Calculating the Risk Premium and Extracting Information from Financial Instruments using the Theory of Expectations and a Prediction Model of the TSIR

Abstract

In this study, we used an alternative methodology to achieve two main objectives: calculate the 'risk premium' and 'extract information' from financial instruments (interest rate options). In the first case, we used a 'weak' version of the 'expectations hypothesis', according to which, quantitatively, this important quantity is obtained from the observed forward rate and the expected value of the spot rate, both from the public securities market, but now with a time-variant risk premium. In the second case, we sought to extract information from the options price. In both cases, we employed a dynamic temporal evolution model of the TSIR, which we developed for a previous work. By way of example, a preliminary empirical evaluation was also performed, which suggested the feasibility and effectiveness of our approach.

1. Introduction.

It is well established that the monetary authorities of countries with organized financial systems have to deal with a dilemma: they have to concern themselves primarily with the stability of the currency, but in doing so they cannot exaggerate in their disincentive to employment, growth and the increase in the yield of economic resources. To this end, real and monetary markets must be monitored efficiently.

If, on the one hand, the real market does not react immediately to the interventions of the monetary authority, the opposite occurs in the securities market, which quickly incorporates the expectations of economic agents, making it a convenient source of information that may enable the monetary authority to make inferences about the inflation expectations in both the short and long term, and also to obtain some information on the expected yield generated by economic resources.

In Brazil, market agents are currently consulted to obtain their inflation expectations and an official predictive model is also used, which provides the monetary authority's own expectations regarding inflation. And although this routine has its plausibility and effectiveness, we propose an alternative methodology in this work, which measures expectations solely and directly from the financial securities market.

The indicator used by the Central Bank (BACEN) for guiding its monetary policy is the IPCA. It allows them to infer whether the past inflation rate fell within the target range and it also serves to guide decisions so that the future inflation, as measured by this index, will converge to a value within the target range established for the market.

One of the challenges facing the BACEN, however, is how to deduct transitional turbulences in the price index. For it runs the risk of destabilizing the market by adopting an excessively reactive policy, since there is always a delay between the interventions in monetary policy and their effects on the market. In addition, by using the IPCA, the Central Bank is making a pre-judgment about the importance that each sector or product has on the evolution of prices. Among other effects in retrospective evaluations, this has the positive result of contributing to an evaluation of the impact of inflation on the loss of income of families. There are two drawbacks, however, when it is used prospectively to guide the predictions on price increases or declines through sectoral analyses of the price evolution of the products that make up the index. The first is the difficulty faced by those setting monetary policy to predict price pressures in local markets - in addition to the intensity of these pressures; and the second is the tendency to assign fixed weights to the contribution of each product or sector for the future effects of inflation.

In turn, the market inflation curve is a measure based on the expectations of the financial market (understood in broad terms), taking advantage of the perception of economic agents regarding potential market directions and inflationary pressures. In this case, the assessment of inflation expectations becomes independent of the definition of pre-defined inflation indicators.

In this study, we used an alternative methodology to achieve two main objectives: calculate the 'risk premium' and 'extract information' from financial instruments (options). In the first case, we used a 'weak' version of the 'expectations hypothesis', according to which, quantitatively, this important quantity is obtained from the observed forward rate and the expected value of the spot rate, both from the public securities market, but now with a risk premium that is variable in time. In the second case, we sought to extract information from the options price. In both cases, we employed a dynamic temporal evolution model of the TSIR, which was developed in a previous work [1] - the reading of which we strongly recommend, because its details will not be reproduced here.

The article is organized as follows. In the following section, the main concepts we employ in the subsequent sections will be laid out. Section 3 is devoted to 'risk premium' and section 4 deals with the extraction of information from financial instruments. A first empirical evaluation is presented in section 5 and section 6 summarizes the main conclusions.

2. Fundamental Concepts and Considerations.

Let us consider the simplest contract being traded in the fixed income markets: a zero-coupon bond (default-free) with maturity on date T which pays its holder 1 cash unit at T, where $P(t,T)$ denotes the price of this bond at time t , with $t \leq T$. Therefore, $P(T,T) = 1$, and the following relationship must be met:

$$P(t,T) = \exp\left(-\int_t^T f(t,s) \cdot ds \right), \quad \forall T \geq t, \tag{1}$$

for some locally integrable function $f(t,\cdot):[t,\infty)\to\Re$ (hereafter referred to as the **forward curve at time t**).

The number $f(t,T)$ is the **forward interest rate at time t** for a risk-free bond with a start date of $T \geq t$. In the particular case where $T = t$, $f(t,t)$ coincides with the short-term interest rate.

The logical evolution of this simple contract is a bonus with a series of intermediate coupons $C_1, C_2, \ldots, C_J$ that are paid in the future periods $m_1, m_2, \ldots, m_J$, respectively. The principal is included in the final coupon. The price of this bonus can be written as:

$$P = \sum_{j=1}^{J} C_j \exp(-m_j S_j) \qquad (2)$$

where S_j is the 'spot' rate of the corresponding coupon term (payment interval) m_j. By definition, the term structure of interest rates (TSIR) is obtained by executing S_j as a function of m_j.

- *The Expectations Hypothesis.*

Based on the concepts that we have just seen, two distinct approaches emerge as possible interpretations of the TSIR. The first and most widespread one is called the 'expectations hypothesis', according to which the long-term interest rate should be equal to the mean future expectation of short-term rates plus a maturity premium. When this premium (also known as risk premium) is constant, we speak of the 'strong' version of the theory. The other, less well-known concept is called the 'market segmentation hypothesis', according to which the agents, based on their personal characteristics, pursue maturity 'segments' - sometimes called '*habitats*' in market terminology - in which they can act more effectively.[1] Depending on the risk appetite of the agents, a 'migration' between the terms may also occur.

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[1] A typical example is the case of Pension Funds which, for obvious reasons, prefer securities with longer terms.

During the 1990s, several empirical studies based on regression statistics showed that the 'strong' version of the 'expectations hypothesis' was rejected in most cases with longer terms. What was specifically observed was that investors actually require varying risk premiums to the extent that the terms change. In Brazil, studies along these lines were done by Lima & Issler [7] and Tabak & Andrade [8], who diverge in their conclusions about the validity of the application of the expectations hypothesis to the Brazilian securities market, however. Instead of a purely statistical analysis, we use a dynamic model for the TSIR in this work with a time-dependent risk premium. This will be done in the next section.

3. Risk Premium in the Bond Market.

As we have seen, based on their definition, forward rates can be interpreted as averages future 'spot' rates. This is what has become known as the 'expectations hypothesis'. It is perfectly fine to weaken this hypothesis a little, rewriting it as:

$$E_t^m\left[S(t,T)\right] = f(0,T-t) - \varphi(t,T) \tag{3}$$

where $E_t^m\left[S(t,T)\right]$ is the interest rate expectation of the market (which may differ from the theory of expectations), $f(0,T-t)$ is the forward rate of maturity T- t observed at t = 0 and $\varphi(t,T)$ is the "risk premium", of paramount importance for many applications. From the previous equation,

$$\varphi(t,T) = f(0,T-t) - E_t^m\left[S(t,T)\right] \tag{4}$$

which shows us that the risk premium at any instant t (i.e., non-constant at maturity), ($t \geq 0$), can be assessed based on the forward rate and the expected value of the spot rate at the instant in question.

As we could see in the previous equations, we are dealing with "future" rate values, or put more succinctly, at different moments of t > 0. In other words, with the 'movement’ of the curves in time, or better yet, with the 'dynamic' of the rates. In the previously cited work [1], a unified model for the TSIR dynamics was developed in detail, which will not be reproduced here, where we will limit ourselves to presenting the main results below. The fundamental equation of the model is developed in:

$$\overline{S(t,T)} = \left(\frac{T}{T-t}\right)\cdot\overline{S(0,T)} + \left(-\frac{1}{T-t}\right)\cdot\left[I_1(t)+I_2(t)+I_3(t)+I_4(t,T)\right] \quad (5)$$

where,

- $\overline{S(t,T)}$ is the 'spot' rate of maturity T observed at time t = t.
- $\overline{S(0,T)} = \beta_o + \beta_1\cdot\left(\frac{1-\exp(-T/\tau_1)}{(T/\tau_1)}\right) + \beta_2\cdot\left(\frac{1-\exp(-T/\tau_1)}{(T/\tau_1)} - \exp(-T/\tau_1)\right) \quad (6)$

is the 'spot' rate of maturity T observed at time t = 0.

- $f(0,t) = \beta_o + \beta_1\cdot\exp(-t/\tau_1) + \beta_2\cdot\left(\frac{t}{\tau_1}\right)\cdot\exp(-t/\tau_1) \quad (7)$

is the 'forward' rate of maturity T observed at time t = 0

$$I_1(t) \equiv \int_0^t f(0,t)dt$$

$$I_2(t) \equiv \int_0^t \left(\frac{\sigma^2}{2\beta^2}\left(1-e^{-2\beta t}\right)^2\right)dt$$

$$I_3(t) \equiv \cdot\int_0^t \left(M + Ne^{-\beta\cdot t}\right)dt$$

$$I_4(t,T) = \int_0^t A(t,T) \cdot dt$$

$$A(t,T) = -\int_t^T \alpha(t,s)ds$$

with

$$\alpha(t,T) = \nu(t,T) \cdot \int_t^T \nu(t,s) \cdot ds$$

M and N are constants obtained from the following stochastic evolution of the short-term rate:

$$dr(t) = [\theta(t) - \beta \cdot r(t)] \cdot dt + \sigma \cdot dW(t)$$

- commonly known as the 'Hull-White model' [3] - in the following way:

- $M = -\dfrac{\sigma^2}{2\beta^2}$
- $N = \dfrac{\sigma^2}{2\beta^2}$

For an arbitrary, but fixed maturity $T \in \Re_+$, the temporal evolution of $f(t,T)$ according to the HJM model is given by:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T) \cdot ds + \int_0^t \nu(s,T) \cdot dW_s \quad , \quad t \in [0,T] \tag{8}$$

Or, in the differential version,

$$df(t,T) = \alpha(t,T) \cdot dt + \nu(t,T) \cdot dW, t \in [0,T] \tag{9}$$

The equation for the bond price is:

$$dp(t,T) = p(t,T) \cdot \left\{ r(t) + A(t,T) + \frac{1}{2} \left\| \Omega(t,T) \right\|^2 \right\} \cdot dt + p(t,T) \cdot \left\| \Omega(t,T) \right\| \cdot dW \tag{10}$$

with:

$$\Omega(t,T) = -\int_t^T \nu(t,s) ds$$

As shown in [1], (5) is the *fundamental equation for the 'spot' rate dynamics*, i.e., it is the equation that determines the term structure at any given moment, with the term structure and the volatility term structure given at the first moment. Of these, the first has already been discussed above. As for the second, surprisingly, there are simple functional forms for the volatility of the forward rate that ensure easy mathematical treatment. Considering the following structures for the volatility:

$$\nu(t,T) = b_o + \left(a_o^{'} + a_1^{'} \cdot (T-t) \right) \cdot h\left(r(t) \right) e^{-k \cdot (T-t)} \tag{11}$$

(b_o , $a_o^{'}$, $a_1^{'}$ e k are constants and *h(r(t))* is a function of the short-term rate), Bhar and others (2000) showed that a finite markovian representation for the term structure is possible if the coefficient of the exponentially amortized term is a finite-degree polynomial at maturity T.

More precisely, we could say that equation (5) is the expression of the average 'spot' rate reflecting the "market expectations", since it contemplates all the bonds and terms that are available through the 'Nelson-Siegel parametrization', according to which the expression for the 'spot' rate is:

$$S(m)_{\{\}} = \beta_o + \beta_1 \cdot \left(\frac{1 - \exp(-m/\tau_1)}{(m/\tau_1)} \right) + \beta_2 \cdot \left(\frac{1 - \exp(-m/\tau_1)}{(m/\tau_1)} - \exp(-m/\tau_1) \right) \tag{12}$$

From the definition of the forward rate, we immediately get,

$$f(m)_{\{\ \}} = \beta_o + \beta_1 \cdot \exp(-m/\tau_1) + \beta_2 \cdot \left(\frac{m}{\tau_1}\right) \cdot \exp(-m/\tau_1) \quad (13)$$

where $\{\ \} = \{\beta_o, \beta_1, \beta_2, \tau_1\}$ represents the set of parameters that determine the shape of the curves for r and f.

Based on this expression, we immediately get,

$$\varphi(t,T) = f(0,T-t) - \left\{ \left(\frac{T}{T-t}\right) \cdot \overline{S(0,T)} + \left(-\frac{1}{T-t}\right) \cdot \left[I_1(t) + I_2(t) + I_3(t) + I_4(t,T)\right] \right\} \quad (14)$$

Substituting the already known expressions in (11), we get the risk premium as result:

$$\begin{aligned} \varphi(t,T) &= \beta_o + \beta_1 \cdot \exp(-(T-t)/\tau_1) + \beta_2 \cdot \left(\frac{T-t}{\tau_1}\right) \cdot \exp(-(T-t)/\tau_1) - \\ &- \left\{ \left(\frac{T}{T-t}\right) \cdot \left[\beta_o + \beta_1 \cdot \left(\frac{1-\exp(-T/\tau_1)}{(T/\tau_1)}\right) + \beta_2 \cdot \left(\frac{1-\exp(-T/\tau_1)}{(T/\tau_1)} - \exp(-T/\tau_1)\right)\right] + \right. \quad (15) \\ &\left. + \left(-\frac{1}{T-t}\right) \cdot \left[I_1(t) + I_2(t) + I_3(t) + I_4(t,T)\right] \right\} \end{aligned}$$

The theory of expectations can easily be extended to other economic variables besides the interest rate.

As such, in general, equation(15) is the *risk premium of any future market index* (be it the interest rate, the exchange rate, the inflation rate, etc.) To put it more precisely, if the return of a given economic variable Ψ between the instants t and T (T>t) is expressed by:

$$\Re(t,T) = \left(\frac{1}{T-t}\right) \cdot \left(\ln \Psi(T) - \ln \Psi(t)\right) \quad (16)$$

Then, for example, if Ψ is the exchange rate, the *expected value* at time t of $\Re_{FX}(t,T)$ is given by:

$$E_t\left[\Re_{FX}(t,T)\right] = E_t\left[S(t,T) - S^*(t,T)\right] - \varphi_{FX}(t,T) \tag{17}$$

where $S(t,T)$ represents the nominal interest rate (spot) and $S^*(t,T)$ is the external nominal interest rate. $\varphi_{FX}(t,T)$ is here the *risk premium of the exchange rate.*

If now *IPCA(T)* and *IPCA(t)* are the consumer price index on the dates T and t, respectively. Then,

$$\Re_\pi(t,T) = \left(\frac{1}{T-t}\right)\cdot\left(\ln[IPCA(T)] - \ln[IPCA(t)]\right) \tag{18}$$

is the inflation rate between the dates t and T. Therefore, the expected inflation rate between the dates t and T is given by:

$$E_t\left[\Re_\pi(t,T)\right] = E_t\left[S(t,T) - r(t,T)\right] - \varphi_\pi(t,T) \tag{19}$$

where $S(t,T)$ and $r(t,T)$ denote the nominal and real interest rates, respectively. $\varphi_\pi(t,T)$ is now the *risk premium of the inflation rate.*

The expectation of inflation can be measured based on the difference between the profitability of public securities indexed to inflation and non-indexed securities. Just for example, in Brazil, assuming that the profitability of the non-indexed and pre-fixed LTN bond is around 12% p.a., depending on the maturity, but with little oscillation, while the NTN-B bond, indexed to the IPCA, yields around 7% p.a., also with little oscillation, depending on the maturity. The inflation expectation of the market, therefore, is 12% -7% = 5% p.a.[2] However, if the inflation expectation,

[2] In fact, 1.12/1.07 –1 = 4.7.

according to the ANBIMA, is 0.3% p. m., which would add up to approximately 3.7% p.a. In general, therefore, if one takes the ANBIMA estimate as very likely, the market is charging a risk premium of approximately 1.3% (p. yr).

So to extract the inflation expectation from the difference between the profitability between indexed and non-indexed securities, the risk premium must be deducted, otherwise the expectation will be skewed upwards. In fact, no matter how low these premiums are, the cumulative effect that occurs over the long term becomes significant.

As shown in the beginning of this work, the combination of the theory of expectations with the liquidity preference theory (which we will discuss below), together with the TSIR dynamics, provides a methodology to estimate the risk premium with some degree of plausibility.

As already discussed, according to the theory of expectations, the forward rate for a given period is equal to the expected value (in the statistical sense) of the spot rate for that period. In addition, the liquidity preference theory states that and seeks to explain why the forward rate is higher than the average future spot rate. To see this, let us first consider the role of financial institutions that have the ability to raise funds in the short term and lend them in the long term. On the one hand, investors prefer liquidity, on the other, borrowers prefer longer maturities. To balance supply and demand, therefore, the institutions charge a surcharge of the borrowers and use it, in part, to increase the amount of investors willing to part with resources and, in part, to cope with the risk of an increase in the interest rate.

The same phenomenon occurs in relation to the risk of inflation rate fluctuations, which is assumed by a long position in non-indexed securities, which makes the inflation between today and say, every month for a future period, less than the inflation in this period.

Following this description, we can consistently obtain the risk premium for indexed securities from the spot rate curves for indexed securities and the projected forward rate.

However, let's assume here that the risk premium of the difference between indexed and non-indexed securities coincides with the risk premium of indexed securities. This tends to be an overestimation of the risk premium of the difference because there is a positive risk premium related to non-indexed securities that should be discounted[3], but which, due to the difficulty of separating it from the inflation expectation itself, seems to be an unfeasible calculation.

A critique that can be made of our approach to developing the price curve is that, on the one hand, markets (such as the one in Brazil) are not completely efficient and, therefore, the changes in expectations would not be so quick to occur as assumed, and on the other hand, that inflation expectations are endogenous in the sense that the expectations of the market depend on the actions of the central bank and vice versa. However, it seems reasonable to assume that these factors tend to focus more sharply on the risk premium and that they will be eliminated, at least in part, when the risk premium is discounted.

4. Extracting Information from Financial Instruments (Interest rates options).

It is no exaggeration to say that interest rate derivatives are the most important assets in the global financial scenario. As could hardly be otherwise, the literature on this subject is vast, ranging from purely abstract approaches to the treatment of typical day-to-day questions of trading desks. In the paradigm of the HJM model [4], the theoretical price of any derivative of this category depends solely on the knowledge of the volatility structure of the forward rate.

[3] If Rt=rt+pt and Rn=rn+pn, where Rt is the profitability of the indexed securities, Rn of the non-indexed securities, pt and pn are the respective risk premiums and rt, rn are the discounted returns with the subtraction of the premiums, then Rn-Rt=rn-rt + (pn-pt).

For the sake of simplicity, let us consider the case of a call option on a bond. By definition, the price on date t of this contract on a bond that matures in T is:

$$Call(t) = Máx[P(t,T) - X, 0]$$

where,

$$P(t,T) = e^{-\int_t^T f(t,s)\cdot ds} \tag{20}$$

where $f(t,s)$ is given by (7).

In the light of the methodology we just described, the price of an interest rate derivative is therefore identical to the one obtained purely with the HJM methodology [4].

One of the main results of the application of HJM to the formation of derivative prices is revealed in the following proposition, which we will posit without proof[4]:

Proposition :

If the volatility structure is given by (see eq. (8)):

$$v(T) = b_o + \left(a_o + a_1 \cdot (T-t)\right) \cdot e^{-k\cdot(T-t)}$$

and the forward rate evolves according to (8) and (10), then *the price today of a Call with maturity t* over a bond with maturity T is :

4 See [5].

$$Call(0) = P(0,T) \cdot N(d_1) - X \cdot P(0,t) \cdot N(d_2) \tag{21}$$

where,

$$d_1 = \frac{\ln\left(\frac{K(t,T)}{X}\right) + \delta^2(t,T)}{\delta(t,T)}$$

$$d_2 = d_1 - \delta(t,T) \tag{22}$$

with

$$K(t,T) = \frac{P(0,T)}{P(0,t)} e^{-H_1(t,T)} \tag{23}$$

$$H_1(t,T) = \int_t^T h_1(t,s) \cdot ds$$

$$h_1(t,T) = \int_0^t \alpha(t,T) \cdot dt \tag{24}$$

$\delta^2(t,T)$ is the matrix

$$\delta^2(t,T) = \sum_{i=0}^{2}\sum_{j=0}^{2} D_i D_j w_{i,j} \tag{25}$$

whose elements are given by:

$$D_0(t,T) = b_o(T-t)$$

$$D_1(t,T) = \frac{1}{k^2}\left[a_1 + a_0 k - \left(a_1 k(T-t) + a_o k + a_1\right) \cdot e^{-k(T-t)}\right] \tag{26}$$

$$D_2(t,T) = \frac{a_1}{k}\left[1 - e^{-k(t-t)}\right]$$

and

$$
\begin{aligned}
w_{00} &= t \\
w_{11} &= \frac{1}{2k}\left[1-e^{-2kt}\right] \\
w_{22} &= \frac{1}{4k^2}\left[1-\left(1-2kt+2k^2t^2\right)\cdot e^{-2kt}\right] \\
w_{01} &= \frac{1}{k}\left[1-e^{-kt}\right] \\
w_{02} &= \frac{1}{k^2}\left[1-\left(1+kt\right)\cdot e^{-kt}\right] \\
w_{12} &= \frac{1}{4k^2}\left[1-\left(1+2kt\right)\cdot e^{-2kt}\right]
\end{aligned}
\tag{27}
$$

In terms of the bond price logarithm, $b=\ln(P(t,T))$, the Call price can also be written as:

$$
Call(X)=e^{-i(t,T)\cdot(T-t)}\int_{\ln(X)}^{\infty}\left(e^{b}-X\right)\cdot\tilde{h}(b) \tag{28}
$$

where $\tilde{h}(b)$ is the risk-neutral probability density function. Differentiating (28) with respect to the strike and rearranging the terms, we get a *risk-neutral distribution function*:

$$
\Pr(P\leq X)=1+e^{i(t,T)\cdot(T-t)}\frac{\partial Call(X)}{\partial X} \tag{29}
$$

Differentiating one more time, and changing the variable to b, we get the *risk-neutral probability density function:*

$$
\tilde{h}(b)=e^{[S(t,T)\cdot(T-t)+b]}\frac{\partial^2 Call(X)}{\partial X^2} \tag{30}
$$

Using equation (21) and performing the differentiations,

$$
\tilde{h}(b)=\frac{e^{[S(t,T)\cdot(T-t)+b]}}{\delta(t,T)}\cdot\frac{P(0,t)}{X}\cdot N'(d_2) \tag{31}
$$

with

$$N'(d_2) = \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2} \tag{32}$$

and d_2 given by (22).

For an option with maturity τ evaluated at time t, we get:

$$\tilde{h}(b) = \frac{e^{[S(t,\tau)\cdot(\tau-t)+b]}}{\delta(t,T)} \cdot \frac{P(t,\tau)}{X} \cdot N'(d_2) \tag{33}$$

which can be simplified as:

$$\tilde{h}(b) = \frac{e^{b}}{\delta(t,T)} \cdot \frac{1}{X} \cdot N'(d_2) \tag{34}$$

or,

$$\tilde{h}(b) = \frac{e^{b-\ln(X)}}{\delta(t,T)} \cdot N'(d_2) \tag{35}$$

Making the abbreviations:

$$\begin{aligned} y &\equiv \ln\left[\frac{P(t,T)}{X}\right] \\ \lambda &\equiv \ln\left[\frac{e^{-H_1(\tau,T)}}{P(t,\tau)}\right] \end{aligned} \tag{36}$$

(35) becomes:

$$\tilde{h}(b) = \frac{e^{y}}{\sqrt{2\pi}\cdot\delta(t,T)} \cdot e^{-\frac{1}{2}\cdot\frac{(y+\lambda)^2}{\delta^2(t,T)}} \tag{37}$$

Performing the square of this last expression, and after some algebraic manipulations, we arrive at:

$$\widetilde{h}(b) = \frac{(cte)}{\sqrt{2\pi}\cdot\delta(t,T)}\cdot e^{-\frac{1}{2\delta^2(t,T)}\cdot(y-\mu)^2} \tag{38}$$

with

$$\mu \equiv \delta^2(t,T) - \lambda \tag{39}$$

$$cte \equiv e^{-\frac{1}{2\delta^2(t,T)}\left[\lambda^2-\mu^2\right]} \tag{40}$$

Disregarding the multiplicative constant (cte), (38) is a gaussian function with mean μ and variance $\delta^2(t,T)\cdot$

The evaluation of expression (38) depends explicitly on the price of the security with maturity τ or T, in t = t. In the common applications, these prices are observed in the market, so at t = 0, and (30) is then evaluated numerically.

However, it would be very useful to obtain the distribution for any moment in the future. To this end, we need to have a price estimate of the security in the respective maturities at that moment. The definition of the average spot rate,

$$\left(\left[\ln p(t,T)\right]\right)_t = -\overline{S(t,T)}\cdot(T-t)$$

is the average price of the security with maturity T on date t. As a first approximation, it is therefore valid to assess λ the following way:

$$\lambda \equiv \ln\left[\frac{e^{-H_1(\tau,T)}}{P(t,\tau)}\right] \cong -H_1(\tau,T) - \overline{\ln(P(t,\tau))} \tag{41}$$

or,

$$\overline{\lambda} = -H_1(\tau,T) - \overline{S(t,\tau)} \cdot (\tau - t) \tag{42}$$

where $\overline{S(t,\tau)}$ is given by (5).

In this approximation, we can calculate *the risk-neutral probability distribution at any time t* by replacing λ by $\overline{\lambda}$ in (39) and finally using (38)[5].

5. Preliminary Empirical Evaluations.

4.2. Volatility Structure of the Forward Rate.

As we have seen, according to (11), the volatility structure depends on the functional specification $h(r(t))$. For the sake of simplicity, we will take the simplest of them, that is, a linear function:

$$h(r(t)) = \mu_1 r(t) + \mu_2 \qquad\qquad \mu_1 \ , \ \mu_2 \rightarrow ctes$$

With this specification, (11) can be written as:

$$v(0, T-t) = b_o + \left(a_o^{'} + a_1^{'} \cdot (T-t)\right) \cdot [\ \mu_1 r(t=0) + \mu_2\ ]\, e^{-k \cdot (T-t)}$$

In accordance with what was specified in [1], we get for t = 0 (moment of the volatility assessment),

[5] One example of the applicability of the method of equation (38) in national markets can be found in Tabak & Chang: "Extração de Informação de Opções Cambiais no Brasil" – Banco Central do Brasil- *Trabalhos para Discussão,* no. 104 (2006).

$$r(0) = \gamma(0) = f(0,0) = \beta_o + \beta_1$$

and, therefore,

$$v(0,T) = b_o + \left(a_o' + a_1' \cdot T\right) \cdot \left[\, \mu_1\left(\beta_o + \beta_1\right) + \mu_2 \,\right] e^{-k \cdot T}$$

With the data from Appendix 3 of [1] in hand, the volatility term structure of the forward rate (volatility rate with respect to the apex of the TSIR (***1,21,42,63,84,105,126,252 e 504***) was estimated as the standard deviation of the observed daily returns of the respective rates (in the period under analysis - Dec/1997 to May/2001). Subsequently, the parameters of the equation for $v(0,T)$ were obtained through an adjustment by ordinary least squares. The results are shown in table III [6].

- The dynamics of the 'spot' rate.

This section follows directly from the procedure and the results of the reference [1], which we faithfully reproduce here.

The practical use of equation (5) assumes the evaluation of at least 12 parameters: Four of the Nelson-Siegel parametrization $\{\beta_o, \beta_1, \beta_2, \tau_1\}$, two of the modeling of the short-term rate $\{\sigma, \beta\}$, and six of the version used in the HJM model $\{a_o', a_1', b_o, k, \mu_1, \mu_2\}$.

Assuming, for simplicity's sake, that *no new security has been issued in the period under analysis*, the following tables summarize the typical values of the parameters for each parametrization (determined on date t=0).

[6] In the other tables, these same parameters are merely plausible estimates of the actual values (which were not calculated in these cases).

Table I - NBC-E and NTN-D (time in years)

Parameter	Final value	Parameter	Final value
β_o (*)	0.1216	$a_o^{'}$	-0.00715
β_1 (*)	-0.0714	$a_1^{'}$	0.46271
β_2 (*)	-0.1843	b_o	0.01061
τ_1 (*)	0.3970	k	19.57
σ (**)	0.1465	μ_1	0.03733
β (**)	1.2955	μ_2	0.64910

(*) extracted from (**) extracted from [9].

Table I - TR Coupon (time in years)

Parameter	Final value	Parameter	Final value
β_o (*)	0.08946	$a_o^{'}$	-0.003575
β_1 (*)	0.00389	$a_1^{'}$	0.2313
β_2 (*)	0.01949	b_o	0.0054
τ_1 (*)	1.0129	k	27.75
σ (**)	0.1465	μ_1	0.03733
β (**)	1.2955	μ_2	0.64910

(*) extracted from [2] (**) extracted from [9].

Table I - Exchange Coupon (time in years)

Parameter	Final value	Parameter	Final value
β_o (*)	0.06882	$a_o^{'}$	-0.003575
β_1 (*)	-0.04635	$a_1^{'}$	0.2313
β_2 (*)	-0.04866	b_o	0.0054
τ_1 (*)	1.3965	k	27.75
σ (**)	0.1465	μ_1	0.03733
β (**)	1.2955	μ_2	0.64910

(*) extracted from [2 (**) extracted from [9].

With these parameters in hand, equation (5) was evaluated at some time intervals. The results are shown in the following figures, in which the time horizon reaches only the first maturity so as to consider all securities at the same time.

- Risk Premium.

Equation (15) gives the risk premium as a function of maturity and time. This behavior is illustrated in figures 3 and 4. A brief comment is needed here. Froot [6], analyzing the securities market in the United States, found sufficient evidence that the risk premium is important for short duration securities (especially 3 months), becoming increasingly negligible for those of longer duration. Equation (15) shows a very similar behavior, as can be easily seen by analyzing Fig.4.

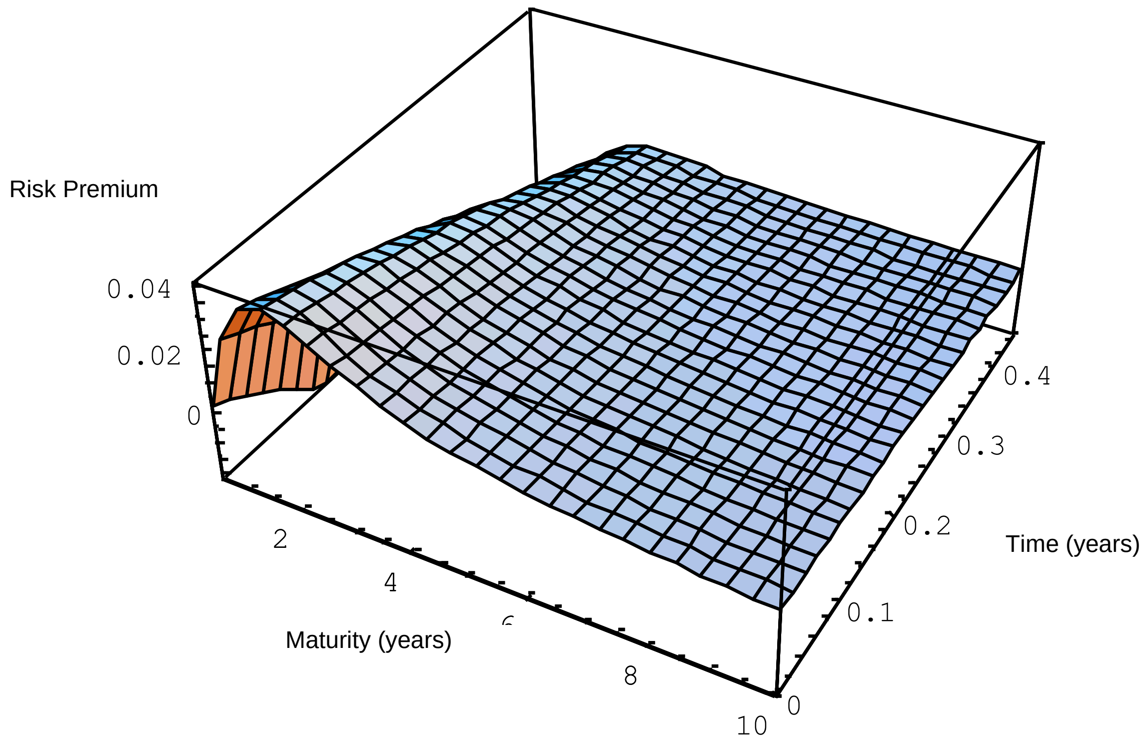

Fig 3 - Risk premium (equation (15)) with the parameters of Table I

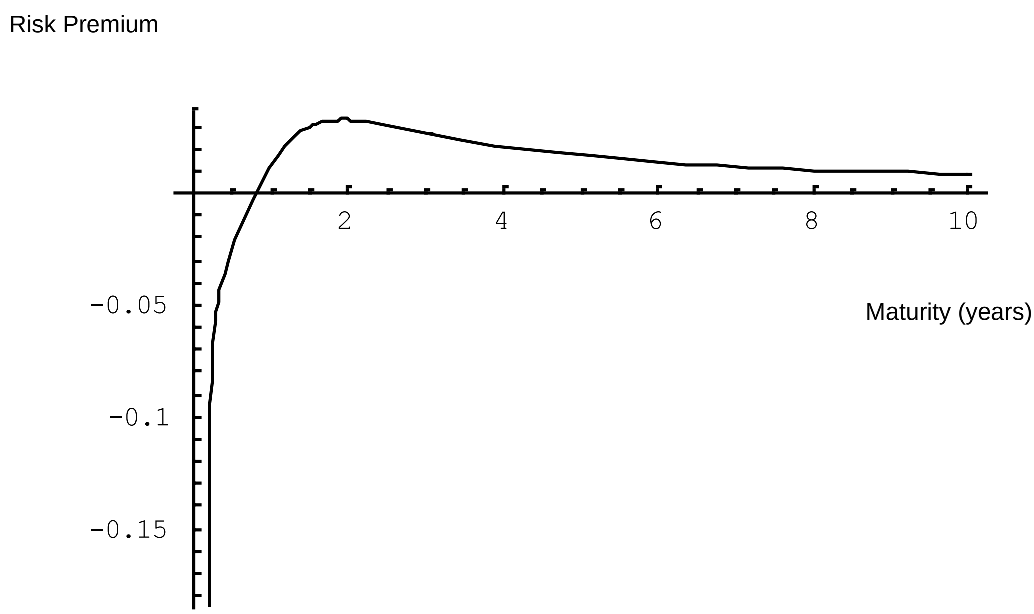

Fig. 4 - Risk premium as a function of maturity with the parameters of Table I for t = 0.15 years

6. Conclusions and Comments.

The risk premium is without a doubt a much less known quantity than the term structure itself. In general, there seems to be consensus today that it is not as constant as previously expected, especially for an economy like the Brazilian one. In the words of Lima and Issler [7] "in an economy so prone to shocks as the Brazilian one, the models should perhaps include risk premiums that vary in time, perhaps based on structural CCAPM models (...)". In fact, the result we obtained here confirms the existence of a varying risk premium, especially for short maturities.

With respect to the evaluation of expectations we carried out here (equation (30) and subsequent equations), they represent - by definition - the expectations of the market, which, as noted, *are not necessarily rational in the strict sense*, that is, they may contain biases in the distribution of the probabilities that may not be justified *a priori*. Market expectations as an aggregate of promise are relevant to the market and for Petropoulos et. Al [10], "The combining, or aggregation, of forecasts, which is not a new idea, has received increased attention in the forecasting community recently and has been shown to perform well (...)". However, in line with Cassettari, and Chiappin [1], it is unlikely that a single theoretical framework can encompass, with sufficient effectiveness, all situations encountered in the day-to-day of the markets. Despite this, it can be useful in the daily life of market participants.

In spite of the deficiency of the data available in the Brazilian market, the use of equation (5) would be a very effective tool to the extent that it gives the Central Bank the projection of the interest rate curves for any time horizon (within a certain tolerance); in principle, these curves reflect the expectations of the market participants (rational or otherwise) in a fairly straightforward manner. And a monetary policy that is based on the assumption that the financial markets are always rational is doomed to failure in many situations, especially over very short or very long periods. This is an argument against the purely mechanical reactions of the Central Bank and in favor of a monetary policy that has also been designed to be effective in situations of "irrationality". It is our opinion that the methods suggested here serve this purpose.

7. References.

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