.07	0.08	0.93	1.81	2.67	3.52	4.38
Managed Futures	0.03	0.22	0.37	0.52	0.67	0.81	0.00	0.09	0.24	0.38	0.53	0.68
Multi Strat	0.16	1.95	3.97	6.03	8.13	10.28	0.17	2.04	4.08	6.15	8.26	10.41

Table 7: Cents per dollar fees for hedge fund investments, when $\delta = 2$, up to 50% short sales allowed.

$\gamma = 5$	Asymmetric regime						Gaussian regime					
	T=1	**T=2**	**T=3**	**T=6**	**T=9**	**T=12**	**T=1**	**T=2**	**T=3**	**T=6**	**T=9**	**T=12**
Index	0.05	0.10	0.15	0.31	0.47	0.64	0.08	0.16	0.23	0.45	0.66	0.87
Convertible Arb	0.04	0.08	0.12	0.23	0.35	0.47	0.05	0.09	0.13	0.26	0.38	0.50
Emerging Mkts	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.03	0.04	0.08	0.11	0.14
Eq Mkt Neutral	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Event Driven	0.10	0.19	0.29	0.58	0.88	1.18	0.11	0.22	0.33	0.64	0.95	1.25
ED Distressed	0.17	0.35	0.53	1.06	1.60	2.15	0.21	0.42	0.62	1.22	1.80	2.37
ED Multi Strat	0.05	0.11	0.16	0.32	0.49	0.66	0.07	0.13	0.20	0.38	0.56	0.73
ED Risk Arb	0.02	0.05	0.07	0.14	0.21	0.28	0.02	0.04	0.06	0.12	0.18	0.25
Fixed Income Arb	0.01	0.03	0.04	0.08	0.11	0.14	0.00	0.01	0.01	0.02	0.04	0.05
Global Macro	0.17	0.35	0.52	1.05	1.58	2.11	0.18	0.36	0.53	1.07	1.61	2.15
Long/Short Eq	0.06	0.13	0.19	0.39	0.58	0.78	0.09	0.17	0.25	0.50	0.74	0.97
Managed Futures	0.03	0.04	0.05	0.06	0.05	0.05	0.00	0.00	0.00	0.00	0.00	0.00
Multi Strat	0.12	0.23	0.35	0.70	1.05	1.40	0.11	0.21	0.32	0.65	0.98	1.31

$\gamma = 10$	Asymmetric regime						Gaussian regime					
	T=1	**T=2**	**T=3**	**T=6**	**T=9**	**T=12**	**T=1**	**T=2**	**T=3**	**T=6**	**T=9**	**T=12**
Index	0.06	0.13	0.19	0.39	0.59	0.79	0.08	0.17	0.25	0.48	0.71	0.93
Convertible Arb	0.05	0.11	0.16	0.32	0.48	0.64	0.05	0.10	0.15	0.31	0.47	0.63
Emerging Mkts	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.03	0.05	0.09	0.12	0.16
Eq Mkt Neutral	0.01	0.01	0.02	0.03	0.05	0.07	0.00	0.00	0.01	0.02	0.03	0.04
Event Driven	0.12	0.24	0.36	0.72	1.09	1.45	0.14	0.27	0.40	0.79	1.16	1.54
ED Distressed	0.21	0.41	0.62	1.26	1.91	2.56	0.26	0.52	0.77	1.50	2.21	2.91
ED Multi Strat	0.07	0.14	0.22	0.43	0.65	0.87	0.08	0.15	0.23	0.45	0.68	0.90
ED Risk Arb	0.06	0.12	0.18	0.35	0.53	0.71	0.06	0.11	0.17	0.33	0.50	0.66
Fixed Income Arb	0.03	0.06	0.09	0.18	0.26	0.34	0.02	0.04	0.06	0.13	0.19	0.26
Global Macro	0.13	0.27	0.41	0.83	1.26	1.69	0.15	0.31	0.46	0.92	1.38	1.83
Long/Short Eq	0.06	0.12	0.18	0.37	0.55	0.74	0.07	0.14	0.21	0.41	0.62	0.82
Managed Futures	0.03	0.05	0.06	0.08	0.08	0.08	0.00	0.00	0.00	0.00	0.00	0.00
Multi Strat	0.16	0.32	0.47	0.94	1.42	1.89	0.16	0.31	0.46	0.93	1.39	1.86

Table 8: Cents per dollar fees for hedge fund investments with a T months lockup period.

Appendix

A Quantile and tail dependence

Quantile dependence measures the degree of association in the tails of a distribution. If X and Y are random variables with distribution functions F_X and F_Y, then there is quantile dependence in the lower tail at threshold α, whenever $P[Y < F_Y^{-1}(\alpha)|X < F_X^{-1}(\alpha)]$ is different from zero. *Tail dependence* obtains as the limit of this probability, as we go arbitrarily far out into the tails. The coefficient of lower tail dependence of X and Y is:

$$\lim_{\alpha \to 0^+} P[Y < F_Y^{-1}(\alpha)|X < F_X^{-1}(\alpha)] = \lambda_L \ ,$$

provided a limit $\lambda_L \in [0, 1]$ exists. If $\lambda_L \in (0, 1]$, X and Y are said to be asymptotically dependent in the lower tail; if $\lambda_L = 0$ they are asymptotically independent. If the marginal distributions of random variables X and Y are continuous, then the tail dependence of these random variables is a function only of their copula, and hence the amount of tail dependence is invariant under strictly increasing transformations of the marginals. If a bivariate copula C is such that the limit

$$\lim_{u \to 0^+} C(u, u)/u = \lambda_L$$

exists, then C has lower tail dependence if $\lambda_L \in (0, 1]$ and no lower tail dependence if $\lambda_L = 0$. Similarly, if a bivariate copula C is such that

$$\lim_{u \to 1^-} \bar{C}(u, u)/(1 - u) = \lambda_U$$

exists, then C has upper tail dependence if $\lambda_U \in (0, 1]$ and no upper tail dependence if $\lambda_U = 0$. $\bar{C}(u, v) = 1 - u - v + C(u, v)$ denotes the survivor function of copula C.

B Bivariate copulas

B.1 Gaussian copula

The distribution function of the Gaussian copula is:

$$C_N(u_1, \ldots, u_n; \Sigma) = \Phi_\Sigma(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)),$$

where Φ^{-1} denotes the inverse cumulative distribution function of the standard normal and $\Phi_\Sigma(x_1, \ldots, x_n; \Sigma)$ denotes the standard multivariate normal cumulative distribution:

$$\Phi_\Sigma(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2} v' \Sigma^{-1} v\right) dv,$$

where $v = (v_1, \ldots, v_n)$ and Σ is a correlation matrix, that is symmetric, semi-definite positive with ones on the diagonal and off diagonal terms between -1 and 1. The corresponding density is:

$$c_N(u_1, \ldots, u_n; \Sigma) = |\Sigma|^{-1/2} exp\left[-\frac{1}{2}\left(x'\Sigma^{-1}x - x'x\right)\right],$$

where $x = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n))$. The bivariate version that we use in the canonical vine copulas is:

$$c_\rho(u_1, u_2) = \frac{1}{\sqrt{1-\rho^2}} \exp\left[\frac{-[\Phi^{-1}(u_1)^2 + \Phi^{-1}(u_2)^2 - 2\rho\Phi^{-1}(u_1)\Phi^{-1}(u_2)]}{2(1-\rho^2)} + \frac{\Phi^{-1}(u_1)^2 + \Phi^{-1}(u_2)^2}{2}\right],$$

where ρ is a correlation coefficient that lies between -1 and 1.

The Gaussian copula has zero upper and lower tail dependence, $\lambda_U = \lambda_L = 0$, except in the case of perfect correlation, $\rho = 1$.

B.2 Bivariate Gumbel and rotated Gumbel copula

The Gumbel copula has the following distribution:

$$C_G(u_1, u_2, \theta) = exp\left(-((-\log u_1)^\theta + (-\log u_2)^\theta)^{1/\theta}\right),$$

and the following density:

$$c_G(u_1, u_2, \theta) = \frac{C_G(u_1, u_2, \theta)(\log u_1 . \log u_2)^{\theta-1}}{u_1 u_2((-\log u_1)^\theta + (-\log u_2)^\theta)^{2-1/\theta}} \left(((-\log u_1)^\theta + (-\log u_2)^\theta)^{1/\theta} + \theta - 1\right),$$

where $\theta \in [1, \infty)$.

We use the rotated version of the Gumbel defined as: $C_{RG}(u_1, u_2, \theta) = u_1 + u_2 - 1 + C_{RG}(1 - u_1, 1 - u_2, \theta)$ and $c_{RG}(u_1, u_2, \theta) = c_G(1 - u_1, 1 - u_2, \theta)$. For the rotated version of the Gumbel, $\lambda_L = 2 - 2^{1/\theta}$, $\lambda_U = 0$.

B.3 Bivariate Clayton copula

The Clayton copula has the following distribution

$$C_C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta},$$

and the following density:

$$c_C(u_1, u_2; \theta) = (1 + \theta)(u_1 u_2)^{-\theta-1}(u_1^{-\theta} + u_2^{-\theta} - 1)^{-2-1/\theta},$$

where $\theta \in [-1, \infty)\backslash 0$.

The Clayton copula has lower but not upper tail dependence: $\lambda_L = 2^{-1/\theta}$, $\lambda_U = 0$.