The Efficiency vs. Pricing Accuracy Trade-Off in GMM Estimation of Multifactor Linear Asset Pricing Models

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ABSTRACT

Even though a multifactor linear asset pricing model can be equivalently represented in a Beta or in a stochastic discount factor (SDF) form, its inferential efficiency and pricing accuracy features may differ when estimated by the generalized method of moments (GMM), both in small and in large samples. Using a multifactor linear asset pricing model, we use bootstrapped simulations and analytical approximations to compare and test the estimated variances of the GMM estimators of parameters under the equivalent Beta vs. the SDF representations. We find that the SDF approach is likely to be less efficient but to yield more accurate pricing than the Beta method. We show that the main drivers of this trade-off are the higher-order moments of the factors that play an important role in the estimation process, and that the increased efficiency of the out-of-sample Beta risk premia estimation dominates the SDF increased pricing accuracy. The increased efficiency yielded by the Beta representation risk premia estimation in small samples translates into an increase of the out-of-sample Sharpe ratio in a trading exercise.

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I. Introduction

Any asset pricing model can be formally characterized either under a Beta or a stochastic discount factor (SDF) representation.[1] Even though the two characterizations are theoretically equivalent, the parameters of interest carry a different meaning under the two setups. In particular, the Beta representation is formulated to analyze the factor risk premia, δ, and as a residual, the Jensen's alpha, α (that can be interpreted as a measure of mean abnormal returns when the factors are tradeable). In contrast, the SDF representation is intended to analyze the parameters that enter into the assumed stochastic discount factor, henceforth called λ, and the resulting pricing errors, π. As a matter of fact, only when the factors are standardized to have zero mean and a unit variance, and to be mutually uncorrelated, the parameters of interest will coincide, i.e., $\delta = \lambda$, and $\alpha = \pi$; however, these conditions are rather primitive and hardly ever observed in practice.[2] The fact that the two representations are equivalent implies that there is a one-to-one mapping between δ and λ, and between α and π, which may facilitate the comparison of the estimators, although this theoretical equivalence does not necessarily entail an empirical or numerical equivalence.[3] Therefore, the experimental questions that naturally arise are: (i) *Is it better to produce inferences on δ or on λ?* and, analogously, (ii) *Is it better to make inferences on α or on π?*; finally, given that there are precise links between δ and λ, and α and π, one naturally wonders whether (iii) *Is it better to perform estimation of the Beta representation (i.e., recover δ and α), or of the SDF representation (i.e., of λ and π?).*

Our analytical and simulation results show that, in general, the Beta representation of a linear multifactor model is more **efficient** but less **accurate in pricing** than the SDF one. That is, the estimators of δ and π have lower simulated, asymptotic relative standard errors vs. the corresponding estimators for λ and α. The main objective of our paper is therefore to document how the choice between the Beta and the SDF methods to implement a multifactor pricing model can be addressed in the light of this trade-off between estimation vs. pricing accuracy. We provide empirically motivated evidence about what drives this trade-off, which is valuable to researchers and practitioners because it provides them with an *a priori* idea about the benefits and costs of adopting either representation in empirical work.[4]

[1] The SDF representation states that the value of any asset equals the expected value of the product of the (stream of) payoffs yielded by the asset and the SDF. In a Beta representation, the expected return on an asset is instead a linear function of its factor exposures (betas). The Beta approach is widespread in the finance literature and usually implemented through the two-stage cross-sectional regression methodology advocated by Black et al. (1972), Fama and MacBeth (1973), and Kan et al. (2013). The relatively more recent SDF characterization can be traced back to Dybvig and Ingersoll (1982), Hansen and Richard (1987), and Ingersoll (1987), who derive it for a number of theoretical asset pricing models formerly available only in the classical Beta framework. See Ferson and Jagannathan (1996) for a general discussion of the equivalence and differences between the two representations.

[2] Of course, factors may be built by the econometrician to satisfy these properties but these will be then derivative factors stemming from the observed ones and the necessary transformations are likely to severely impact their economic interpretation.

[3] Yet, such a theoretical equivalance is crucial to us because we shall exploit the one-to-one mapping between Beta and SDF estimators to transform the Beta estimators into SDF units. By doing so, we are able to perform a fair comparison of the simulated standard errors because even though the values do not coincide numerically, they will have the same units of measurement.

[4] For instance, researchers and practitioners employ linear pricing models to estimate the cost of capital associated with investment and takeover decisions, which is a recurring task in accounting and corporate finance. Also, asset pricing models are used in comparative analyses of the success of different investors or to implement the performance evaluation of investment funds' managers. Therefore the relative performance of the Beta vs. the SDF methods become

Our contribution to the literature in asset pricing and financial econometrics is threefold. First, we provide an analytical derivation of the asymptotic variance (efficiency) of the risk premia estimator underthe Beta vs. the SDF representations when the factors are many (i.e., for a linear multifactor model) and have higher-order (co-)moments that deviate from the Gaussian distribution, extending Chen and Kan (2004) multivariate normal results. Second, we generate empirical, bootstrapped simulations to test the statistical significance of the differences between the Beta and SDF representations. Third, we use an out-of-sample (OOS) experiment to measure the economic significance of the first two sets of results. In this exercise, we translate the trade-off between efficiency of the risk premia and pricing accuracy of both representations, Beta and SDF, into a Sharpe ratio performance measure.

In our first contribution, we produce two important extensions of Jagannathan and Wang (2002)'s analytical results: (i) to the case of multifactor asset pricing models, and (ii) to the case of non-Gaussian factors (Kan and Zhou, 2001 did consider this extension but just in the single-factor case). On the one hand, Kan and Zhou (2001) already provided a very reasonable intuition as to why an SDF representation may be ideal in a pricing perspective, but they do not develop formal inferential methods applicable to the risk premia estimates under the SDF method. The reason is that the SDF method does not place any restriction on λ, whereas a Beta representation explicitly incorporates the definition of δ as a subset of the moment restrictions. Therefore, it is reasonable to expect that the Beta method should be more efficient than the SDF method at estimating λ and, conversely, that the SDF method may be more accurate than the Beta approach at pricing the cross-section of test assets. On the other hand, in this paper we also find that the source of the higher estimation efficiency of the Beta method over the SDF method, is rooted in the higher-order moments that are likely to characterize the commonly used factors, as they adversely impact more the SDF than the Beta estimation. In particular, we prove analytically and document through appropriate simulations that negative skewness in the factors – which is usually an empirical feature displayed by the returns on the momentum portfolio – pose an obstacle to the accurate estimation of λ. This occurs even in the rather long samples typical of research with US data. Of course, the loss of precision in the inference on the risk premia caused by non-zero skewness grows as the samples become smaller. Moreover, such problematic estimation of λ turns into an advantage when it comes to estimating the pricing errors π, given that the SDF method is essentially based on the idea of minimizing the pricing errors.

In the asset pricing literature, it has now become common to compare the performance of different econometric procedures within either within the Beta framework or the SDF method. For example, Jagannathan and Wang (1998) compare the asymptotic efficiency of the two-stage cross-sectional regression and of the Fama and MacBeth's (1973) procedure.[5] Shanken and Zhou (2007), analyze the finite sample properties and empirical performance of the maximum likelihood estimator (MLE) applied to the implementation of Fama–MacBeth's approach and of the GMM for Beta pricing representations. Other important examples are Amsler and Schmidt (1985), Velu and Zhou (1999), Farnsworth et al. (2002), Chen and Kan (2004), Kan and Robotti (2008), and Kan and Robotti (2009). However, only recently there have been attempts to evaluate the inferential

relevant because if an investigator's choice were to fall on the technique that delivers the most precise estimators, her calculations and hypothesis test results would be more reliable. For a related discussion, see Kan and Zhou (1999), Kan and Zhou (2001), Jagannathan and Wang (2002), and Lozano and Rubio (2011).

[5]See Kan et al. 2013 for recent methodological advances concerning Fama–MacBeth's approach.

performance in finite samples of the Beta vs. the SDF approaches. This is where our main interest lies.

Ang et al. (2020) provided asymptotic results on the efficiency – both for the MLE and the GMM – in tests of asset pricing models when using individual stocks vs. portfolios. They find that the process of forming portfolios "destroys information" by shrinking the Betas. However, their GMM analytical results on efficiency when estimating risk premia under the Beta method use the samples distribution. In our case, we use the moments information of the known factors and assume no sample structure for Betas (as in Jagannathan and Wang (2002)). Nevertheless, in our empirical simulations, we provide results based on the sampled parameters (including the sampled Beta).

In a first attempt to evaluate the finite sample efficiency of the Beta vs. the SDF approaches, using a standardized single-factor model, Kan and Zhou (1999) show that the SDF method may be less efficient than the Beta method. Jagannathan and Wang (2002), Cochrane (2001), and Cochrane (2005) have debated this conclusion in a non-standardized single-factor model but assuming joint normality for both the asset returns and the factors; they conclude that the SDF method is as efficient as the Beta method for estimating the risk premia. In addition, they find that standard specification tests are equally powerful in either of the two frameworks. Yet, Kan and Zhou (2001) have shown that, under more general distributional assumptions and considering non-standardized factors, the inference based on λ may be less reliable than the based on δ, especially in realistic situations where the factors are leptokurtic. Ferson (2005) has reported that when the two representations correctly exploit the same moments, they will deliver nearly identical results. The interest in the topic has recently attracted additional research. For example, Lozano and Rubio (2011) show evidence suggesting that the inference on δ and π is more reliable than the inference on their corresponding estimators of λ and α but fail to recognize the existence of a precise mapping between λ and δ as well as π and α so that the comparisons do not occur on comparable scales. On the other hand, Peñaranda and Sentana (2015) show that a particular GMM procedure leads to numerically identical Beta and SDF estimates.

The financial econometrics literature has provided numerous attempts to improve the performance of the GMM estimators in finite samples. From first-stage to second-stage estimation improvements (Cumby et al., 1983), to finite sample corrections (Windmeijer, 2005), to principal components (Doran and Schmidt, 2006). Our asymptotic results shed some light on the finite sample efficiency comparisons in the case the distribution of the variables in the model identification have certain properties (deviations from the Gaussian case): higher-order moments of the explanatory variables do matter in the GMM identification process, and their impact will depend on the identification method selected – in our case Beta vs. SDF representations.

Our paper provides a second contribution with an application in which the inference on risk premia and pricing errors under the Beta pricing and SDF representations is performed with reference to the single-factor CAPM of Sharpe (1964), Lintner (1965), and Mossin (1966), the three-factor of Fama and French (1993), the three-factor of Asness et al. (2013), and the four-factor of Carhart (1997) models in an application to US data. The application has an important role because it builds on our empirical results to offer two additional contributions that derive from differences in our finite sample approach with respect to previous, similar studies. The first is that we assume factors and returns are drawn from their marginal empirical distribution. The second is that we evaluate not only single, but multifactor linear asset pricing models. By doing so, we examine the performance of the estimation methods in the presence of highly non-normal distributions, as actually happens in realistic applications. Furthermore, we use a wide range of sets of test assets in order to address the tight factor structure problem described by Lewellen et al. (2009).

The debate on the relative, finite sample performance of the GMM as applied to alternative ways to write linear factor pricing models appears to be open. Given the findings from Jagannathan and Wang (2002) to Kan and Zhou (1999), common wisdom is that alternative ways to represent standard linear pricing models does not matter much in terms of inferential efficiency, see for example, Cochrane (2001), Smith and Wickens (2002), Cochrane (2005), Nieto and Rodríguez (2005), Vassalou et al. (2006), Wang and Zhang (2006), Balvers and Huang (2007), Jagannathan et al. (2008), Cai and Hong (2009), and Brandt and Chapman (2018). With this background, our driving motivation is that in the existing literature the comparisons between Beta and SDF representations are conducted under sensible but rather specific conditions that appear to be insufficient to differentiate the relative performance of GMM inferences, in particular concerning the number of factors (typically one, as in the CAPM) and their distributional departures from normality (typically, modest). Once we relax these conditions, we show that important differences between the strength and usefulness of the inferences on the two otherwise equivalent methods of representation emerge. We find evidence in this finite samples setting suggesting that the Beta method leads to more precise risk premia GMM estimators while the SDF method leads to better pricing error estimators, in terms of efficiency. We evaluate the magnitude of the resulting efficiency losses and illustrate what are the drivers of such differential performance. In this sense, our results are closer to those on Kan and Zhou (1999), Kan and Zhou (2001), and Lozano and Rubio (2011).

In a third contribution of our paper – for the empirical finance literature – we provide an economic significance of our results, by testing the out-of-sample (OOS) performance of a mean–variance portfolio. In the test, we use the Ferreira and Santa-Clara's (2011) approach but with updates on the expected mean and expected variances, by incorporating the inferential properties with a pricing filter approach derived from Peñaranda and Sentana (2012) and Peñaranda (2016), and using factor models in the estimation of the expected returns and covariances. Our filter approach relates to Kim et al.'s (2020) orthogonal firm-characteristics approach, as the factors carry firm characteristics that are used to improve the estimation of the factor models (in our case, the estimation of the mean–covariance parameters).

Our OOS trading exercise results show that in most of the cases, the increased efficiency of the risk premia estimation by the Beta method translates into a better estimation of the expected mean of the portfolio, while the more precise inference of the SDF method translates into a better approximation of the expected covariance. However, by using the Sharpe ratio and the certainty equivalent of the portfolios as measures of performance, we show the increased precision in identifying the expected mean by the Beta method surpasses the increased precision of identifying the expected covariance by the SDF method.

The outline of the rest of the paper is as follows. Section II presents the methodology, Section III reports our analytical results, Section IV provides simulations and empirical results, Section V shows the economic significance with a portfolio exercise, while Section VI concludes.

II. A Review of the Outstanding Methodological Issues

In order to estimate and evaluate the Beta and SDF representations of a generic asset pricing model, we follow the GMM procedure by Hansen (1982). This guarantees that we can retrieve valid inferences even if the assumptions of independence, conditional homoskedasticity, and/or normality are not imposed, which seems to be rather realistic in practice.

A. *The Beta Representation*

Denote r_t a vector of N stock returns in excess of the risk-free rate and f_t a vector of K economy-wide pervasive risk factors observed at time t. The mean and the covariance matrix of the factors are denoted by $E[f_t] = \mu$ and $\text{Cov}(f_t) = \mathbf{\Sigma}$. Under the Beta representation, a standard linear pricing model can be written as

$$\mathrm{E}[r_t] = \mathbf{B}\delta, \tag{1}$$

where δ is the vector of factor risk premia, and $\mathbf{B}$ is the matrix of $N \times K$ factor loadings which measure the sensitivity of asset returns to the factors, defined as

$$\mathbf{B} \equiv \mathrm{E}[r_t(f_t - \mu)']\mathbf{\Sigma}^{-1}. \tag{2}$$

Equivalently, we can identify $\mathbf{B}$ as a matrix of parameters in the time-series regression

$$r_t = \phi + \mathbf{B} f_t + \epsilon_t, \tag{3}$$

where the residual ϵ_t has zero mean and covariance $\mathbf{\Omega}$, and it is uncorrelated with the factors f_t. We consider the general case where the factors may have higher-order moments that deviate from the Gaussian case. We define $E[f_t \otimes (f_t f_t')] = m_3$ as the third-order uncentered co-moment tensor of f_t (related to co-skewness, defined as $E[(f_t - \mu) \otimes ((f_t - \mu)(f_t - \mu)')] = \kappa_3$), and $E[(f_t f_t') \otimes (f_t f_t')] = m_4$ as the fourth-order uncentered co-moment tensor of f_t (related to co-kurtosis, defined as $E[(f_t - \mu)(f_t - \mu)' \otimes ((f_t - \mu)(f_t - \mu)')] = \kappa_4$).[6] The specification of the asset pricing model under the Beta representation in equation (1) imposes a number of restrictions on the time-series intercept, $\phi = (\delta - \mu)\mathbf{B}$. By substituting this restriction in the regression equation, we obtain:

$$r_t = \mathbf{B}\left(\delta - \mu + f_t\right) + \epsilon_t \quad \text{where:} \quad \begin{cases} \mathrm{E}\left[\epsilon_t\right] = 0_N \\ \mathrm{E}\left[\epsilon_t f_t'\right] = 0_{N\times K} \end{cases}. \tag{4}$$

Hence, the Beta representation in equation (1) gives rise to the factor model in equation (4). The associated set of moment conditions g implied by the factor model are:

$$\begin{array}{lcl} \mathrm{E}\left[r_t - \mathbf{B}(\delta - \mu + f_t)\right] & = & 0_N, \\ \mathrm{E}\left[\left[r_t - \mathbf{B}(\delta - \mu + f_t)\right] f_t'\right] & = & 0_{N\times K}, \\ \mathrm{E}\left[f_t - \mu\right] & = & 0_{K\times 1}, \\ \mathrm{E}\left[\left(f_t - \mu\right)\left(f_t - \mu\right)' - \mathbf{\Sigma}\right] & = & 0_{K\times K}, \end{array} \tag{5}$$

[6] Coskewness and co-kurtosis have been investigated in asset pricing studies such as Harvey and Siddique (2000), Dittmar (2002) and Guidolin and Timmermann (2008). A tensor is an N-dimensional array: co-skewness is then a 3-dimensional array while co-kurtosis is a 4-dimensional array.

and the corresponding unknown parameters are $\theta^* = [\delta^{*\prime}, \text{vec}(\mathbf{B}^*)', (\mu^*)', \text{vec}(\mathbf{\Sigma}^*)]'$, where the vec$(\cdot)$ operator 'vectorizes'the $\mathbf{B}^*_{N\times K}$ and the $\mathbf{\Sigma}^*$ matrices by stacking their columns. The observable variables are $x_t = [r_t', \; f_t']'$.

B. *The SDF Representation*

To derive the SDF representation from the Beta representation we follow Ferson and Jagannathan (1996), and Jagannathan and Wang (2002). First, we substitute the expression for $\mathbf{B}$ in equation (2) into equation (1) and rearrange the terms, to obtain

$$\text{E}[r_t] - \text{E}[r_t\delta'\mathbf{\Sigma}^{-1}f_t - r_t\delta'\mathbf{\Sigma}^{-1}\mu'] = \text{E}[r_t(1+\delta'\mathbf{\Sigma}^{-1}\mu - \delta'\mathbf{\Sigma}^{-1}f_t)] = 0_N.$$

The covariance matrix between r_t and the term $(1+\delta'\mathbf{\Sigma}^{-1}\mu - \delta'\mathbf{\Sigma}^{-1}f_t)$ is different from zero, then for equality in the expected value we can divide each side of the inner expression by $1+\delta'\mathbf{\Sigma}^{-1}\mu$, to obtain,[7]

$$\text{E}\left[r_t\left(1 - \frac{\delta'\mathbf{\Sigma}^{-1}}{1+\delta'\mathbf{\Sigma}^{-1}\mu}f_t\right)\right] = 0_N.$$

If we transform the vector of risk premia, δ, into a vector of new parameters λ as follows,

$$\lambda = \frac{\mathbf{\Sigma}^{-1}\delta}{1+\delta'\mathbf{\Sigma}^{-1}\mu}, \tag{6}$$

then we obtain the following SDF representation, which serves at the same time the as set of moment restrictions h used to estimate the linear asset-pricing model,

$$\text{E}[r_t(1-\lambda' f_t)] = 0_N, \tag{7}$$

where the random variable $m_t \equiv 1 - \lambda' f_t$ is the SDF defined as usual as $\text{E}[r_t m_t] = 0_N$.[8] In this case, the corresponding unknown parameter is $\widehat{\theta} = [\widehat{\lambda}]$, while $[r_t', \; f_t']'$ are observable.

C. *Comparison of the Representations*

There is a one-to-one mapping between the factor risk premia collected in δ and the SDF parameter vector λ, which facilitates the comparison of the two methods and that exploits the possibility to derive an indirect estimator of λ by the Beta method.[9] By the same token, we can derive an estimate of δ not only by the Beta method but also indirectly, by the SDF method. From the previous definition of λ in (6), we have:

$$\lambda = \delta'\left(\mathbf{\Sigma} + \delta\mu'\right)^{-1}, \qquad \text{or} \qquad \delta = \frac{\mathbf{\Sigma}\lambda}{1-\mu'\lambda}. \tag{8}$$

[7] It is common to assume $1+\delta'\mathbf{\Sigma}^{-1}\mu \neq 0$.

[8] Alternatively, we could derive the Beta representation from the SDF representation by expanding m and rearranging the terms, thus going in reverse compared to steps that have led us from the Beta representation to the SDF representation.

[9] We thank to Raymond Kan for kindly sharing complementary notes on Kan and Zhou (2001) that are at the roots of what follows.

In a similar way, by substituting (8) into π, we can find a one-to-one mapping between π and α, estimated from the Beta method.

$$\pi = \left(1 + \delta' \boldsymbol{\Sigma}^{-1} \mu\right)^{-1} \alpha, \qquad \text{or} \qquad \alpha = \left(1 + \delta' \boldsymbol{\Sigma}^{-1} \mu\right) \pi. \tag{9}$$

Yet, we cannot directly compare λ and δ, neither π and α because they are measured in different units. An alternative to allow direct comparisons is to transform δ into λ units, and α into π units following equations (8) and (9). For convenience, we will decorate all Beta estimators with '$*$' to easily emphasize that they are Beta estimators.

In a first formal attempt to compare both methods, Kan and Zhou (1999) assumed that the only factor had zero mean and unit variance, that is $\mu = 0$ and $\boldsymbol{\Sigma} = 1$. In this standardized, single-factor model, equations (8) and (9) simplify to $\lambda = \delta$ and $\pi = \alpha$. By assuming that the mean and the variance of the factor are predetermined without a need for estimating them, Kan and Zhou (1999) ignored the sampling error associated with the estimation of μ and $\boldsymbol{\Sigma}$ and concluded that the estimators obtained through the Beta method were more efficient. Jagannathan and Wang (2002) and Cochrane (2001) further investigated the effects of standardizing the factors, showing that in general, predetermining the factor moments reduces the sampling error in the estimates from the Beta method but not from the SDF method.[10] Predetermining the values of μ and $\boldsymbol{\Sigma}$ to be known constants, not necessarily $\mu = 0$ and $\boldsymbol{\Sigma} = 1$, gives an informational advantage to the Beta method in terms of efficiency. Predetermining without estimation implies ignoring the sampling errors associated with μ^* and $\boldsymbol{\Sigma}^*$: as a result, λ^* becomes considerably more efficient than when we solve the GMM problem with the Beta method. In our analysis to follow, we therefore consider the case where μ and $\boldsymbol{\Sigma}$ must be estimated.

To summarize, the Beta method gives the GMM estimate δ^* that is transformed into a corresponding λ^*, while the SDF method gives the GMM estimate $\widehat{\lambda}$.[11] But before solving the empirical identification problem through a set of simulation experiments, in the next Section we provide necessary analytical background to our key Monte Carlo results to follow.

III. Asymptotic, Analytical Results

In this Section, we generalize the results in Jagannathan and Wang (2002), in the sense that the vector of factors f_t is multivariate and is allowed to have a (joint) non-Gaussian distribution.[12]

[10] However, under the moment restrictions derived from the Beta representation, (23), we only can make inference on δ, not on λ. Yet to compare the methods using equation (8) requires an estimator of $\boldsymbol{\Sigma}$. One solution is to add an additional moment condition to (23) to estimate $\boldsymbol{\Sigma}$. An alternative is to estimate μ and $\boldsymbol{\Sigma}$ outside the GMM. In simulation results not reported in this paper, we find that the efficiency of both alternatives is the same. Hence, in what follows, we elect to estimate $\boldsymbol{\Sigma}$ outside the GMM.

[11] In this paper, we plan to we transform the estimate δ^* into an estimate of λ and then compare the variances of the sampling distribution of λ^* and $\widehat{\lambda}$. In the same way, we shall transform α^* into an estimate of π and then compare the efficiency of π^* and $\widehat{\pi}$. We also compare the distributions of the Hansen (1982) classical test of overidentification using the J-statistic of the transformed Beta J^* and $\widehat{J}$ from the SDF method. The null hypothesis is that all pricing errors are zero.

[12] Our results will be useful to understand that the small-sample simulation findings on the trade-off between estimation vs. pricing accuracy when using the Beta vs. the SDF representation in the next section is rooted in the asymptotic case; nevertheless, we have to highlight that maximum likelihood estimation is a more efficient method in such a case: our calculations are used to confirm that the simulations results differences between Beta vs. the SDF representation estimation do not disappear with the size of the sample.

THEOREM 1: ***Part A (factors on non-traded assets, risk premia):*** *Let f_t represent the multivariate, systematic risk factors with mean μ, covariance $\mathbf{\Sigma}$, third-order central moment κ_3, and fourth-order central moment κ_4, and consider the Beta representation in equation (1) with risk premia δ on the factors f_t, and the SDF representation in equation (7). Then, the asymptotic covariance matrix of the $\hat{\lambda}$ estimators obtained under the first-stage ($\mathbf{W} = \mathbf{I}$) uncentered GMM for the SDF case is,*

$$Acov(\hat{\lambda}) = \left(\left(\mathbf{\Sigma} + \mu\mu'\right)'\mathbf{B}'\right)^{-1}\left(\frac{1}{a_{\epsilon_t}}\mathbf{\Omega}^{-1} - \frac{1}{a_{\epsilon_t}^2}\mathbf{\Omega}^{-1}\mathbf{B}\left(\mathbf{A}_s^{-1} + \frac{1}{a_{\epsilon_t}}\mathbf{B}'\mathbf{\Omega}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{\Omega}^{-1}\right) \times \left(\mathbf{B}\left(\mathbf{\Sigma} + \mu\mu'\right)\right)^{-1}, \quad (10)$$

where $a_{\epsilon_t} = 1 - 2\lambda'\mu + (\lambda \odot \lambda)'\mathrm{diag}\left(\mathbf{\Sigma} + \mu\mu'\right) + \mathrm{triu\_vec}\left(\lambda\lambda'\right) + 2\left(\mathrm{triu\_vec}\left(\mathbf{\Sigma} + \mu\mu'\right)\right)$, *and* $\mathbf{A}_s$ *is,*

$$\mathbf{A}_s = m_4^{reduced} + m_3^{reduced} + m_2^{reduced} + m_1^{reduced} + (\delta - \mu)(\delta - \mu)',$$

where $m_2 = \mathbf{\Sigma} + \mu\mu'$ is the second-order uncentered moment of f_t ($E[f_t f_t']$), $m_{2(:,i)}$ the i-th column of m_2, $\odot$ is the element wise multiplication, $\mathrm{diag}(\cdot)$ is the operator that returns the diagonal of a matrix as a vector, $\mathrm{triu\_vec}(\cdot)$ is the operator that return the upper triangular matrix (without the diagonal) in a vector form, and $m_4^{reduced}, m_3^{reduced}, m_2^{reduced}$, and $m_1^{reduced}$ are matrices resulting from tensor operations (see Appendix A for details). The expression of the asymptotic covariance in equation (10) turns out to depend on the (co-)skewness and (co-)kurtosis (co-moments of higher order) coefficients of factors. The asymptotic covariance matrix of the λ^* estimators obtained under the first-stage uncentered GMM estimator for the Beta case is

$$Acov(\lambda^*) = \left(|\mathbf{\Sigma}| \times \left|\left(\mathbf{\Sigma} + \mu\delta'\right)^{-1}\right| \times \left(\mathbf{\Sigma} + \mu\delta'\right)^{-1}\right)^2 V_{1,1}, \quad (11)$$

with $V_{1,1}$ a matrix of dimension $K \times K$ with the asymptotic covariance of δ (see Appendix A for details). In the case of a single-factor, the asymptotic variance of the GMM risk premia estimate from the SDF representation is,

$$Avar(\hat{\lambda}) = \frac{\sigma^2(\sigma^2 + \delta^2)}{(\sigma^2 + \mu\delta)^4}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1} + \frac{\sigma^2(\sigma^4 + \delta^4)}{(\sigma^2 + \mu\delta)^4} + \frac{2\kappa_3(\delta^3 - \delta\sigma^2) + \delta^2\left(\kappa_4 - 3\sigma^4\right)}{(\sigma^2 + \mu\delta)^4}, \quad (12)$$

where σ^2 is the variance of the single-factor f_t, and κ_3, κ_4 are the skewness and kurtosis. The equivalent asymptotic covariance for the Beta representation is,

$$Avar(\lambda^*) = \frac{\sigma^2(\sigma^2 + \delta^2)}{(\sigma^2 + \mu\delta)^4}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1} + \frac{\sigma^6}{(\sigma^2 + \mu\delta)^4}, \quad (13)$$

when using a first-order Delta approximation, and,

$$Avar(\lambda^*) \;=\; \frac{\sigma^2(\sigma^2+\delta^2)}{(\sigma^2+\mu\delta)^4}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1} + \frac{\sigma^6}{(\sigma^2+\mu\delta)^4} - 2\frac{\left(\sigma^4\mu\right)}{(\sigma^2+\mu\delta)^5}\kappa_{3,\delta}, \tag{14}$$

$$Avar(\lambda^*) \;=\; \frac{\sigma^2(\sigma^2+\delta^2)}{(\sigma^2+\mu\delta)^4}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1} + \frac{\sigma^6}{(\sigma^2+\mu\delta)^4} - 2\frac{\left(\sigma^4\mu\right)}{(\sigma^2+\mu\delta)^5}\kappa_{3,\delta} + 2\frac{\sigma^4\mu^4}{(\sigma^2+\mu\delta)^6}\kappa_{4,\delta}. \tag{15}$$

when using a second- and a third-order Delta approximation, where $\kappa_{3,\delta}$ and $\kappa_{4,\delta}$ are the asymptotic third- and fourth-order central moments of the estimator of δ.

Part B (factors on non-traded assets, pricing errors): *The corresponding SDF and Beta pricing errors asymptotic variance estimators are,*

$$Acov\left(\widehat{\pi}\right) = S_s - D_s\left(D_s' S_s^{-1} D_s\right)D_s', \tag{16}$$

where $S_s = \mathbf{B}\mathbf{A}_s\mathbf{B}' + a_{\epsilon_t}\mathbf{\Omega}$, $D_s = -\mathbf{B}\left(\mathbf{\Sigma}+\mu\mu'\right)$, and,

$$Acov\left(\pi^*\right) \;=\; \left(1+\delta'\mathbf{\Sigma}\mu\right)^{-2} V_{1,1}^{\pi,stacked}, \tag{17}$$

where $Q = [I_n, \mathbf{0}_{n\times n}, \mathbf{B}, \boldsymbol{0}_{n\times 1}]$, and $V_{1,1}^{\pi,stacked}$ is a matrix of dimension $N\times N$ with the asymptotic covariance of the pricing errors of δ (see Appendix A for details).

Part C (factors on traded assets, $\mu=\delta$, risk premia): *In this case, the Beta representation parameters asymptotic covariance is,*

$$Avar(\lambda^*) \;=\; Avar(\lambda_b^*) = \left(\mathbf{\Sigma}+\mu\mu'\right)^{-1} S_b\left(\left(\mathbf{\Sigma}+\mu\mu'\right)^{-1}\right)'. \tag{18}$$

with $S_b = (\mathbf{\Sigma}+\mu\mu') + m_3^{b,\text{reduced}} + m_4^{b,\text{reduced}}$, where $m_3^{b,\text{reduced}}$ and $m_4^{b,\text{reduced}}$ and third- and fourth order tensor moment functions of the distribution of $g_b(r_t, f_t, \theta)$ (see Appendix A for details). In the single-factor case equation (18) reduces to,

$$Avar(\lambda^*) \;=\; \frac{\left((\sigma^2+\mu^2) - 2\lambda_b E[f_t^3] + \lambda^2 E[f_t^4]\right)}{(\sigma^2+\mu^2)^2}, \tag{19}$$

with $E[f_t^3] = \kappa_3 + 3\sigma^2\mu + \mu^3$ and $E[f_t^4] = \kappa_4 + 4\kappa_3\mu + 6\sigma^2\mu^2 + \mu^4$.

Proof. See Appendix A. □

By analyzing equation (3), and considering that the error ϵ_t will be assumed– at least in general – to display a standard normal distribution, we note that the source of non-normality of the returns will find its origins in the non-normality of the factors. However, even if we were in the more general case in which ϵ_t may be itself non-Gaussian and hence display non zero skewness and excess kurtosis, by Theorem 1, only the higher-order moments of f_t would be important for the efficiency properties of the Beta- vs. the SDF-based risk premia estimates.

COROLLARY 1: *Consider a single-factor linear asset pricing model (as in* (1) *but f_t univariate), where f_t returns are from a portfolio of non-traded assets, and consider these returns to have a Gaussian distribution. Then, when GMM is used for the estimation of the parameters (obtained under the uncentered first-stage ($\mathbf{W} = \mathbf{I}$)), the Beta representation has a higher efficiency than the SDF representation when measuring the risk premia, but the difference is negligible for asset pricing tests.*

Proof. In this case,

$$Avar(\hat{\lambda}) - Avar(\lambda^*) = \sigma^2\delta^4 / \left(\sigma^2 + \mu\delta\right)^4, \tag{20}$$

that is always positive. However, for the 'standard' moment values typical values of factors, this term is small. For example, consider single-factor models with Gaussian returns, where the mean and variance follow the values of the factors in Table I of the next section (market risk, size, value, and momentum factors). Then, the differences in equation (20), proportional to the SDF representation asymptotic variances, for the 'Gaussian' single-factor model loaded with the market risk, size, value, and momentum factors, are equal to 0.021, 0.0016, 0.0037, 0.0088 respectively (in percentage), calibrating the calculations with the mean and variance of the factors in Table I of the next section.[13]. These values are 'negligible' given they will go unnoticed in any test with significance of 1% or over (Even unnoticed for test of 0.1% significance or over for the size, value, and momentum factors). Even when considering the asymptotic case of Table II in Jagannathan and Wang (2002) (columns 1 and 2), the relative difference in equation (20) for the sample size 60 is about 0.0019 (correct value of the Beta asymptotic variance is 2.2020) that can not be captured in any test with significance of 1% or over. $\square$

COROLLARY 2: *Consider a single-factor linear asset pricing model (as in* (1) *but f_t univariate), where f_t returns are from a portfolio of non-traded assets, and consider these returns to have higher-order moments that deviate from a Gaussian distribution, but where the first-order Delta approximation is precise (error of the approximation is non-detectable in statistical tests). Consider the returns of the factors to have stylized facts such as daily returns (negative skewness and heavy tailed). Then, when GMM is used for the estimation of the parameters (obtained under the uncentered first-stage ($\mathbf{W} = \mathbf{I}$)), the Beta representation has a higher efficiency than the SDF representation when measuring the risk premia.*

Proof. Consider that the first-order Delta method provides an accurate approximation to the asymptotic variances of the estimators, i.e. the third-, fourth- and higher-order moments in the Taylor expansions for the Delta approximation are negligible.[14] In this case, the difference between

[13] The *'betas'* in this case are estimated with the time-series regression of the returns of a US 10 size-sorted portfolio on the factor, and the *'deltas'* with the cross-section regression of the *'betas'* with the average decile returns

[14] For instance, higher-order moments are much lower than the second-order moments: $|\kappa_{i,\delta} - \hat{\kappa}_{i,delta}| < 1e^{-10}\sigma^2$ for $i \geq 3$ with $\kappa_{i,\delta}$ the i-th central moment of the $\hat{\delta}$ distribution, and $\hat{\kappa_{i,\delta}}$ the i-th central moment of the $\hat{\delta}$ distribution in the Gaussian case.

the asymptotic variance of the SDF and the Beta methods can be expressed as,

$$Avar(\hat{\lambda}) - Avar(\lambda^*) = \frac{\sigma^2(\delta^4)}{(\sigma^2+\mu\delta)^4} + \frac{\mu^2\left(\kappa_4 - 2\kappa_3(\frac{\sigma^2-\delta^2}{\delta}) - 3\sigma^2\right)}{(\sigma^2+\mu\delta)^4}. \tag{21}$$

Consider the following conditions based on typical stylized facts for the higher-frequency returns on the market portfolio (see, e.g., Pagan, 1996; Christoffersen, 2012) : $\delta < \sigma$ (volatility is higher than expected returns), $3\sigma^2 < \kappa_4$ (heavy tailed returns), and $\kappa_3 < 0$ (negatively skewed returns), then the second term of right-hand side of equation (21) is positive and the result is yield. □

COROLLARY 3: *Consider a multifactor linear asset pricing model as in equation (3), where f_t returns are from a portfolio of non-traded assets, and consider these returns are Gaussian. Then, when GMM is used for the estimation of the parameters (obtained under the uncentered first-stage ($\mathbf{W} = \mathbf{I}$)), the Beta representation has equal efficiency than the SDF representation when measuring the risk premia.*

Proof. Considering Jagannathan and Wang (2002), and by observing equations (12), and (15), we can infer that the differences in the efficiency of the SDF and the Beta method are driven by deviations from the Gaussian distribution. When these deviations do not exist, the asymptotic variance of both methods can be approximated by,

$$\begin{aligned} Avar(\lambda^*) \approx Avar(\hat{\lambda}) = |\mathbf{\Sigma}| \times \left|\left(\mathbf{\Sigma}+\mu\delta'\right)^{-1}\right| \times \left(\mathbf{\Sigma}+\mu\delta'\right)^{-1}\Big(\mathbf{\Sigma}^{-1}\times \\ \left(\mathbf{\Sigma}+\delta\delta'\right)\left(\mathbf{B}'\mathbf{\Omega}^{-1}\mathbf{B}\right)^{-1} + \mathbf{\Sigma} + \mathbf{E}\Big), \end{aligned} \tag{22}$$

where $\mathbf{E} = \mathbf{\Sigma}\delta'\delta$ for the SDF method, and $\mathbf{E} = \mathbf{0}$ for the Beta method. The differences in the asymptotic variance, for 'Gaussian' multifactor Fama-French is $(0.0061, 0.0506, 0.0247)$ (in percentage – with the moments of the Gaussian distribution calibrated with Table I of the next section. In the next section (empirical simulations), we provide further intuition in this result, and in the Online Appendix we have a simulation case for $T = 5000$ with the multivariate Gaussian distribution where differences equal to 0 cannot be rejected. □

It must be acknowledged that the results from the Corollary 2 depend on the assumption that the Delta approximation is "very accurate".[15] Nevertheless, when we consider factors as portfolios of traded assets – as in this paper – it is possible to find a precise result independent of the Delta approximation.

A. Factors as Portfolios of Traded Assets

In the previous section we showed that the Beta representation has a lower efficiency that the SDF representation, but a lower pricing accuracy, for a general case such as in Jagannathan and Wang (2002) (factors of portfolios with non-traded assets). However, when the factor is the return

[15]In actual applications the systematic risk factors may possess distributions that are complex enough to interfere with the quality of such approximation. Even though the Corollary may not apply in an exact sense, the simulation experiments that follow show that in a qualitative sense, the implications of the Corollary hold in all (at least, most) interesting cases.

on a portfolio of traded assets, as in the single and multifactor models analyzed in this paper – the CAPM, Fama-French, Asness-Moskowitz-Pedersen, and the Carhart's factor models – it can be easily shown that the estimate of μ (the sample mean vector of the factors) is also the estimate of the risk premia δ. Therefore, given $\delta = \mu$, the moment conditions given in equation (5) simplify to:

$$\begin{array}{lcl} \mathrm{E}[r_t - \mathbf{B} f_t] & = & 0_N, \\ \mathrm{E}[(r_t - \mathbf{B} f_t) f_t'] & = & 0_{N\times K}, \\ \mathrm{E}[f_t - \mu] & = & 0_K, \\ \mathrm{E}\left[(f_t - \mu)(f_t - \mu)' - \mathbf{\Sigma}\right] & = & 0_{K\times K}. \end{array} \tag{23}$$

It is also possible to estimate the last two moments restriction of equation (23) outside the GMM framework by computing $\mu = E[f_t]$ and $\mathbf{\Sigma} = \mathrm{E}\left[(f_t - \mu)(f_t - \mu)'\right]$.[16] Hence, the efficiency of equation (5) when $\mu = \delta$ and equation (23) is not affected by imposing the additional $K + K^2$ moment restrictions in $\mathrm{E}[f_t - \mu] = 0_K$ and $\mathrm{E}\left[(f_t - \mu)(f_t - \mu)' - \mathbf{\Sigma}\right] = 0_{K\times K}$. Following this logic, we can drop the factor-mean and factor-covariance moment conditions without ignoring that they have to be estimated.

We define two vectors of unknown parameters, $\theta_1 = [\mathrm{vec}(\mathbf{B})']'$, and $\theta_1 = [\mu'\mathrm{vec}(\mathbf{\Sigma})']'$. The observable variables are $x_t = [r_t' \; f_t']'$. Then, the functions g_1, g_2 that capture the moment conditions required by the GMM can be written as:

$$g_{b1}(x_t, \theta) = \begin{pmatrix} r_t - \mathbf{B} f_t \\ \mathrm{vec}[(r_t - \mathbf{B} f_t) f_t'] \end{pmatrix}_{(N+NK)\times 1}, \tag{24}$$

and,

$$g_{b2}(x_t, \theta) = \begin{pmatrix} f_t - \mu \\ (f_t - \mu)(f_t - \mu)' - \mathbf{\Sigma} \end{pmatrix}_{(K+K^2)\times 1}, \tag{25}$$

being the system of equations (25) the one that has to be solved to estimate the efficiency of δ (or the efficiency of μ). We derive the asymptotic covariance of the Beta method in this case (The SDF method is unaffected).

COROLLARY 4: *Consider a single-linear asset pricing model (as in (3) but f_t is univariate), where f_t has higher-order moments that deviate from a Gaussian distribution, and its returns are represented by a portfolio of traded assets ($\mu = \delta$). Then, when GMM is used for the estimation of the parameters (obtained under the uncentered first-stage ($\mathbf{W} = \mathbf{I}$)), the Beta representation has a higher efficiency than the SDF representation*

Proof. We can rewrite the asymptotic variances to show that (see Appendix A),

$$Avar(\hat{\lambda}) - Avar(\lambda^*) = \frac{\sigma^2(\sigma^2 + \mu^2)}{(\sigma^2 + \mu^2)^4} \left(\beta' \mathbf{\Omega}^{-1} \beta\right)^{-1}, \tag{26}$$

[16]This is because the number of added moment restrictions in equation (23) compared with equation (5) when $\mu = \delta$ is the same as the number of added unknown parameters (We provide an analytical demonstration of this "efficiency equivalence" in the Online Appendix).

that is always positive. $\square$

In this section we have provided asymptotic results for asymptotic variance in the general non-Gaussian multifactor case, but, (i) in empirical applications the sample size is usually small (larger sample sizes will favour maximum likelihood estimation), and (ii) most of the conclusions were made on single-factor models, given that the interaction of higher-order moments (co-skewness and co-kurtosis) can add or subtract to the efficiency of the method in a complex manner. To address this issue, and provide more general results on the efficiency of the methods, following the literature (see Hansen et al., 1996) we provide numerical simulations to empirically test the efficiency of the methods for small samples.

IV. Simulation Experiments

We use bootstrap simulations to study whether the GMM estimators and test statistics carry any biases and their relative efficiency. In particular, we are interested in evaluating the standard deviations of λ^*, $\widehat{\lambda}$, π^*, $\widehat{\pi}$, denoted as $\sigma(\cdot)$ and also the thickness of the tails of the distribution of the J-statistic used to conduct specification tests.

A. Bootstrap Simulations

To artificially generate the excess returns we use the factor model, equation (4) where $t = 1, ..., T$. We develop an empirical simulation, where the returns are generated by bootstrapping the observed historical returns,[17] and factors are generated[18] by bootstrapping the observed historical factors:[19]

$$r_t = \mathbf{B}(\delta - \mu + f_t) + \epsilon_t, \qquad \epsilon_t | f_t \sim N(\mathbf{0}, \Omega), \qquad f_t \sim \mathbf{F}, \tag{27}$$

where $\mathbf{F}$ is the sample factor matrix observed $(T \times K)$. In our case, we focus in the case $\delta = \mu$. We provide additional results for factors on portfolios of non-traded assets in the Online Appendix, but we demonstrated that results in this case are similar to the traded case.

[17]The simulations represent a (nonparametric) bootstrap in the sense that we assume that the factors f_t are drawn from their empirical distribution which allows for non-normalities, autocorrelation, heteroskedasticity and dependence of factors and residuals.

[18]The simulations were executed using the THOR Grid computational cluster provided by the Department of Economics, Finance and Accounting at the Maynooth University, the THOR2 Grid computational cluster provided by the School of Business at the University College of Dublin, and the QUEST computational cluster provided by the Northwestern University.

[19]In addition, we tested two other sampling methods and results are provided in the Online Appendix: (i) the Monte Carlo approximate higher-order simulation method of Arismendi and Kimura (2016), that allows us to reduce the sampling error of the bootstrap, but that adds a bias, and (ii) independent simulations that eliminate the sampling error without bias, but that ignores the covariance matrix. We explain briefly the Arismendi and Kimura (2016) tensor exact moment simulations: Let f_t be the $K \times 1$ vector of known factors, and $\tilde{\mathbf{F}} = \left[\tilde{f}_1, \ldots, \tilde{f}_T\right]'$ a matrix of dimension $T \times K$ with T-sample observations of the K known factors, i.e., $\tilde{f}_i$, for $i = 1, \ldots, T$, are sample observations of f_t. Assume the first four non-centered moments of f_t are known: $m_1 = E\left[f_t\right], m_2 = E\left[f_t f_t'\right], m_3 = E\left[\otimes_E(f_t, f_t f_t')\right], m_4 = E\left[\otimes_E(f_t f_t', f_t f_t')\right]$, and denote $\tilde{M}(\tilde{f}_i, 1), \tilde{M}(\tilde{f}_i, 2), \tilde{M}(\tilde{f}_i, 3)$, and $\tilde{M}(\tilde{f}_i, 4)$ the corresponding sample moments of $\tilde{\mathbf{F}}$. The Arismendi and Kimura's (2016) method finds a random sample that minimizes the $\mathcal{L}^1$ norm of their differences, i.e. minimizes $\left|\tilde{M}(\tilde{f}_i^A, 1) - M(f_t, 1)\right|$, $\left|\tilde{M}(\tilde{f}_i^A, 2) - m_2\right|$, $\left|\tilde{M}(\tilde{f}_i^A, 3) - m_3\right|$, and $\left|\tilde{M}(\tilde{f}_i^A, 4) - m_4\right|$.

As far the overall sample size T is concerned, we consider four alternative time spans: 60, 360, 600, and 1000 months. As Shanken and Zhou (2007) argue, varying T is useful in order to understand the small-sample properties of the tests and the validity of any asymptotic approximations invoked. For instance, we examine a 5 year, 60-observation window because this may show how distorted any inferences may potentially be from taking a really short sample, whilst this is a commonly adopted horizon when using rolling window recursive estimation schemes. Instead, a 30-year window corresponds approximately to the sample sizes in Fama and French (1992, 1993) and Jagannathan and Wang (1996) while the 600-month long sample matches the largest sample sized examined by Jagannathan and Wang (2002). We also examine $T = 1000$ months since this approximates the current size of the largest sample available in Kenneth French's public data library [January 1927 to December 2018 – 1104 months as of the writing of this paper], and could be considered as an approximation to the true asymptotic variance. The estimators and specification tests are calculated based on the T-long samples of factors and returns generated from the model. We repeat such simulation experiments independently to obtain 10,000 draws of the estimators of λ, π (the pricing errors) and J (the over-identifying restriction statistic). Considering that we found in Section III that the difference of the asymptotic variance of the Beta method to be slightly small but 'negligible'[20] compared to the SDF method for the traditional statistical tests of overidentification, we estimate a ratio of the relative standard errors of the method,

$$\sigma_r(\hat{\lambda}) = \sigma(\hat{\lambda})/E(\hat{\lambda}), \qquad \text{and,} \qquad \sigma_r(\lambda^*) = \sigma(\lambda^*)/E(\lambda^*), \tag{28}$$

and with them we measure the four ratios, $\sigma_r(\hat{\lambda}_1^U)/\sigma_r(\lambda^*), \sigma_r(\hat{\lambda}_2^U)/\sigma_r(\lambda^*), \sigma_r(\hat{\lambda}_1^C)/\sigma_r(\lambda^*)$, and $\sigma_r(\hat{\lambda}_2^C)/\sigma_r(\lambda^*)$, where the U and C indicate estimators obtained by the GMM from the uncentered and centered SDF representations, and with 1 and 2 represent estimators obtained by the first and second-stage methods. With the bootstrap of the ratios, we test the null hypothesis that the interval of confidence (measured at p-values of 10%, 5% and 1%) is equal to 1, by rejecting the test when both extremes of the interval are superior to 1, or both extremes are inferior to 1.

We evaluate the GMM estimators efficiency with reference to standard multifactor models and in particular the Fama-French and Asness-Moskowitz-Pedersen three factor models, and Carhart four-factor model (in comparison to earlier studies (e.g., Kan and Zhou (1999, 2001), Jagannathan and Wang (2002) and Cochrane (2005)) that simply focused on the CAPM model to compare the efficiency of the Beta and SDF methods), which means that $K = 3$ and $K = 4$. We additionally consider the CAPM model to disentangle the effects of skewness and kurtosis in the efficiency. The factors as the excess market return (RMRF), size (SMB), value (HML) and momentum (UMD). To generate the excess returns from equation (4) we first need the $N \times K$ matrix $\mathbf{B}$, capturing the sensitivity of returns to the factor(s). The $\mathbf{B}$ matrix previously defined in equation (2), represents the slope coefficients in the OLS regressions of each N-test portfolio and K-factor model. We use $N = 10$ to generate $\mathbf{B}$, i.e., the value weighted returns of the 10 US size-sorted portfolios. Additionally, we use $N = 25$ and $N = 30$ when considering the 25 Fama-French size/value portfolios (the intersections of the 5 size and 5 book-to-market portfolios) and the 30 industry portfolios (Results for the 25 size/value and 30 industry sorted portfolios are presented in the Online Appendix). As Lewellen et al. (2009) suggest, the traditional tests portfolio used in empirical work such as the

[20]Defined in the previous section.

size and 25 size/value sorted portfolios frequently present a strong factor structure, hence it seems reasonable to adopt other criteria (industry) for sorting. The combination of three different values for K and three values for N give rise to nine $\mathbf{B}$ matrices, allowing us to add another criteria for evaluating the method's performance, in this case measured by efficiency.[21]

In Table I we report the descriptive statistics for the time series of factors and test portfolios. These values are used to calibrate the simulations of the two sets of simulations experiments. As can be noted in Table I for US data, the additional factors characterizing the multifactor models display rather different statistical properties vs. the classical, excess market return factor. In particular, with a sample kurtosis in excess of 30, the momentum factor is almost three times more leptokurtic than the excess market return (10.8). Moreover, while the market factor implies an unconditional distribution which is essentially symmetric (sample skewness is 0.2), the size and value portfolio returns are strongly right-skewed (1.9 and 2.2, respectively), while momentum returns exhibit massive left-skewness (-3.1). Thus, it is important to consider inferential and testing methods for multifactor asset pricing models that are able to reflect relative efficiency and pricing accuracy in the light of the empirical properties of the data, such as extreme asymmetries and excess kurtosis, when factors over and beyond market risk are considered. Studies such as Kan and Zhou (2017) have considered asset pricing models under a Student-t distribution, even though the magnitude of kurtosis is still limited for a t-distribution when asymptotics requires a finite fourth moment.[22] For instance, in unreported simulations, we have experimented with a Student-t distribution with five degrees of freedom and it yields a kurtosis of 6 for the RMRF factor, which is still much lower than the empirical value of 11. Therefore, in this paper we entertain the empirical distribution as the most realistic alternative to the classical, multivariate normal and resort to a bootstrap design.

[Place Table I about here]

We find that the choice of following either the Beta or the SDF method to empirically estimate and evaluate an asset pricing model can be framed in terms of a choice between efficiency and pricing accuracy. In particular, we show that frequently the Beta method dominates in terms of efficiency whereas the SDF method dominates in terms of pricing accuracy. While in a celebrated paper Jagannathan and Wang (2002) argue that the Beta and SDF approaches lead to parameter estimates with similar precision even in finite samples, in what follows we illustrate that their conclusions are only valid under rather specific conditions that cannot be uncritically generalized.

B. Bootstrap Simulations: Comparison of λ Estimators

Tables II and III compare the performance of Beta and SDF methods at estimating λ by the single-factor CAPM model using US data, and by computing ratios of relative standard errors such as $\sigma_r(\lambda) = \sigma(\lambda)/\mathrm{E}(\lambda)$. As one would expect, our results are qualitatively and quantitatively similar to those in Jagannathan and Wang (2002). In particular, Table II shows that the expected value and the standard error of λ^* and $\widehat{\lambda}$ are indeed similar. In fact, we cannot reject the null

[21] The covariance matrix $\mathrm{E}[\epsilon_t \epsilon_t' \mid f_t]$ in equation (4), is set equal to the sample covariance matrix of the residuals obtained in the N OLS regressions.

[22] The asymptotic distribution theory for the GMM requires that returns and factors have finite fourth moments. Hence, any marginal t-Student distribution for errors and factors must be characterized by more than four degrees of freedom, which limits considerably tail thickness

hypothesis that the standard errors of the Beta estimators are equal to the standard errors of the SDF estimators in most of the cases. Therefore, under a specific, single-factor framework there are virtually no differences in terms of efficiency between the Beta and SDF methods in this empirical assessment when using simple standard errors. One of the key implications of this result is that there are no significant advantages to applying the Beta method to nonlinear asset pricing models formerly expressed in SDF representation through linear approximations.

[Place Table II about here]

[Place Table III about here]

In order to conduct a clearer comparison of the standard errors of λ we present Table III.[23] Ratios close to one represent a high degree of similarity of the efficiency of both methodologies at estimating λ. Ratios in excess of one suggest that applying the Beta method provides more accurate estimators of risk premia (even in finite samples and net of scaling effects) vs. the SDF method.

Given that one is generally interested in testing the null hypothesis that the estimator is equal to zero, ratios such as those reported in Table III offer a good indication of which method may lead to the most accurate inferences. In particular, all ratios in Table III are slightly in excess of one and significant, which means that the Beta method leads to lower standard errors when estimating the CAPM market risk premia. In general, the values of the ratios decline as we increase the size of the sample while, as expected, second-stage SDF estimators of λ are more efficient than first-stage estimators. In addition, the uncentered SDF specification reveals a marginal advantage when compared to the centered specification, probably as a result of the additional moment restrictions that derive from centering.

To compare the empirical results of this section, with the analytical results in Section III, we provide an estimated asymptotic ratio of the relative standard errors. This asymptotic estimate is defined as *'asymptotic first-order approximation'* as the results from Section III depend on a first-order Taylor approximation of the asymptotic covariances of the GMM method (see Hansen (1982) appendix). We can observe in Table III that the ratios of the empirical simulations approaches the asymptotic first-order approximation.

Even though our set of simulation experiments offer a clear-cut perspective on efficiency matters, our original contribution regarding the comparison of $\sigma_r(\lambda)$ across representations of the asset pricing models focuses on multifactor asset pricing models. Tables IV, V, and VI report the ratios of relative standard errors for the risk premia estimators derived from the Beta and SDF methods in the case of the Fama-French, Asness-Moskowitz-Pedersen, and Carhart models respectively.[24]

[Place Table IV about here]

[Place Table V about here]

[23]Table II allows us to compare $\sigma(\lambda^*)$ versus $\sigma(\widehat{\lambda})$ instead of $\sigma(\widehat{\delta})$ versus $\sigma(\widehat{\lambda})$ in order to avoid misleading conclusions driven by a scaling issue. However, if we consider the possibility of intrinsic differences among the methods, the expected values of λ^* and $\widehat{\lambda}$ do not necessarily would be similar in general. For this reason it is convenient to compute ratios of relative standard errors such as $\sigma_r(\lambda) = \sigma(\lambda)/\mathrm{E}(\lambda)$. By doing so, we would have an accurate measure of the relative efficiency of the methods.

[24]The correspondent expected values and standard errors for the multifactor asset pricing models are in the Online Appendix.

[Place Table VI about here]

The first (upper) panel of Table IV asymptotic variance estimation is somehow comparable to the Table III because in both cases the estimated λ corresponds to the market factor – although in this case under a multifactor model. Thus, it is not surprising to find a similar pattern which reinforces the conclusion that there are small differences (about 4% of the ratio for the case of Beta vs. SDF second-stage methods) when estimating the parameter associated with the market factor. The uncentered and second-stage SDF methods are again more efficient than the centered and first-stage SDF methods at estimating λ.

Contrary to the market factor case, the standard error of Beta estimators linked to the size and value factors are statistically and significantly smaller than the corresponding standard errors of SDF estimators. This becomes evident in the higher ratios of second and third panels of Table IV (about 27% more efficient when comparing the first-order GMM asymptotic ratio difference for the size factor, and about 62% more efficient when comparing the case of the value factor). These results suggest that the empirical equivalence of both methods is subject to the loaded factor in the asset pricing model. In particular, the market factor does not represent a challenge to the SDF method whilst the value factor can lead to significant differences according to the $\sigma_r(\lambda)$ ratios. For instance, the relative standard error of the uncentered first-stage SDF method can be more than twice as big as the relative standard error of the Beta method. Beta estimators are even more efficient than second-stage SDF estimators, which by construction are intended to increase the estimation efficiency of $\widehat{\lambda}$.

The second multifactor asset pricing model is the Asness-Moskowitz-Pedersen, which factors are market, momentum and value. The estimation of the Asness-Moskowitz-Pedersen model allows us to compare the efficiency of the estimator associated with the market and value factors in previous tables and introduces the result for momentum.

The ratios of relative standard errors $\sigma_r(\lambda^*)/\sigma_r(\hat{\lambda})$ linked to the market factor are statistically significant larger than one. The second panel of Table V shows the ratios for the momentum factor, which are somewhat greater than the ratios for the other two factors in this model. On the other hand, the magnitudes for the value factor (third panel) are of a similar order of magnitude to the ones of the value factor of the Fama-French model in Table IV. We also find that the SDF method may have difficulties in small samples on the momentum factor risk premia which is reflected in ratio values of 25.2 and 12.98.

The third and last multifactor asset pricing model estimated is the Carhart model, that features as factors the market, size, value, and momentum. The estimation of Fama-French, Asness-Moskowitz-Pedersen, and Carhart models represent the core contribution to the field, which lead to the main original results. For now, we show the relative standard errors of Carhart model λ estimators in Table VI.

The results for Carhart's risk premia estimated on US data support the argument that the efficiency of the different methods/representations depends on the specification of the factors. The lower ratios of relative standard errors of the λ estimators are those related to the size factor, followed by market, value, and momentum are the highest for the momentum factor. In this case we note that the ratio of the size factor seems to converge empirically to 0, which means that the SDF method with the bootstrapping sampling might be 'underperforming' all the risk premia parameters. The asymptotic first-order approximation reveals that the ratio is equal to 1.46, which reveals the sampling method might be failing to be precise. For this reason we provide in the Online Appendix results with the Arismendi and Kimura (2016) tensor moment simulations where the ratio appears to clearly converge to the asymptotic estimation in a harmonic way (crossing the convergence point several times).

The size/composition of the portfolios might have an effect on the results. To check this effect we produce additional tests (see the Online Appendix) for $N = 25$ (The 5×5 size/value-sorted portfolios), and $N = 30$ (The 30 industry-sorted portfolio), extracted from the Kenneth French website for the same period of the 10 US size-sorted portfolios. We found that increasing the size of the portfolio reduce the effects of the higher-order moments of the factors in the inferential process when estimating risk premia by the Beta or the SDF method. Still, in most of the cases the inferential difference in regards to efficiency of the risk premia. In regards to the pricing errors accuracy, the difference is still maintained, but in some cases it changes the result depending on the combination of factors/type of portfolio. Recalling Jagannathan and Wang (2002), the pricing errors accuracy of the Beta method is superior. When the factors have higher-order moments that deviate from the Gaussian distribution, this relation is reduced and inverted. Our initial conclusion with this result on portfolios of larger size and different composition, is that the trade-off is reduced. In fact, where the pricing errors accuracy relation (SDF superior to Beta) is inverted, the risk premia efficiency is reduced. Nevertheless, in the majority of cases the trade-off result is maintained.

C. Specification Tests

Tables VII and VIII provide results of the J–statistic of Hansen (1982) (size and power test statistics, $W = \mathbf{S}$) of overidentification for the four factor models tested in this paper. We can observe that for larger samples (1000) the statistic provides completely different results when comparing the λ estimated with the Beta method, or the one estimated with any of the versions of the SDF method for the CAPM and Fama-French at the greatest significance level (1%). For small samples (60) the statistics reveal that the λ estimated with the Beta method might lead to completely different overidentification results than the one estimated with any of the SDF methods for the case of the Asness-Moskowitz-Pedersen and the Carhart models. Power J-test statistics reveal a similar behavior in most of the cases, but it seems the misspecification generated by including a nonzero Jensen's alpha, reduces the trade-off in some cases and benefits the SDF method. Still, there exists an inferential difference when using both methods that might lead to different inferential conclusions.

[Place Table VII about here]

[Place Table VIII about here]

D. Inference on Pricing Errors

We now turn our attention to the pricing errors estimates $\widehat{\pi}$, π^*, to their corresponding standard errors $\sigma(\widehat{\pi})$, $\sigma(\pi^*)$, and the associated $\widehat{J}$, and J^* statistics.[25] A given representation of an asset

[25] $\widehat{J}$ and J^* statistics results are presented in the Online Appendix, and reveal a similar result than the analysis with the bootstrap ratio tests in this section: there is a significant difference between the inference of two methods that might lead to oppose inferential conclusions.

pricing model would turn out to display a superior pricing accuracy vs. the other if the simulated standard error of the pricing errors were lower.

To better understand where the advocated trade-off between efficiency and pricing accuracy comes from, we need to briefly describe the set of moments used in estimation under each representation. On the one hand, the traditional Beta GMM restrictions incorporate three sets of moments: (i) the N asset pricing restriction which define the α vector; (ii) the $N \times K$ zero covariances between the errors and the factors; and (iii) the K definitions of δ, which equals the mean of the traded factors. Therefore, by imposing the definition of δ, the Beta approach increases its relative estimation efficiency. On the other hand, the SDF representation is simply based on the N asset pricing restrictions for the uncentered specification which defines π; and on the N asset pricing restriction plus the K mean definitions for each of the factor risk premia in the case of the centered specification. However, the inferences based on the SDF representation fail to impose the definition of λ. As a result, they allow for freely varying risk premia estimates in order to achieve lower mean pricing errors, favoring pricing accuracy over efficiency. By the same token, the specification tests derived under the Beta representation tend to under-reject in finite samples while the SDF-based tests approximately display the correct size.

Table IX shows the relative standard errors of the pricing errors for the single and multifactor models. Clearly, most of the ratios of the normalized standard errors of the pricing errors are now below one and this tends to be stronger in the case of smaller vs. larger sample sizes.[26] We note in this case the asymptotic approximation of first-order for the market risk factor seems not to be precise when compared to the empirical results. Further analyses reveal that given the strong correlation between the portfolio used (US 10 size-sorted) and the market risk factor, the second moments of these two variables ($E[r_t f_t']$) generates additional terms that are captured in the first-order approximation of the asymptotic covariance of the risk premia, but they are not in the asymptotic covariance of the pricing errors.[27] The extension of the Hansen (1982) first-order approximation to a second-order approximation is beyond the scope of this work, and it is suggested as a further improvement, not only of our analytical results, but for econometric theory in general.

[Place Table IX about here]

V. Economic Significance: Parameters Estimation in a Mean-Variance Optimal Portfolio

In this section we follow a mean-variance (Markowitz optimal) trading OOS exercise as in Ferreira and Santa-Clara (2011), but changing the estimation of the expected returns and the expected covariance matrix by incorporating a filtered returns approach based on the SDF representation as in Peñaranda and Sentana (2012); Peñaranda (2016).

[26] In the Online Appendix we present tables with the values used in the construction of Table IX.

[27] In the Online Appendix we provide additional results for the CAPM market risk factor pricing error ratios, with a 30 industry-sorted portfolio that has lower second cross-moments between the portfolio returns and the factor returns, where the asymptotic results match the empirical estimation, indicating the second moments between the portfolio returns and the factor returns affects the asymptotic estimation.

A. *Pricing Model Filter Approach*

The pricing model filter can be illustrated as follows. Consider a linear asset pricing model as in equation (3), from where we have a panel of sample asset returns observations $\mathbf{R}$ of dimension $T \times N$ from the asset returns r_t. From $\mathbf{R}$, using the SDF representation as in equation (7), we estimate the K-factor pricing model using the known factors f_t from which we have a sample factor matrix $\mathbf{F}$ of dimension $K \times N$, and estimate the corresponding sample error $\tilde{\mathbf{E}}$ with dimension $T \times N$. The filter consist in subtracting an estimated error term $\tilde{\mathbf{E}}$ from the asset returns $\mathbf{R}$, i.e., the new filtered returns are $\tilde{\mathbf{R}} = \mathbf{R} - \tilde{\mathbf{E}}$, that with the SDF representation is estimated as,

$$\tilde{\mathbf{R}} = \mathbf{R} - \underbrace{\mathbf{R}(\mathbf{1} - \tilde{\lambda}'\mathbf{F})}_{\text{estimated error } \tilde{\mathbf{E}}}, \tag{29}$$

where $\tilde{\lambda}$ is the dimension $K \times 1$ estimated risk premia, and $\mathbf{1}$ is the unit vector. With the resulting filtered excess returns $\tilde{\mathbf{R}}$ we compute the required inputs for the optimal mean-variance portfolio, the sample mean and sample covariance as estimates of the expected mean and variance $\mu_{r_t} = E[r_t] = \bar{\tilde{R}}$, $\mathbf{\Sigma}_{r_t} = E\left[(r_t - \mu_{r_t})(r_t - \mu_{r_t})'\right] = COV(\tilde{R})$ and obtain optimal allocation rules $\tilde{\omega}$.

Given that the pricing model filter approach alters the original information set, we can implement it in combination with other existing alternatives, enhancing the potential risk management gains. The idea behind the filter is that, under a linear asset pricing model as in equation (3), the error should be close to 0, $E\left[\tilde{\mathbf{E}}\right] = 0$. Differences in the estimation of the error $\tilde{\mathbf{E}}$ are translated into errors in the estimation of the mean and covariance matrices from the new filtered returns $\tilde{\mathbf{R}}$, and ***ceteris paribus*** in the mean-variance method, the estimation error is a result of the pricing errors of the method applied.

Using exactly the same procedure for estimation of the sample mean and sample covariance,[28] we apply the two different representations for the estimation of $\tilde{\lambda}$, the Beta estimated ($\tilde{\lambda}^*$), and the SDF estimated ($\tilde{\lambda}_1^U$, $\tilde{\lambda}_2^U$, $\tilde{\lambda}_1^C$, and $\tilde{\lambda}_2^C$ for the corresponding uncentered first- and second-stage, and centered first- and second-stage respectively).

B. *Noisy vs. Smooth Expected Covariance Initialization Inference*

The inference of the expected mean and expected covariance can be performed under many different scenarios of partial knowledge about the properties of the model. In our OOS trading exercise, following the literature (Ferreira and Santa-Clara, 2011), we consider an expanding window, that resembles the learning process of an investor on the underlying properties on the distribution of the portfolio returns – in this case, the US 10 size-sorted portfolios extracted from the Kenneth French website that spans the period January 1927 – December 2018 ($T = 1104$). The data of the factors corresponds to the same library a time span.

To add an additional layer of partial knowledge of the system, we divide our trading experiments in two: one where the investor is poorly knowledgeable of the properties of the returns, in particular, the expected covariance. By using expanding windows, with an initial window size of 60, 120, and

[28] We use the Ledoit and Wolf (2017) shrinkage method for estimating the covariance matrix in both representations, Beta and SDF, to avoid problems with small sample covariance estimates. Still, we need to use an initial window of 360, 480 and 600 months for stable estimations of the covariance matrix and to reduce the effects of the error of the estimation of the covariance into the error of estimation of the risk premia and the pricing errors.

240 months, we find that the OOS portfolio variance (after trading) peaks initially given the poor knowledge on the expected covariance. That initial 'noisy' covariance, cannot be improved even after applying the shrinking (sparse) method of Ledoit and Wolf (2017). We define this setup as a 'noisy expected covariance initialization' model. In contrast to poor knowledge of the properties, the 'smooth expected covariance initialization' has an initial window of 360, 480 and 600 months, and we are able to verify the OOS portfolio variance returns bounded values close to the values of the whole sample.

C. Results

Tables X and XI present the results of the absolute difference of the Sharpe ratios, between the model that estimates the filtered returns with the expected mean and expected covariance using the Beta method (SR^*) and the one that uses the SDF method ($SR_1^U, SR_2^U, SR_1^C, SR_2^C$). We show absolute values of the Sharpe ratio of the portfolio with parameters estimated with Beta in the first column. The first observation is that the inferential method used to estimate the expected mean and expected covariance, has no effects when considering the single-factor CAPM model. This reinforces our previous asymptotic results in regards to the 'little' (or null in this case) significance that the choice of representation (Beta or SDF) might have when using the CAPM single factor model. This result is aligned with Jagannathan and Wang (2002) and similar literature. Our empirical results show that there is a difference in pricing accuracy in favor of the SDF representation, but that result is biased given the use of US 10 size-sorted portfolios strongly correlated with the market risk factor. None of the portfolios had Sharpe ratio superior to the 50% of the naive $1/N$ portfolio, but this just a consequence of the models not being able to provide predictability beyond the market (additional tests with predictability factors are suggested).

[Place Table X about here]

[Place Table XI about here]

However, a different perspective is observed when using multifactor models. We note that in the case of Fama-French the Sharpe ratio is more favorable when using the SDF method for inference, but that advantage is reduced when the initial window size is increased. We look into the dimension of the trade-off, and the ratio of the relative standard errors for the pricing accuracy in the Fama-French model case, is about 78% while the increased efficiency improvement from using Beta representation averages 30%. (From 4%, 27% and 61% improvement for market, size and value factors improvement). In this scenario the trade-off seems to partially favour SDF representation.

In the of the case of the Asness-Moskowitz-Pedersen and Carhart models, the difference is overwhelming: all different scenarios (noisy and smooth initialization) and all different versions of the absolute difference between the Sharpe ratio when filtering the returns with the Beta method minus the Sharpe ratio filtering the returns with the SDF method, provide an increased Sharpe ratio of about 6.4% for the noisy initialization model case, and of about 7.2% in the smooth initialization case. Exploring deeper into this results, we find that, on the one hand the Beta representation is better in predicting the expected mean: learns faster by having an improved efficiency in the risk premia estimation (when measuring the norm of the distance between the 'full model expected mean' – the expected mean of the full period); on the other hand the SDF representation is better predicting the expected covariance: learns faster by having an improved pricing accuracy, but this improvement in the expected covariance is insufficient to overcome the loss in the expected mean estimation efficiency. All in all, Beta representation seems to be superior in inferential properties, when there exists higher-order moments in the factors, or when there are linear multifactor models in consideration.

VI. Conclusion

The interest in learning about the asymptotic and finite sample properties of parameter estimators (and their functions) in asset pricing models, like risk premia and pricing errors, has attracted the attention of researchers for decades. This attention is motivated by an extensive list of theoretical and empirical applications mainly – but not exclusively – in economics and finance. It is not uncommon to find examples in which different econometric approaches involve a trade-off between efficiency and pricing accuracy since the most efficient estimators may possess this property at the cost of higher pricing errors and vice versa. However, to the best of our knowledge, this is the first time in which such a formal dichotomy is explicitly used to better understand the difference between the statistical properties of inferences derived from the Beta and SDF representations in linear multifactor models. We find that the Beta representation has superior efficiency when estimating risk premia, but lower pricing accuracy than the SDF representation. The analytical asymptotic results and the simulation evidence that we have presented in this paper is useful to researchers and practitioners because they could choose a proper procedure given the goals of their application, i.e., whether accurate pricing and model overidentification testing may be more relevant vs. the goal of producing accurate estimates of the risk premia. For instance, in cost of capital estimation or when multiple asset pricing models need to be compared, a choice for pricing accuracy and testing guaranteed in many occasions by a SDF framework may be sensible, while in portfolio choice and asset management applications where to have a clear idea of what risks are compensated and in what amount, appears of primary importance and is probably best obtained from a Beta representation of the model.

As always, there are a number of possible extensions that it may be worthwhile to pursue. Chiefly, it may be of high relevance to the literature to explore what happens to GMM estimators when one considers also the non-traded factors. In principle, there is no reason to expect a similar pattern to hold. For example, Kan and Robotti (2008) show that the standard errors under correctly specified vs. potentially misspecified models are similar for traded factors, while they can differ substantially for non-traded factors such as a scaled market return factor and the lagged state variable CAY. One additional extension would consist of providing examples of the economic value that can be generated, especially in the presence of many factors deviating from joint normality, in financial applications (for instance, capital budgeting vs. portfolio selection) in which accurate pricing vs. efficient estimation of the risk premia may carry differential importance.

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Appendix A. Proof of Theorem 1 (And Corollaries)

Part A (factors on non-traded assets, risk premia)

A.1 Asymptotic covariance of the SDF method

In the case of the SDF representation, we calculate the GMM estimator asymptotic covariance. First, we consider the general case where δ, and μ in equation (6) can be different. Before calculating the asymptotic covariance of the SDF and the Beta representations, we define tensor operations.

Definition 1. Let $S1$ be a tensor of dimension $N_1 \times N_2 \times \cdots \times N_p$, and $S2$ a tensor of dimension $K_1 \times K_2 \times \cdots \times K_o$, with all the elements $N_1, \ldots, N_p, K_1, \ldots, K_o$ greater than one and $p > o$ without loss of generality, we define the **expansion tensor product** by each of its elements as,

$$\otimes_E(S1, S2)_{i_1,\ldots,i_p,i_{p+1},\ldots,i_{p+o}} = S1_{i_1,\ldots,i_p} \times S2_{i_1,\ldots,i_o}$$

as the result of the expansion of tensors $S1$ and $S2$ in a tensor of dimension $N_1 \times N_2 \times \cdots \times N_p \times K_1 \times K_2 \times \cdots \times K_o$, where $\{i_1, \ldots, i_p, i_{p+1}, \ldots, i_{p+o}\}$ denotes the elements of the tensor. The tensor expansion product is a tensor version of the Kronecker product ($\otimes$).

Definition 2. Let $S1$ be a tensor of dimension $N_1 \times N_2 \times \cdots \times N_p$, and $1 \leq i_{r1}, i_{r2} \leq p$ two indices. The **reduction tensor operator** (of a tensor into a matrix or vector for tensors of fourth- and third-order) is defined as,

$$\otimes_R(S1, i_{r1}, i_{r2}) = S1_{i_1,\ldots,i_{r1-1},i_{r1+1},\ldots,i_{r2-1},i_{r2+1},\ldots,i_p},$$

the tensor of reduced dimension $N_1 \times \ldots N_{i_{r1}-1} \times N_{i_{r1}+1} \ldots N_{i_{r2}-1} \times N_{i_{r2}+1} \ldots N_p$. In the case $p \leq 4$, this operator returns a matrix or a vector, and we can use the notation $\mathrm{mat}(\cdot)$ or $\mathrm{vec}(\cdot)$ to refer to this tensor reduction operator.

Definition 3. Third- and fourth-order moments: Consider a multifactor linear asset pricing model as in equation (3). The fourth- and third-order uncentered moments (co-moments) of f_t are defined as the tensors $E[f_t^3] = m_3 = \otimes_E(f_t f_t', f_t)$ and $E[f_t^4] = m_4 = \otimes_E(f_t f_t', f_t f_t')$.

Definition 4. Let λ be as in equation (6). Let define the resulting reduced tensors,

$$\begin{aligned}
m_4^{\text{reduced}} &= \sum_i \sum_j (\lambda \odot \lambda)' \mathrm{diag}\left(\otimes_R(m_4, i, j)\right) + 2\left(\mathrm{triu_vec}\left(\lambda\lambda'\right)' \mathrm{triu_vec}\left(\otimes_R(m_4, i, j)\right)\right), \\
m_3^{\text{reduced}} &= \sum_i \sum_j -2\lambda' \otimes_R(m_3, i, j) + \\
&\qquad (\delta_j - \mu_j)\left((\lambda \odot \lambda)' \mathrm{diag}\left(\otimes_R(m_3, i)\right) + 2\left(\mathrm{triu_vec}\left(\lambda\lambda'\right)' \mathrm{triu_vec}\left(\otimes_R(m_3, i)\right)\right)\right) + \\
&\qquad (\delta_i - \mu_i)\left((\lambda \odot \lambda)' \mathrm{diag}\left(\otimes_R(m_3, j)\right) + 2\left(\mathrm{triu_vec}\left(\lambda\lambda'\right)' \mathrm{triu_vec}\left(\otimes_R(m_3, j)\right)\right)\right), \\
m_2^{\text{reduced}} &= m_2 + \sum_i \sum_j -2\left((\delta_j - \mu_j)\left(\lambda'\left(m_{2(:,i)}\right)\right) + (\delta_i - \mu_i)\left(\lambda'\left(m_{2(:,j)}\right)\right)\right) + (\delta - \mu)(\delta - \mu)' + \\
&\qquad 2(\lambda \odot \lambda)' \mathrm{triu_vec}\left(m_2\right), \\
m_1^{\text{reduced}} &= \mu(\delta - \mu)' + (\delta - \mu)\mu' - 2(\delta - \mu)(\delta - \mu)'\left(\lambda'\mu\right),
\end{aligned}$$

where m_4^{reduced}, m_3^{reduced}, and m_2^{reduced} are matrices resulting from fourth-, third-, and second-order tensor reduction operations of the $g_s(r_t, f_t, \lambda)g_s(r_t, f_t, \lambda)'$ tensor.

To calculate the asymptotic covariance of the SDF representation, define $g_s(r_t, f_t, \lambda) = r_t(1 - \lambda f_t)$, then, the covariance matrix of $g_s(r_t, f_t, \lambda)$, in the case the factors f are non-Gaussian and have higher-order moments that deviate from the Gaussian distribution, is,

$$\begin{aligned} S_s &= E[g_s(r_t, f_t, \lambda)g_s(r_t, f_t, \lambda)'] \\ &= \mathbf{B}\Big(m_4^{\text{reduced}} + m_3^{\text{reduced}} + m_2^{\text{reduced}} + m_1^{\text{reduced}} + (\delta - \mu)(\delta - \mu)'\Big)\mathbf{B}' + \\ &\quad \left(1 - 2\lambda'\mu + (\lambda \odot \lambda)'\text{diag}\left(\mathbf{\Sigma} + \mu\mu'\right) + \text{triu\_vec}\left(\lambda\lambda'\right) + 2\left(\text{triu\_vec}\left(\mathbf{\Sigma} + \mu\mu'\right)\right)\right)\mathbf{\Omega}. \end{aligned} \tag{A1}$$

The elements in (A1) are sorted from the more complex (matrices resulting from reducing tensors of fourth-order, to the most simple (a tensor of second order – a matrix). Higher-order moments inside (A1) are the result of higher-order expected values of the multivariate factor f_t. These elements will not appear in a single-factor analysis such as Kan and Zhou (1999) or Jagannathan and Wang (2002). We split the elements of (A1). Define:

$$\mathbf{A}_s = m_4^{\text{reduced}} + m_3^{\text{reduced}} + m_2^{\text{reduced}} + m_1^{\text{reduced}} + (\delta - \mu)(\delta - \mu)',$$

and

$$a_{\epsilon_t} = 1 - 2\lambda'\mu + (\lambda \odot \lambda)'\text{diag}\left(\mathbf{\Sigma} + \mu\mu'\right) + \text{triu\_vec}\left(\lambda\lambda'\right) + 2\left(\text{triu\_vec}\left(\mathbf{\Sigma} + \mu\mu'\right)\right),$$

then the covariance of $g_s(r_t, f_t, \lambda)$ can be written as:

$$S_s = \mathbf{B}\mathbf{A}_s\mathbf{B}' + a_{\epsilon_t}\mathbf{\Omega} \tag{A2}$$

The inverse of (A2) is:

$$S_s^{-1} = \frac{1}{a_{\epsilon_t}}\mathbf{\Omega}^{-1} - \frac{1}{a_{\epsilon_t}^2}\mathbf{\Omega}^{-1}\mathbf{B}\left(\mathbf{A}_s^{-1} + \frac{1}{a_{\epsilon_t}}\mathbf{B}'\mathbf{\Omega}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{\Omega}^{-1}. \tag{A3}$$

The partial derivatives of g_s respect to λ will produce a matrix:

$$D_s = E\left[\frac{\partial g_s}{\partial \lambda}\right] = -\mathbf{B}\left(\mathbf{\Sigma} + \mu\delta'\right), \tag{A4}$$

Then, using (A3) and (A42), we have the asymptotic covariance of the SDF representation is,

$$Acov(\hat{\lambda}) = \left(D_s'S_s^{-1}D_s\right)^{-1}. \tag{A5}$$

In the case of single-factor models, and defining σ^2, the variance of the single-factor, and c_4 as the cumulant of fourth-order ($c_4 = \kappa_4 - 3\sigma^4$, or the *excess kurtosis*), the equations (A2), (A3), (A42), and (A5), have their equivalents in:

$$S_s = \frac{\sigma^2(\sigma^4 + \delta^4) + 2\kappa_3(\delta^3 - \delta\sigma^2) + \delta^2\left(\kappa_4 - 3\sigma^4\right)}{(\sigma^2 + \mu\delta)^2}\beta\beta' + \frac{\sigma^2(\sigma^2 + \delta^2)}{(\sigma^2 + \mu\delta)^2}\mathbf{\Omega}, \tag{A6}$$

$$
\begin{aligned}
S_s^{-1} &= \frac{(\sigma^2+\mu\delta)^2}{\sigma^2(\sigma^2+\delta^2)}\boldsymbol{\Omega}^{-1} - \frac{(\sigma^2+\mu\delta)^2}{\sigma^2(\sigma^2+\delta^2)} \times \\
& \left(\beta'\boldsymbol{\Omega}^{-1}\beta + \frac{\sigma^2(\sigma^4+\delta^4)+2\kappa_3(\delta^3-\delta\sigma^2)+\delta^2 c_4}{\sigma^2(\sigma^2+\delta^2)}\right)^{-1}\boldsymbol{\Omega}^{-1}\beta\beta'\boldsymbol{\Omega}^{-1}, && \text{(A7)} \\
D_s &= E\left[\frac{\partial g_s}{\partial\lambda}\right] = -(\sigma^2+\mu\delta)\beta, && \text{(A8)} \\
Avar(\hat{\lambda}) &= \frac{\sigma^2(\sigma^2+\delta^2)}{(\sigma^2+\mu\delta)^4}\left(\beta'\boldsymbol{\Omega}^{-1}\beta\right)^{-1} + \frac{\sigma^2(\sigma^4+\delta^4)}{(\sigma^2+\mu\delta)^4} + \frac{2\kappa_3(\delta^3-\delta\sigma^2)+\delta^2 c_4}{(\sigma^2+\mu\delta)^4} \\
& && \text{(A9)}
\end{aligned}
$$

The equivalent asymptotic variance (A9) in the single-factor Gaussian case is (Jagannathan et al., 2002):

$$
Avar(\hat{\lambda}) = \frac{\sigma^2(\sigma^2+\delta^2)}{(\sigma^2+\mu\delta)^4}\left(\beta'\boldsymbol{\Omega}^{-1}\beta\right)^{-1} + \frac{\sigma^2(\sigma^4+\delta^4)}{(\sigma^2+\mu\delta)^4}. \tag{A10}
$$

The difference in asymptotic variance of the SDF method when modeling a Gaussian factor, and a non-Gaussian vector comes from the higher-order moments terms:

$$
\frac{2\kappa_3\delta(\delta^2-\sigma^2)+\delta^2 c_4}{\sigma^2(\sigma^2+\delta^2)}. \tag{A11}
$$

The additional term (A11) will increase the asymptotic variance for heavy tailed distributions (greater κ_4 equals excess kurtosis, c_4, greater than 0), and will decrease by higher negative skewness (lower values of κ_3).

A.2 Asymptotic covariance of the Beta method - joint estimation

We calculate the GMM asymptotic variance of the risk premia for the case of multifactor Beta models. First, we solve for the general case where the parameters, $\theta^* = \left(\delta^*, \mathbf{B}^*, \mu^*, \sigma^{*2}\right)$, are estimated jointly as in Jagannathan and Wang (2002). Define

$$
\begin{aligned}
g_b(r_t, f_t, \theta) = \begin{pmatrix} g_b(1) \\ g_b(2) \\ g_b(3) \\ g_b(4) \end{pmatrix} &= \begin{pmatrix} r_t - \mathbf{B}(\delta + f_t - \mu) \\ (r_t - \mathbf{B}(\delta + f_t - \mu))f_t' \\ f_t - \mu \\ (f_t-\mu)(f_t-\mu)' - \boldsymbol{\Sigma} \end{pmatrix} \\
&= \begin{pmatrix} \epsilon_t \\ \epsilon_t f_t' \\ f_t - \mu \\ (f_t-\mu)(f_t-\mu)' - \boldsymbol{\Sigma} \end{pmatrix}, && \text{(A12)}
\end{aligned}
$$

where $\theta = (\delta, \mathbf{B}, \mu, \boldsymbol{\Sigma})$. The covariance of g_b (A12) is $E\left[g_b(r_t, f_t, \theta)g_b(r_t, f_t, \theta)'\right]$,

$$
S_b = \begin{pmatrix} \boldsymbol{\Omega} & \otimes_E(\boldsymbol{\Omega}, \mu) & 0 & 0 \\ \otimes_E(\boldsymbol{\Omega}, \mu) & \otimes_E(\boldsymbol{\Omega}, \boldsymbol{\Sigma} + \mu\mu') & 0 & 0 \\ 0 & 0 & \boldsymbol{\Sigma} & \kappa_3 \\ 0 & 0 & \kappa_3 & \kappa_4 - \otimes_E(\boldsymbol{\Sigma}, \boldsymbol{\Sigma}) \end{pmatrix}. \tag{A13}
$$

We need to calculate the partial derivatives of g_b respect to the parameters θ. The first partial derivative, $\frac{\partial g_b(1)}{\partial \delta} = \mathbf{B}$. The partial derivative $\frac{\partial g_b(2)}{\partial \delta}$ will produce the third-order tensor,

$$\frac{\partial g_b(2)}{\partial \delta} = - \otimes_E (\mathbf{B}, \mu) .$$

The following partial derivatives are null: $\frac{\partial g_b(3)}{\partial \delta} = \frac{\partial g_b(4)}{\partial \delta} = \frac{\partial g_b(3)}{\partial \mathbf{B}} = \frac{\partial g_b(4)}{\partial \mathbf{B}} = \frac{\partial g_b(1)}{\partial \mu} = \frac{\partial g_b(2)}{\partial \mu} = \frac{\partial g_b(1)}{\partial \mathbf{\Sigma}} = \frac{\partial g_b(2)}{\partial \mathbf{\Sigma}} = \frac{\partial g_b(3)}{\partial \mathbf{\Sigma}} = 0$.

In the case of $\frac{\partial g_b(1)}{\partial \mathbf{B}}$, it will produce the following $N \times N \times K$ third-order tensor:

$$\frac{\partial g_b(1)}{\partial \mathbf{B}} = - \left\{ \begin{pmatrix} & \delta' & & \\ 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \end{pmatrix}_{N \times K}, \dots, \begin{pmatrix} 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \\ & \delta' & & \\ 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \\ & \delta' & & \end{pmatrix} \right\}_{N \times N \times K} .$$

We need to define some tensor notations. Let us define the canonical basis vector:

$$\mathbf{e}' = [\mathbf{e}_{1,:}, \mathbf{e}_{2,:}, \dots, \mathbf{e}_{N,:}]_{1 \times N} .$$

where every element of this vector is a matrix,

$$\mathbf{e}_{i,:} = \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \end{pmatrix}_{N \times K}}_{i-th \text{ row equal to one.}} ,$$

A identity tensor can be denoted using tensor notation as $\mathbf{I}_{N \times N \times K} = \otimes_E (\mathbf{e}, \mathbf{1}_{N \times 1})$. Then, we can denote $\frac{\partial g_b(1)}{\partial \mathbf{B}}$ as,

$$-\delta \mathbf{I}_{N \times N \times K} = - \otimes_E (\delta \mathbf{e}, \mathbf{1}_{N \times 1}) , \tag{A14}$$

where $\delta \mathbf{e} = \left[[\delta', \dots, \delta']' \odot \mathbf{e}_{1,:}, \dots, [\delta', \dots, \delta']' \odot \mathbf{e}_{N,:} \right]$. The partial derivative $\frac{\partial g_b(2)}{\partial \mathbf{B}}$ will produce a fourth-order tensor:

$$\frac{\partial g_b(2)}{\partial \mathbf{B}} =$$
$$-\left\{\begin{array}{ccc}
\begin{pmatrix} \delta_1\mu_1+\mathbf{\Sigma}_{1,1} & \dots & \delta_1\mu_k+\mathbf{\Sigma}_{1,k} \\ 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{pmatrix}_{N\times K} & \dots & \begin{pmatrix} \delta_k\mu_1+\mathbf{\Sigma}_{k,1} & \dots & \delta_k\mu_k+\mathbf{\Sigma}_{k,k} \\ 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & \dots & 0 \\ \delta_1\mu_1+\mathbf{\Sigma}_{1,1} & \dots & \delta_1\mu_k+\mathbf{\Sigma}_{1,k} \\ 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{pmatrix}_{N\times K} & \dots & \begin{pmatrix} 0 & \dots & 0 \\ \delta_k\mu_1+\mathbf{\Sigma}_{k,1} & \dots & \delta_k\mu_k+\mathbf{\Sigma}_{k,k} \\ 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{pmatrix} \\
 & \vdots & \\
\begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \\ \delta_1\mu_1+\mathbf{\Sigma}_{1,1} & \dots & \delta_1\mu_k+\mathbf{\Sigma}_{1,k} \end{pmatrix}_{N\times K} & \dots & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \\ \delta_k\mu_1+\mathbf{\Sigma}_{k,1} & \dots & \delta_k\mu_k+\mathbf{\Sigma}_{k,k} \end{pmatrix}
\end{array}\right\}_{N\times K\times N\times K} \tag{A15}$$

Using a similar notation as in (A14), we denote the fourth-order identity tensor and the corresponding partial derivative $\frac{\partial g_b(2)}{\partial \mathbf{B}}$ as,

$$\mathbf{I}_{N\times N\times K\times K} = \otimes_E\left(\mathbf{e}, \mathbf{1}_{N\times K}\right), \tag{A16}$$
$$\frac{\partial g_b(2)}{\partial \mathbf{B}} = -\left(\mathbf{\Sigma}+\mu\delta'\right)\mathbf{I}_{N\times N\times K\times K} = -\otimes_E\left(\left(+\mathbf{\Sigma}+\mu\delta'\right)\mathbf{e}, \mathbf{1}_{N\times N}\right). \tag{A17}$$

The partial derivatives of $g_b(3)$ respect to $\delta, \mathbf{B}$, and μ are $\frac{\partial g_b(3)}{\partial \delta} = \mathbf{B}$, $\frac{\partial g_b(3)}{\partial \mathbf{B}} = \otimes_E(\mathbf{B}, \mu)$, and $\frac{\partial g_b(3)}{\partial \mu} = -\mathbf{I}_{K\times K}$, respectively. In a similar way we can calculate $\frac{\partial g_b(4)}{\partial \mathbf{\Sigma}} = -\mathbf{I}_{K\times K\times K\times K}$. Then, the expected value of the partial derivatives $D_b = E\left[\frac{\partial g_b}{\partial \theta}\right]$ is the following matrix

$$D_b = E\left[\frac{\partial g_b}{\partial \theta'}\right] = \begin{pmatrix} -\mathbf{B} & -\delta\mathbf{I}_{N\times N\times K} & \mathbf{B} & 0 \\ -\otimes_E(\mathbf{B},\mu) & -\left(\mathbf{\Sigma}+\mu\delta'\right)\mathbf{I}_{N\times N\times K\times K} & \otimes_E(\mathbf{B},\mu) & 0 \\ 0 & 0 & -\mathbf{I}_{K\times K} & 0 \\ 0 & 0 & 0 & -\mathbf{I}_{K\times K\times K\times K} \end{pmatrix}. \tag{A18}$$

The matrices S_b and D_b are *'tensor'* matrices: the elements of S_b and D_b are tensors. The calculations with these tensor matrices might generate problems for calculations (For example, the tensor $\kappa_4 - \otimes_E(\mathbf{\Sigma}, \mathbf{\Sigma})$ is not a full rank matrix). For this reason, we simplify the calculations by estimating the covariance matrix only of the parameter of interest for this paper (δ). Define

$$S_b^{-1} = \text{inv}(S_b), \tag{A19}$$

then, from the resulting matrix $V = \left(D_b' S_b^{-1} D_b\right)^{-1}$, the asymptotic variance of the δ^* parameter is equal to the top left corner element of the matrix, $Avar(\delta^*) = V_{1,1}$. In this case, $V_{1,1}$ is equal to,

$$\begin{aligned} V_{1,1} &= \Big(S_{b,3,3} D^2_{b,1,2} D^2_{b,2,3} + S_{b,2,2} D^2_{b,1,2} D^2_{b,3,3} + S_{b,3,3} D^2_{b,1,3} D^2_{b,2,2} + S_{b,1,1} D^2_{b,2,2} D^2_{b,3,3} \\ &\quad -2 S_{b,3,3} D_{b,1,2} D_{b,1,3} D_{b,2,2} D_{b,2,3} - 2 S_{b,1,2} D_{b,1,2} D_{b,2,2} D^2_{b,3,3} \Big) \times \\ &\quad \left(D^2_{b,3,3} \left(D_{b,1,1} D_{b,2,2} - D_{b,1,2} D_{b,2,1} \right)^2 \right)^{-1}, \end{aligned} \tag{A20}$$

where $S_{i,j}$ and $D_{b,i,j}$ are the elements i, j of the matrices S_b and D_b. The equation (A20) returns a matrix, and the problematic tensors ($S_{b,4,4}$) is not anymore in the calculation. Then, we can stack the tensors such as $S1_{N_1 \times N_2 \times N_3}$ as matrices by using the operator $\text{mat}\left(S1_{N_1 \times N_2 \times N_3}\right)_{N_1 \times N_2, N_3}$ that reorganizes the tensor as a matrix of dimension $(N_1 \times N_2, N_3)$: $D_{b,2,1} = \text{mat}\left(- \otimes_E (\mathbf{B}, \mu)\right)_{N \times K, K}$, $D_{b,1,2} = \text{mat}\left(-\delta \mathbf{I}_{N \times N \times K}\right)_{N, N \times K}$, $D_{b,2,2} = \text{mat}\left(-\left(\mathbf{\Sigma} + \mu\delta'\right) \mathbf{I}_{N \times N \times K \times K}\right)_{N \times K, N \times K}$, and $D_{b,2,3} = \text{mat}\left(\otimes_E (\mathbf{B}, \mu)\right)_{N \times K, K}$. The resulting stacked matrix D_b^{stacked} (without the last row and last column) is of dimension $(N + N \times K + K, K + N \times K + K)$. We apply the same stacking to the matrix S_b (without the last row nor the last column) to get S_b^{stacked}, and the final dimension of this stacked matrix $V^{\text{stacked}} = \left(D_b^{\text{stacked}\prime} (S_b^{\text{stacked}})^{-1} D_b^{\text{stacked}} \right)^{-1}$ is $(K + N \times K + K, K + N \times K + K)$. The resulting sub-matrix with the asymptotic covariances, $V_{1,1} = V^{\text{stacked}}_{(1:K, 1:K)}$, is of dimensions $K \times K$.

Using the Delta method, and the definition of λ in equation (6), the asymptotic covariance for the risk premia of the Beta method with multiple non-Gaussian factors is,

$$\begin{aligned} Acov(\lambda^*) &= \left(\frac{\partial \lambda}{\partial \delta} \right) \left(\frac{\partial \lambda}{\partial \delta} \right)' Avar(\delta^*) \\ &= \left(|\mathbf{\Sigma}| \times \left| \left(\mathbf{\Sigma} + \mu\delta' \right)^{-1} \right| \times \left(\mathbf{\Sigma} + \mu\delta' \right)^{-1} \right)^2 V_{1,1}. \end{aligned}$$

The calculation of the asymptotic variance of the risk premia by using the Beta method, for the case a single non-Gaussian factor has the corresponding equations to the multifactor equivalents (A12), (A13), (A18), (A19), in

$$g_b(r_t, f_t, \theta) = \begin{pmatrix} r_t - (\delta + f_t - \mu)\beta \\ (r_t - (\delta + f_t - \mu)\beta) f_t \\ f_t - \mu \\ (f_t - \mu)^2 - \sigma^2 \end{pmatrix} = \begin{pmatrix} \epsilon_t \\ \epsilon_t f_t \\ f_t - \mu \\ (f_t - \mu)^2 - \sigma^2 \end{pmatrix}, \tag{A21}$$

$$S_b = \begin{pmatrix} \mathbf{\Omega} & \mu\mathbf{\Omega} & 0 & 0 \\ \mu\mathbf{\Omega} & (\mu^2 + \sigma^2)\mathbf{\Omega} & 0 & 0 \\ 0 & 0 & \sigma^2 & \kappa_3 \\ 0 & 0 & \kappa_3 & \kappa_4 - \sigma^4 \end{pmatrix}, \tag{A22}$$

$$S_b^{-1} = \frac{1}{\sigma^2} \begin{pmatrix} (\mu^2 + \sigma^2)\mathbf{\Omega}^{-1} & -\mu\mathbf{\Omega}^{-1} & 0 & 0 \\ -\mu\mathbf{\Omega}^{-1} & \mathbf{\Omega}^{-1} & 0 & 0 \\ 0 & 0 & -\frac{\sigma^2(\kappa_4 - \sigma^4)}{\sigma^6 - \kappa_4\sigma^2 + \kappa_3^2} & \frac{\kappa_3\sigma^2}{\sigma^6 - \kappa_4\sigma^2 + \kappa_3^2} \\ 0 & 0 & \frac{\kappa_3\sigma^2}{\sigma^6 - \kappa_4\sigma^2 + \kappa_3^2} & \frac{\sigma^4}{\sigma^6 - \kappa_4\sigma^2 + \kappa_3^2} \end{pmatrix}, \tag{A23}$$

$$D_b = E\left[\frac{\partial g_b}{\partial \theta'}\right] = \begin{pmatrix} -\beta & -\delta I_n & \beta & 0 \\ -\mu\beta & -(\sigma^2+\mu\delta)I_n & \mu\beta & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A24}$$

We calculate $\left(D_b' S_b^{-1} D_b\right)^{-1}$

$$\left(D_b' S_b^{-1} D_b\right)^{-1} = \begin{pmatrix} \frac{\sigma^2+\delta^2}{\sigma^2}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1}+\sigma^2 & -\frac{\delta}{\sigma^2}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1}\beta' & \sigma^2 & \kappa_3 \\ -\frac{\delta}{\sigma^2}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1}\beta & \frac{1}{\sigma^2+\delta^2}\mathbf{\Omega}+\frac{\delta^2\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1}\beta\beta'}{\sigma^2(\sigma^2+\delta^2)} & 0 & 0 \\ \sigma^2 & 0 & \sigma^2 & \kappa_3 \\ \kappa_3 & 0 & \kappa_3 & \kappa_4-\sigma^4 \end{pmatrix} \tag{A25}$$

The asymptotic variance of the GMM estimation of δ^* for the single-factor non-Gaussian case is

$$Avar(\delta^*) = \frac{\sigma^2+\delta^2}{\sigma^2}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1}+\sigma^2. \tag{A26}$$

Applying the Delta method,[29] the corresponding asymptotic variance of the λ^* parameter –equivalent to (A21)– is[30,31]

$$Avar(\lambda^*) = \frac{\sigma^2(\sigma^2+\delta^2)}{(\sigma^2+\mu\delta)^4}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1}+\frac{\sigma^6}{(\sigma^2+\mu\delta)^4}. \tag{A27}$$

Considering (A5), (A21), and by applying some algebra with the support of equations (A10) and (A27) the asymptotic variance of the risk premia estimator result is yield. The asymptotic variance in equation (A27) depends on the *linearity* of the first-order Delta approximation. We can increase the order of the Delta approximation: a second- and third-order Delta approximation considers third- and fourth-order moments terms and incorporates the skewness and kurtosis of the estimator distribution,

$$\begin{aligned} Avar(\lambda^*) &= \frac{\sigma^2(\sigma^2+\delta^2)}{(\sigma^2+\mu\delta)^4}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1}+\frac{\sigma^6}{(\sigma^2+\mu\delta)^4}-2\frac{\left(\sigma^4\mu\right)}{(\sigma^2+\mu\delta)^5}\kappa_{3,\delta}, \\ Avar(\lambda^*) &= \frac{\sigma^2(\sigma^2+\delta^2)}{(\sigma^2+\mu\delta)^4}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1}+\frac{\sigma^6}{(\sigma^2+\mu\delta)^4}-2\frac{\left(\sigma^4\mu\right)}{(\sigma^2+\mu\delta)^5}\kappa_3+2\frac{\sigma^4\mu^4}{(\sigma^2+\mu\delta)^6}\kappa_{4,\delta}, \end{aligned}$$

where $\kappa_{3,\delta}$ and $\kappa_{4,\delta}$ are the asymptotic third- and fourth-order central moments of the distribution of δ^*.

[29] The use of the delta method requires that the parameter estimation–given the sequence X_t–converges to a normal distribution, $\sqrt{T}\left|X_T-\theta\right| \xrightarrow{D} N(0,\sigma^2)$. In the case the distribution of the factor deviates from the normal distribution, the estimated parameters might deviate from the normal, and the Delta approximation might underestimate the asymptotic variance. In our case, as we estimate δ separately from $\mathbf{B}$ in the next subsection, we use the GMM asymptotic results to provide an exact estimate of the asymptotic variance of δ without using the Delta method.

[30] The equation (A27) corrects the Jagannathan et al. (2002) approximation of the asymptotic variance by using the Beta method, that has a difference of $\frac{\sigma^2\delta^4}{(\sigma^2+\mu\delta)^4}$ between the Beta and the SDF methods.

[31] We can observe that in the non-Gaussian case, when using the Beta method, the higher-order moments do not affect the asymptotic variance of the risk premia estimation. This is consistent with the Beta method being a first- and second-order only asset pricing model. In the case of the SDF model, higher-order moments will discount risk premia, therefore they will affect the asymptotic variance.

Part B (factors on non-traded assets, pricing errors)

We provide the asymptotic covariances of the pricing errors. In both cases, SDF and Beta methods, the asymptotic variance of the pricing error is found by defining a sample mean of the estimator, for the SDF method,

$$e_s(\hat{\lambda}) = \frac{1}{T}\left(\sum_{i=1}^{T} g_s(r_t, f_t, \lambda)\right), \tag{A28}$$

and for the Beta method,

$$e_b(\theta^*) = \frac{1}{T}\left(\sum_{i=1}^{T} g_b(r_t, f_t, \theta)\right). \tag{A29}$$

The SDF pricing error $\hat{\pi}$ will be equal to (A28) (Jagannathan and Wang, 2002), then, by Hansen (1982) results,

$$Acov\left(\hat{\pi}\right) = Acov\left(e_s(\hat{\lambda})\right) = S_s - D_s\left(D_s' S_s^{-1} D_s\right) D_s'. \tag{A30}$$

In the empirical section, for our numerical simulations, the pricing errors presented are averaged as $\widehat{\pi}^{\text{average}} = \left(N^{-1}\sum_{i=1}^{N}\widehat{\pi}_i^2\right)^{1/2}$. In this case, an approximation of the asymptotic variance of $\widehat{\pi}^{\text{average}}$, not considering the covariances of the pricing errors, is,

$$Avar\left(\widehat{\pi}^{average}\right) = N^{-1}\sum_{i}^{N} Acov\left(\widehat{\pi}\right)_{i,i}, \tag{A31}$$

where $Acov\left(\widehat{\pi}\right)_{i,i}$ is the (i,i)-th element of the covariance matrix. Now consider the Beta method. Using Hansen (1982) results, $Acov\left(\theta^*\right) = Acov\left(e_b(\theta^*)\right)$, and,

$$Acov\left(e_b(\theta^*)\right) = V^{\pi,\text{stacked}} = S_b^{\text{stacked}} - D_b^{\text{stacked}}\left(D_b^{\text{stacked}\prime}(S_b^{\text{stacked}})^{-1} D_b^{\text{stacked}}\right) D_b^{\text{stacked}\prime}. \tag{A32}$$

The equivalent Jensen's α is:

$$\alpha^* = Q^* e(\theta^*) = \left[I_n, \mathbf{0}_{n\times n}, \beta^*, \mathbf{0}_{n\times 1}\right] e(\theta^*). \tag{A33}$$

Define,

$$\left(\frac{\partial\pi}{\partial\alpha}\right) = \left(1 + \delta'\mathbf{\Sigma}\mu\right)^{-1}. \tag{A34}$$

Then, the asymptotic variance of the pricing error, using equations (A34) and (9) is,

$$Acov\left(\pi^*\right) = \left(1 + \delta'\mathbf{\Sigma}\mu\right)^{-2} V_{1,1}^{\pi,\text{stacked}}, \tag{A35}$$

where $V_{1,1}^{\pi,\text{stacked}}$ is the left corner superior sub-matrix of dimension $N \times N$ from $V^{\pi,\text{stacked}}$. In the empirical section, for our numerical simulations, the pricing errors presented are averaged as $\pi^{*,\text{average}} = \left(N^{-1}\sum_{i=1}^{N}(\pi_i^*)^2\right)^{1/2}$. In this case, an approximation of the asymptotic variance of

$\pi^{*,\text{average}}$, not considering the covariances of the pricing errors, is,

$$Avar\left(\pi^{*,average}\right) = N^{-1}\sum_{i}^{N} Acov\left(\pi^{*}\right)_{i,i}. \tag{A36}$$

Part C (factors on traded assets, $\mu = \delta$, risk premia)

In our paper, $\mu = \delta$, and the estimation of the parameter $\mathbf{B}$ is separate from the estimation of the parameters $\theta^* = \left(\delta, \sigma^2\right) = \left(\mu, \sigma^2\right)$. Then, we have that to estimate the asymptotic variance of the parameter $\delta^* = \mu^*$, we define

$$g_b(r_t, f_t, \theta) = \begin{pmatrix} f_t - \mu \\ (f_t-\mu)(f_t-\mu)' - \mathbf{\Sigma} \end{pmatrix}, \tag{A37}$$

$$S_b = \begin{pmatrix} \mathbf{\Sigma} & \kappa_3 \\ \kappa_3 & \kappa_4 - \otimes_E\left(\mathbf{\Sigma},\mathbf{\Sigma}\right) \end{pmatrix}, \tag{A38}$$

and

$$D_b = E\left[\frac{\partial g_b}{\partial \theta^{*\prime}}\right] = \begin{pmatrix} -\mathbf{I}_{K\times 1} & 0 \\ 0 & -\mathbf{I}_{K\times K} \end{pmatrix}, \tag{A39}$$

then

$$Avar(\lambda^*) = \left(|\mathbf{\Sigma}| \times \left|\left(\mathbf{\Sigma} + \mu\delta'\right)^{-1}\right| \times \left(\mathbf{\Sigma} + \mu\delta'\right)^{-1}\right)^2 \mathbf{\Sigma}. \tag{A40}$$

This is a Delta (first-order) approximation. A better approximation is made when considering the definition of λ into the GMM,

$$g_b(f_t, \lambda_b) = \mathrm{E}[r_t m_t] = \mathrm{E}[r_t(1-\lambda_b f_t)] = (r_t(1-\lambda_b f_t)) = 0,$$

That can be reduced to

$$g_b(f_t, \lambda_b) = (f_t(1-\lambda_b f_t)) = 0. \tag{A41}$$

The parameter λ_b is not estimated by the SDF method, but by the Beta method and using moment restrictions in equations (8), $\lambda_b = \mu'\left(\mathbf{\Sigma} + \mu\mu'\right)^{-1}$. Consider m_3^b and m_4^b the third- and fourth-order uncentered tensor moments of $g_b(f_t, \lambda_b)$. Define $m_3^{*,\text{reduced}}$ and $m_4^{*,\text{reduced}}$ as the reduced tensor functions of third- and fourth-order moments,

$$m_3^{b,\text{reduced}} = -2\sum_i\sum_j \lambda_b' \otimes_R\left(m_3^b, i, j\right),$$
$$m_4^{b,\text{reduced}} = \sum_i\sum_j (\lambda\odot\lambda)'\text{diag}\left(\otimes_R\left(m_4,i,j\right)\right) + 2\left(\text{triu\_vec}\left(\lambda\lambda'\right)'\text{triu\_vec}\left(\otimes_R\left(m_4,i,j\right)\right)\right).$$

Then, the covariance matrix of $g_b(f_t, \lambda_b)$ is,

$$S_b = \mathrm{E}[g_b(f_t,\lambda_b)g_b(f_t,\lambda_b)'] = \left(\mathbf{\Sigma} + \mu\mu'\right) + m_3^{b,\text{reduced}} + m_4^{b,\text{reduced}}.$$

The partial derivatives of g_b respect to λ_b will produce a matrix,

$$D_b = \mathrm{E}\left[\frac{\partial g_b}{\partial \lambda}\right] = -\left(\mathbf{\Sigma} + \mu\mu'\right). \tag{A42}$$

The asymptotic covariance of λ^* is equal to

$$Acov(\lambda^*) \quad = \quad Acov(\lambda_b^*) = \left(\mathbf{\Sigma} + \mu\mu'\right)^{-1} S_b \left(\left(\mathbf{\Sigma} + \mu\mu'\right)^{-1}\right)'. \tag{A43}$$

In the case of single-factors we have,

$$Avar(\lambda^*) \quad = \quad \frac{\left((\sigma^2+\mu^2) - 2\lambda_b m_3^b + \lambda_b^2 m_4^b\right)}{(\sigma^2+\mu^2)^2}, \tag{A44}$$

with

$$\begin{aligned} m_3^b &= \mathrm{E}[f_t^3] = \kappa_3 + 3\sigma^2\mu + \mu^3, \\ m_4^b &= \mathrm{E}[f_t^4] = c_4 + 4\kappa_3\mu + 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4. \end{aligned}$$

Proof of Corollary 4: Consider the asymptotic variance of the Beta method for the case where the factors are traded (equation A44). We can rewrite this equation as,

$$\begin{aligned} Avar(\lambda^*) \quad &= \quad \frac{(\sigma^2+\mu^2)}{(\sigma^2+\mu^2)^2} - \frac{2\left(\frac{\mu}{(\sigma^2+\mu^2)}\right)}{(\sigma^2+\mu^2)^2} \underbrace{\left(\kappa_3 + 3\mu\sigma^2 + \mu^3\right)}_{\text{third-order moment } m_3^b} + \\ &\quad + \frac{\left(\frac{\mu}{(\sigma^2+\mu^2)}\right)^2}{(\sigma^2+\mu^2)^2} \underbrace{\left(c_4 + 4\kappa_3\mu + 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4\right)}_{\text{fourth-order moment } m_4^b}, \\ &= \quad \sigma^2 \frac{(\sigma^4+\mu^4)}{(\sigma^2+\mu^2)^4} + \frac{\mu^2 c_4 + 2\kappa_3(\mu^3 - \mu\sigma^2)}{(\sigma^2+\mu^2)^4}. \end{aligned}$$

Then,

$$Avar(\hat{\lambda}) - Avar(\lambda^*) = \frac{\sigma^2(\sigma^2+\mu^2)}{(\sigma^2+\mu^2)^4}\left(\beta'\mathbf{\Omega}^{-1}\beta\right)^{-1}, \tag{A45}$$

and this expression is always positive.

Appendix B. Descriptive Statistics

Table I: Descriptive statistics of factors and test portfolios.

The data on US factors and portfolios are taken from Kenneth French's library, and is on percentage. The sample spans the period January 1927 – December 2018 ($T = 1104$).

	Factors				Portfolios		
	Market	Size	Value	Momentum	10 size	25 size-value	30 industry
				US			
Mean	0.6471	0.2096	0.3682	0.6617	0.8642	0.8786	0.7239
Variance	28.58	10.22	12.17	22.02	46.8924	47.9440	31.3499
Std. Dev.	5.34	3.19	3.48	4.69	6.8478	6.9242	5.5991
Skewness	0.19	1.93	2.18	-3.06	1.3791	1.3957	0.3583
Kurtosis	10.81	22.28	22.18	30.90	18.3248	18.2319	11.8010
				US - Recession Periods			
Mean	-0.4320	-0.1292	0.2696	0.6361	-0.4787	-0.4481	-0.3041
Variance	67.35	11.31	25.52	55.30	103.6166	108.4459	71.4597
Std. Dev.	8.20	3.36	5.05	7.43	10.1792	10.4137	8.4534
Skewness	0.40	0.48	3.05	-3.24	1.4465	1.3667	0.5638
Kurtosis	6.83	5.02	22.84	21.98	12.3939	11.4992	7.5031

Table II: **Expected value and standard errors of risk premia for CAPM model: US data, 10 size-sorted portfolios**

The table presents the expected value and the standard error of the GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments being sampled from the univariate empirical distribution of the factor. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped empirical independent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\mathrm{E}[\lambda^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\lambda^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
60	2.37	2.40	3.18	2.59	3.00	2.69	2.93	3.61	3.26	3.58
360	2.17	2.17	2.29	2.22	2.28	1.03	1.11	1.09	1.16	1.11
600	2.16	2.17	2.23	2.21	2.24	0.80	0.85	0.83	0.89	0.85
1000	2.15	2.15	2.19	2.19	2.21	0.62	0.66	0.64	0.69	0.66

Table III: **Relative standard errors of risk premia estimated from the CAPM model: US data, 10 size-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments being sampled from the univariate empirical distribution of each factor. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped empirical independent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 − December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
60	1.0735***	1.0003	1.1101***	1.0522***
360	1.0750***	1.0047***	1.1010***	1.0306***
600	1.0613***	1.0026***	1.0871***	1.0280***
1000	1.0652***	1.0020**	1.0912***	1.0279***
Asymptotic (1st-Ord)	1.0863			

Table IV: **Relative standard errors of risk premia estimated from the Fama-French model: US data, 10 size-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, and Value) moments being sampled from the empirical distribution applying bootstrapping. Estimators decorated with a * are obtained by GMM from a Beta representation of the Fama-French model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	1.1432***	1.0292***	1.1832***	1.0957***
360	1.1262***	1.0538***	1.1362***	1.0631***
600	1.1166***	1.0476***	1.1243***	1.0548***
1000	1.1191***	1.0377***	1.1263***	1.0456***
Asymptotic (1st-Ord)	1.0460			
		Size		
60	0.4084**	0.3901**	0.2338***	0.2128***
360	0.9521	0.8670***	0.9052***	0.7844***
600	1.0609**	0.9492**	1.0298	0.8921***
1000	1.0910***	0.9737*	1.0719***	0.9378***
Asymptotic (1st-Ord)	1.3673			
		Value		
60	2.2088***	1.8250***	2.7403***	3.0190***
360	2.9783***	2.3351***	3.0486***	2.5326***
600	3.0086***	2.3650***	3.0523***	2.4803***
1000	3.0798***	2.3943***	3.1119***	2.4688***
Asymptotic (1st-Ord)	2.6219			

Table V: **Relative standard errors of risk premia estimated from the Asness-Moskowitz-Pedersen model: US data, 10 size-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Momentum, and Value) moments being sampled from the empirical distribution applying bootstrapping. Estimators decorated with a * are obtained by GMM from a Beta representation of the Asness-Moskowitz-Pedersen model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	1.1442***	0.9799*	1.4931***	1.3178***
360	1.4210***	1.2566***	1.7415***	1.5107***
600	1.4773***	1.2893***	1.7597***	1.5149***
1000	1.5400***	1.3293***	1.7963***	1.5370***
Asymptotic (1st-Ord)	1.4004			
		Momentum		
60	2.8205***	2.0929***	25.1919***	12.9843***
360	4.7186***	3.8545***	8.7920***	6.9631***
600	5.1621***	4.2124***	7.6393***	6.2049***
1000	5.4884***	4.4432***	7.0397***	5.7554***
Asymptotic (1st-Ord)	4.6360			
		Value		
60	1.4554***	1.0929***	5.8393***	6.3630***
360	2.9517***	2.4302***	4.8139***	4.2775***
600	3.2012***	2.6355***	4.3874***	3.7936***
1000	3.4854***	2.8286***	4.2691***	3.5995***
Asymptotic (1st-Ord)	3.6640			

Table VI: Relative standard errors of risk premia estimated from the Carhart model: US data, 10 size-sorted portfolios

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, Value, and Momentum) moments being sampled from the empirical distribution applying bootstrapping. Estimators decorated with a * are obtained by GMM from a Beta representation of the Carhart model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	1.1823***	1.0553***	1.5959***	1.4850***
360	1.3860***	1.2834***	1.7107***	1.5805***
600	1.4097***	1.2949***	1.6881***	1.5452***
1000	1.4567***	1.3483***	1.7079***	1.5760***
Asymptotic (1st-Ord)	1.4157			
		Size		
60	2.9360***	2.6395***	1.8828***	1.9595***
360	0.3931***	0.3665***	0.2863***	0.2762***
600	0.2440***	0.2245***	0.1891***	0.1794***
1000	0.1456***	0.1334***	0.1204***	0.1133***
Asymptotic (1st-Ord)	1.4635			
		Value		
60	1.5132***	1.2923***	12.6684***	10.5390***
360	3.2375***	2.8995***	6.8216***	5.6345***
600	3.4535***	3.1444***	5.4927***	4.7474***
1000	3.6626***	3.3574***	4.9674***	4.3756***
Asymptotic (1st-Ord)	4.4710			
		Momentum		
60	2.5915***	2.1321***	17.5486***	13.2040***
360	4.3878***	4.0154***	8.4040***	7.2284***
600	4.7195***	4.3607***	7.2355***	6.4094***
1000	4.8740***	4.6044***	6.5327***	5.9530***
Asymptotic (1st-Ord)	4.9860			

Table VII: GMM estimation specification tests (size, $W = \mathbf{S}$): US data, 10 size-sorted portfolios

The table presents the results of the bootstrapped simulations on the rejection rate of the Hansen (1982) $\hat{J}$-statistic under the Beta and the SDF methods. The specification tests are size tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments being sampled from the empirical distribution applying bootstraping. Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
CAPM															
60	3.30	5.21	5.21	5.91	5.91	9.84	14.48	14.48	15.54	15.54	16.06	22.17	22.17	23.45	23.45
360	0.79	1.52	1.52	1.54	1.54	3.86	6.30	6.30	6.38	6.38	7.76	12.23	12.23	12.30	12.30
600	0.70	1.40	1.40	1.43	1.43	3.68	6.07	6.07	6.08	6.08	7.50	11.85	11.85	11.85	11.85
1000	0.51	1.05	1.05	1.03	1.03	3.20	5.18	5.18	5.23	5.23	6.72	10.95	10.95	11.04	11.04
Fama-French															
60	0.64	2.95	2.95	3.54	3.54	2.45	9.53	9.53	10.63	10.63	4.72	16.20	16.20	17.52	17.52
360	0.17	1.09	1.09	1.20	1.20	0.68	4.83	4.83	4.92	4.92	1.69	9.76	9.76	9.94	9.94
600	0.10	1.05	1.05	1.13	1.13	0.77	5.28	5.28	5.23	5.23	1.71	9.85	9.85	9.88	9.88
1000	0.08	1.11	1.11	1.12	1.12	0.63	5.21	5.21	5.26	5.26	1.68	10.44	10.44	10.43	10.43
Asness-Moskowitz-Pedersen															
60	0.50	1.71	1.71	2.25	2.25	2.20	6.70	6.70	7.84	7.84	3.92	11.72	11.72	13.33	13.33
360	0.12	0.86	0.86	0.95	0.95	0.70	3.89	3.89	4.21	4.21	1.85	7.82	7.82	8.58	8.58
600	0.09	0.75	0.75	0.92	0.92	0.63	4.08	4.08	4.74	4.74	1.69	8.28	8.28	9.04	9.04
1000	0.06	0.82	0.82	1.01	1.01	0.66	4.26	4.26	4.75	4.75	1.79	8.88	8.88	9.67	9.67
Carhart															
60	0.17	1.59	1.59	2.32	2.32	0.75	6.14	6.14	7.72	7.72	1.62	11.15	11.15	12.76	12.76
360	0.03	0.71	0.71	0.81	0.81	0.31	3.55	3.55	4.12	4.12	0.73	7.64	7.64	8.41	8.41
600	0.04	0.77	0.77	0.96	0.96	0.28	3.95	3.95	4.52	4.52	0.80	8.43	8.43	9.10	9.10
1000	0.05	0.83	0.83	1.09	1.09	0.37	4.69	4.69	5.19	5.19	0.88	9.07	9.07	9.79	9.79

Table VIII: GMM estimation specification tests (power, $W = S$): US data, 10 size-sorted portfolios

The table presents the results of the bootstrapped simulations on the rejection rate of the Hansen (1982) $\hat{J}$-statistic under the Beta and the SDF methods. The specification tests are power tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying bootstraping. Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
CAPM															
60	4.01	6.04	6.04	6.86	6.86	10.95	15.59	15.59	16.62	16.62	17.14	23.89	23.89	24.95	24.95
360	2.05	3.37	3.37	3.29	3.29	7.84	12.25	12.25	12.25	12.25	14.40	20.34	20.34	20.32	20.32
600	3.37	5.23	5.23	5.15	5.15	11.10	16.30	16.30	16.14	16.14	19.30	26.27	26.27	26.14	26.14
1000	6.67	9.94	9.94	9.78	9.78	18.69	25.32	25.32	25.13	25.13	28.98	37.45	37.45	37.21	37.21
Fama-French															
60	0.76	3.54	3.54	4.08	4.08	2.86	10.48	10.48	11.59	11.59	4.81	16.99	16.99	18.73	18.73
360	0.42	2.56	2.56	2.66	2.66	1.83	9.66	9.66	9.79	9.79	3.92	16.83	16.83	17.01	17.01
600	0.75	4.08	4.08	4.08	4.08	3.01	13.64	13.64	13.63	13.63	5.60	22.59	22.59	22.63	22.63
1000	1.46	8.20	8.20	8.32	8.32	6.08	21.68	21.68	21.85	21.85	10.91	32.84	32.84	32.99	32.99
Asness-Moskowitz-Pedersen															
60	0.47	1.94	1.94	2.74	2.74	2.12	7.12	7.12	8.19	8.19	4.15	12.64	12.64	13.91	13.91
360	0.50	1.88	1.88	1.79	1.79	2.27	7.21	7.21	7.22	7.22	4.45	13.02	13.02	13.17	13.17
600	0.77	3.06	3.06	2.92	2.92	3.20	11.12	11.12	10.67	10.67	6.51	18.65	18.65	18.61	18.61
1000	1.82	6.10	6.10	5.98	5.98	7.02	18.26	18.26	18.11	18.11	12.49	28.93	28.93	28.69	28.69
Carhart															
60	0.20	1.88	1.88	2.62	2.62	0.92	6.65	6.65	7.70	7.70	2.01	11.65	11.65	13.49	13.49
360	0.17	1.88	1.88	1.77	1.77	1.07	7.76	7.76	7.47	7.47	2.19	14.27	14.27	13.67	13.67
600	0.35	3.13	3.13	2.97	2.97	1.65	11.42	11.42	10.99	10.99	3.61	19.49	19.49	19.10	19.10
1000	0.85	6.39	6.39	6.06	6.06	4.07	19.27	19.27	18.52	18.52	7.71	29.90	29.90	28.98	28.98

Table IX: **Relative standard errors of pricing errors for four alternative asset pricing models: US data, 10 size-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\pi}) = \sigma(\widehat{\pi})/\mathrm{E}(\widehat{\pi})$. The returns are generated by equation (3) under the null hypothesis with the combination of factors (Market, Momentum, Size, and Value – depending on model) being sampled from the empirical distribution applying bootstrapping. Estimators decorated with a * are obtained by GMM from a Beta representation of the corresponding model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\pi}_1^U)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_2^U)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_1^C)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_2^C)/\sigma_r(\pi^*)$
		CAPM		
60	0.8430***	1.1292***	0.8425***	1.0438***
360	0.8337***	0.9962	0.8334***	0.9901**
600	0.8286***	1.0018	0.8286***	0.9987
1000	0.8319***	0.9992	0.8317***	0.9987
Asymptotic$^{\text{ave}}$ (1st Ord)	0.9951			
		Fama-French		
60	0.6475***	1.5690***	0.6539***	1.6775***
360	0.7197***	1.1929***	0.7198***	1.1480***
600	0.7415***	1.1035***	0.7420***	1.0298*
1000	0.7575***	1.0008	0.7576***	0.9768**
Asymptotic$^{\text{ave}}$ (1st Ord)	0.7779			
		Asness-Moskowitz-Pedersen		
60	0.6307***	1.3723***	0.6282***	1.4370***
360	0.6233***	1.1571***	0.6200***	1.1372***
600	0.6414***	1.1250***	0.6436***	1.1055***
1000	0.6670***	1.1054***	0.6684***	1.0843***
Asymptotic$^{\text{ave}}$ (1st Ord)	0.6644			
		Carhart		
60	0.6053***	1.4469***	0.6065***	1.4902***
360	0.7910***	1.6388***	0.7868***	1.4804***
600	0.9292***	1.6034***	0.9204***	1.4802***
1000	1.1151***	1.6688***	1.1073***	1.5735***
Asymptotic$^{\text{ave}}$ (1st Ord)	0.7337			

Table X: Sharpe ratio of mean-variance optimal portfolios with parameters estimated with Beta and SDF representations (noisy expected covariance initialization model)

The table presents the Sharpe ratio difference of portfolios estimated with (i) the Beta representation (SR^*), and (ii) the SDF representation (SR_1^U, SR_2^U, SR_1^C, and SR_2^C). The portfolio weights for each period are estimated by optimizing a Markowitz mean-variance optimal portfolio, which uses the Peñaranda and Sentana's (2012), and Peñaranda's (2016) filtered returns approach described in Section V. The returns are obtained from US 10 size-sorted portfolios from the Kenneth French's public data library, adjusted with the risk-free return. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different initial window estimation sizes (T). The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

Model difference	Initial window size		
	60	120	240
	CAPM		
SR^*	11.49	12.67	16.05
$SR^* - SR_1^U$	-1.54×10^{-9}	5.05×10^{-10}	-7.23×10^{-10}
$SR^* - SR_2^U$	-1.32×10^{-9}	-4.78×10^{-10}	-2.27×10^{-9}
$SR^* - SR_1^C$	-1.00×10^{-9}	-3.79×10^{-9}	-1.29×10^{-9}
$SR^* - SR_2^C$	-8.40×10^{-9}	-4.06×10^{-9}	-2.73×10^{-9}
	Fama-French		
SR^*	14.63	14.06	17.11
$SR^* - SR_1^U$	−1.542	−1.729	−2.006
$SR^* - SR_2^U$	−1.818	−0.721	−0.617
$SR^* - SR_1^C$	−1.485	−1.681	−1.953
$SR^* - SR_2^C$	−1.395	−0.672	−0.584
	Asness-Moskowitz-Pedersen		
SR^*	23.13	25.61	30.91
$SR^* - SR_1^U$	6.235	8.596	10.390
$SR^* - SR_2^U$	2.980	5.481	6.845
$SR^* - SR_1^C$	8.500	9.365	11.469
$SR^* - SR_2^C$	2.828	6.003	7.303
	Carhart		
SR^*	25.74	25.66	31.09
$SR^* - SR_1^U$	7.411	5.357	5.979
$SR^* - SR_2^U$	3.370	4.330	5.855
$SR^* - SR_1^C$	7.912	5.417	6.149
$SR^* - SR_2^C$	5.763	4.541	6.263

Table XI: Sharpe ratio of mean-variance optimal portfolios with parameters estimated with Beta and SDF representations (smooth expected covariance initialization model)

The table presents the Sharpe ratio difference (in percentage) of portfolios estimated with (i) the Beta representation (SR^*), and (ii) the SDF representation (SR_1^U, SR_2^U, SR_1^C, and SR_2^C). The portfolio weights for each period are estimated by optimizing a Markowitz mean-variance optimal portfolio, which uses the Peñaranda and Sentana's (2012), and Peñaranda's (2016) filtered returns approach described in Section V. The returns are obtained from US 10 size-sorted portfolios from the Kenneth French's public data library, adjusted with the risk-free return. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different initial window estimation sizes (T). The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

Model difference	Initial window size		
	360	480	600
	CAPM		
SR^*	5.97	8.21	8.70
$SR^* - SR_1^U$	-6.73×10^{-10}	-9.91×10^{-10}	8.54×10^{-11}
$SR^* - SR_2^U$	-1.37×10^{-9}	-1.33×10^{-9}	-2.37×10^{-9}
$SR^* - SR_1^C$	-1.82×10^{-9}	-3.50×10^{-9}	-1.68×10^{-9}
$SR^* - SR_2^C$	-2.73×10^{-9}	-3.32×10^{-9}	-2.25×10^{-9}
	Fama-French		
SR^*	7.15	9.23	10.32
$SR^* - SR_1^U$	-1.876	-0.812	-0.176
$SR^* - SR_2^U$	-0.603	0.449	1.124
$SR^* - SR_1^C$	-1.818	-0.747	-0.105
$SR^* - SR_2^C$	-0.563	0.498	1.175
	Asness-Moskowitz-Pedersen		
SR^*	19.65	20.90	25.36
$SR^* - SR_1^U$	9.590	7.568	9.672
$SR^* - SR_2^U$	7.052	6.445	8.060
$SR^* - SR_1^C$	10.497	8.228	10.566
$SR^* - SR_2^C$	7.162	6.347	8.168
	Carhart		
SR^*	19.85	20.85	25.18
$SR^* - SR_1^U$	5.981	5.250	6.733
$SR^* - SR_2^U$	6.004	5.427	7.099
$SR^* - SR_1^C$	5.919	5.224	6.822
$SR^* - SR_2^C$	6.259	5.671	7.416

The Efficiency vs. Pricing Accuracy Trade-Off in GMM Estimation of Multifactor Linear Asset Pricing Models

ONLINE APPENDIX

Appendix A. Model Tests – J–statistics

Appendix A. The Beta Representation

In our particular case, with the moments conditions function $g(x_t$ is represented by:

$$g(x_t, \theta) = \begin{pmatrix} r_t - \mathbf{B} f_t \\ \text{vec}[(r_t - \mathbf{B} f_t) f_t'] \\ f_t - \mu \end{pmatrix}_{(N+NK+K)\times 1}, \tag{A1}$$

In which, for any θ, the sample analogue of $\text{E}[g(x_t, \theta)]$ is

$$g_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} g(x_t, \theta). \tag{A2}$$

Therefore a natural estimation strategy for θ is to choose the values that make $g_T(\theta)$ as close to the zero vector as possible. For that reason, we choose θ to solve

$$\min_{\theta} \; g_T(\theta)' \mathbf{W}^{-1} g_T(\theta). \tag{A3}$$

To compute the first-stage GMM estimator θ_1 we consider $\mathbf{W} = \mathbf{I}$ in the minimization (A3). The second-stage GMM estimator θ_2 is then the solution to the problem (A3) when the weighting matrix $\mathbf{W}$ is the spectral density matrix of $g(x_t, \theta_1)$:

$$\mathbf{S} = \sum_{j=-\infty}^{\infty} \text{E}[g(x_t, \theta_1) g(x_t, \theta_1)'], \tag{A4}$$

i.e., $\mathbf{W} = \mathbf{S}$, where $\mathbf{S}$ is of size $N \times N$. Moreover, to examine the validity of the pricing model derived from the moment restrictions in equation (23), we can test whether the vector of N Jensen's alphas, given by $\alpha = \text{E}[r_t] - \delta \mathbf{B}$ is jointly equal to zero. This approach is known as the restricted test, see MacKinlay and Richardson (1991). This can be done using the J-statistic which turns out to have an asymptotic χ^2 distribution. The covariance matrix of the pricing errors, $\text{Cov}(g_T)$, is given by

$$\text{Cov}(g_T) = \frac{1}{T} \left[(I - \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}') \mathbf{S} (I - \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}') \right], \tag{A5}$$

and the test is a quadratic form in the vector of pricing errors. In particular, the Hansen (1982) J-statistic is computed as

$$\begin{array}{ll} \text{First-stage:} & g_T(\theta_1)' \text{Cov}(g_T)^{-1} g_T(\theta_1) \sim \chi^2_N, \\ \text{Second-stage:} & T g_T(\theta_2)' S^{-1} g_T(\theta_2) \sim \chi^2_N. \end{array} \tag{A6}$$

Both the first and second-stage statistics in equation (A6) lead to the same numerical value. However, if we weight equations (A5) and (A6) by any other matrix different from $\mathbf{S}$, such as $\text{E}[r_t r_t']$ or

Cov$[r_t]$, this result will no longer hold. Given that there are $N + NK + K$ equations and $NK + K$ unknown parameters in the vector equation (A1), the degrees of freedom are equal to N. In the main paper we provided specification tests results when the weighting matrix $\mathbf{S}$ is used, and in Section

Appendix B. The SDF representation

From the moment restrictions and equation (7), we obtain the vector of N pricing errors defined as $\pi \equiv$E$[r_t]-$E$[r_t f_t']\lambda$. The numerical estimation of the parameters implied by equation (7) can once more be obtained by GMM. Let's start by writing the sample pricing errors as

$$h_T(\lambda) = \frac{1}{T}\sum_{t=1}^{T}(-r_t + r_t f_t')\lambda, \tag{A7}$$

and by defining $\mathbf{D}^U = -\frac{\partial h_T(\lambda)}{\partial \lambda'} = \frac{1}{T}\sum_{t=1}^{T} r_t f_t'$, the second-moment matrix of returns and factors. The first-order condition to minimize the quadratic form of the sample pricing errors, equation (A3), is $-\left(\mathbf{D}^U\right)' \mathbf{W}[\frac{1}{T}\sum_{t=1}^{T} r_t - \mathbf{D}^U \lambda'] = 0$, where $\mathbf{W}$ is the GMM weighting matrix of size $N \times N$, equal to the identity matrix in the first-stage estimator and equal to the spectral density matrix $\mathbf{S}$, equation (A4), in the second-stage estimator. Therefore, the GMM estimates of λ are:

$$\begin{aligned} \widehat{\lambda}_1^U &= \left(\left(\mathbf{D}^U\right)' \mathbf{D}^U\right)^{-1} \left(\mathbf{D}^U\right)' \tfrac{1}{T}\textstyle\sum_{t=1}^{T} r_t, \\ \widehat{\lambda}_2^U &= \left(\left(\mathbf{D}^U\right)' \mathbf{S}^{-1}\mathbf{D}^U\right)^{-1} \left(\mathbf{D}^U\right)' \mathbf{S}^{-1} \tfrac{1}{T}\textstyle\sum_{t=1}^{T} r_t. \end{aligned} \tag{A8}$$

For illustrative purposes, we add an apex an U to $\widehat{\lambda}$ to indicate when the estimator is obtained from the uncentered specification and with a C to indicate when it comes from the centered specification.

Specifying the SDF as a linear function of the factors as in equation (7) is very popular in the empirical literature. However, Kan and Robotti (2008) point out that equation (7) is problematic because the specification test statistic will not be invariant to an affine transformation of the factors; Burnside (2007) reaches a similar conclusion. Therefore, we also consider an alternative specification that defines the SDF as a linear function of the de-meaned factors. Examples of this representation can be found in Julliard and Parker (2005) and Yogo (2006). The alternative centered version of equation (7) is therefore defined as:

$$\text{E}[r_t[1 - \lambda(f_t - \mu)]] = 0_N. \tag{A9}$$

According to Jagannathan and Wang (2002) and Jagannathan et al. (2008), it is also possible to estimate μ in equation (A9) outside of the GMM estimation by the mean, $\mu =$E$[f_t]$. This is because the number of added moment restrictions is the same as the number of added unknown parameters. Hence, the efficiency of the estimator remains the same. By following this logic, we can drop the factor-mean moment condition without ignoring that it has to be estimated, to obtain analytical expressions for $\widehat{\lambda}_1^C$ and $\widehat{\lambda}_2^C$. In fact, the procedure to enforce the moment restrictions in equation (A9) and to solve the GMM minimization is similar to that for the uncentered SDF^U method. In particular, we substitute $\text{E}[r_t f_t]$ for $\text{Cov}[r_t f_t]$ in equation (A7) and define $\mathbf{D}^C = -\frac{\partial h_T(\lambda)}{\partial \lambda'}$ as the covariance matrix of returns and factors. As a result, under SDF^C, the first and second stage GMM estimates are given by:

$$
\begin{aligned}
\widehat{\lambda}_1^C &= \left((\mathbf{D}^C)'\mathbf{D}^C\right)^{-1}(\mathbf{D}^C)'\tfrac{1}{T}\textstyle\sum_{t=1}^T r_t,\\
\widehat{\lambda}_2^C &= \left((\mathbf{D}^C)'\mathbf{S}^{-1}\mathbf{D}^C\right)^{-1}(\mathbf{D}^C)'\mathbf{S}^{-1}\tfrac{1}{T}\textstyle\sum_{t=1}^T r_t.
\end{aligned} \tag{A10}
$$

Valid specification tests can be conducted by using (A2) and (A6), the only difference being that we substitute $\mathbf{B}$ by $\mathbf{D}^U = \text{E}[r_t f_t]$ (the second moment matrix of returns and factors) in the SDF^U case, and by $\mathbf{D}^C = \text{Cov}[r_t f_t]$ (the covariance matrix of returns and factors) in the SDF^C case. The degrees of freedom in equation (A6) are specific to the Beta method, as under the SDF method the degrees of freedom is equal to $N - K$, because there are N equations and K unknown parameters in both equations (7) and (A9).[32]

[32]Equations (A5) and (A6) are weighted by (A4), and this is known to be optimal. This approach was first suggested by Hansen (1982) because it maximizes the asymptotic elicitation of information in the sample about a model, given the choice of moments. However, there are also alternatives for the weighting matrix which are suitable for model comparisons because they are invariant to the nature of the model and their parameters. For instance, Hansen and Jagannathan (1997) suggest the use of the second moment matrix of excess returns $\mathbf{W} = \text{E}[r_t r_t']$ instead of $\mathbf{W} = \mathbf{S}$. Also, Burnside (2007), Balduzzi and Yao (2007), and Kan and Robotti (2008) suggest that the SDF^C method should use the covariance matrix of excess returns $\mathbf{W} = \text{Cov}[r_t]$. We shall investigate the implications of using these alternative weighting matrices later.

Appendix B. Higher-order Moments Variation

In this section, we propose to perform an alternative simulation exercise different from the one explained in subsection IV.B, to empirically reveal how the statistical characteristics of factors and their relation to the test portfolios relates to the variance of λ^* and $\widehat{\lambda}$. In particular, we simulate 200 series of size $T = 996$ for the market, size, value and momentum factors. The particularity of this new factor simulation is that the mean, skewness and kurtosis are set to be less than 5% different from the original historical series (see Table I for the descriptive statistics of the original samples of returns and factors). The new resulting simulated series have different variances but virtually identical mean, skewness and kurtosis, whilst the previous factors generated from the empirical distribution allow for greater variation in the mean, variance, skewness and kurtosis since we do not impose any restriction to the data-generating process.

Figures A1, A2, A3, and A4 show the results. For each of these Figures we plot the values of λ^* and $\widehat{\lambda}_1^U$ with respect to the factor variance and the second moment of returns and factors. We do not present the comparison of the centered SDF method since results are quite similar; however, they are available upon request.

[Place Figure A1 about here]

[Place Figure A2 about here]

[Place Figure A3 about here]

[Place Figure A4 about here]

Figure A1 corresponds to the results of the market factor. The mean of the factor is equal to the δ estimator in the Beta method, so any fluctuation on $\widehat{\delta}$ is independent of the higher order factor moments. Therefore, the negative relation between λ^* and the variance of the factor is straightforward explained by the definition of λ^* in equation 8. In other words, the negative relation on the upper-left panel of Figure A1 is mainly driven by the transformation from $\widehat{\delta}$ to λ^*. On the other hand, the lower-left panel shows that the values of $\widehat{\lambda}_1^U$ are extremely sensitive to variations in the factor variance $\mathbf{\Sigma}$. This high sensitivity of $\sigma(\widehat{\lambda}_1^U)$ is present for low or high values of $\mathbf{\Sigma}$. In sum, the variance of the factor does not seem to have a significant role at explaining the fluctuations of Beta estimators since $\widehat{\delta}$ is independent of $\mathbf{\Sigma}$. However, the fluctuations of SDF estimators seem to be quite sensible to $\mathbf{\Sigma}$.

According to Table I, the variance of the market factor for the US is around 30. A value of $\mathbf{\Sigma} \approx 30$ in Figure A1 may correspond to values of $\widehat{\lambda}_1^U$ between -200 to 300. However, we know from Table II that the expected value of λ^* and $\widehat{\lambda}_1^U$ are around 2.15. The question that arises is why market factors with variance around 30 may lead to such dramatic fluctuations on $\widehat{\lambda}_1^U$ given that we know that the value of $\sigma(\widehat{\lambda}_1^U)$ is only 0.63? The answer relies on the high covariance, or the high second moment of returns and factor.

Right panel of Figure A1 relates the values of Beta and SDF λ estimators to the second moment of returns and factor. As expected, Beta estimators are independent of the covariance of returns and factors. Contrary, SDF estimators reveal a clear relation with respect to the second moment of returns and factors. To better understand this relation we can refer to equation (A7) in which we illustrate the moment conditions of the uncentered SDF method which is minimized by GMM. The equation (A7) imply that $\mathrm{E}[r_t] \approx \lambda \mathrm{E}[r_t f_t]$ where $\mathrm{E}[r_t f_t]$ represent the second moment matrix of returns and factor. Thus, relative high values of $\mathrm{E}[r_t f_t]$ represent an advantage to the SDF method at estimating $\widehat{\lambda}_1^U$ with greater precision. By the same token, values of $\mathrm{E}[r_t f_t]$ close to zero would represent a drawback of the SDF method at estimating $\widehat{\lambda}_1^U$ because the method would allow all the necessary variation on the estimator in order to validate $\mathrm{E}[r_t] \approx \lambda \mathrm{E}[r_t f_t]$.

According to the historical series of market returns and the 10 size sorted portfolios, the value of $\mathrm{E}[r_t f_t] = 37.19$. This value is identified in the black squared data cursor. The corresponding values of 2.008 and 2.189 are not identical to those on Table II since the values on Figure A1 correspond to one single estimation whereas results on Table II correspond to the expected value of 10,000 realizations.

This evidence reveals that results such as the ones presented on Jagannathan and Wang (2002) which argue that there are no differences in the efficiency of the Beta and SDF methods, are not driven by intrinsic similarities of the methods. In fact, the similarity in terms of efficiency is driven by the relative high value of $\mathrm{E}[r_t f_t]$. In other words, if the Beta and SDF methods lead to similar levels of efficiency at estimating λ can be mostly explained by the statistical properties of both the returns and factors rather than any empirical equivalence of the methods.

It is worthwhile to clarify that $\mathrm{E}[r_t f_t]$ is actually a $N \times K$ matrix, which in our case is a 10×1 vector. The values of $\mathrm{E}[r_t f_t]$ in Figure A1 are in fact the average of each vector value.

Figures A2 and A3 correspond to the size and value factors. The values of $\mathrm{E}[r_t f_t]$ are 7.521 and 6.428 for size and value factors, which are lower than the correspondent value of $\mathrm{E}[r_t f_t]$ for the market factor of 37.19. Therefore, it is more likely that the SDF method would lead to less accurate λ estimators. Graphically, the largest value of $\mathrm{E}[r_t f_t]$, the less volatile $\widehat{\lambda}$ should be, so that $\lambda \mathrm{E}[r_t f_t]$ would asymptotically converge to $\mathrm{E}[r_t]$, which has a value of 0.86 for the 10 size-sorted portfolios according to Table I. If the covariance is rather low, as shown in Figures A2 and A3, the method would vary $\widehat{\lambda}$ in order to $\lambda \mathrm{E}[r_t f_t]$ could asymptotically converge to $\mathrm{E}[r_t]$, and this is traduced in high values of $\widehat{\lambda}$.

Figure A4 deserves special attention since the momentum factor has a skewness equal to -3.03, and this high and negative value has important implications in the comparison of both methods.

As long as the factor exhibits a large and negative skewness, the chances that the covariance of returns and factor become negative increases. A negative value of $\mathrm{E}[r_t f_t]$ would force the SDF method to deliver a negative λ in order to validate $\mathrm{E}[r_t] \approx \lambda \mathrm{E}[r_t f_t]$ because in general $\mathrm{E}[r_t] > 0$. This is less likely to happen in the Beta method since the value of the estimator basically depends on the first moment of the factor. Figure A4 show that the value of $\mathrm{E}[r_t f_t]$ for momentum factor is 10.83, which is higher than those on size and value factor, but less than the correspondent of the market factor. For one side, the higher covariance with respect to the size and value factors tend to decreases the fluctuations of $\widehat{\lambda}$, but the marked negative skewness is the responsible of the negative value of the estimator, which is equal to -10.88, and equal to 2.932 for the Beta method.

Our results show that the efficiency of the SDF method represented by the value of $\sigma(\widehat{\lambda})$ is highly sensitive to the low covariance of the returns and factors. Besides, the higher order moments of factors such a negative skewness may cause negative values of λ which are at least not easily interpreted in economic terms. The second-stage estimators naturally increase the efficiency of the SDF method, however these gains are not enough to outperform the efficiency of Beta method. We could force the SDF method to deliver λ estimators with greater accuracy by including its definition – see equation (8) – into the moment restrictions of the uncentered SDF method in equation (A7), or into the centered moment restrictions of the SDF method in equation (A9). Nevertheless, this alternative is not the way the SDF method is usually implemented, and our main interest is evaluating the Beta and SDF methods as existing approaches.

Results such as in Jagannathan and Wang (2002) are conceivable because the covariance of the returns and factors is especially large for the market factor and because this particular factor is almost centered. However, we show that factors with a negative skewness, and a low covariance with respect to the returns such as momentum, entail significant differences in the estimation efficiency of λ and δ estimators.

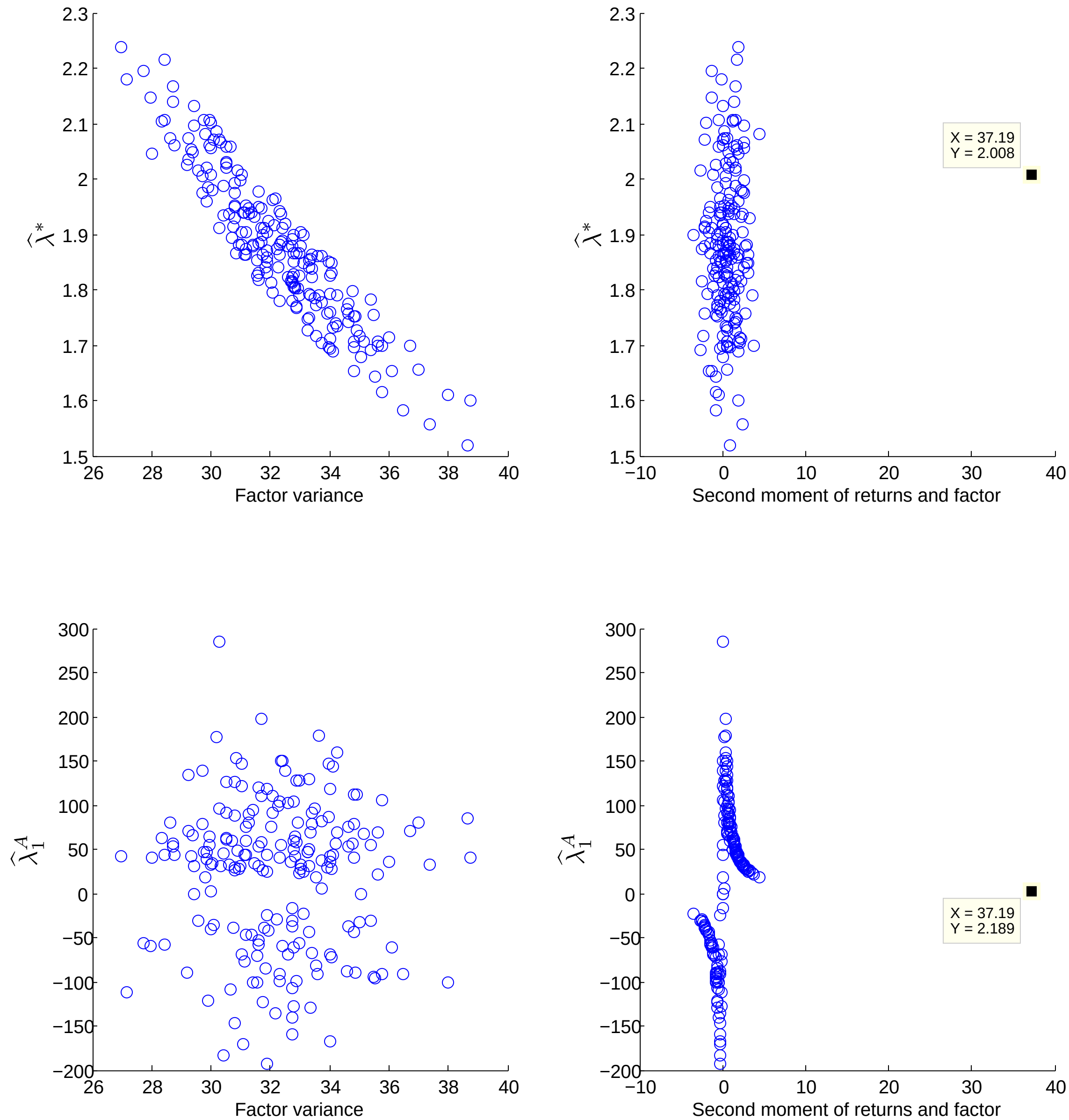


Figure A1. US: Estimators resulting from market factor draws with alternative variances and nearly independent of the portfolio returns. Actual values represented by the squared data cursor.

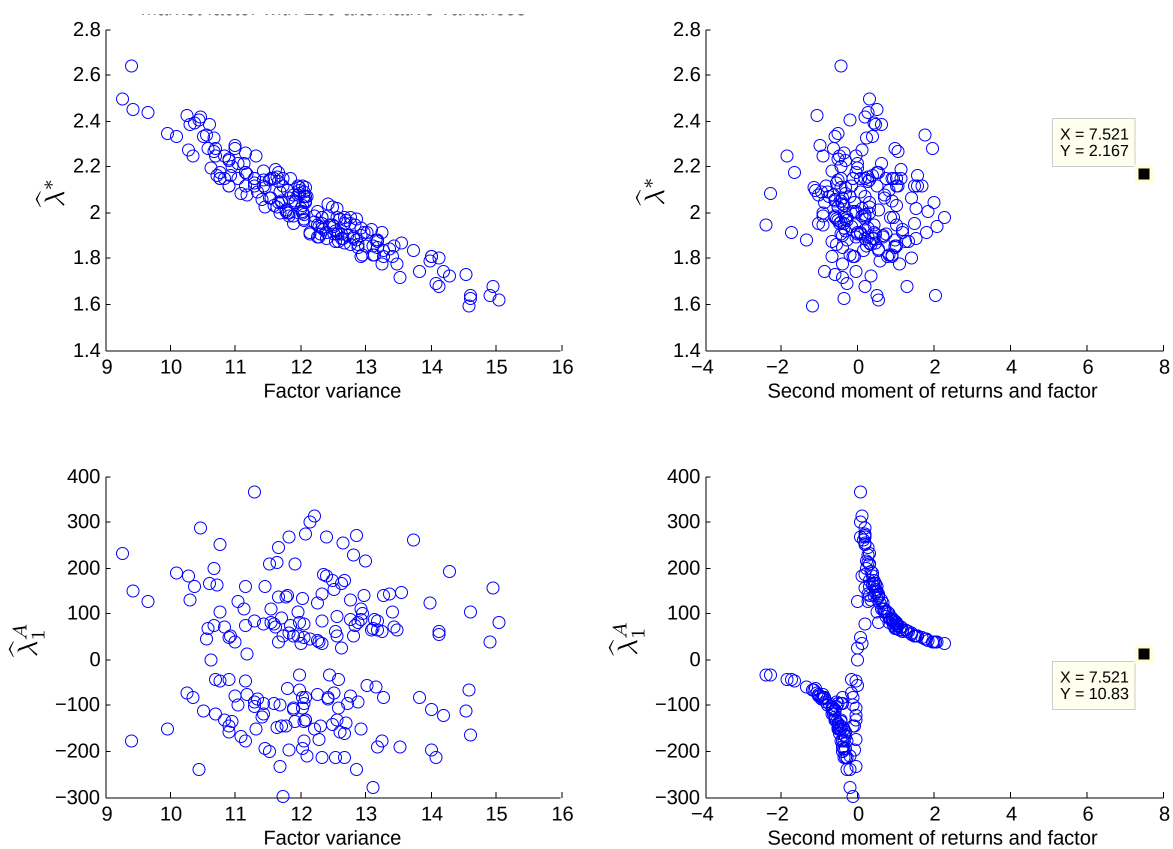


Figure A2. US: Estimators resulting from size factor draws with alternative variances and nearly independent of the portfolio returns. Actual values represented by the squared data cursor.

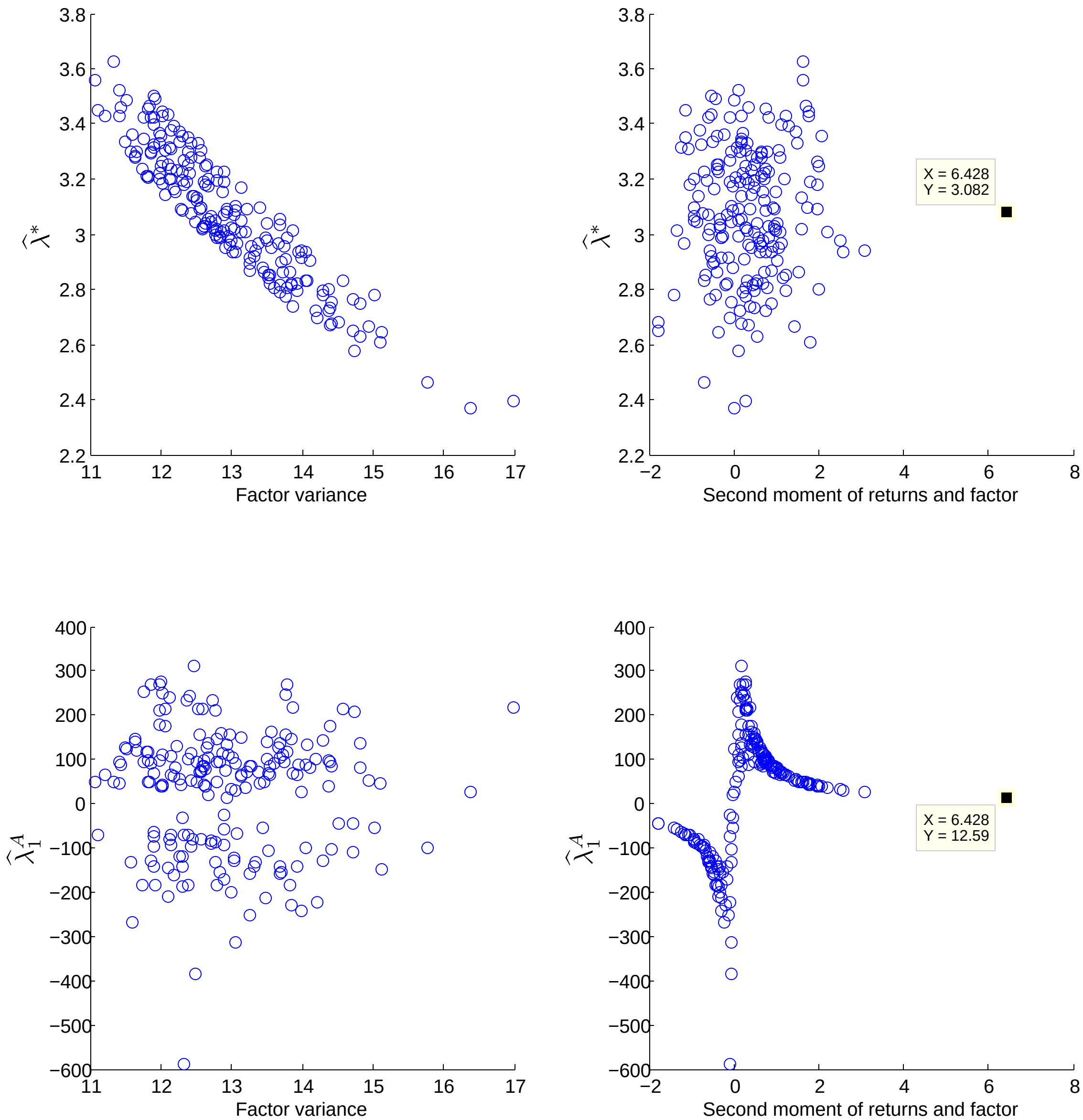


Figure A3. US: Estimators resulting from value factor draws with alternative variances and nearly independent of the portfolio returns. Actual values represented by the squared data cursor.

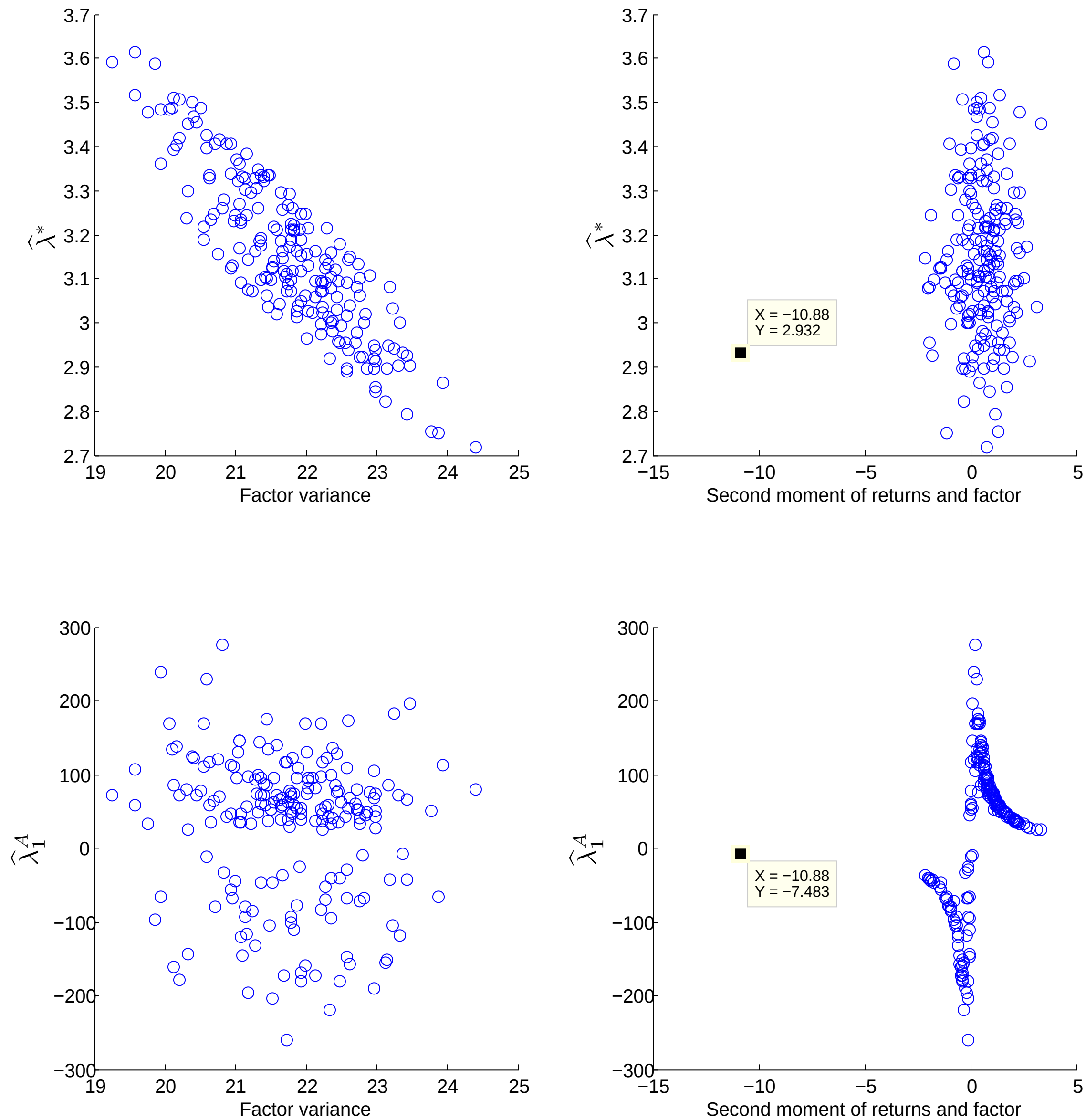


Figure A4. US: Estimators resulting from momentum factor draws with alternative variances and nearly independent of the portfolio returns. Actual values represented by the squared data cursor.

Appendix C. Further Robustness Checks

In a first additional inquiry, we explore the specific role of the sign of the skewness of the factors. We find that such skewness is likely to determine the sign of λ rather than the sign of δ. To clarify this finding, it is practical to describe the uncentered SDF method as a cross-sectional regression of mean excess returns on the second moment matrix of returns and factors. Thus, the N moment restrictions which define π are equal to the product of λ and the second moment matrix of returns and factors minus the expected returns. It turns out that if the (single, for simplicity) factor is left-skewed, it is more likely that the second moment covariance pairs between returns and factors would be negative. When this occurs, λ should be negative in order to minimize the pricing errors. Naturally, a negative λ is not what we normally expect, and it usually remains difficult to give it an economically meaningful interpretation. This is unlikely to occur under the Beta method because a subset of the moment restrictions define the value of δ. However, what does decrease the efficiency in the estimation of the λ risk premia, actually increases the pricing accuracy of the SDF representation-based inferences.

Perhaps more important is the effect of the magnitude of the second moment matrix of returns and factors over λ. Our (unreported, but available upon request) experiments confirm that there is a strikingly close relationship between a low covariance between factors and returns (in pairs), and highly volatile estimators of λ. This is easy to grasp when such covariances are slightly in excess of zero: then λ should be considerably large in order for the product between risk premia and the risk exposures implied by such covariances to equate the expected returns. In the same way, if the value of the covariance between the factors and returns is slightly negative, then λ should be considerably negative to allow the product described above to satisfy the pricing restrictions in the sample. This of course represents a valid reason to favor the inclusion of factors which display large covariation with asset returns in SDF models.

Because we have reported results for two different implementations, centered and uncentered, of the SDF representation, it is of interest to also analyze their relative performance, even though this is not the main core of our paper. The standard error of λ is consistently lower for the uncentered representation across model representations and sample sizes. This may reflect the additional K moment restrictions that appear in the centered characterization that evidently decreases estimation efficiency of λ relative to the uncentered specification. Regarding the standard deviation of the pricing errors estimates of π, the first-stage uncentered representation also delivers lower standard errors relative to the first-stage centered representation; however this is less evident for the second-stage.

Appendix D. Single-factor models with higher-order moments

One might counter that the results in Tables IV, V and VI may be partially driven by the additional factors that had failed to be investigated in earlier literature, and not by any structural differences of GMM estimators of risk premia across alternative representations of the linear factor models. To address this possibility, we estimate four alternative single-factor asset pricing models. In each case, the model just includes one of the Fama-French-Carhart risk factors at the time, i.e., the market, size, value, and momentum. The ratios of the relative, normalized standard errors of the Beta- vs. the SDF-based inferences for these single-factor models are reported in Table I. The corresponding expected values and standard errors are shown in the Online Appendix.

The evidence in Table I shows that the Beta representation leads to more accurate inferences than the SDF one, at least in terms of inferences on the risk premia. The key implication of Table I is that, because it is built by setting $K = 1$ in all the single factor models we experiment with artificially fixing every element in the assessment of the pricing models except for the assumed risk factor. Given that ratios across panels are different, the statistical characteristics of each factor as well as their relation to the test portfolios are presumably the main drivers behind the differences of the methods at estimating λ.

[Place Table XVI about here]

Table I: **Relative standard errors of risk premia estimated from four alternative single-factor models: US data, 10 size-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, Value, and Momentum) moments being sampled from the univariate empirical distribution of each factor applying bootstrapping. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively.

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
	Single-factor model loaded with market factor			
60	1.0687***	0.9995	1.1096***	1.0570***
360	1.0710***	1.0040***	1.0982***	1.0305***
600	1.0688***	1.0057***	1.0951***	1.0311***
1000	1.0630***	1.0023***	1.0892***	1.0278***
10000	1.0600***	1.0004	1.0854***	1.0258***
Asymptotic (1st-Ord)	1.0863			
	Single-factor model loaded with size factor			
60	1.7177***	1.1241***	1.6900***	1.1488***
360	1.6938***	1.0733***	1.6936***	1.0820***
600	1.6569***	1.0605***	1.6596***	1.0683***
1000	1.6501***	1.0629***	1.6549***	1.0707***
10000	1.6356***	1.0674***	1.6430***	1.0744***
Asymptotic (1st-Ord)	1.1348			
	Single-factor model loaded with value factor			
60	3.5123***	1.9236***	3.3027***	2.8719***
360	3.0388***	1.9885***	3.0202***	2.1303***
600	2.8944***	1.9656***	2.8998***	2.0550***
1000	2.9078***	1.9959***	2.9233***	2.0600***
10000	2.8930***	1.9999***	2.9249***	2.0297***
Asymptotic (1st-Ord)	2.4514			
	Single-factor model loaded with momentum factor			
60	3.0813***	1.5185***	3.1677***	2.5774***
360	1.8525***	1.5276***	1.9036***	1.6744***
600	1.7884***	1.5072***	1.8333***	1.5930***
1000	1.7856***	1.5401***	1.8268***	1.6054***
10000	1.8007***	1.5492***	1.8369***	1.5832***
Asymptotic (1st-Ord)	1.8027			

Appendix E. Convergence: Monte Carlo Simulation with Known Factors and Gaussian Error

Appendix A. Monte Carlo Simulation: Convergence

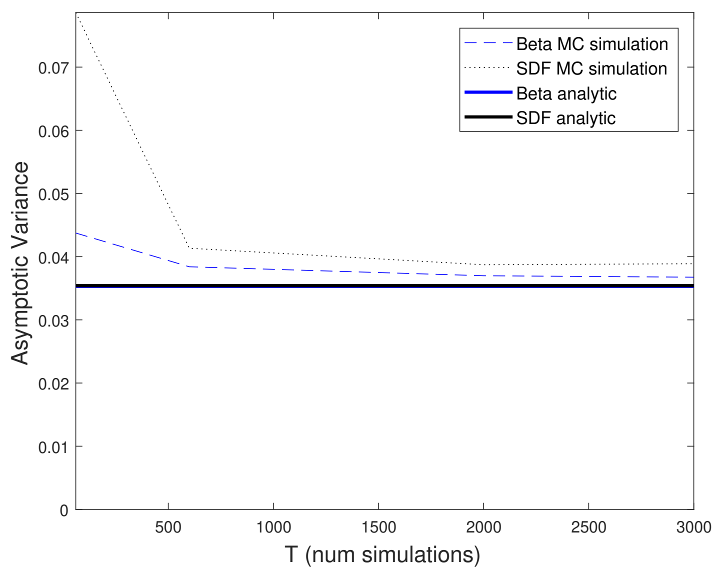
Figures A5a, A5b, A6a, and A6b present the asymptotic variance results from estimating λ^* by GMM under the Beta method and λ_2^U by GMM under the SDF method. The Monte Carlo simulation parameters used are those in Table II of the Online Appendix (16 Tables), respectively. We observe that the asymptotic variance obtained from writing the asset pricing model in a Beta framework ($Avar(\lambda^*)$) is always lower than the asymptotic variance from GMM applied under the second-stage non-centered SDF method ($Avar(\lambda_2^U)$).[33]

[Place Figure A5 about here]

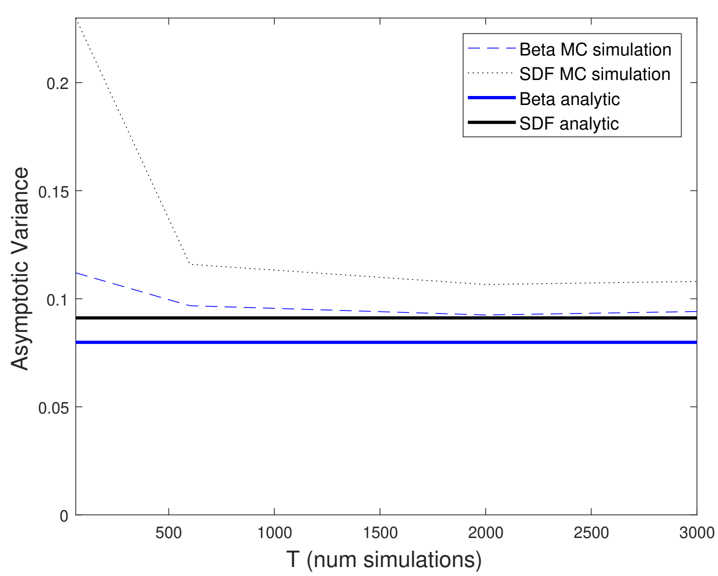
[Place Figure A6 about here]

The results in Figure A5 show that, in the case of a single-factor model based on the market risk factor, for which sample skewness is closer to zero and the kurtosis is the lower vs. all single-factor models considered in our simulations, the Beta and the SDF Monte Carlo simulations and the Beta and SDF analytic estimated asymptotic variances converge towards the same value, consistent with Jagannathan and Wang (2002). Nevertheless, in the case of the size, value, and momentum single-factor models, the SDF estimated asymptotic variance is always higher than the one estimated in a Beta framework: the higher third-order central moment (κ_3) produces an increase in the asymptotic variance, consistent with analytical results in the Section III.

[33] Results for the first-stage SDF ($Avar(\lambda_1^U)$) and first-, and second-stage centered methods ($Avar(\lambda_1^C)$, $Avar(\lambda_2^C)$) are not reported, but the resulting variances are all uniformly higher than the second-stage non-centered SDF method ($Avar(\lambda_2^U)$).

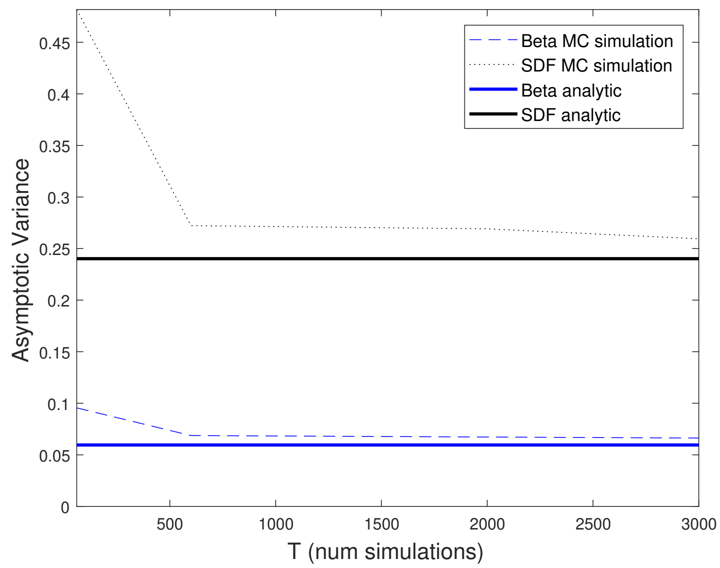


(a) Market risk factor

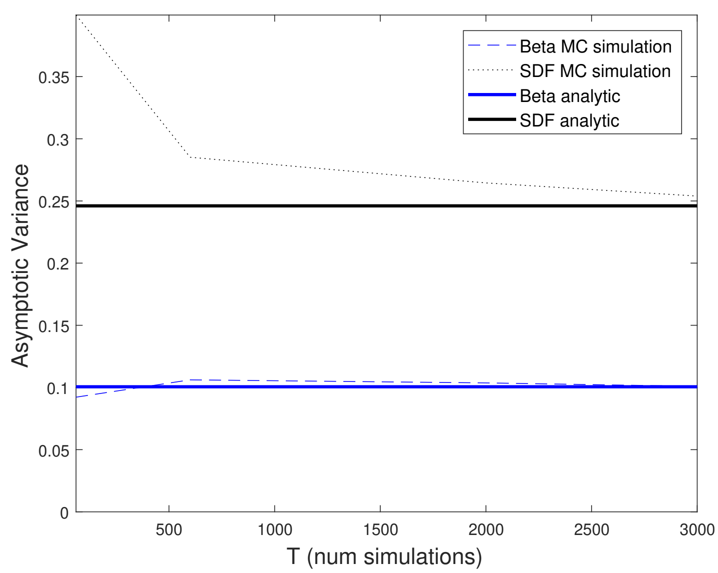


(b) Size factor

Figure A5. Asymptotic variance of the analytically and empirically estimated GMM under the Beta and SDF methods, from a set of 10,000 Monte Carlo simulation based on parameters calibrated to the observed market risk, and size factors on a sample from January 1927 to December 2018. Data are downloaded from Kenneth French's library.



(a) Value factor



(b) Momentum factor

Figure A6. Asymptotic variance of the analytically and empirically estimated GMM under the Beta and SDF methods, from a set of 10,000 Monte Carlo simulation based on parameters calibrated to the observed value, and momentum factors on a sample from January 1927 to December 2018. Data are downloaded from Kenneth French's library..

Table II: **Parameter Values used in Monte Carlo Simulation: US data, 10 size-sorted portfolios**

The table presents the parameters used for the Monte Carlo simulation to estimate GMM under Beta and SDF methods. Parameters are estimated from single US factors and portfolios taken from Kenneth R. French library, January 1927 to December 2018 (T = 1104).

Panel A: Market Factor

μ =0.6471, σ =5.3457, δ =0.6471, λ =0.0223, κ_3 =28.4438, κ_4 =8830.8782

Decile Portfolios

Small	2	3	4	5	6	7	8	9	Large
				β					
1.4187	1.3854	1.3285	1.2572	1.2287	1.2037	1.1514	1.1143	1.0628	0.9311
				α					
0.1951	0.0750	0.1006	0.1094	0.0772	0.1037	0.0708	0.0675	0.0288	-0.0051
				$\boldsymbol{\Sigma}_\epsilon$					
40.9683	26.4672	19.2810	15.9265	11.5231	8.7891	6.9172	3.6288	1.7423	-2.9924
26.4672	21.0432	14.8231	12.5335	9.4669	7.1121	5.5034	3.0992	1.2590	-2.5093
19.2810	14.8231	12.7128	10.2017	7.7451	6.0527	4.5158	2.9101	1.2151	-2.1518
15.9265	12.5335	10.2017	9.9662	6.9964	5.6539	4.3061	2.7373	1.1963	-1.9928
11.5231	9.4669	7.7451	6.9964	6.3529	4.4908	3.4261	2.2858	1.0150	-1.6416
8.7891	7.1121	6.0527	5.6539	4.4908	4.5537	2.9280	2.1043	1.0706	-1.3824
6.9172	5.5034	4.5158	4.3061	3.4261	2.9280	3.2914	1.7930	0.9149	-1.1456
3.6288	3.0992	2.9101	2.7373	2.2858	2.1043	1.7930	2.1077	0.8522	-0.8831
1.7423	1.2590	1.2151	1.1963	1.0150	1.0706	0.9149	0.8522	1.2128	-0.4942
-2.9924	-2.5093	-2.1518	-1.9928	-1.6416	-1.3824	-1.1456	-0.8831	-0.4942	0.6792

Panel B: Size Factor

μ =0.2096, σ =3.1976, δ =0.2096, λ =0.0204, κ_3 =63.0820, κ_4 =2329.3833

Decile Portfolios

Small	2	3	4	5	6	7	8	9	Large
				β					
2.1964	1.9059	1.6321	1.4843	1.3097	1.1070	0.9908	0.8134	0.6319	0.3032
				α					
0.6529	0.5722	0.6183	0.6119	0.5979	0.6507	0.6082	0.6182	0.5841	0.5339
				$\boldsymbol{\Sigma}_\epsilon$					
49.1522	39.8210	36.4736	33.5470	31.9046	32.7070	31.3220	30.5135	30.6135	27.9171
39.8210	38.7365	35.5968	33.3644	32.5677	33.1717	31.7548	31.3391	30.9958	28.4157
36.4736	35.5968	35.8935	33.1421	32.5133	33.2535	31.6684	31.6150	30.9928	28.1090
33.5470	33.3644	33.1421	32.5886	31.2411	32.0761	30.6112	30.4023	29.7651	26.8311
31.9046	32.5677	32.5133	31.2411	31.9316	31.9058	30.5595	30.4927	29.8437	26.9634
32.7070	33.1717	33.2535	32.0761	31.9058	33.4033	31.2926	31.2015	30.4507	27.1876
31.3220	31.7548	31.6684	30.6112	30.5595	31.2926	31.1114	30.1903	29.4565	26.3927
30.5135	31.3391	31.6150	30.4023	30.4927	31.2015	30.1903	30.7998	29.4141	26.2191
30.6135	30.9958	30.9928	29.7651	29.8437	30.4507	29.4565	29.4141	29.3838	25.8014
27.9171	28.4157	28.1090	26.8311	26.9634	27.1876	26.3927	26.2191	25.8014	24.4915

Table I: **Parameter Values used in Monte Carlo Simulation: US data, 10 size-sorted portfolios (Cont.)**

The table presents the parameters used for the Monte Carlo simulation to estimate GMM under Beta and SDF methods. Parameters are estimated from single US factors and portfolios taken from Kenneth R. French library, January 1927 to December 2018 (T = 1104).

Panel C: Value Factor

μ =0.3682, σ =3.4880, δ =0.3682, λ =0.0299, κ_3 =92.6417, κ_4 =3282.2015

Decile Portfolios

Small	2	3	4	5	6	7	8	9	Large
				β					
1.3215	1.0101	0.8781	0.7841	0.6631	0.6651	0.5611	0.5238	0.4976	0.2951
				α					
0.6266	0.5996	0.6370	0.6343	0.6282	0.6377	0.6093	0.5958	0.5333	0.4888
				$\mathbf{\Sigma}_\epsilon$					
77.2036	66.3564	58.9870	54.2554	50.6370	46.8608	44.5395	40.3482	36.7966	29.9806
66.3564	63.4395	56.5904	52.6359	49.9237	46.5579	44.1559	40.7427	37.1873	30.6963
58.9870	56.5904	53.7316	49.5204	47.2705	44.6110	42.1987	39.5844	36.2158	30.0150
54.2554	52.6359	49.5204	47.6225	44.7794	42.5227	40.2872	37.7428	34.6031	28.6167
50.6370	49.9237	47.2705	44.7794	44.1080	41.3551	39.2925	37.1520	34.2859	28.6418
46.8608	46.5579	44.6110	42.5227	41.3551	40.5448	37.9611	36.1645	33.5728	28.2310
44.5395	44.1559	42.1987	40.2872	39.2925	37.9611	37.3131	34.8500	32.4578	27.4493
40.3482	40.7427	39.5844	37.7428	37.1520	36.1645	34.8500	34.2225	31.4955	26.8597
36.7966	37.1873	36.2158	34.6031	34.2859	33.5728	32.4578	31.4955	30.4520	25.9737
29.9806	30.6963	30.0150	28.6167	28.6418	28.2310	27.4493	26.8597	25.9737	24.3724

Panel D: Momentum Factor

μ =0.6617, σ =4.6930, δ =0.6617, λ =0.0295, κ_3 =-315.7859, κ_4 =14986.8391

Decile Portfolios

Small	2	3	4	5	6	7	8	9	Large
				β					
-0.8723	-0.7138	-0.6682	-0.5838	-0.5502	-0.5373	-0.4702	-0.4723	-0.4548	-0.3434
				α					
1.6904	1.4439	1.4025	1.3093	1.2364	1.2382	1.1270	1.1012	1.0175	0.8247
				$\mathbf{\Sigma}_\epsilon$					
81.6857	68.8806	60.2655	55.6434	50.7276	47.2306	44.5265	39.6965	36.0591	28.1299
68.8806	64.6311	56.8770	53.0932	49.4241	46.2851	43.6598	39.7564	36.1533	28.9266
60.2655	56.8770	53.2794	49.3051	46.2587	43.8098	41.2739	38.2313	34.8395	28.1163
55.6434	53.0932	49.3051	47.5951	44.0312	41.9589	39.5941	36.6677	33.5025	27.0181
50.7276	49.4241	46.2587	44.0312	42.7920	40.2110	38.1225	35.6563	32.7902	26.8634
47.2306	46.2851	43.8098	41.9589	40.2110	39.5691	36.9378	34.8153	32.2180	26.5570
44.5265	43.6598	41.2739	39.5941	38.1225	36.9378	36.2747	33.5359	31.1455	25.9092
39.6965	39.7564	38.2313	36.6677	35.6563	34.8153	33.5359	32.6494	29.9370	25.1703
36.0591	36.1533	34.8395	33.5025	32.7902	32.2180	31.1455	29.9370	28.9096	24.3221
28.1299	28.9266	28.1163	27.0181	26.8634	26.5570	25.9092	25.1703	24.3221	22.8367

Appendix F. Expected Value And Standard Errors Bootstrap Simulations

Table II: **Expected value and standard errors of risk premia for CAPM model: US data, 10 size-sorted portfolios**

The table presents the expected value and the standard error of the GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the univariate empirical distribution of the factor. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped empirical simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\mathrm{E}[\lambda^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\lambda^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
60	2.37	2.40	3.18	2.59	3.00	2.69	2.93	3.61	3.26	3.58
360	2.17	2.17	2.29	2.22	2.28	1.03	1.11	1.09	1.16	1.11
600	2.16	2.17	2.23	2.21	2.24	0.80	0.85	0.83	0.89	0.85
1000	2.15	2.15	2.19	2.19	2.21	0.62	0.66	0.64	0.69	0.66

Table III: Expected value and standard errors of risk premia for Fama-French model: US data, 10 size-sorted portfolios

The table presents the expected value and the standard error of the GMM estimates under the Beta and the SDF methods. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, and Value) moments being sampled from the empirical distribution applying bootstrapping. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\mathrm{E}[\lambda^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\lambda^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
Market										
60	2.26	2.18	2.71	2.37	2.64	3.23	3.57	3.99	4.01	4.14
360	1.83	1.80	1.88	1.86	1.91	1.17	1.30	1.27	1.35	1.30
600	1.79	1.77	1.81	1.82	1.84	0.88	0.97	0.94	1.01	0.96
1000	1.78	1.76	1.78	1.80	1.82	0.68	0.75	0.70	0.77	0.72
Size										
60	-0.13	0.46	0.51	0.86	0.95	4.76	6.99	7.40	7.46	7.45
360	0.47	0.76	0.76	0.82	0.85	1.67	2.57	2.36	2.63	2.39
600	0.51	0.74	0.76	0.78	0.82	1.26	1.95	1.78	2.00	1.81
1000	0.54	0.76	0.77	0.79	0.81	0.96	1.48	1.34	1.52	1.36
Value										
60	2.03	2.63	2.83	2.25	1.75	4.44	12.73	11.31	13.54	11.56
360	2.13	2.38	2.40	2.41	2.25	1.52	5.07	4.00	5.25	4.08
600	2.13	2.36	2.35	2.40	2.29	1.16	3.87	3.03	3.99	3.10
1000	2.12	2.30	2.30	2.34	2.29	0.89	2.98	2.32	3.07	2.38

Table IV: **Expected value and standard errors of risk premia for Asness-Moskowitz-Pedersen model: US data, 10 size-sorted portfolios**

The table presents the expected value and the standard error of the GMM estimates under the Beta and the SDF methods. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Momentum, and Value) moments being sampled from the empirical distribution applying bootstrapping. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\mathrm{E}[\lambda^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\lambda^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
					Market					
60	2.53	3.14	3.74	2.83	3.07	2.92	4.15	4.24	4.89	4.68
360	2.63	2.97	3.07	2.81	2.88	1.07	1.71	1.57	1.99	1.77
600	2.64	2.96	3.02	2.92	2.97	0.81	1.34	1.20	1.58	1.38
1000	2.66	2.96	3.01	3.02	3.06	0.62	1.06	0.93	1.26	1.09
					Momentum					
60	5.15	6.52	7.76	0.81	1.39	3.50	12.51	11.05	13.93	12.24
360	4.80	5.58	5.84	3.36	3.53	1.32	7.21	6.17	8.10	6.74
600	4.74	5.54	5.68	4.19	4.28	0.99	5.99	5.01	6.71	5.56
1000	4.71	5.42	5.56	4.75	4.82	0.76	4.82	4.00	5.42	4.50
					Value					
60	2.26	4.84	5.41	1.35	1.01	4.44	13.83	11.60	15.43	12.56
360	3.44	4.69	4.82	3.20	3.00	1.65	6.66	5.63	7.41	6.17
600	3.56	4.73	4.81	3.84	3.70	1.27	5.39	4.51	6.01	5.00
1000	3.64	4.68	4.79	4.27	4.20	0.97	4.37	3.63	4.89	4.05

Table V: **Expected value and standard errors of risk premia for Carhart model: US data, 10 size-sorted portfolios**

The table presents the expected value and the standard error of the GMM estimates under the Beta and the SDF methods. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, Value, and Momentum) moments being sampled from the empirical distribution applying bootstrapping. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\mathrm{E}[\lambda^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\lambda^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
					Market					
60	2.80	3.05	3.54	2.70	2.86	3.27	4.21	4.36	5.02	4.96
360	2.66	2.79	2.88	2.60	2.68	1.17	1.70	1.62	1.96	1.86
600	2.65	2.79	2.85	2.73	2.79	0.88	1.31	1.23	1.54	1.44
1000	2.66	2.80	2.84	2.83	2.88	0.67	1.03	0.97	1.22	1.15
					Size					
60	-1.44	0.74	0.83	1.27	1.20	5.20	7.80	7.93	8.61	8.47
360	-0.24	0.95	0.95	1.38	1.32	1.77	2.77	2.58	2.90	2.69
600	-0.14	0.91	0.92	1.24	1.21	1.33	2.09	1.95	2.19	2.03
1000	-0.09	0.92	0.92	1.17	1.14	1.01	1.59	1.47	1.67	1.54
					Value					
60	2.35	4.73	5.20	0.63	0.71	4.74	14.44	13.54	16.02	15.05
360	3.40	4.56	4.68	2.38	2.65	1.71	7.43	6.83	8.17	7.52
600	3.52	4.70	4.76	3.27	3.50	1.31	6.03	5.56	6.68	6.17
1000	3.59	4.70	4.74	3.85	4.05	1.01	4.82	4.46	5.35	4.96
					Momentum					
60	5.37	6.69	7.68	1.12	1.41	3.84	12.40	11.72	14.09	13.36
360	4.80	5.55	5.78	3.22	3.55	1.36	6.92	6.60	7.70	7.31
600	4.72	5.54	5.68	4.06	4.35	1.02	5.66	5.37	6.37	6.04
1000	4.68	5.49	5.57	4.64	4.87	0.79	4.48	4.30	5.08	4.86

Table VI: **Expected value and standard errors of risk premia for four alternative single-factor models: US data, 10 size-sorted portfolios**

The table presents the expected value and the standard error of λ GMM estimates under the Beta and the SDF methods. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. The first estimator decorated with * are from the Beta method; the second and third correspond to the first and second-stage uncentered SDF method; and the fourth and fifth to the first and second-stage centered SDF method. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 (T = 1104).

T	$\mathrm{E}[\lambda^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\lambda^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
					Single-factor model loaded with market factor					
60	2.37	2.40	3.18	2.59	3.00	2.69	2.93	3.61	3.26	3.58
360	2.17	2.17	2.29	2.22	2.28	1.03	1.11	1.09	1.16	1.11
600	2.16	2.17	2.23	2.21	2.24	0.80	0.85	0.83	0.89	0.85
1000	2.15	2.15	2.19	2.19	2.21	0.62	0.66	0.64	0.69	0.66
					Single-factor model loaded with size factor					
60	1.60	1.67	2.02	1.79	1.86	4.34	7.77	6.16	8.19	5.79
360	1.82	1.82	1.90	1.85	1.87	1.66	2.81	1.86	2.86	1.85
600	1.83	1.81	1.88	1.83	1.86	1.27	2.09	1.38	2.11	1.39
1000	1.82	1.82	1.86	1.83	1.85	0.98	1.61	1.06	1.63	1.06
					Single-factor model loaded with value factor					
60	2.84	2.90	3.39	3.32	2.15	3.98	14.30	9.15	15.36	8.65
360	2.91	2.84	2.99	2.93	2.80	1.43	4.25	2.92	4.36	2.94
600	2.92	2.92	3.01	2.99	2.92	1.09	3.15	2.21	3.23	2.24
1000	2.93	2.94	2.98	2.99	2.94	0.83	2.41	1.67	2.46	1.70
					Single-factor model loaded with momentum factor					
60	4.26	4.33	5.75	5.43	3.25	3.94	12.32	8.06	15.88	7.73
360	3.06	3.10	3.35	3.26	3.04	1.76	3.30	2.95	3.58	2.93
600	2.87	2.91	3.05	3.02	2.90	1.36	2.47	2.18	2.63	2.19
1000	2.77	2.77	2.87	2.85	2.80	1.03	1.85	1.65	1.94	1.68

Table VII: **Expected value and standard errors of pricing errors for four alternative asset pricing models: US data, 10 size-sorted portfolios**

The table presents the expected value and the standard error of π GMM estimates under the Beta and the SDF methods. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. The first estimator decorated with * are from the Beta method; the second and third correspond to the first and second-stage uncentered SDF method; and the fourth and fifth to the first and second-stage centered SDF method. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	E[π^*]	E[$\widehat{\pi}_1^U$]	E[$\widehat{\pi}_2^U$]	E[$\widehat{\pi}_1^C$]	E[$\widehat{\pi}_2^C$]	$\sigma(\pi^*)$	$\sigma(\widehat{\pi}_1^U)$	$\sigma(\widehat{\pi}_2^U)$	$\sigma(\widehat{\pi}_1^C)$	$\sigma(\widehat{\pi}_2^C)$
					CAPM					
60	35.45	23.60	46.44	24.38	40.73	20.13	11.30	29.78	11.66	24.14
360	14.68	9.76	15.19	9.92	15.09	8.22	4.56	8.47	4.63	8.36
600	11.38	7.55	11.55	7.66	11.57	6.46	3.55	6.56	3.60	6.56
1000	8.72	5.82	8.82	5.90	8.88	4.94	2.74	5.00	2.78	5.02
					Fama-French					
60	23.06	11.50	33.19	12.37	31.50	11.62	3.75	26.25	4.08	26.63
360	9.17	4.59	7.53	4.73	7.27	3.84	1.38	3.77	1.43	3.50
600	7.40	3.56	5.30	3.66	5.19	2.99	1.07	2.36	1.10	2.16
1000	6.16	2.77	3.89	2.84	3.87	2.43	0.83	1.54	0.85	1.49
					Asness-Moskowitz-Pedersen					
60	39.18	12.21	40.42	13.54	41.46	21.71	4.27	30.74	4.71	33.02
360	17.99	4.94	11.19	5.30	11.39	9.32	1.59	6.71	1.70	6.71
600	15.19	3.82	8.16	4.11	8.36	7.59	1.23	4.59	1.32	4.62
1000	13.32	2.96	5.94	3.18	6.21	6.32	0.94	3.12	1.01	3.19
					Carhart					
60	37.66	9.85	33.18	11.11	33.23	21.23	3.36	27.07	3.80	27.91
360	22.14	4.07	8.15	4.38	7.88	9.05	1.32	5.46	1.41	4.76
600	20.33	3.17	5.56	3.40	5.52	7.04	1.02	3.09	1.08	2.83
1000	19.30	2.48	3.95	2.67	4.02	5.57	0.80	1.90	0.85	1.83

Appendix G. GMM Estimation Specification Tests ($W = Cov\left[r_t\right]$ and $W = E\left[r_t r_t'\right]$)

Table VIII: **GMM estimation specification tests (size, $W = E[r_t r_t']$): US data, 10 size-sorted portfolios**

The table presents the results of the bootstrapped simulations on the rejection rate of the Hansen (1982) $\hat{J}$-statistic under the Beta and the SDF methods. The specification tests are size tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying bootstraping. Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
CAPM															
60	0.14	0.35	0.75	0.56	1.21	1.85	3.08	4.20	4.57	6.18	4.57	7.78	9.61	10.49	12.17
360	0.44	0.83	0.88	1.05	1.10	2.82	4.47	4.52	5.37	5.51	6.19	9.50	9.65	10.96	11.21
600	0.47	0.91	0.93	1.24	1.26	3.02	4.70	4.76	5.52	5.62	6.50	9.96	10.04	11.42	11.50
1000	0.44	0.77	0.81	0.99	1.00	2.79	4.54	4.58	5.15	5.19	6.11	9.64	9.68	10.96	11.01
Fama-French															
60	0.00	0.39	1.13	1.11	2.64	0.21	3.52	5.82	6.88	10.19	0.86	8.07	11.38	13.94	17.77
360	0.09	0.73	0.79	1.13	1.22	0.57	3.94	4.13	5.52	5.69	1.24	8.55	8.84	10.63	10.92
600	0.07	0.90	0.91	1.29	1.33	0.65	4.50	4.62	5.70	5.81	1.47	8.61	8.76	10.76	11.01
1000	0.06	0.96	0.98	1.33	1.33	0.57	4.66	4.69	5.86	5.89	1.55	9.43	9.48	11.36	11.40
Asness-Moskowitz-Pedersen															
60	0.01	0.23	1.11	0.93	2.85	0.21	2.32	4.74	6.74	10.64	0.77	6.05	9.89	13.64	18.13
360	0.04	0.56	0.62	1.38	1.60	0.47	3.07	3.43	6.20	6.67	1.44	6.93	7.45	12.01	12.54
600	0.05	0.58	0.65	1.47	1.60	0.50	3.26	3.52	6.76	6.99	1.37	7.17	7.53	12.86	13.16
1000	0.06	0.62	0.66	1.63	1.68	0.60	3.42	3.52	6.86	6.93	1.63	7.43	7.57	12.98	13.15
Carhart															
60	0.00	0.19	1.08	1.46	3.62	0.06	2.57	5.59	7.72	12.01	0.20	6.61	10.28	14.89	20.13
360	0.01	0.53	0.62	1.31	1.47	0.18	3.04	3.35	6.16	6.74	0.54	6.96	7.53	11.94	12.62
600	0.00	0.56	0.60	1.50	1.62	0.23	3.36	3.58	6.46	6.72	0.64	7.33	7.56	12.80	13.08
1000	0.03	0.58	0.63	1.60	1.63	0.34	3.77	3.82	7.30	7.48	0.81	7.85	8.03	12.76	12.91

Table IX: **GMM estimation specification tests (size, W =Cov$[r_t]$): US data, 10 size-sorted portfolios**

The table presents the results of the bootstrapped simulations on the rejection rate of the Hansen (1982) $\hat{J}$-statistic under the Beta and the SDF methods. The specification tests are size tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying bootstraping. Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 (T = 1104).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
CAPM															
60	3.30	4.66	4.96	6.67	7.15	9.84	13.02	13.57	16.54	17.33	16.06	20.62	21.32	24.95	25.75
360	0.79	1.34	1.36	1.66	1.70	3.86	5.81	5.87	6.83	6.88	7.76	11.41	11.52	13.22	13.31
600	0.70	1.30	1.33	1.56	1.58	3.68	5.50	5.54	6.43	6.47	7.50	11.17	11.23	12.70	12.74
1000	0.51	0.91	0.92	1.15	1.16	3.20	4.94	4.95	5.69	5.71	6.72	10.29	10.34	11.69	11.71
Fama-French															
60	0.64	3.41	4.38	6.72	8.33	2.45	10.18	11.81	16.61	18.69	4.72	16.83	18.72	24.87	27.15
360	0.17	1.10	1.16	1.58	1.66	0.68	4.83	4.95	6.46	6.65	1.69	9.78	9.99	12.18	12.38
600	0.10	1.03	1.07	1.50	1.53	0.77	5.13	5.18	6.31	6.46	1.71	9.56	9.65	11.90	11.99
1000	0.08	1.09	1.09	1.50	1.51	0.63	5.02	5.07	6.21	6.26	1.68	9.96	10.00	11.80	11.84
Asness-Moskowitz-Pedersen															
60	0.50	2.21	3.40	6.59	8.55	2.20	7.94	10.35	16.69	19.28	3.92	13.48	16.47	24.60	27.85
360	0.12	0.73	0.81	1.88	2.06	0.70	3.77	4.18	7.43	7.78	1.85	7.88	8.50	13.55	14.00
600	0.09	0.68	0.74	1.87	1.96	0.63	3.77	3.93	7.49	7.66	1.69	7.89	8.15	13.82	14.02
1000	0.06	0.71	0.74	1.84	1.88	0.66	3.71	3.79	7.27	7.36	1.79	7.77	7.90	13.36	13.49
Carhart															
60	0.17	1.93	3.21	6.26	8.52	0.75	7.23	9.61	15.94	19.24	1.62	12.30	15.46	24.20	27.95
360	0.03	0.67	0.79	1.66	1.81	0.31	3.61	3.97	7.26	7.64	0.73	7.82	8.34	13.28	13.83
600	0.04	0.67	0.71	1.75	1.84	0.28	3.70	3.83	7.10	7.32	0.80	7.78	8.08	13.65	13.96
1000	0.05	0.69	0.73	1.73	1.81	0.37	3.96	4.14	7.64	7.74	0.88	8.11	8.26	13.18	13.37

Table X: GMM estimation specification tests (power, $W = E\left[r_t r_t'\right]$): US data, 10 size-sorted portfolios

The table presents the results of the bootstrapped simulations on the rejection rate of the Hansen (1982) $\hat{J}$-statistic under the Beta and the SDF methods. The specification tests are power tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying bootstraping. Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
CAPM															
60	0.16	0.43	1.01	0.75	1.75	1.88	3.78	5.35	5.61	7.45	5.40	8.40	10.83	11.48	13.99
360	1.33	2.10	2.21	2.53	2.71	6.00	9.28	9.56	10.88	11.13	12.02	16.83	17.16	18.99	19.20
600	2.75	3.94	4.01	4.67	4.77	9.67	13.79	13.95	15.56	15.75	17.37	23.63	23.83	25.54	25.70
1000	5.88	8.28	8.39	9.79	9.87	17.44	23.09	23.18	25.17	25.28	27.29	34.64	34.76	37.69	37.79
Fama-French															
60	0.00	0.47	1.44	1.32	3.09	0.28	3.96	6.56	7.45	10.74	1.05	8.89	12.03	14.64	18.60
360	0.30	1.90	2.03	2.68	2.82	1.39	8.30	8.61	10.18	10.52	3.07	15.27	15.68	18.35	18.68
600	0.61	3.39	3.43	4.29	4.44	2.58	12.05	12.35	14.48	14.60	4.90	20.56	20.74	23.68	23.97
1000	1.26	7.18	7.22	8.94	9.00	5.54	20.11	20.27	22.87	23.01	10.32	31.14	31.25	34.24	34.40
Asness-Moskowitz-Pedersen															
60	0.02	0.16	1.11	1.12	2.99	0.21	2.58	5.05	6.84	10.69	0.60	6.30	9.82	14.01	18.94
360	0.24	1.47	1.71	2.73	2.95	1.61	6.59	7.14	9.95	10.40	3.64	12.54	13.20	17.78	18.42
600	0.55	2.56	2.72	4.21	4.38	2.78	9.89	10.23	14.03	14.25	5.78	17.50	18.02	23.37	23.71
1000	1.57	5.23	5.33	7.86	7.96	6.47	16.56	16.80	22.21	22.43	11.81	27.22	27.42	33.55	33.76
Carhart															
60	0.00	0.33	1.30	1.59	3.60	0.09	2.79	5.76	7.94	12.43	0.23	7.00	11.33	15.67	21.26
360	0.08	1.56	1.87	2.94	3.15	0.75	7.24	7.90	10.81	11.39	1.71	13.71	14.52	18.75	19.52
600	0.26	2.85	3.12	4.45	4.66	1.36	10.69	11.20	15.00	15.31	3.11	18.86	19.36	24.55	25.01
1000	0.76	5.48	5.67	8.28	8.47	3.72	17.80	18.07	23.15	23.38	7.29	28.48	28.75	34.51	34.74

Table XI: **GMM estimation specification tests (power, W =Cov$[r_t]$): US data, 10 size-sorted portfolios**

The table presents the results of the bootstrapped simulations on the rejection rate of the Hansen (1982) $\hat{J}$-statistic under the Beta and the SDF methods. The specification tests are power tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying bootstraping. Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
								CAPM							
60	4.01	5.50	5.94	7.77	8.30	10.95	14.29	14.99	18.23	19.19	17.14	21.97	22.89	26.92	27.86
360	2.05	3.10	3.17	3.97	4.02	7.84	11.55	11.64	13.07	13.21	14.40	19.41	19.54	21.56	21.72
600	3.37	4.88	4.94	5.78	5.82	11.10	15.46	15.52	17.60	17.71	19.30	25.26	25.32	27.50	27.63
1000	6.67	9.24	9.30	10.79	10.84	18.69	24.19	24.26	26.47	26.55	28.98	36.08	36.25	38.93	38.97
								Fama-French							
60	0.76	3.88	4.95	7.35	8.93	2.86	10.92	12.69	17.56	19.65	4.81	17.91	20.00	26.05	28.34
360	0.42	2.61	2.66	3.60	3.71	1.83	9.63	9.90	11.89	12.10	3.92	17.08	17.43	20.15	20.41
600	0.75	3.94	4.03	5.06	5.15	3.01	13.25	13.36	15.62	15.74	5.60	21.90	21.99	25.14	25.25
1000	1.46	7.85	7.88	9.48	9.51	6.08	21.09	21.25	23.88	23.92	10.91	32.12	32.17	35.29	35.38
								Asness-Moskowitz-Pedersen							
60	0.47	2.54	3.61	6.77	8.62	2.12	8.07	10.31	16.93	20.08	4.15	13.98	17.02	25.25	28.51
360	0.50	2.09	2.35	3.59	3.85	2.27	7.78	8.24	11.73	12.14	4.45	14.00	14.60	19.76	20.32
600	0.77	2.96	3.12	4.93	5.06	3.20	10.96	11.11	15.22	15.44	6.51	18.69	19.06	24.60	24.88
1000	1.82	5.78	5.89	8.42	8.55	7.02	17.46	17.60	23.26	23.36	12.49	28.16	28.35	34.41	34.56
								Carhart							
60	0.20	2.29	3.52	6.59	8.67	0.92	7.80	10.44	16.88	20.25	2.01	13.58	17.11	25.23	29.04
360	0.17	2.07	2.36	3.67	4.03	1.07	8.45	9.00	12.24	12.81	2.19	15.03	15.78	20.38	20.99
600	0.35	3.33	3.56	5.17	5.38	1.65	11.55	11.87	15.89	16.15	3.61	19.82	20.44	25.64	26.05
1000	0.85	6.07	6.23	8.91	9.06	4.07	18.60	18.83	23.88	24.07	7.71	29.23	29.52	35.27	35.48

Appendix H. Relative Standard Errors of Risk Premia And Pricing Errors With Arismendi and Kimura's (2016) Tensor Moment Simulations

Table XII: Relative standard errors of risk premia estimated from the CAPM model: US data, 10 size-sorted portfolios

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 empirical correlated simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively.

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
60	1.1650***	1.0890***	1.1670***	1.0926***
360	1.0857***	1.0071***	1.0851***	1.0042*
600	1.0777***	1.0037***	1.0780***	1.0038***
1000	1.0831***	1.0024***	1.0838***	1.0032***
Asymptotic (1st-Ord)	1.0863			

Table XIII: Relative standard errors of risk premia estimated from the Fama-French model: US data, 10 size-sorted portfolios

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, and Value) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the Fama-French model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively.

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	2.2835***	1.6661***	3.1779***	6.9512***
360	1.6878***	1.3697***	1.6682***	1.2677***
600	1.7460***	1.3676***	1.7369***	1.3209***
1000	1.7532***	1.4033***	1.7465***	1.3766***
Asymptotic (1st-Ord)	1.0460			
		Size		
60	1.4206***	1.3449***	1.3090***	1.2542***
360	1.3305***	1.1968***	1.3107***	1.1640***
600	1.3594***	1.2068***	1.3463***	1.1830***
1000	1.3780***	1.2225***	1.3695***	1.2074***
Asymptotic (1st-Ord)	1.3673			
		Value		
60	4.0466***	2.8796***	4.3327***	3.7397***
360	2.7503***	2.0834***	2.7699***	2.1755***
600	2.6829***	2.0943***	2.6935***	2.1514***
1000	2.6847***	2.1446***	2.6914***	2.1789***
Asymptotic (1st-Ord)	2.6219			

Table XIV: **Relative standard errors of risk premia estimated from the Asness-Moskowitz-Pedersen model: US data, 10 size-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Momentum, and Value) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the Asness-Moskowitz-Pedersen model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively.

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	1.0169	0.9177	2.6497**	1.0283
360	1.6657***	1.6511***	1.1663	1.0703
600	1.9383***	1.5923***	1.6583***	1.3710***
1000	2.2604***	1.7126***	2.0046***	1.5402***
Asymptotic (1st-Ord)	1.4004			
		Momentum		
60	3.0121***	2.0806***	11.3844***	27.1998***
360	7.3145***	5.7086***	9.6193***	7.2650***
600	7.5122***	6.2704***	9.1698***	7.5849***
1000	7.3603***	6.4619***	8.3892***	7.4720***
Asymptotic (1st-Ord)	4.6360			
		Value		
60	4.7366***	3.6604***	5.8230***	5.6400***
360	4.8897***	4.3255***	3.6215***	3.3582***
600	5.6468***	4.0432***	3.4851***	2.7740***
1000	5.5245***	2.4637**	2.6332**	1.5529
Asymptotic (1st-Ord)	3.6640			

Table XV: **Relative standard errors of risk premia estimated from the Carhart model: US data, 10 size-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, Value, and Momentum) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the Carhart model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively.

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	2.9680***	2.2481***	3.5925***	3.3381***
360	3.1206***	2.3523***	1.6980**	1.3372
600	1.4482*	1.3837*	1.2206	1.1326
1000	2.0155***	1.6411***	1.8518***	1.5052***
Asymptotic (1st-Ord)	1.4157			
		Size		
60	1.7373***	1.5464***	1.6783***	1.5504***
360	2.1006***	1.8263***	2.3753***	2.0157***
600	2.1930***	2.0577***	2.4536***	2.2611***
1000	1.8797***	1.7575***	2.0146***	1.8556***
Asymptotic (1st-Ord)	1.4635			
		Value		
60	3.2278***	2.8505***	3.2210***	3.2537***
360	4.5180***	5.6192***	2.5822***	3.0845***
600	4.1898***	3.7239***	2.6739***	2.6084***
1000	15.7342***	10.8345***	7.2138***	6.4949***
Asymptotic (1st-Ord)	4.4710			
		Momentum		
60	12.0459***	8.3712***	14.9666***	12.1458***
360	6.5128***	5.7430***	8.7229***	7.4313***
600	7.7401***	7.0820***	9.9305***	8.8132***
1000	6.2194***	6.1846***	6.9426***	6.8570***
Asymptotic (1st-Ord)	4.9860			

Table XVI: **Relative standard errors of risk premia estimated from four alternative single-factor models: US data, 10 size-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, Value, and Momentum) moments being sampled from the univariate empirical distribution of each factor applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively.

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
	Single-factor model loaded with market factor			
60	1.1650***	1.0890***	1.1670***	1.0926***
360	1.0857***	1.0071***	1.0851***	1.0042*
600	1.0777***	1.0037***	1.0780***	1.0038***
1000	1.0831***	1.0024***	1.0838***	1.0032***
Asymptotic (1st-Ord)	1.0863			
	Single-factor model loaded with size factor			
60	1.7317***	1.4302***	1.7161***	1.4384***
360	1.6367***	1.0598***	1.6390***	1.0681***
600	1.5855***	1.0513***	1.5910***	1.0595***
1000	1.5570***	1.0527***	1.5640***	1.0612***
Asymptotic (1st-Ord)	1.1348			
	Single-factor model loaded with value factor			
60	3.0091***	2.7394***	2.8900***	3.9371***
360	2.7805***	1.8948***	2.7740***	1.9749***
600	2.6201***	1.8053***	2.6256***	1.8521***
1000	2.5086***	1.7466***	2.5190***	1.7779***
Asymptotic (1st-Ord)	2.4514			
	Single-factor model loaded with momentum factor			
60	2.5770***	3.0863***	2.4866***	4.4503***
360	2.5737***	2.0780***	2.5649***	2.1790***
600	2.4843***	2.0274***	2.4857***	2.0857***
1000	2.4278***	1.9878***	2.4348***	2.0262***
Asymptotic (1st-Ord)	1.8027			

Table XVII: Relative standard errors of pricing errors for four alternative asset pricing models: US data, 10 size-sorted portfolios

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\pi}) = \sigma(\widehat{\pi})/\mathrm{E}(\widehat{\pi})$. The returns are generated by equation (3) under the null hypothesis with the combination of factors (Market, Momentum, Size, and Value – depending on model) being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the corresponding model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively.

T	$\sigma_r(\widehat{\pi}_1^U)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_2^U)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_1^C)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_2^C)/\sigma_r(\pi^*)$
		CAPM		
60	0.8326***	1.0971***	0.8328***	1.0121
360	0.8256***	0.9874***	0.8256***	0.9858***
600	0.8283***	0.9914***	0.8283***	0.9916***
1000	0.8252***	0.9919***	0.8253***	0.9926***
Asymptotic$^{\text{ave}}$ (1st Ord)	0.9951			
		Fama-French		
60	0.8059***	1.9665***	0.8076***	1.9020***
360	0.8107***	1.1723***	0.8107***	1.0984***
600	0.8081***	1.0548***	0.8083***	1.0293***
1000	0.8143***	1.0204**	0.8143***	1.0135
Asymptotic$^{\text{ave}}$ (1st Ord)	0.7779			
		Asness-Moskowitz-Pedersen		
60	0.6039***	1.2973***	0.6059***	1.2790***
360	0.6024***	0.9957	0.6019***	0.9806*
600	0.5957***	0.9357***	0.5962***	0.9300***
1000	0.5835***	0.9105***	0.5833***	0.9072***
Asymptotic$^{\text{ave}}$ (1st Ord)	0.6644			
		Carhart		
60	0.7673***	1.8104***	0.7691***	1.7846***
360	0.8233***	1.2487***	0.8236***	1.2042***
600	0.8523***	1.1466***	0.8527***	1.1247***
1000	0.8612***	1.0913***	0.8608***	1.0843***
Asymptotic$^{\text{ave}}$ (1st Ord)	0.7337			

Table XVIII: Expected value and standard errors of risk premia for CAPM model: US data, 10 size-sorted portfolios

The table presents the expected value and the standard error of the GMM estimates under the Beta and the SDF methods. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations.

T	$\mathrm{E}[\lambda^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\lambda^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
60	0.33	0.33	0.43	0.34	0.39	1.36	1.53	1.96	1.57	1.74
360	0.36	0.36	0.38	0.36	0.37	0.58	0.63	0.62	0.63	0.61
600	0.37	0.37	0.38	0.37	0.37	0.43	0.47	0.45	0.47	0.44
1000	0.39	0.39	0.40	0.39	0.39	0.32	0.35	0.33	0.35	0.32

Table XIX: Expected value and standard errors of risk premia for Fama-French model: US data, 10 size-sorted portfolios

The table presents the expected value and the standard error of the GMM estimates under the Beta and the SDF methods. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, and Value) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\mathrm{E}[\lambda^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\lambda^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
					Market					
60	0.12	0.11	0.14	0.08	0.03	1.19	2.41	2.34	2.47	2.28
360	0.13	0.13	0.13	0.13	0.14	0.75	1.27	1.07	1.29	1.06
600	0.14	0.14	0.15	0.14	0.15	0.60	1.03	0.86	1.04	0.85
1000	0.11	0.11	0.11	0.11	0.12	0.48	0.84	0.70	0.84	0.70
					Size					
60	0.69	0.78	0.93	0.86	0.97	1.41	2.28	2.57	2.32	2.49
360	0.35	0.37	0.38	0.38	0.39	0.96	1.35	1.26	1.36	1.25
600	0.23	0.24	0.25	0.24	0.25	0.76	1.07	0.99	1.08	0.98
1000	0.16	0.16	0.16	0.16	0.17	0.59	0.82	0.75	0.82	0.75
					Value					
60	1.46	1.47	1.84	1.40	1.43	1.39	5.67	5.07	5.81	5.11
360	0.79	0.80	0.83	0.80	0.79	1.01	2.81	2.21	2.84	2.21
600	0.57	0.58	0.58	0.58	0.56	0.81	2.22	1.74	2.24	1.74
1000	0.40	0.41	0.41	0.41	0.40	0.64	1.74	1.39	1.75	1.38

Table XX: **Expected value and standard errors of risk premia for Asness-Moskowitz-Pedersen model: US data, 10 size-sorted portfolios**

The table presents the expected value and the standard error of the GMM estimates under the Beta and the SDF methods. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Momentum, and Value) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations.

T	$\mathrm{E}[\lambda^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\lambda^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
					Market					
60	0.04	0.11	0.12	0.04	0.11	1.08	2.78	2.69	2.83	2.64
360	0.05	0.06	0.06	0.09	0.08	0.78	1.57	1.38	1.58	1.38
600	0.09	0.09	0.10	0.11	0.11	0.65	1.28	1.08	1.28	1.08
1000	0.08	0.07	0.07	0.08	0.08	0.52	1.03	0.87	1.03	0.86
					Momentum					
60	1.49	1.63	1.96	1.34	1.28	1.55	7.99	7.45	8.12	7.49
360	0.36	0.37	0.35	0.50	0.45	1.22	6.08	5.16	6.11	5.15
600	0.15	0.14	0.17	0.23	0.24	1.01	5.68	4.73	5.70	4.72
1000	0.04	0.05	0.08	0.10	0.13	0.83	4.98	4.03	4.99	4.03
					Value					
60	0.12	0.27	0.37	0.07	0.03	1.11	7.55	7.00	7.61	7.02
360	0.53	0.51	0.55	0.39	0.44	0.77	5.36	4.57	5.39	4.58
600	0.45	0.45	0.44	0.37	0.37	0.67	4.94	4.09	4.95	4.09
1000	0.36	0.37	0.34	0.33	0.30	0.56	4.27	3.44	4.27	3.44

Table XXI: **Expected value and standard errors of risk premia for Carhart model: US data, 10 size-sorted portfolios**

The table presents the expected value and the standard error of the GMM estimates under the Beta and the SDF methods. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, Value, and Momentum) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations.

T	$\mathrm{E}[\lambda^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\lambda^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
					Market					
60	1.97	2.34	2.65	2.74	2.87	2.87	3.18	3.53	3.69	3.86
360	1.91	2.12	2.18	2.26	2.29	1.00	1.09	1.11	1.17	1.17
600	1.91	2.10	2.13	2.22	2.24	0.76	0.82	0.83	0.89	0.88
1000	1.90	2.07	2.10	2.18	2.19	0.59	0.63	0.63	0.67	0.67
					Size					
60	0.16	1.53	1.71	2.33	2.37	4.48	6.44	6.73	6.99	7.12
360	0.89	1.75	1.78	2.08	2.05	1.62	2.48	2.33	2.57	2.40
600	0.99	1.77	1.80	2.00	2.00	1.23	1.87	1.75	1.93	1.80
1000	1.05	1.74	1.76	1.92	1.92	0.95	1.48	1.37	1.52	1.41
					Value					
60	1.46	2.90	3.01	1.26	1.14	4.12	12.88	11.75	13.89	12.76
360	2.18	2.71	2.81	2.08	2.24	1.42	5.63	4.93	5.98	5.25
600	2.24	2.83	2.86	2.47	2.56	1.08	4.31	3.77	4.57	4.00
1000	2.28	2.91	2.91	2.73	2.77	0.82	3.37	2.90	3.57	3.08
					Momentum					
60	3.75	4.30	5.12	0.66	1.12	4.02	11.11	10.73	12.23	11.67
360	2.66	2.94	3.15	1.93	2.16	1.61	5.53	5.36	5.78	5.55
600	2.57	2.85	2.95	2.27	2.40	1.24	4.28	4.09	4.52	4.27
1000	2.48	2.73	2.81	2.42	2.54	0.95	3.26	3.14	3.46	3.30

Table XXII: **Expected value and standard errors of risk premia for four alternative single-factor models: US data, 10 size-sorted portfolios**

The table presents the expected value and the standard error of λ GMM estimates under the Beta and the SDF methods. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. The first estimator decorated with * are from the Beta method; the second and third correspond to the first and second-stage uncentered SDF method; and the fourth and fifth to the first and second-stage centered SDF method. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\mathrm{E}[\lambda^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\lambda^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
Single-factor model loaded with market factor										
60	2.38	2.40	3.19	2.59	3.00	2.67	2.92	3.60	3.26	3.57
360	2.17	2.17	2.29	2.22	2.28	1.03	1.10	1.09	1.16	1.12
600	2.18	2.18	2.25	2.22	2.25	0.78	0.83	0.81	0.87	0.83
1000	2.14	2.14	2.18	2.18	2.20	0.62	0.66	0.63	0.68	0.65
Single-factor model loaded with size factor										
60	1.66	1.69	2.02	1.82	1.84	4.31	7.90	6.19	8.34	5.81
360	1.80	1.77	1.87	1.79	1.85	1.64	2.78	1.85	2.82	1.84
600	1.83	1.86	1.90	1.88	1.88	1.26	2.06	1.37	2.09	1.38
1000	1.85	1.87	1.88	1.89	1.88	0.97	1.60	1.05	1.61	1.06
Single-factor model loaded with value factor										
60	2.84	2.79	3.33	3.22	2.09	3.94	14.17	8.97	15.24	8.51
360	2.93	2.90	3.05	2.99	2.87	1.44	4.19	2.93	4.30	2.95
600	2.92	2.91	2.97	2.98	2.88	1.09	3.16	2.18	3.24	2.21
1000	2.91	2.96	2.98	3.01	2.94	0.84	2.41	1.67	2.46	1.70
Single-factor model loaded with momentum factor										
60	4.30	4.39	5.79	5.51	3.25	3.96	12.52	8.15	15.44	7.85
360	3.01	3.06	3.29	3.22	3.00	1.75	3.26	2.91	3.53	2.91
600	2.89	2.91	3.05	3.02	2.89	1.35	2.45	2.16	2.61	2.17
1000	2.77	2.76	2.87	2.84	2.80	1.02	1.82	1.61	1.91	1.63

Table XXIII: **Expected value and standard errors of risk premia for four alternative asset pricing models: US data, 10 size-sorted portfolios**

The table presents the expected value and the standard error of π GMM estimates under the Beta and the SDF methods. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. The first estimator decorated with * are from the Beta method; the second and third correspond to the first and second-stage uncentered SDF method; and the fourth and fifth to the first and second-stage centered SDF method. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\mathrm{E}[\pi^*]$	$\mathrm{E}[\widehat{\pi}_1^U]$	$\mathrm{E}[\widehat{\pi}_2^U]$	$\mathrm{E}[\widehat{\pi}_1^C]$	$\mathrm{E}[\widehat{\pi}_2^C]$	$\sigma(\pi^*)$	$\sigma(\widehat{\pi}_1^U)$	$\sigma(\widehat{\pi}_2^U)$	$\sigma(\widehat{\pi}_1^C)$	$\sigma(\widehat{\pi}_2^C)$
					CAPM					
60	35.39	23.50	47.11	24.27	41.30	20.05	10.98	30.04	11.34	24.39
360	14.63	9.74	15.13	9.90	15.01	8.23	4.57	8.52	4.64	8.40
600	11.25	7.50	11.48	7.61	11.48	6.35	3.48	6.48	3.54	6.45
1000	8.80	5.86	8.90	5.94	8.96	5.00	2.76	5.06	2.80	5.08
					Fama-French					
60	24.10	11.50	30.48	12.41	27.23	12.37	3.69	21.36	4.04	17.88
360	10.12	4.61	7.26	4.77	6.92	4.54	1.39	3.25	1.44	2.80
600	8.37	3.57	5.25	3.69	5.11	3.61	1.06	2.14	1.10	1.95
1000	7.07	2.76	3.85	2.84	3.81	2.89	0.82	1.49	0.84	1.42
					Asness-Moskowitz-Pedersen					
60	36.68	12.38	36.33	13.49	35.80	20.01	4.22	25.02	4.61	23.71
360	15.46	5.02	10.59	5.27	10.66	8.28	1.62	5.80	1.70	5.68
600	12.33	3.89	7.79	4.07	7.93	6.65	1.24	4.01	1.29	4.00
1000	10.15	3.03	5.87	3.16	6.02	5.31	0.98	2.94	1.02	2.95
					Carhart					
60	32.97	9.99	27.49	11.08	26.98	19.16	3.37	19.41	3.75	18.34
360	16.28	4.17	6.87	4.39	6.78	7.98	1.36	3.44	1.43	3.14
600	14.49	3.24	4.89	3.40	4.91	6.31	1.03	2.13	1.08	2.02
1000	13.28	2.52	3.59	2.63	3.65	4.99	0.81	1.49	0.84	1.47

Table XXIV: GMM estimation specification tests (size, spectral matrix $W = \mathbf{S}$)

The table presents the results of the tensor moment simulations on the rejection rate of the Hansen (1982) $\hat{J}$-statistic under the Beta and the SDF methods. The specification tests are size tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 (T = 1104).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
CAPM															
60	3.82	5.49	5.49	5.67	5.67	11.01	14.81	14.81	15.19	15.19	17.70	23.56	23.56	23.92	23.92
360	0.84	1.47	1.47	1.47	1.47	4.33	6.54	6.54	6.54	6.54	8.34	11.99	11.99	12.05	12.05
600	0.61	1.15	1.15	1.15	1.15	3.45	5.48	5.48	5.49	5.49	7.18	10.84	10.84	10.88	10.88
1000	0.54	1.13	1.13	1.12	1.12	3.49	5.33	5.33	5.33	5.33	6.91	10.52	10.52	10.51	10.51
Fama-French															
60	0.58	3.79	3.79	4.14	4.14	2.21	11.38	11.38	11.96	11.96	4.20	18.48	18.48	18.89	18.89
360	0.10	1.14	1.14	1.14	1.14	0.61	5.72	5.72	5.73	5.73	1.53	10.76	10.76	10.90	10.90
600	0.09	1.10	1.10	1.15	1.15	0.55	5.40	5.40	5.41	5.41	1.46	10.49	10.49	10.53	10.53
1000	0.07	1.04	1.04	1.04	1.04	0.57	5.25	5.25	5.26	5.26	1.43	10.55	10.55	10.52	10.52
Asness-Moskowitz-Pedersen															
60	0.68	2.62	2.62	2.81	2.81	2.83	8.32	8.32	8.82	8.82	5.09	15.01	15.01	15.26	15.26
360	0.15	0.83	0.83	0.86	0.86	0.90	4.55	4.55	4.54	4.54	1.89	9.00	9.00	8.97	8.97
600	0.14	0.84	0.84	0.83	0.83	0.79	4.17	4.17	4.22	4.22	1.93	8.53	8.53	8.40	8.40
1000	0.08	0.81	0.81	0.79	0.79	0.74	4.39	4.39	4.39	4.39	1.92	8.71	8.71	8.65	8.65
Carhart															
60	0.16	2.39	2.39	2.78	2.78	0.77	8.33	8.33	8.75	8.75	1.65	14.26	14.26	14.91	14.91
360	0.01	0.82	0.82	0.90	0.90	0.26	4.29	4.29	4.37	4.37	0.65	9.20	9.20	9.22	9.22
600	0.04	0.80	0.80	0.80	0.80	0.22	3.93	3.93	3.95	3.95	0.47	8.60	8.60	8.60	8.60
1000	0.00	0.78	0.78	0.78	0.78	0.10	3.96	3.96	3.97	3.97	0.41	8.47	8.47	8.47	8.47

Table XXV: **GMM estimation specification tests (size, spectral matrix $W = E\left[r_t r_t'\right]$)**

The table presents the results of the tensor moment simulations on the rejection rate of the Hansen (1982) $\hat{J}$-statistic under the Beta and the SDF methods. The specification tests are size tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 (T = 1104).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
								CAPM							
60	0.14	0.35	0.57	0.43	0.62	2.06	3.57	4.32	3.99	4.69	5.22	8.79	9.66	9.52	10.29
360	0.57	0.90	0.92	0.98	0.98	3.07	5.11	5.11	5.23	5.24	6.78	9.90	9.96	10.22	10.25
600	0.46	0.83	0.83	0.83	0.83	2.83	4.67	4.67	4.77	4.78	6.42	9.74	9.75	9.88	9.90
1000	0.44	0.95	0.95	0.96	0.97	3.09	4.80	4.82	4.91	4.91	6.40	9.77	9.78	9.88	9.90
								Fama-French							
60	0.01	0.61	1.17	0.92	1.59	0.25	4.45	5.98	5.59	7.35	0.81	9.92	12.26	11.77	14.21
360	0.07	0.89	0.91	1.03	1.08	0.42	5.05	5.09	5.48	5.57	1.19	10.07	10.16	10.70	10.76
600	0.06	0.94	0.95	1.02	1.03	0.37	5.00	5.03	5.30	5.32	1.26	9.90	9.96	10.36	10.37
1000	0.04	0.93	0.93	1.00	1.00	0.47	4.91	4.92	5.08	5.08	1.26	10.21	10.24	10.50	10.51
								Asness-Moskowitz-Pedersen							
60	0.00	0.53	1.09	0.74	1.38	0.27	4.46	6.42	5.30	7.39	0.94	9.83	12.50	11.41	14.16
360	0.07	0.91	0.96	1.03	1.08	0.57	5.02	5.16	5.47	5.60	1.55	10.20	10.41	10.81	11.01
600	0.11	1.02	1.02	1.05	1.05	0.62	4.70	4.74	5.10	5.15	1.56	9.97	10.04	10.49	10.56
1000	0.07	0.99	0.99	1.03	1.04	0.65	5.10	5.12	5.20	5.21	1.81	10.12	10.16	10.34	10.39
								Carhart							
60	0.01	0.64	1.48	1.09	2.25	0.05	4.80	7.48	6.71	9.77	0.26	10.87	14.16	13.60	17.37
360	0.01	0.96	1.01	1.14	1.19	0.15	5.02	5.20	5.60	5.73	0.51	10.40	10.54	11.02	11.23
600	0.01	0.83	0.87	0.94	0.95	0.15	4.72	4.74	5.01	5.07	0.37	9.68	9.71	10.13	10.22
1000	0.00	0.91	0.92	0.95	0.95	0.09	4.59	4.64	4.80	4.85	0.34	9.45	9.50	9.80	9.83

Table XXVI: GMM estimation specification tests (size, spectral matrix $W = \mathbf{Cov}[r_t]$)

The table presents the results of the tensor moment simulations on the rejection rate of the Hansen (1982) $\hat{J}$-statistic under the Beta and the SDF methods. The specification tests are size tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 (T = 1104).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
CAPM															
60	3.82	5.37	5.59	5.96	6.17	11.01	14.50	14.74	15.48	15.76	17.70	23.02	23.35	24.20	24.66
360	0.84	1.47	1.47	1.49	1.49	4.33	6.49	6.49	6.59	6.59	8.34	11.97	11.97	12.18	12.19
600	0.61	1.12	1.12	1.16	1.16	3.45	5.48	5.48	5.55	5.55	7.18	10.76	10.76	10.96	10.98
1000	0.54	1.13	1.13	1.13	1.13	3.49	5.31	5.31	5.35	5.35	6.91	10.46	10.46	10.57	10.58
Fama-French															
60	0.58	4.28	4.97	5.45	6.05	2.21	12.70	13.68	14.58	15.77	4.20	19.98	21.12	22.64	24.05
360	0.10	1.36	1.36	1.52	1.55	0.61	6.04	6.11	6.63	6.69	1.53	11.18	11.24	12.06	12.08
600	0.09	1.16	1.16	1.29	1.30	0.55	5.58	5.58	5.87	5.88	1.46	10.80	10.82	11.19	11.22
1000	0.07	1.07	1.07	1.07	1.08	0.57	5.33	5.33	5.53	5.53	1.43	10.76	10.78	11.07	11.09
Asness-Moskowitz-Pedersen															
60	0.68	4.37	5.28	5.17	6.10	2.83	12.28	13.65	14.20	15.67	5.09	20.10	21.79	22.19	23.89
360	0.15	1.42	1.43	1.56	1.57	0.90	6.13	6.19	6.62	6.72	1.89	11.59	11.74	12.21	12.36
600	0.14	1.22	1.23	1.29	1.29	0.79	5.37	5.44	5.81	5.83	1.93	10.62	10.71	11.15	11.22
1000	0.08	1.10	1.10	1.16	1.16	0.74	5.37	5.39	5.54	5.55	1.92	10.53	10.57	10.91	10.95
Carhart															
60	0.16	3.85	4.73	5.39	6.58	0.77	11.93	13.75	14.71	16.71	1.65	18.77	20.99	22.42	24.85
360	0.01	1.31	1.34	1.54	1.57	0.26	5.93	6.02	6.53	6.63	0.65	11.52	11.64	12.29	12.46
600	0.04	1.14	1.14	1.22	1.22	0.22	5.10	5.20	5.54	5.58	0.47	10.38	10.47	10.79	10.87
1000	0.00	0.97	0.98	1.05	1.07	0.10	4.85	4.86	5.03	5.05	0.41	9.88	9.90	10.16	10.21

Appendix I. Relative Standard Errors of Risk Premia And Pricing Errors With Multivariate Independent Factors Simulations

Table XXVII: Relative standard errors of risk premia estimated from the Fama-French model: US data, 10 size-sorted portfolios

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, and Value) moments being sampled from the univariate empirical distribution of each factor. Estimators decorated with a * are obtained by GMM from a Beta representation of the Fama-French model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped empirical independent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	0.9881**	0.9724***	1.0176***	1.0131**
360	1.0183***	1.0010	1.0410***	1.0246***
600	1.0242***	1.0075***	1.0464***	1.0308***
1000	1.0203***	1.0060***	1.0408***	1.0284***
Asymptotic (1st-Ord)	1.0460			
		Size		
60	0.6903***	0.6292***	0.5747***	0.5343***
360	1.0943***	0.9899	1.0673***	0.9563***
600	1.1506***	1.0386***	1.1331***	1.0175*
1000	1.1797***	1.0681***	1.1681***	1.0548***
Asymptotic (1st-Ord)	1.3673			
		Value		
60	2.0741***	1.7221***	2.3896***	2.5585***
360	3.2432***	2.4355***	3.3059***	2.6280***
600	3.1755***	2.4181***	3.2214***	2.5400***
1000	3.1909***	2.4528***	3.2280***	2.5314***
Asymptotic (1st-Ord)	2.6219			

Table XXVIII: **Relative standard errors of risk premia estimated from the Asness-Moskowitz-Pedersen model: US data, 10 size-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Momentum, and Value) moments being sampled from the univariate empirical distribution of each factor. Estimators decorated with a * are obtained by GMM from a Beta representation of the Asness-Moskowitz-Pedersen model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped empirical independent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	0.9469***	0.9175***	0.9442***	0.9345***
360	0.9928*	0.9778***	1.0010	0.9878***
600	0.9948	0.9852***	1.0075**	0.9975
1000	0.9918***	0.9831***	1.0112***	
Asymptotic (1st-Ord)	1.4004			
		Momentum		
60	2.6173***	1.9170***	39.4630***	12.2398***
360	3.2179***	2.6564***	5.0584***	4.1168***
600	3.4313***	2.8366***	4.4947***	3.7244***
1000	3.5421***	2.8872***	4.1374***	3.3929***
Asymptotic (1st-Ord)	4.6360			
		Value		
60	1.7724***	1.3215***	3.0123***	4.0724***
360	2.5930***	2.1256***	2.9961***	2.6709***
600	2.7485***	2.2194***	3.0221***	2.5730***
1000	2.7482***	2.2343***	2.9265***	2.4610***
Asymptotic (1st-Ord)	3.6640			

Table XXIX: Relative standard errors of risk premia estimated from the Carhart model: US data, 10 size-sorted portfolios

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, Value, and Momentum) moments being sampled from the univariate empirical distribution of each factor. Estimators decorated with a * are obtained by GMM from a Beta representation of the Carhart model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped empirical independent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	0.9379***	0.9170***	0.9254***	0.9279***
360	0.9841***	0.9744***	0.9896**	0.9810***
600	0.9822***	0.9716***	0.9968	0.9875***
1000	0.9814***	0.9738***	0.9928**	0.9866***
		Size		
60	0.1464***	0.1372***	0.1046***	0.1047***
360	0.7722***	0.7148***	0.6758***	0.6397***
600	0.8546***	0.7864***	0.7782***	0.7294***
1000	0.9319***	0.8527***	0.8734***	0.8091***
		Value		
60	1.5723***	1.3820***	3.8910***	3.9486***
360	3.1936***	2.6915***	4.4208***	3.6043***
600	3.1488***	2.7186***	3.8231***	3.2284***
1000	3.2321***	2.7796***	3.6420***	3.0976***
		Momentum		
60	2.4065***	1.9549***	17.2529***	9.6863***
360	3.1085***	2.8092***	4.9611***	4.2450***
600	3.1019***	2.8641***	4.1104***	3.6712***
1000	3.1059***	2.9053***	3.7084***	3.3792***

Table XXX: **Relative standard errors of pricing errors for four alternative asset pricing models: US data, 10 size-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\pi}) = \sigma(\widehat{\pi})/\mathrm{E}(\widehat{\pi})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, Value, and Momentum) being sampled from the univariate empirical distribution of each factor. Estimators decorated with a * are obtained by GMM from a Beta representation of the corresponding model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped empirical independent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\pi}_1^U)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_2^U)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_1^C)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_2^C)/\sigma_r(\pi^*)$
		CAPM		
60	0.8430***	1.1292***	0.8425***	1.0438***
360	0.8337***	0.9962	0.8334***	0.9901**
600	0.8286***	1.0018	0.8286***	0.9987
1000	0.8319***	0.9992	0.8317***	0.9987
		Fama-French		
60	0.6247***	1.3886***	0.6330***	1.2918***
360	0.6760***	1.0155	0.6764***	0.9253***
600	0.6912***	0.9578***	0.6913***	0.8946***
1000	0.7419***	0.9661***	0.7425***	0.9301***
		Asness-Moskowitz-Pedersen		
60	0.6201***	1.2266***	0.6260***	1.1963***
360	0.5893***	0.9825	0.5917***	0.9670***
600	0.5864***	0.9610***	0.5893***	0.9375***
1000	0.6017***	0.9502***	0.6042***	0.9279***
		Carhart		
60	0.5890***	1.2209***	0.5883***	1.1614***
360	0.6652***	0.9958	0.6635***	0.9296***
600	0.7399***	1.0055	0.7386***	0.9554***
1000	0.8441***	1.0778***	0.8429***	1.0460***

Table XXXI: **GMM estimation specification tests (size, spectral matrix $W = \mathbf{S}$)**

The table presents the $\hat{J}$ statistic under the Beta and the SDF methods. The specification tests are size tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the univariate empirical distribution of the factor (marginal distribution, no dependence). Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
CAPM															
60	3.30	5.21	5.21	5.91	5.91	9.84	14.48	14.48	15.54	15.54	16.06	22.17	22.17	23.45	23.45
360	0.79	1.52	1.52	1.54	1.54	3.86	6.30	6.30	6.38	6.38	7.76	12.23	12.23	12.30	12.30
600	0.70	1.40	1.40	1.43	1.43	3.68	6.07	6.07	6.08	6.08	7.50	11.85	11.85	11.85	11.85
1000	0.51	1.05	1.05	1.03	1.03	3.20	5.18	5.18	5.23	5.23	6.72	10.95	10.95	11.04	11.04
Fama-French															
60	1.03	3.19	3.19	3.89	3.89	3.29	9.74	9.74	11.04	11.04	5.65	16.58	16.58	18.18	18.18
360	0.13	1.18	1.18	1.19	1.19	1.22	5.27	5.27	5.43	5.43	2.71	10.52	10.52	10.60	10.60
600	0.20	1.11	1.11	1.14	1.14	1.11	5.36	5.36	5.38	5.38	2.60	10.48	10.48	10.53	10.53
1000	0.20	1.18	1.18	1.18	1.18	1.12	5.31	5.31	5.34	5.34	2.58	9.93	9.93	9.95	9.95
Asness-Moskowitz-Pedersen															
60	0.64	1.96	1.96	2.56	2.56	2.87	7.02	7.02	8.24	8.24	5.15	12.72	12.72	13.91	13.91
360	0.16	0.85	0.85	0.90	0.90	0.95	4.01	4.01	4.33	4.33	2.32	8.77	8.77	9.18	9.18
600	0.16	0.80	0.80	0.87	0.87	0.87	4.28	4.28	4.51	4.51	2.14	8.90	8.90	9.29	9.29
1000	0.10	0.94	0.94	1.03	1.03	0.91	4.46	4.46	4.80	4.80	2.27	8.96	8.96	9.23	9.23
Carhart															
60	0.29	1.83	1.83	2.35	2.35	1.19	6.19	6.19	7.57	7.57	2.28	11.40	11.40	13.33	13.33
360	0.06	0.89	0.89	0.90	0.90	0.52	4.08	4.08	4.50	4.50	1.26	8.63	8.63	9.31	9.31
600	0.03	0.83	0.83	0.92	0.92	0.41	4.22	4.22	4.47	4.47	1.08	8.71	8.71	8.97	8.97
1000	0.04	0.88	0.88	0.93	0.93	0.41	4.67	4.67	4.86	4.86	1.12	9.25	9.25	9.68	9.68

Table XXXII: **GMM estimation specification tests (size, $W = E[r_t r_t']$): US data, 10 size-sorted portfolios**

The table presents the results of the bootstrapped simulations on the rejection rate of the Hansen (1982) $\hat{J}$-statistic under the Beta and the SDF methods. The specification tests are size tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the univariate empirical distribution of the factor (marginal distribution, no dependence). Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T =$ 1104).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
CAPM															
60	0.14	0.35	0.75	0.56	1.21	1.85	3.08	4.20	4.57	6.18	4.57	7.78	9.61	10.49	12.17
360	0.44	0.83	0.88	1.05	1.10	2.82	4.47	4.52	5.37	5.51	6.19	9.50	9.65	10.96	11.21
600	0.47	0.91	0.93	1.24	1.26	3.02	4.70	4.76	5.52	5.62	6.50	9.96	10.04	11.42	11.50
1000	0.44	0.77	0.81	0.99	1.00	2.79	4.54	4.58	5.15	5.19	6.11	9.64	9.68	10.96	11.01
Fama-French															
60	0.00	0.43	1.22	1.36	2.90	0.44	3.46	5.54	7.25	10.46	1.34	8.11	11.16	14.31	18.41
360	0.06	0.81	0.87	1.30	1.39	0.83	4.34	4.48	5.78	5.99	2.07	8.80	9.07	11.51	11.83
600	0.11	0.88	0.92	1.29	1.31	0.95	4.45	4.54	6.02	6.14	2.23	9.01	9.16	11.65	11.76
1000	0.16	1.00	1.00	1.32	1.33	1.03	4.66	4.69	5.93	5.98	2.30	8.99	9.10	11.17	11.26
Asness-Moskowitz-Pedersen															
60	0.01	0.26	1.23	1.25	3.02	0.25	3.06	5.75	6.97	10.64	0.98	7.19	10.68	13.80	18.33
360	0.08	0.66	0.75	1.27	1.39	0.66	3.83	4.15	6.23	6.59	1.71	8.30	8.77	11.81	12.29
600	0.13	0.61	0.73	1.32	1.36	0.73	3.92	4.12	6.15	6.37	1.82	8.43	8.72	11.69	11.99
1000	0.08	0.80	0.82	1.38	1.41	0.83	4.18	4.27	6.04	6.15	2.00	8.34	8.57	11.77	11.87
Carhart															
60	0.01	0.31	1.34	1.41	3.62	0.10	3.10	5.61	7.89	12.28	0.39	7.08	10.95	15.37	19.99
360	0.04	0.62	0.80	1.27	1.45	0.34	3.81	4.25	6.33	6.70	0.96	8.48	9.08	12.41	13.07
600	0.02	0.73	0.75	1.32	1.37	0.37	3.81	3.99	6.13	6.29	0.88	8.13	8.49	11.96	12.18
1000	0.03	0.70	0.72	1.23	1.27	0.34	4.21	4.39	6.38	6.48	1.02	8.73	8.86	12.01	12.12

Table XXXIII: **GMM estimation specification tests (size, W =Cov$[r_t]$): US data, 10 size-sorted portfolios**

The table presents the results of the bootstrapped simulations on the rejection rate of the Hansen (1982) $\hat{J}$-statistic under the Beta and the SDF methods. The specification tests are size tests. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the univariate empirical distribution of the factor (marginal distribution, no dependence). Estimators decorated with a * are obtained by GMM from a Beta representation of the CAPM model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 (T = 1104).

	1 %					5 %					10 %				
	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$	λ^*	$\widehat{\lambda}_1^U$	$\widehat{\lambda}_2^U$	$\widehat{\lambda}_1^C$	$\widehat{\lambda}_2^C$
CAPM															
60	3.30	4.66	4.96	6.67	7.15	9.84	13.02	13.57	16.54	17.33	16.06	20.62	21.32	24.95	25.75
360	0.79	1.34	1.36	1.66	1.70	3.86	5.81	5.87	6.83	6.88	7.76	11.41	11.52	13.22	13.31
600	0.70	1.30	1.33	1.56	1.58	3.68	5.50	5.54	6.43	6.47	7.50	11.17	11.23	12.70	12.74
1000	0.51	0.91	0.92	1.15	1.16	3.20	4.94	4.95	5.69	5.71	6.72	10.29	10.34	11.69	11.71
Fama-French															
60	1.03	3.40	4.15	7.14	8.24	3.29	10.23	11.75	17.10	19.19	5.65	16.95	18.86	25.88	28.26
360	0.13	1.26	1.32	1.76	1.84	1.22	5.16	5.27	6.93	7.04	2.71	10.12	10.28	13.02	13.25
600	0.20	1.07	1.08	1.66	1.69	1.11	5.06	5.16	6.62	6.68	2.60	9.70	9.79	12.47	12.54
1000	0.20	1.07	1.08	1.45	1.45	1.12	4.96	4.98	6.25	6.28	2.58	9.48	9.54	11.59	11.66
Asness-Moskowitz-Pedersen															
60	0.64	2.99	4.34	6.82	8.72	2.87	9.19	11.27	16.43	19.20	5.15	15.34	18.12	25.08	28.27
360	0.16	0.82	0.98	1.66	1.79	0.95	4.70	4.99	7.36	7.57	2.32	9.60	10.05	13.51	13.93
600	0.16	0.85	0.95	1.51	1.54	0.87	4.38	4.57	6.85	7.01	2.14	9.09	9.37	12.70	12.97
1000	0.10	0.87	0.90	1.51	1.54	0.91	4.44	4.55	6.45	6.57	2.27	8.83	9.02	12.16	12.25
Carhart															
60	0.29	2.49	3.55	6.32	8.36	1.19	7.77	10.09	16.37	19.14	2.28	13.45	16.56	24.49	27.84
360	0.06	0.84	0.97	1.67	1.80	0.52	4.41	4.82	7.30	7.70	1.26	9.54	9.93	13.76	14.26
600	0.03	0.83	0.85	1.55	1.61	0.41	4.25	4.42	6.71	6.93	1.08	8.83	9.06	12.75	12.92
1000	0.04	0.78	0.80	1.38	1.39	0.41	4.52	4.61	6.74	6.85	1.12	9.10	9.19	12.40	12.54

Appendix J. Factors as Portfolio of Non-Traded Assets

Table XXXIV: Relative standard errors of risk premia estimated from the CAPM model: US data, 10 size-sorted (non-traded, Gaussian) portfolios

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from a Normal (Gaussian) distribution, with first- and second-order moments obtained from the empirical distribution. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 empirical correlated simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios for the calibration of the Gaussian factor returns is done with data taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
60	0.9948***	0.9677***	1.0507***	0.9961
360	0.9888***	0.9263***	1.0368***	0.9696***
600	0.9949***	0.9349***	1.0436***	0.9810***
1000	0.9919***	0.9374***	1.0405***	0.9849***
2000	1.0000	0.9315***	1.0483***	0.9794***
3000	1.0000	0.9335***	1.0483***	0.9815***
4000	0.9999	0.9321***	1.0480***	0.9801***
5000	1.0000	0.9306***	1.0478***	0.9784***
15000	0.9999***	0.9364***	1.0481***	0.9850***
Asymptotic (1st-Ord)	1			

Appendix K. Size/Value-Sorted Portfolios

Table XXXV: Relative standard errors of risk premia estimated from the CAPM model: US data, 25 (5×5) size/value-sorted portfolios

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 empirical correlated simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
60	1.1283***	1.1926***	1.1277***	1.1760***
360	1.0782***	1.0103***	1.0792***	1.0100***
600	1.0749***	1.0049***	1.0762***	1.0061***
1000	1.0756***	1.0043***	1.0770***	1.0058***
Asymptotic (1st-Ord)	1.0863			

Table XXXVI: Relative standard errors of risk premia estimated from the Fama-French model: US data, 25 (5×5) size/value-sorted portfolios

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, and Value) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the Fama-French model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	1.1209***	1.0039	1.4634***	1.1647***
360	1.0165	1.0127	0.9960	1.0137
600	1.0305***	1.0061	1.0218*	1.0066
1000	1.0421***	1.0051	1.0372***	1.0055
Asymptotic (1st-Ord)	1.0460			
		Size		
60	1.0491***	1.0362***	1.0530***	1.0611***
360	1.0100	0.9580***	1.0092	0.9579***
600	1.0057	0.9660***	1.0046	0.9652***
1000	1.0136	0.9880**	1.0131	0.9877**
Asymptotic (1st-Ord)	1.3673			
		Value		
60	1.0644***	1.0918***	1.0761***	1.1328***
360	1.0226***	1.0090***	1.0245***	1.0124***
600	1.0250***	1.0095***	1.0258***	1.0116***
1000	1.0288***	1.0147***	1.0291***	1.0158***
Asymptotic (1st-Ord)	2.6219			

Table XXXVII: **Relative standard errors of risk premia estimated from the Asness-Moskowitz-Pedersen model: US data, 25 (5×5) size/value-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Momentum, and Value) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the Asness-Moskowitz-Pedersen model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	2.0496**	2.8342***	6.9060***	2.9197*
360	2.4130***	1.2152*	1.1742	0.6863***
600	1.6336***	1.1800***	1.3030***	0.9302
1000	2.5470***	1.4352***	2.0860***	1.2125***
Asymptotic (1st-Ord)	1.4004			
		Momentum		
60	9.6422***	1.4593***	4.6083***	10.3833***
360	5.3467***	3.2456***	7.1039***	4.7745***
600	6.8724***	3.6902***	8.9022***	4.9611***
1000	4.9287***	3.1963***	5.5251***	3.7502***
Asymptotic (1st-Ord)	4.6360			
		Value		
60	2.8964***	1.6609***	3.2504***	2.1886***
360	3.2655***	1.7400***	2.3770***	1.3049***
600	2.5708***	1.6128***	1.8245***	1.1049
1000	5.7467***	3.0098***	9.1512***	1.1599
Asymptotic (1st-Ord)	3.6640			

Table XXXVIII: Relative standard errors of risk premia estimated from the Carhart model: US data, 25 (5×5) size/value-sorted portfolios

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, Value, and Momentum) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the Carhart model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	2.5450***	1.6942***	4.3504***	2.6045***
360	1.9443*	1.3157	0.9255	0.5689***
600	1.7260***	1.2951**	1.3948**	0.9707
1000	1.5417***	1.2218**	1.4074***	1.0580
Asymptotic (1st-Ord)	1.4157			
		Size		
60	1.1729***	1.0860***	1.1999***	1.1854***
360	1.2977***	1.1052***	1.3853***	1.2317***
600	1.3672***	1.1369***	1.4344***	1.2344***
1000	1.4354***	1.1587***	1.4861***	1.2342***
Asymptotic (1st-Ord)	1.4635			
		Value		
60	1.7712***	1.2325***	1.5643***	1.2400***
360	2.7815***	1.7672***	1.8403***	1.0947
600	2.4394***	1.7364***	1.7875***	1.1269
1000	2.3297***	1.7520***	2.0197***	1.3850**
Asymptotic (1st-Ord)	4.4710			
		Momentum		
60	8.3767***	3.7108***	23.2957***	12.8090***
360	5.1968***	3.1221***	6.6837***	4.5453***
600	5.7033***	3.2731***	6.7877***	4.2218***
1000	6.8100***	3.5643***	7.6996***	4.2554***
Asymptotic (1st-Ord)	4.9860			

Table XXXIX: Relative standard errors of pricing errors for four alternative asset pricing models: US data, 25 (5×5) size/value-sorted

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\pi}) = \sigma(\widehat{\pi})/\mathrm{E}(\widehat{\pi})$. The returns are generated by equation (3) under the null hypothesis with the combination of factors (Market, Momentum, Size, and Value – depending on model) being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the corresponding model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\pi}_1^U)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_2^U)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_1^C)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_2^C)/\sigma_r(\pi^*)$
		CAPM		
60	0.7509***	1.8679***	0.7517***	1.6579***
360	0.7436***	0.9987	0.7436***	0.9679***
600	0.7490***	0.9800***	0.7490***	0.9690***
1000	0.7533***	0.9900***	0.7533***	0.9844***
Asymptotic$^{\text{ave}}$ (1st Ord)	0.9951			
		Fama-French		
60	1.0058	2.6716***	1.0064	2.7000***
360	1.0302***	1.3790***	1.0303***	1.1449***
600	1.0276***	1.1356***	1.0275***	1.0531***
1000	1.0324***	1.0483***	1.0324***	1.0334***
Asymptotic$^{\text{ave}}$ (1st Ord)	0.7779			
		Asness-Moskowitz-Pedersen		
60	0.7709***	1.8782***	0.7684***	1.8553***
360	0.7870***	1.1361***	0.7879***	1.0870***
600	0.7726***	1.0564***	0.7723***	1.0331***
1000	0.7854***	1.0300***	0.7860***	1.0222***
Asymptotic$^{\text{ave}}$ (1st Ord)	0.6644			
		Carhart		
60	0.9158***	2.4760***	0.9169***	2.5145***
360	0.9366***	1.4532***	0.9358***	1.3097***
600	0.9582***	1.2131***	0.9577***	1.1544***
1000	0.9480***	1.0983***	0.9487***	1.0813***
Asymptotic$^{\text{ave}}$ (1st Ord)	0.7337			

Appendix L. Industry-Sorted Portfolios

Table XL: **Relative standard errors of risk premia estimated from the CAPM model: US data, 30 industry-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis that the market risk factor moments are being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the single-factor model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 empirical correlated simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
60	1.0418***	1.1901***	1.0422***	1.1662***
360	1.0310***	1.0172***	1.0321***	1.0176***
600	1.0175***	1.0087***	1.0190***	1.0103***
1000	1.0213***	1.0067***	1.0229***	1.0086***
Asymptotic (1st-Ord)	1.0863			

Table XLI: **Relative standard errors of risk premia estimated from the Fama-French model: US data, 30 industry-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, and Value) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the Fama-French model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	1.7294***	1.3307***	16.1118***	202.2165***
360	1.2154***	1.1038***	1.0803*	0.9941
600	1.3339***	1.1487***	1.2690***	1.1044***
1000	1.3476***	1.1288***	1.3111***	1.1056***
Asymptotic (1st-Ord)	1.0460			
		Size		
60	1.6500***	1.4547***	1.8112***	2.0413***
360	1.3083***	1.2022***	1.3103***	1.2402***
600	1.2611***	1.1853***	1.2578***	1.2038***
1000	1.2994***	1.2018***	1.2966***	1.2125***
Asymptotic (1st-Ord)	1.3673			
		Value		
60	1.9698***	1.5037***	2.2417***	2.1443***
360	1.4837***	1.2293***	1.5071***	1.2762***
600	1.4602***	1.2158***	1.4725***	1.2415***
1000	1.4550***	1.2350***	1.4620***	1.2503***
Asymptotic (1st-Ord)	2.6219			

Table XLII: Relative standard errors of risk premia estimated from the Asness-Moskowitz-Pedersen model: US data, 30 industry-sorted portfolios

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Momentum, and Value) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the Asness-Moskowitz-Pedersen model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	0.6928*	1.0306	0.5476***	0.9507
360	1.9183***	1.4978***	1.0104	0.8540*
600	1.4165***	1.3035***	1.1197*	1.0285
1000	1.3978***	1.3303***	1.2080***	1.1386**
Asymptotic (1st-Ord)	1.4004			
		Momentum		
60	2.0186***	1.2278	15.6964***	45.6447***
360	3.3694***	2.8215***	4.2093***	3.7788***
600	3.5269***	2.9416***	4.2000***	3.6915***
1000	3.5232***	3.0054***	3.9415***	3.4859***
Asymptotic (1st-Ord)	4.6360			
		Value		
60	2.5367***	1.8452***	3.2920***	3.1945***
360	2.5495***	2.1287***	1.9553***	1.6596***
600	2.6748***	2.1392***	1.6338***	1.2904**
1000	4.8733***	3.1221***	1.9618**	1.3606
Asymptotic (1st-Ord)	3.6640			

Table XLIII: **Relative standard errors of risk premia estimated from the Carhart model: US data, 30 industry-sorted portfolios**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda})/\mathrm{E}(\widehat{\lambda})$. The returns are generated by equation (3) under the null hypothesis with the factors (Market, Size, Value, and Momentum) moments being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the Carhart model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\lambda}_1^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^U)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_1^C)/\sigma_r(\lambda^*)$	$\sigma_r(\widehat{\lambda}_2^C)/\sigma_r(\lambda^*)$
		Market		
60	2.0547***	2.0316***	13.5836***	14.0904***
360	1.1746	0.9391	0.5503***	0.4573***
600	1.7037***	1.6292***	1.2172*	1.1375
1000	1.1573	1.2713**	1.0173	1.0864
Asymptotic (1st-Ord)	1.4157			
		Size		
60	1.4634***	1.3312***	1.6152***	1.8136***
360	1.4286***	1.3265***	1.5174***	1.4831***
600	1.3562***	1.2648***	1.4184***	1.3718***
1000	1.3326***	1.2879***	1.3721***	1.3629***
Asymptotic (1st-Ord)	1.4635			
		Value		
60	1.8141***	1.2944***	1.8072***	1.6328***
360	2.2396***	1.8925***	1.4285***	1.1833*
600	4.2790***	3.4380***	2.4612***	1.9343***
1000	2.6886***	2.5969***	2.2008***	2.0371***
Asymptotic (1st-Ord)	4.4710			
		Momentum		
60	4.6904***	3.3961***	6.9733***	7.7408***
360	3.5475***	2.9247***	4.5610***	4.0606***
600	3.0681***	2.5405***	3.5287***	3.0500***
1000	3.8727***	2.9288***	4.3313***	3.3306***
Asymptotic (1st-Ord)	4.9860			

Table XLIV: **Relative standard errors of pricing errors for four alternative asset pricing models: US data, 30 industry-sorted**

The table presents the relative standard errors of GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\pi}) = \sigma(\widehat{\pi})/\mathrm{E}(\widehat{\pi})$. The returns are generated by equation (3) under the null hypothesis with the combination of factors (Market, Momentum, Size, and Value – depending on model) being sampled from the empirical distribution applying Arismendi and Kimura's (2016) tensor moment simulation. Estimators decorated with a * are obtained by GMM from a Beta representation of the corresponding model. The U and C are obtained by GMM from the uncentered and centered SDF representations; and with 1 and 2 to the first and second-stage respectively. The results are presented for different sample sizes (T), and they are based on 10,000 bootstrapped dependent simulations. The $*$, $**$, and $***$ represents statistical significance at a p-value of $0.1, 0.05$ and 0.01, respectively. The data on US factors and portfolios are taken from Kenneth French's library. The sample spans the period January 1927 – December 2018 ($T = 1104$).

T	$\sigma_r(\widehat{\pi}_1^U)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_2^U)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_1^C)/\sigma_r(\pi^*)$	$\sigma_r(\widehat{\pi}_2^C)/\sigma_r(\pi^*)$
		CAPM		
60	0.9794***	3.3427***	0.9787***	2.9198***
360	0.9815***	1.0841***	0.9816***	1.0161***
600	0.9797***	1.0261***	0.9800***	1.0008
1000	0.9886***	1.0128***	0.9886***	1.0022
Asymptotic (1st-Ord)	0.9951			
		Fama-French		
60	1.0128***	3.2421***	1.0110**	2.9017***
360	1.0047	1.2288***	1.0043	1.0824***
600	1.0037	1.0743***	1.0035	1.0311***
1000	1.0003	1.0355***	1.0003	1.0228***
Asymptotic (1st-Ord)	0.7779			
		Asness-Moskowitz-Pedersen		
60	0.9845***	3.2685***	0.9834***	2.8902***
360	0.9808***	1.2385***	0.9814***	1.0923***
600	0.9888**	1.0882***	0.9887**	1.0381***
1000	0.9803***	1.0351***	0.9802***	1.0209***
Asymptotic (1st-Ord)	0.6644			
		Carhart		
60	1.0072	3.2757***	1.0068	3.0252***
360	1.0130**	1.2618***	1.0125**	1.1170***
600	1.0169***	1.1032***	1.0166***	1.0568***
1000	1.0182***	1.0504***	1.0178***	1.0402***
Asymptotic (1st-Ord)	0.7337			