XIII SIMMEC
Simpósio de Mecânica Computacional
29 de Outubro a 1º de Novembro de 2018
UFES - Campus Goiabeiras - Vitória, ES

NUMERICAL EVALUATION OF STRESS INTENSITY FACTORS IN TWO-DIMENSIONAL ANISOTROPIC DOMAINS USING THE DUAL BOUNDARY ELEMENT METHOD

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Abstract: The use of materials with anisotropic mechanical properties (e.g., composites) has increased in many engineering domains such as civil, aeronautical and aerospace industries. Thus, the accurate assessment of the phenomena that may lead to failure of anisotropic bodies is essential for designing safe structures. In particular, for problems regarding the material fracture, the description of the stress fields near the crack tips is crucial to verify the crack stability. Within this context, the present study aims the development of a numerical model to analyse fracture problems in anisotropic materials. The linear-elastic fracture mechanics (LEFM) theory is considered and the stress intensity factors (SIFs) are computed with the M-integral approach. This approach is based on the conservative J-integral and applies the asymptotic fields of the LEFM to perform the mode decomposition in mixed-mode problems. The mechanical behaviour is obtained numerically by the dual boundary element method (DBEM), in which both singular and hypersingular formulations are used. The DBEM is an efficient numerical method to simulate the mechanical behaviour of cracked bodies, particularly due to the non-requirement of an approximation for the domain responses. This feature allows the accurate representation of the elastic fields along the structure, including in regions next to the crack tips. In this study, the singular integrands in the DBEM anisotropic formulation are evaluated with the singularity subtraction method, whereas the third-degree polynomial transformation is used to improve the accuracy of the near-singular integrals. Two numerical examples are presented to show the efficiency of the proposed model to evaluate the SIFs for crack tips in two-dimensional anisotropic domains. The results are in good agreement with the responses available in the reference works.

Keywords: linear elastic fracture mechanics; anisotropic materials; stress intensity factors; M-integral; dual boundary element method.

1 INTRODUCTION

Anisotropic materials are characterized by a directional dependence of their mechanical properties. The sources of anisotropy are related to internal structure of the material as, e.g., the inherent anisotropy of the crystals or the preferential arrangement of reinforcements/fibres. At a microscale, such materials are usually heterogeneous. However, at a macroscale, they can be treated as a continuous model with anisotropic properties. The anisotropic materials plays a major role in many engineering fields, especially since the advent of high technology composites in the 1960s.

Due to their practical importance, the knowledge of the mechanical behavior of anisotropic materials is of great interest to obtain safe and efficient applications. The works of Lekhnitskii (1963) and Stroh (1958) had remarkable importance for describing the linear-elastic response of these materials. Concerning the fracture phenomena, Sih et al. (1965) determined the asymptotic fields of stresses and displacements in the vicinity of a crack tip immersed in an anisotropic material with one plane of elastic symmetry. Later, Hoenig (1982) extended this study for a material with general anisotropy. Both works were based on the linear-elastic fracture mechanics (LEFM) theory and showed that singular stress fields near the tip can be completely described by the stress intensity factors (SIFs).

Because of the complexity involving the fracture of anisotropic materials, the analytical solutions for this kind of problem are limited, and the ones available usually consider infinite domains. Consequently, the response of such problem can only be achieved by coupling the mechanical models of fracture mechanics with numerical methods. The finite element method (FEM) and some of its variations, such as the extended FEM (XFEM), are the most popular methods for solving fracture problems in anisotropic domains. Some applications of these techniques can be observed in the works of Asadpoure et al. (2006), Banks-Sills et al. (2005), Chu and Hong (1990) and Su and Sun (2003). However, these domain-based methods face some drawbacks for the approximation of the mechanical fields near the crack tip, which compromises the fracture analysis. Alternatively, the boundary element method (BEM) is an efficient numerical technique for modelling this complex mechanical problem. The BEM allows the reduction of the dimension of the problem to be solved since the approximations are restricted to the boundary. Consequently, this boundary-based method is able to offer precise responses for the internal fields, even in singular regions as at the vicinity of crack tips, which leads to the accurate evaluation of the SIFs. Applications of the BEM in anisotropic fracture problems are found in the works of García et al. (2004), Hattori (2017), Sollero and Aliabadi (1995) and Tan and Gao (1992).

Several methodologies have been proposed to extract the SIFs in mixed-mode fracture problems in anisotropic materials, such as the displacement extrapolation method and the M-integral approach. These techniques are usually based on the asymptotic fields near the crack tip and/or energy approaches. Among them, the M-integral strategy proposed by Wang et al. (1980) stands out. This approach is based on the conservative J-integral proposed by Rice (1968) and applies the asymptotic fields to perform the mode decomposition process.

In the work reported here, a numerical strategy for evaluating the SIFs of crack tips lying within two-dimensional anisotropic domains is presented. The dual BEM (DBEM) formulation (Sollero and Aliabadi, 1995) is used for modelling the mechanical behaviour. Besides, the SIFs of mixed-mode problems are evaluated through the M-integral strategy. Two numerical applications are shown to illustrate the accuracy of the proposed BEM model.

2 ANISOTROPIC ELASTICITY

For fully anisotropic linear-elastic domains, the state of strain ε_{ij} can be related to state of stress σ_{ij} through the following constitutive equation expressed in Voigt notation:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{12}}{E_{11}} & -\frac{\nu_{13}}{E_{11}} & \frac{\eta_{23,1}}{E_{11}} & \frac{\eta_{13,1}}{E_{11}} & \frac{\eta_{12,1}}{E_{11}} \\ -\frac{\nu_{21}}{E_{22}} & \frac{1}{E_{22}} & -\frac{\nu_{23}}{E_{22}} & \frac{\eta_{23,2}}{E_{22}} & \frac{\eta_{13,2}}{E_{22}} & \frac{\eta_{12,2}}{E_{22}} \\ -\frac{\nu_{31}}{E_{33}} & -\frac{\nu_{32}}{E_{33}} & \frac{1}{E_{33}} & \frac{\eta_{23,3}}{E_{33}} & \frac{\eta_{13,3}}{E_{33}} & \frac{\eta_{12,3}}{E_{33}} \\ \frac{\eta_{1,23}}{G_{23}} & \frac{\eta_{2,23}}{G_{23}} & \frac{\eta_{3,23}}{G_{23}} & \frac{1}{G_{23}} & \frac{\chi_{13,23}}{G_{23}} & \frac{\chi_{12,23}}{G_{23}} \\ \frac{\eta_{1,13}}{G_{13}} & \frac{\eta_{2,13}}{G_{13}} & \frac{\eta_{3,13}}{G_{13}} & \frac{\chi_{23,13}}{G_{13}} & \frac{1}{G_{13}} & \frac{\chi_{12,13}}{G_{13}} \\ \frac{\eta_{1,12}}{G_{12}} & \frac{\eta_{2,12}}{G_{12}} & \frac{\eta_{3,12}}{G_{12}} & \frac{\chi_{23,12}}{G_{12}} & \frac{\chi_{13,12}}{G_{12}} & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{1}$$

The matrix on the right-hand side of Eq. (1) is symmetric, as well as the strain and stress tensors, and represents the elastic compliance tensor $\mathbf{C} = c_{ij}$. In general, the components of this tensor are defined by 21 elastic constants: the Young's moduli E_{ii} referenced to material axes x_i; the shear moduli G_{ij} along the planes $x_i x_j$; the Poisson's ratios ν_{ij}; the first-kind mutual influence coefficients $\eta_{ij,k}$; the second-kind mutual influence coefficients $\eta_{i,jk}$; and the Chentsov's coefficients $\chi_{ij,kl}$.

Particularly, when dealing with two-dimensional problems (analysis restricted to the plane x_1x_2), Eq. (1) can be simplified to:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\varepsilon_6 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{16} \\ c_{21} & c_{22} & c_{26} \\ c_{61} & c_{62} & c_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \tag{2}$$

where the following index transformation was adopted:

$$1 \leftrightarrow 11 \qquad 2 \leftrightarrow 22 \qquad 6 \leftrightarrow 12 \text{ or } 21 \tag{3}$$

For a plane stress state, the coefficients c_{ij} in Eq. (2) are determined from the elastic constants as presented in Eq. (1). On the other hand, if a plane strain problem is analysed, the components c_{ij} of the elastic compliance tensor in Eq. (2) must be changed to the following values:

$$c_{ij}^{*} = c_{ij} - \frac{c_{i3}c_{j3}}{c_{33}}, \qquad i,j = 1,2,6 \tag{4}$$

From the solution of the differential equation governing the linear-elastic plane problem with the Airy's stress function approach, Lekhnitskii (1963) defined the following characteristic equation:

$$c_{11}\mu^4 - 2c_{16}\mu^3 + \left(2c_{12} + c_{66}\right)\mu^2 - 2c_{26}\mu + c_{22} = 0 \tag{5}$$

As shown by Lekhnitskii (1963), the roots of Eq. (5) are complex or purely imaginary and they correspond to the material complex parameters.

For the fracture analyses performed in this paper, the coordinate system must be defined according to the crack tip orientation (Figure 1). Therefore, the stress and strain states as well as the compliance tensor must also be related to this local reference frame. For this new orientation, Eq. (2) can be rewritten in a compact form as:

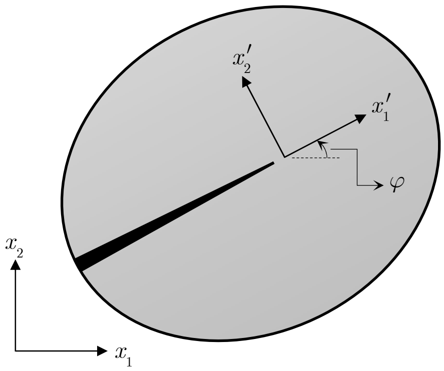

Figure 1. Rotation from the global to the local coordinate system.

$$\left\{\varepsilon'\right\} = \left[C'\right]\left\{\sigma'\right\} \tag{6}$$

where the tensors are now expressed in the local coordinate system (shown in Figure 1) and are obtained as follows:

$$\left\{\varepsilon'\right\} = \left[R\right]^{-T}\left\{\varepsilon\right\} \tag{7}$$

$$\left\{\sigma'\right\} = \left[R\right]\left\{\sigma\right\} \tag{8}$$

$$\left[C'\right] = \left[R\right]^{-T}\left[C\right]\left[R\right]^{-1} \tag{9}$$

in which $\left[R\right]$ is the rotation matrix given by:

$$\left[R\right]=\begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2-s^2 \end{bmatrix} \tag{10}$$

where $c=\cos\varphi$, $s=\sin\varphi$ and φ is the rotation angle.

Moreover, the complex material parameters can also be rewritten with reference to the new coordinate system by:

$$\mu_i'=\frac{\mu_i\cos\varphi-\sin\varphi}{\mu_i\sin\varphi+\cos\varphi} \tag{11}$$

3 DUAL BOUNDARY ELEMENT METHOD

In the DBEM formulation, two boundary integral equations (BIEs) are applied to give rise to a non-singular system of algebraic equations. The first, known as the displacement boundary integral equation (DBIE), allows the evaluation of the displacements at a particular boundary point s (source point) from the displacements and tractions at the points f (field points) along the boundary as follows:

$$C_{ij}(s)u_j(s)+C_{ij}(\overline{s})u_j(\overline{s})+\int_\Gamma P_{ij}^*(s,f)u_j(f)d\Gamma=\int_\Gamma U_{ij}^*(s,f)p_j(f)d\Gamma \tag{12}$$

where u_j and p_j represent the displacement and traction components, respectively. C_{ij} is the free term and is equal to $\delta_{ij}/2$ if s is at a smooth boundary, in which δ_{ij} is the Kronecker delta. $\overline{s}$ represents a potential point at the same position of s but belonging to another surface. This situation occurs, for example, for corresponding points at opposite crack surfaces. When s does not have any corresponding point, the second term in Eq. (12) is nil. Finally, U_{ij}^* and P_{ij}^* stand for the displacements and tractions fundamental solutions, respectively. For anisotropic domains and two-dimensional problems, Cruse and Swedlow (1971) obtained the fundamental solutions based on the Lekhnitskii (1963) formalism.

If only the DBIE is considered in the boundary element formulation to solve crack problems, a degenerate algebraic system of equation is obtained. This occurs since the same equation is generated for the corresponding points s and $\overline{s}$ at the crack surfaces. To overcome this deficiency, a second BIE, denoted as the traction boundary integral equation (TBIE), can be used. Assuming the point s positioned at a smooth boundary, the tractions at this point can be evaluated from the TBIE as follows:

$$\frac{1}{2}\left[p_j(s)-p_j(\overline{s})\right]+n_k(s)\int_\Gamma S_{ijk}^*(s,f)u_i(f)d\Gamma=n_k(s)\int_\Gamma D_{ijk}^*(s,f)p_i(f)d\Gamma \tag{13}$$

where n_k are the components of the outward normal versor. D_{ijk}^* and S_{ijk}^* are fundamental solutions resulting from the U_{ij}^* and P_{ij}^* derivatives and were also derived by Cruse and Swedlow (1971).

Equations (12) and (13) contain improper integrals since the fundamental solutions are singular as the distance between s and f approaches zero. Therefore, the integral kernels must be evaluated in the sense of Cauchy principal value (kernels containing P^*_{ij} and D^*_{ij}) or Hadamard finite part (kernel containing S^*_{ijk}).

The algebraic system of equations provided by the DBEM can be assembled with Eqs. (12) and (13) by applying the collocation method. In this process, the boundary Γ is subdivided into isoparametric elements, in which Lagrange polynomials are used for approximating both the geometry and the mechanical fields. The DBIE is used for collocation on nodes placed at the external boundary and at the upper crack surface, whereas the TBIE is used for collocation on nodes positioned at the lower crack surface. Thus, these nodes become the source point s of their respective boundary integral equation.

The existence of the Hadamard finite part in the TBIE requires the continuity of the displacement derivatives at the collocation points, which is guaranteed with discontinuous elements. In such elements, the collocation points do not coincide with the end nodes but are positioned inside the element. Therefore, this type of element is used along the crack surfaces. Moreover, such elements are used to ensure boundary smoothness at collocation points and to enforce boundary conditions discontinuity between adjacent elements.

The numerical integration of the discretized forms of Eqs. (12) and (13) is performed with the standard Gauss-Legendre quadrature when the integrated element is far from the source point. Otherwise, the Telles' third-degree polynomial transformation (Telles, 1987) is used for integrating quasi-singular elements. To integrate the singular elements, i.e. the element that contains the source point, the scheme based on the singularity-subtraction method presented by Cordeiro and Leonel (2016) is applied. A linear equation is obtained for each collocation point after the integrals involved are numerically evaluated. Then, the resulting system of equations is expressed in the matrix notation as follows:

$$\mathbf{Hu} = \mathbf{Gp} \tag{14}$$

where $\mathbf{H}$ and $\mathbf{G}$ are $2n \times 2n$ matrices containing the influence coefficients, $\mathbf{u}$ and $\mathbf{p}$ are $2n$ vectors with the displacements and tractions at the boundary, respectively, and n is the amount of collocation points into the boundary mesh.

A solution for the mechanical problem is obtained from Eq. (14) after the known boundary conditions are imposed. After the mechanical response at the boundary is evaluated, the displacements and stresses at an internal points can be determined from a post-processing stage by applying the following Somigliana identities:

$$u_i(s) = \int_\Gamma U^*_{ij}(s,f)p_j(f)d\Gamma - \int_\Gamma P^*_{ij}(s,f)u_j(f)d\Gamma \tag{15}$$

$$\sigma_{jk}(s) = \int_\Gamma D^*_{ijk}(s,f)p_i(f)d\Gamma - \int_\Gamma S^*_{ijk}(s,f)u_i(f)d\Gamma \tag{16}$$

4 LINEAR ELASTIC FRACTURE MECHANICS

Regarding anisotropic materials, Sih et al. (1965) developed the mechanical fields near the crack tip for the Griffith crack problems by means of the Lekhnitskii (1963) formalism. It was found that the stress components present a singularity of the order $\rho^{-0,5}$ and they are fully defined by the SIFs, just like the isotropic case. For plane problems, the stress components are given as follows:

$$\sigma_{11} = \frac{1}{\sqrt{2\pi\rho}}\left\{K_I \Re\left[\left(\frac{\mu_1\mu_2}{\mu_1-\mu_2}\right)\left(\frac{\mu_2}{H_2}-\frac{\mu_1}{H_1}\right)\right] + K_{II}\Re\left[\left(\frac{1}{\mu_1-\mu_2}\right)\left(\frac{\mu_2^2}{H_2}-\frac{\mu_1^2}{H_1}\right)\right]\right\} \tag{17}$$

$$\sigma_{22} = \frac{1}{\sqrt{2\pi\rho}}\left\{K_I \Re\left[\left(\frac{1}{\mu_1-\mu_2}\right)\left(\frac{\mu_1}{H_2}-\frac{\mu_2}{H_1}\right)\right] + K_{II}\Re\left[\left(\frac{1}{\mu_1-\mu_2}\right)\left(\frac{1}{H_2}-\frac{1}{H_1}\right)\right]\right\} \tag{18}$$

$$\sigma_{12} = \frac{1}{\sqrt{2\pi\rho}}\left\{K_I \Re\left[\left(\frac{\mu_1\mu_2}{\mu_1-\mu_2}\right)\left(\frac{1}{H_1}-\frac{1}{H_2}\right)\right] + K_{II}\Re\left[\left(\frac{1}{\mu_1-\mu_2}\right)\left(\frac{\mu_1}{H_2}-\frac{\mu_2}{H_1}\right)\right]\right\} \tag{19}$$

where K_I and K_{II} are, respectively, the mode I and mode II SIFs, μ_i are the material complex parameters with positive imaginary part, $\Re[\bullet]$ denotes the real part operator and:

$$H_i = \sqrt{\cos\theta + \mu_i \sin\theta} \tag{20}$$

The displacement components near the crack tip for anisotropic materials are also related to the SIFs and are given by the following expressions:

$$u_1 = \sqrt{\frac{2\rho}{\pi}}\left\{K_I \Re\left[\frac{\mu_1 p_2 H_2 - \mu_2 p_1 H_1}{\mu_1-\mu_2}\right] + K_{II}\Re\left[\frac{p_2 H_2 - p_1 H_1}{\mu_1-\mu_2}\right]\right\} \tag{21}$$

$$u_2 = \sqrt{\frac{2\rho}{\pi}}\left\{K_I \Re\left[\frac{\mu_1 q_2 H_2 - \mu_2 q_1 H_1}{\mu_1-\mu_2}\right] + K_{II}\Re\left[\frac{q_2 H_2 - q_1 H_1}{\mu_1-\mu_2}\right]\right\} \tag{22}$$

in which:

$$p_i = c_{11}\mu_i^2 + c_{12} - c_{16}\mu_i \tag{23}$$

$$q_i = c_{12}\mu_i + c_{22}\mu_i - c_{26} \tag{24}$$

As shown by the asymptotic expansions for the stress and displacements components, the SIFs represent the local behaviour of the elastic fields at the vicinity of the crack tip. However, for linear elastic materials, they can be related to the energy release rate of the body, which is a global parameter. For plane problems, the total energy release rate G is given by the superposition of the energy release rate of each mode of fracture, resulting in:

$$G = G_I + G_{II} \tag{25}$$

Sih et al. (1965) showed that for anisotropic materials the relations between the energy release rate for each mode of fracture and the SIFs are expressed as follows:

$$G_I = -\frac{c_{22}}{2}\Im\left[\frac{K_I^2(\mu_1+\mu_2)+K_I K_{II}}{\mu_1\mu_2}\right] \tag{26}$$

$$G_{II} = \frac{c_{11}}{2}\Im\left[K_{II}^2(\mu_1+\mu_2)+K_I K_{II}\mu_1\mu_2\right] \tag{27}$$

where $\Im[\bullet]$ denotes the imaginary part operator.

The substitution of Eq. (26) and Eq. (27) into Eq. (25) leads to:

$$G = \beta_{11}K_I^2 + \beta_{12}K_I K_{II} + \beta_{22}K_{II}^2 \tag{28}$$

in which:

$$\beta_{11} = -\frac{c_{22}}{2}\Im\left[\frac{\mu_1+\mu_2}{\mu_1\mu_2}\right] \tag{29}$$

$$\beta_{22} = \frac{c_{11}}{2}\Im\left[\mu_1+\mu_2\right] \tag{30}$$

$$\beta_{12} = -\frac{c_{22}}{2}\Im\left[\frac{1}{\mu_1\mu_2}\right] + \frac{c_{11}}{2}\Im\left[\mu_1\mu_2\right] \tag{31}$$

Rice (1968) showed that for linear-elastic materials the path-independent J-integral is equivalent to the energy release rate. This integral is evaluated along a path Γ_j enclosing the crack tip and is expressed by:

$$J = \int_{\Gamma_j}\left(Wn_1 - p_j u_{j,1}\right)d\Gamma \tag{32}$$

where W is the strain energy density given by $\sigma_{ij}u_{i,j}/2$, p_j are the tractions along the integration path given by $\sigma_{ij}n_i$, u_j are displacement components along Γ_j and n_i are the components of the outward normal versor to the path.

In the proposed model, the path Γ_j is assumed as circular and centred at the crack tip. Such path starts in the collocation point of the BEM mesh at one crack surface and finishes at the symmetric collocation point at the opposite surface as illustrated in Figure 2. The integration path must be entirely positioned inside the material and cannot cross any other crack tip. To satisfy such conditions, a simple automatic scheme is used to adjust the length of the path radius, which accounts for the intersection distances.

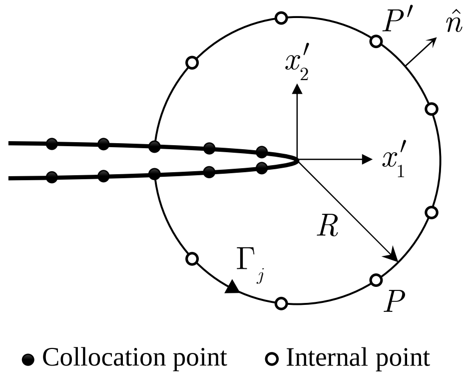

Figure 2. Path used for evaluating the J-integral.

To numerically evaluate the J-integral, a set of internal points is symmetrically positioned at the crack axis along Γ_j. These points define the elements that discretize the integration path. The displacements and stresses of the internal points are obtained into a post-processing phase by using the discretized forms of Eqs. (15) and (16), respectively. Then, the displacement vector and the state of stress for each internal point are rotated considering the crack tip local coordinate system.

Recalling the equality between J and G for linear-elastic materials, Eq. (28) and Eq. (32) can be used to evaluate the SIFs for pure-mode fracture problems, i.e., for problems in which one of the SIFs is nil. However, for mixed-mode problems, a mode decoupling strategy must be applied first. Here, the M-integral technique (Wang et al., 1980) is used to perform the mode decomposition. This approach is based on the definition of a conservative integral for two equilibrium states of a linear-elastic body. By defining a state (0) obtained from the superposition of two equilibrium states, denoted as (1) and (2), the following relations between the mechanical fields can be written:

$$\sigma_{ij}^{(0)} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \tag{33}$$

$$u_i^{(0)} = u_i^{(1)} + u_i^{(2)} \tag{34}$$

$$K_M^{(0)} = K_M^{(1)} + K_M^{(2)} \tag{35}$$

The substitution of Eq. (33) and Eq. (34) into the J-integral expression (Eq. (32)) written for the problem (0) leads to the following:

$$J^{(0)} = \int_{\Gamma_j} \left[\frac{\left(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}\right)\left(u_{i,j}^{(1)} + u_{i,j}^{(2)}\right)}{2} n_1 - \left(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}\right)\left(u_{i,1}^{(1)} + u_{i,1}^{(2)}\right) n_i \right] d\Gamma \tag{36}$$

After some algebraic manipulation, the following relation is obtained:

$$J^{(0)} = J^{(1)} + J^{(2)} + M^{(1,2)} \tag{37}$$

in which $J^{(k)}$ indicates the J-integral for the problem (k) and is given by:

$$J^{(k)} = \int_{\Gamma_j} \left(\frac{\sigma_{ij}^{(k)} u_{i,j}^{(k)}}{2} n_1 - \sigma_{ij}^{(k)} u_{j,1}^{(k)} n_i \right) d\Gamma \tag{38}$$

with no summation on k.

The term $M^{(1,2)}$ in Eq. (37) is defined as the M-integral, which represents an interaction integral between the two equilibrium states (1) and (2). It is given by:

$$M^{(1,2)} = \int_{\Gamma_j} \left[\frac{\sigma_{ij}^{(1)} u_{i,j}^{(2)} + \sigma_{ij}^{(2)} u_{i,j}^{(1)}}{2} n_1 - \left(\sigma_{ij}^{(1)} u_{i,1}^{(2)} + \sigma_{ij}^{(2)} u_{i,1}^{(1)} \right) n_i \right] d\Gamma \tag{39}$$

The J-integral of the state (0) can be related SIFs of states (1) and (2) by using Eq. (28) and Eq. (35) since $J = G$. The resulting expression is given by:

$$J^{(0)} = \beta_{11} \left(K_I^{(1)} + K_I^{(2)} \right)^2 + \beta_{12} \left(K_I^{(1)} + K_I^{(2)} \right) \left(K_{II}^{(1)} + K_{II}^{(2)} \right) + \beta_{22} \left(K_{II}^{(1)} + K_{II}^{(2)} \right)^2 \tag{40}$$

Equation (40) can be organized as follows:

$$J^{(0)} = J^{(1)} + J^{(2)} + 2\beta_{11} K_I^{(1)} K_I^{(2)} + \beta_{12} \left(K_I^{(1)} K_{II}^{(2)} + K_I^{(2)} K_{II}^{(1)} \right) + 2\beta_{22} K_{II}^{(1)} K_{II}^{(2)} \tag{41}$$

in which:

$$J^{(k)} = \beta_{11} \left(K_I^{(k)} \right)^2 + \beta_{12} \left(K_I^{(k)} \right) \left(K_{II}^{(k)} \right) + \beta_{22} \left(K_{II}^{(k)} \right)^2 \tag{42}$$

represents the J-K relation for the state (k).

By comparing Eq. (37) and Eq. (41), the M-integral can also be written in terms of the interaction between the SIFs of the states (1) and (2) as follows:

$$M^{(1,2)} = 2\beta_{11} K_I^{(1)} K_I^{(2)} + \beta_{12} \left(K_I^{(1)} K_{II}^{(2)} + K_I^{(2)} K_{II}^{(1)} \right) + 2\beta_{22} K_{II}^{(1)} K_{II}^{(2)} \tag{43}$$

Equation (39) together with Eq.(43) allow the determination of the SIFs values of a mixed-mode anisotropic fracture problem when the problems (1) and (2) are properly chosen. For this purpose, the state (1) is taken as the analysed problem, for which the values of K_I and K_{II} are desired. The state (2) is chosen as an auxiliary solution, with known mechanical fields.

The first auxiliary solution, denoted here by the superscript a, is taken as a cracked body subjected to a pure mode I loading. Therefore:

$$K_I^{(a)} = 1 \quad \text{and} \quad K_{II}^{(a)} = 0 \tag{44}$$

By combining Eq. (39) and Eq. (43) and by applying the conditions of Eq. (44), the following relation is obtained:

$$2\beta_{11} K_I + \beta_{12} K_{II} = \int_{\Gamma_j} \left[\frac{\sigma_{ij} u_{i,j}^{(a)} + \sigma_{ij}^{(a)} u_{i,j}}{2} n_1 - \left(\sigma_{ij} u_{i,1}^{(a)} + \sigma_{ij}^{(a)} u_{i,1} \right) n_i \right] d\Gamma \tag{45}$$

in which the fields without the superscript are related to the investigated problem and are obtained from the DBEM analysis. The components $\sigma_{ij}^{(a)}$ and $u_i^{(a)}$ correspond to the asymptotic stress and displacement fields determined, respectively, from Eqs. (17)-(19) and Eqs.(21)-(22) after the conditions of Eq. (44) are imposed.

Similarly, the second auxiliary solution, denoted here by the superscript b, is chosen as the problem of a cracked body subjected to a pure mode II loading. This case is represented by the following conditions:

$$K_I^{(b)} = 0 \quad \text{and} \quad K_{II}^{(b)} = 1 \tag{46}$$

For this situation, the combination of Eq. (39) and Eq. (43) after the conditions of Eq. (46) are imposed leads to the following:

$$\beta_{12} K_I + 2\beta_{22} K_{II} = \int_{\Gamma_j} \left[\frac{\sigma_{ij} u_{i,j}^{(b)} + \sigma_{ij}^{(b)} u_{i,j}}{2} n_1 - \left(\sigma_{ij} u_{i,1}^{(b)} + \sigma_{ij}^{(b)} u_{i,1} \right) n_i \right] d\Gamma \tag{47}$$

where $\sigma_{ij}^{(b)}$ and $u_i^{(b)}$ are the components of stress and displacement obtained, respectively, from Eqs. (17)-(19) and Eqs.(21)-(22) after the conditions of Eq. (47) are prescribed.

After the numerical integration of the right-hand sides of Eq. (45) and Eq. (47), which results, respectively, in $M^{(1,a)}$ and $M^{(1,b)}$, the following system of equations is obtained:

$$\begin{bmatrix} 2\beta_{11} & \beta_{12} \\ \beta_{12} & 2\beta_{22} \end{bmatrix} \begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \begin{Bmatrix} M^{(1,a)} \\ M^{(1,b)} \end{Bmatrix} \tag{48}$$

The solution of Eq. (48) gives the SIFs values for the investigated mixed-mode anisotropic fracture problem.

5 NUMERICAL EXAMPLES

5.1 Anisotropic plate with an edge crack

A graphite-epoxy plate containing an edge crack was analysed in this example (Figure 3). The height and the width of the structure were such that $h/w = 2$ and the crack length was $a = w$. The top of the plate was subjected to a uniform shear load τ, whereas the displacements of the bottom edge were fixed. The SIFs for the crack tip were evaluated with the M-integral approach for material orientations γ ranging from –90º to 90º in steps of 10º. To perform the numerical analysis, each crack surface was discretized into 14 discontinuous and quadratic elements, whereas 36 quadratic elements were used for the external boundary.

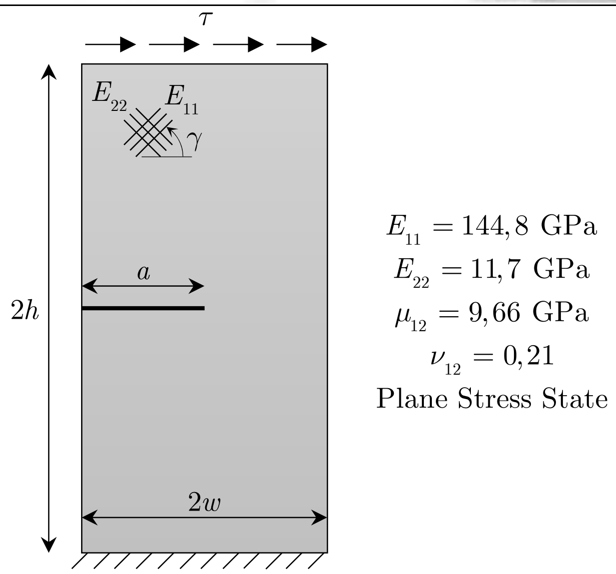

Figure 3. Graphite-epoxy plate containing an edge crack.

Figure 4 shows the variations of the modes I and II SIFs, normalized by $\tau\sqrt{\pi a}$, obtained by the proposed BEM model. A great influence of the material orientation γ on the SIFs was noted. Moreover, the responses provided by Chu and Hong (1990), Ghorashi et al. (2011) and Tan and Gao (1992) are also depicted in Figure 4. Good agreement between the results obtained here and the reference values was observed, with a better approximation to the values found by Tan and Gao (1992). This demonstrates the accuracy of the proposed model to extract the SIFs values in anisotropic domains for different material orientations.

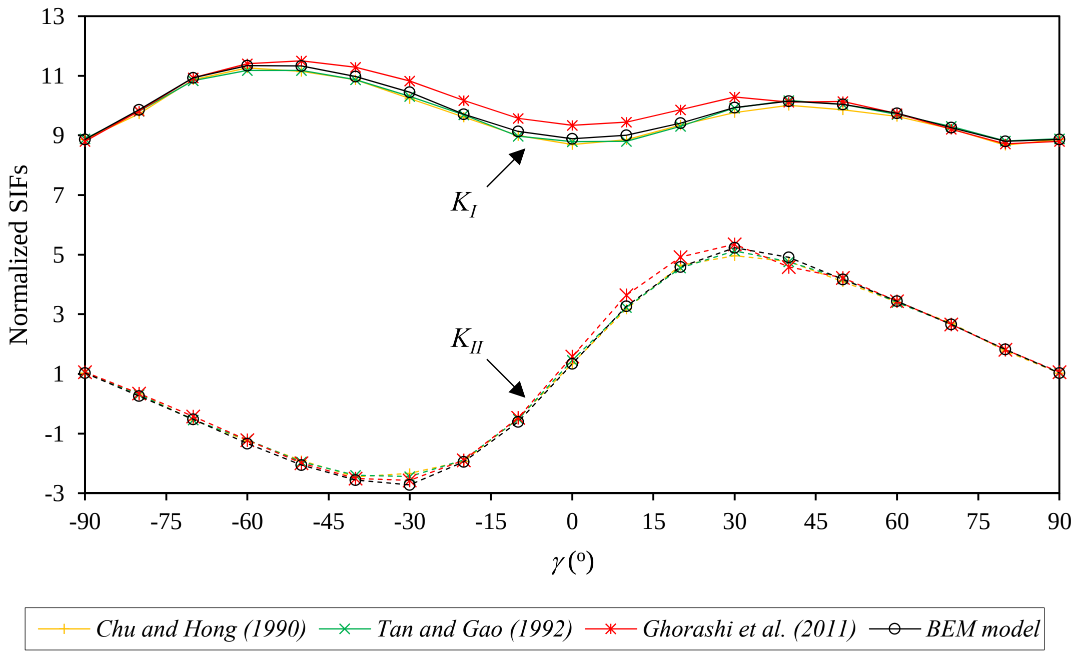

Figure 4. Normalized SIFs for the edge crack in an anisotropic plate.

5.2 Anisotropic plate with a slanted crack

Figure 5 shows a glass-epoxy plate containing a central crack and submitted to a uniform loading. The dimensions of the structure are such that $h/w = 2$ and the crack length is defined as $2a = 0,4w$. The slope of the crack is assumed constant and equal to $\theta = 45^o$. In this example, different values for the material orientation γ were considered and the SIFs values for the crack tips were evaluated for each configuration. To perform the numerical analysis with the DBEM, 36 quadratic elements were used to discretize the external boundary and 10 discontinuous quadratic elements were applied at each crack surface.

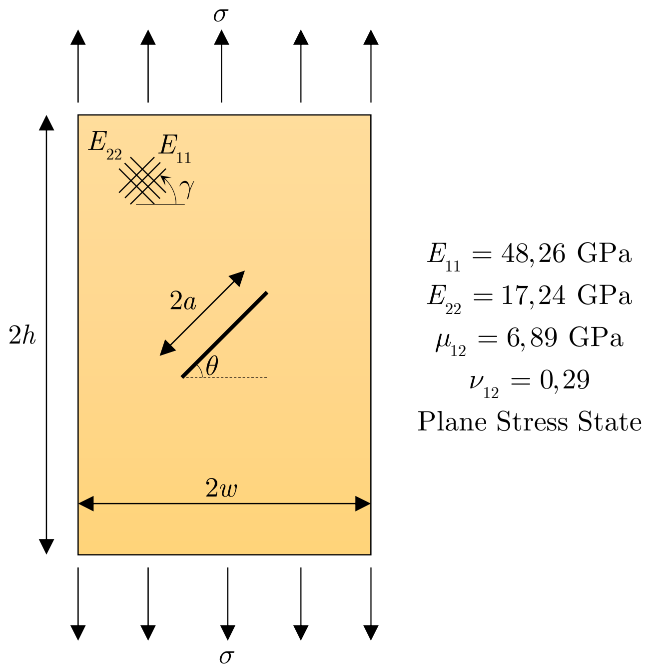

Figure 5. Glass-epoxy plate containing a slanted crack.

Figure 6 presents the computed values of K_I and K_{II}, normalized by $\sigma\sqrt{\pi a}$, for different material orientations. The amplitudes of variation of the SIFs values were not as high as the previous example since the difference between the elastic properties of the material principal directions were smaller. Figure 6 also presents the numerical solutions provided by Sollero and Aliabadi (1995), García et al. (2004) and Hattori et al. (2017). The behaviour of the SIFs values obtained by the proposed BEM model are in good agreement with the reference solutions, in particular with the solutions found in García et al. (2004). This shows that the BEM model can be successfully applied in the analysis of cracked anisotropic domains, including problems with slanted cracks.

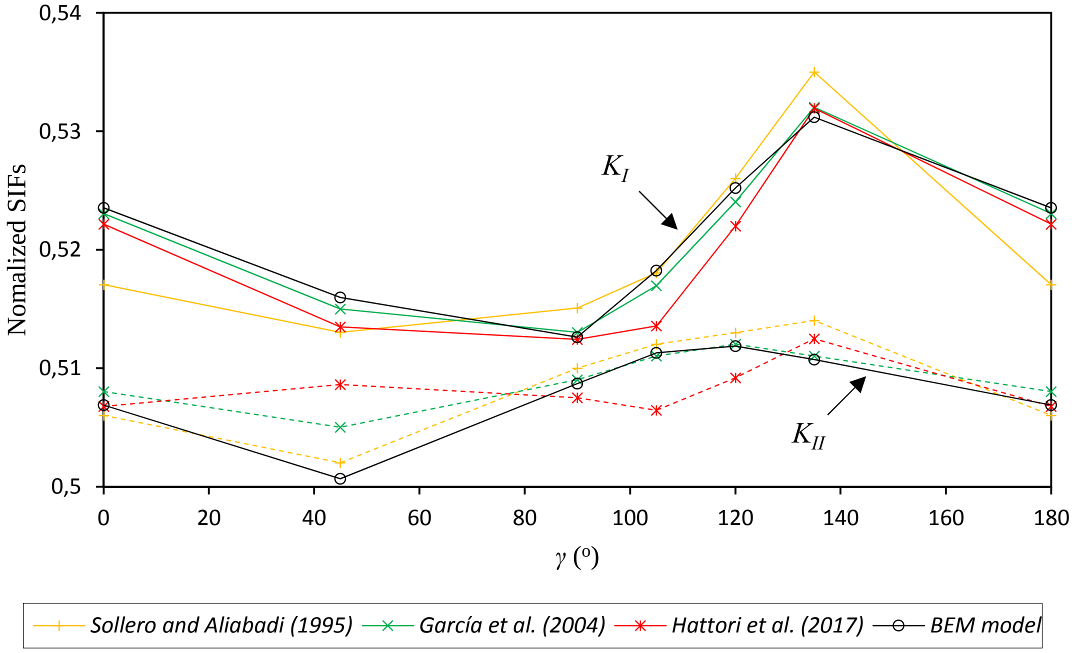

Figure 6. Normalized mode-I SIF for the slanted crack in an anisotropic plate.

6 CONCLUDING REMARKS

A DBEM model able to perform the mechanical analysis of two-dimensional cracked anisotropic domains was presented in this study. The LEFM fundamentals were considered and the M-integral technique was applied to extract the SIFs for mixed-mode problems. Two numerical examples were presented and the responses obtained by proposed model were in good agreement with those found in the literature.

The numerical formulation can be further extended to analyse the crack growth in anisotropic domains, and this is work in progress. To achieve this goal, a mode interaction theory must be used to verify the crack tip stability and to define the propagation angle.

ACKNOWLEDGEMENTS

Sponsorship of this research project by the São Paulo State Foundation for Research (FAPESP), project number 2016/23649-0, is greatly appreciated.

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