Infinitely repeated Cheap-talk*

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Abstract

Decision makers often rely on the same advisees over long periods. Advice should be taken with caution when communication is cheap talk because advisors may have their own agenda. Still, repeated interaction allows for agreements that cannot be reached in static interactions. To explore such possible agreements in strategic information transmission, we study a repeated game version of the Crawford and Sobel (1982) model, with a new state of the world drawn at every period of the game from an identical and independent uniform distribution. We first find a version of the folk theorem where patient enough players reach perfect information transmission. We also look at repeated partition equilibria and find that repeated interaction allows for improvements in communication (more and better-distributed partitions) for impatient players. Moreover, we also show that if the Receiver's action rule favors the Sender, it allows for improvements in communication. Finally, we show that some level of favorable action towards the Sender is welfare improving.

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1 Introduction

In many real-life situations, decision makers do not have all the information about the policy consequences. Information can be gathered from informed parties with deeper knowledge

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than the decision maker. However, communication often takes the form of cheap talk, and the informed parties have distinct preferences over the policy. In such cases, information transmission is restricted or fails entirely, especially in short-term interactions. Still, long-term relations between the decision maker and the informed parties allow more complex arrangements. In this paper, we explore new possibilities for communication with repeated interaction of cheap-talk games.

In cheap talk communication, players can costlessly send any kind of message. There is a long literature on cheap-talk, with Crawford and Sobel (1982) (CS model henceforth) as the cornerstone model. In some situations, the same players engage in cheap-talk communication through a long time span, to convey information about a changing state of the world. As an example, we can think of a lobby group that conveys information to policy makers in every period. Repeated interaction creates opportunities to improve communication, using continuation payoffs to sustain cooperation. Although many papers study strategic information transmission in repeated games, such as Aumann and Hart (2003), Renault et al. (2017, 2013), the structure of CS model was not studied in a repeated game setting. The goal of this paper is to fill this gap.

The CS model presents a situation where a Sender has superior information than a Receiver, who undertakes an action that affects both players' payoffs. The Receiver's optimal policy is state contingent, and more information allows him to make a better decision. However, the two players have a conflict of interest. They desire different actions given the same state of the world. The results show that perfect communication is not possible when the state space is continuous. The intuition is that, if the Receiver believes in a message reporting the precise state of the world, the Sender wishes to lie in order to induce his her desired outcome. So, there is no equilibria where the Receiver trusts an exact information. However, some communication can be achieved with partition equilibrium. This class of equilibria consists of limiting communication to coarse partitions of the state space. Messages do not inform the precise state of the world, but instead an interval (partition) where true state lies. The partitions are set to allow for truthful communication. The decision making processes is improved with superior (but still imperfect) information.

We study this setup in a repeated game context, where at every period the state of the world changes (drawn from an independent distribution) and an action is chosen. The CS model becomes the stage game of our repeated game. We explore two classes of equilibria. First, we study perfect information transmission. Using trigger strategies, we find that if players are patient enough there can be perfect information transmission. The logic follows that of the folk theorem. Players adopt triggers strategies with credible punishment and agree on a fixed action rule for the receiver following a truthful precise message from the sender. The continuation payoff and the threat of punishment prevent the sender from deviating truthful communication when players are sufficiently patient.

The second class we study is the repeated partition equilibria. We use the continuation payoff to compute the partitions that allow truthful communication. Using the logic of trigger strategies, we find that repeated interaction changes the structure of communication, allowing a greater number of partitions, with different positions along the state space. Moreover, the improvements in communication are reached for any level of patience (even near zero).

Another dimension we explore in repeated game are the Receiver's action rules. In the static game, the Receiver chooses his best action given the information available. In repeated games, however, the action that can be shifted to favor the Sender's preferences. An action shift decrease the Receiver instantaneous payoff, but can also improve communication. When the action rule favors the Sender, the Receiver faces a "trade-off" between instantaneous payoff and improved communication. A local analysis of this trade-off shows that favoring the Sender is welfare improving, when the action shift is small. That is, not favoring the Sender is a dominated equilibrium.

One of the many applications of the CS model is lobbying. Translating our model to lobbying makes sense, since a part of the literature views lobbies as information providers, but with different objectives as in Austen-Smith (1993) and Schnakenberg (2016). Also, repeated interactions are important for as shown by i Vidal et al. (2012). So, our model combines these two aspects and finds that repeated information provision can be related to influence (favorable policies). However, favoring the lobby can be welfare improving, at least for low levels of influence.

The importance of our paper comes from understanding what level of cooperation can be built from repeated interaction in the CS model. Many papers have studies models of repeated information transmission, but surprisingly, none of them adopt a setting where the stage game can be described as direct version of the CS model. For example Renault et al. (2013) and Renault et al. (2017) describe Sender-Receiver games where the state of the world follows a Markovian chain, so the game's history matters in every given stage of the game (and the stage game is also different from ours). Also, Forges and Koessler (2008) presents a model where the information is verifiable, or at least, partly verifiable, thus moving away from the original CS model. In all their results, communication is improved with repeated interaction, but the results hold to patient players, making them a version of a folk theorem, while our in our paper, we find improvements in communication regardless of the players patience.

Another paper that follows closely the CS model is that of Golosov et al. (2014). However, they assume the state of the world is the same throughout the game's many periods (with many decisions). So they find a complex and rich communication equilibrium. Yet, their game structure depicts a different type of strategic situation than ours. Golosov et al. (2014) communicates better with Aumann and Hart (2003) and Aumann and Maschler (1995), which describe a frontier of long cheap talk communication for a class of games of incomplete information. But all these models look at a long (and sometimes open) period of communication before actions are chosen on games of incomplete information. We, on the other hand, look at the repeated game version of cheap talk, where the parameter that makes information incomplete is drawn at every period.

The remainder of the paper is organized as follows. The next section presents the model and the benchmark results of the static game. Section 3 presents the results for perfect information transmission. Section 4 presents the repeated game partition equilibrium with and without an action shift. Section 5 discusses our results in the spirit of lobbying literature and finally, Section 6 concludes.

2 The model

The model features two players, the Sender and the Receiver. The Receiver must undertake an action, $y_t \in \mathbb{R}$, but is uninformed about the consequences of this action. His payoff depends on a random variable, m_t, that comprises the state of the world. The true state m_t belongs to an interval $[\underline{m}, \overline{m}]$ and is know to the Sender. She, in turn, has a conflict of interest with the Receiver, expressed in a different bliss point given the realization of the state m_t. The Sender can offer a non-verifiable signal, $s \in S$, so communication is cheap-talk, meaning the message space is not ex ante specified. Plus, the Sender cannot offer hard information (or any other actions) to back his claims.

The receiver's instantaneous payoff given by

$$u\left(y_t, m_t\right) = -\left|y_t - m_t\right|,$$

So that u is a module loss function that reaches a bliss point when $y_t = m_t$.[1] The Sender's instantaneous preferences are given by

$$v(y_t, m_t) = -\left|y_t - m_t - b\right|,$$

with $b > 0$. This are similar to the Receiver's preferences, but with a different bliss point of $y_t = m_t + b$. The intuition is that, given a state of the world m_t, the Sender desires a higher

[1]The original CS model has a general approach that does not specify a functional form. Their running example, however, uses a quadratic loss function.

action y_t, by a size b.

This stage game is repeated over time, stating from time one until infinity. At every period t a new state of the world, m_t, is drawn from the same uniform distribution over $[\underline{m}, \overline{m}]$, which is constant and independent over time. The players' preferences are also time independent and they have an inter temporal discount factor of δ. Thus, the Receiver's preferences over a sequence of actions and states, (y, m) are given by

$$U(y, m) = \sum_{t=1}^{\infty} \delta^t u(y_t, m_t)$$

and the Receiver's preferences over a sequence of actions and states are given by

$$V(y, m) = \sum_{t=1}^{\infty} \delta^t v(y_t, m_t).$$

To sum up, the repeated cheap-talk game has two players, the Sender and the Receiver. The Sender's strategies are a sequence of messages, $s = (s_1, s_2, \ldots)$, with each $s_t \in S$, while the Receiver's strategies are a sequence of actions $y = (y_1, y_2, \ldots)$, with each $y_t \in \mathbb{R}$. Finally, the payoffs are given by $V(y, m)$ and $U(y, m)$ for the Sender and Receiver respectively. We will study stationary perfect bayesian equilibrium of this game.

2.1 Benchmarks

We begin looking at the equilibrium of the stage game as these will be useful for the construction of equilibrium of the repeated game. The first benchmark result is the non-existence of a perfect revelation equilibrium. A perfect revelation equilibrium is one where the Sender's message is exactly the true state, $s = m$, and the receiver believes the true message and sets the action equal to the state, $y = s$. The problem with such strategy is that, once the Sender anticipates the Receiver will believe the signal s, he benefits from sending a non truthful message $s = m + b$. Therefore, the Sender's signal cannot be trusted and this cannot be an equilibrium. This reasoning is summarized in the following lemma below

Lemma 1. *Perfect revelation (signal $s = m$ and action $y = s$) is never an equilibrium of the static game.*

On the other hand, no information revelation is an equilibrium of the stage game. If we suppose the Receiver does not believe any of the Sender's signals, he will choose an action that maximizes his uninformed utility. The Sender then, knowing no signal will be trusted, chooses not to send any message at all. CS call this the babbling equilibrium.

Lemma 2. *There exists a babbling equilibrium of the stage game, where no message is sent and the Receiver chooses the optimal action without information. Such action is given by* $\overline{y} = {}^{(\underline{m}+\overline{m})}/_2$. *We denote the static payoff of this equilibrium by* $\overline{v}$ *and* $\overline{u}$ *for the Sender and Receiver respectively.*

The babbling equilibrium is one where no communication takes place. It is a lower bound for players' payoffs.

Finally, we have another equilibrium where some communication takes place, even if imperfectly, called the partition equilibrium. In this equilibrium, the state space is divided into small segments. The communication takes place by reporting the partition where the true state lies. That is, if there are n partitions, a^1, ..., $a^{n-1}, a^n = 1$, and the true state $m \in [a^{j-1}, a^j]$, the signal informs "the true state lies between a^{j-1} and a^j". Upon receiving the truthful signal, the Receiver chooses the optimal policy within the partition, which is $y^j = {}^{(a^j+a^{j-1})}/_2$. The partitions are chosen in order to satisfy conditions that impose the communication will be truthful, these are called arbitrage conditions by CS. They impose that given $\tilde{m} \in [a^{j-1}, a^j]$, the Sender must prefer to report the truthful partition

$$-\left|\frac{a^{j-1}+a^j}{2} - \tilde{m} - b\right| \geq -\left|\frac{a^j+a^{j+1}}{2} - \tilde{m} - b\right|$$

and

$$\left|\frac{a^{j-1}+a^j}{2} - \tilde{m} - b\right| \geq -\left|\frac{a^{j-2}+a^{j-1}}{2} - \tilde{m} - b\right|$$

When we approximate the true state $\tilde{m}$ to the partition a^j, from above or below, we get the following inequality

$$a^{j+1} - a^j = a^j - a^{j-1} + 4b \tag{1}$$

This is a difference equation that imposes the partitions must be increasing in size (provided b is positive) in order to achieve truthful communication. The resulting partitions are smaller to the left and larger to the right of the state space. This difference equation has a solution, that leads to the following lemma.

Lemma 3. *There exists a* $n-$*partition equilibrium of the stage game where the Sender's signals are the partitions,* a^j *and the Receiver's action rule is the midpoint of the reported partition,* $y(a^j) = {}^{a^j+a^{j-1}}/_2$. *The maximum number of partitions* n^*, *satisfies*

$$2n^*\left(n^*-1\right)b < \overline{m} - \underline{m}$$

and the partitions given by the following rule

$$a^j = \frac{j}{n^*}\overline{m} + \frac{(n^* - j)}{n^*}\underline{m} - 2bj\,(n^* - j)\,.$$

Lemma 3 shows the solution is exactly the same os in the CS example with quadratic preferences. One important aspect from Lemma 3 is that if you have a partition equilibrium with $n^* > 2$, then you also have other partitions equilibria with $n < n^*$ partitions (they satisfy all conditions). Therefore, n^* is the maximum number of equilibrium partitions. From Lemma 3 we find that when the conflict of interest b decreases, the number of partitions decrease (weakly) and also the position of partitions on the state space change. To understand the change in position, imagine the number of partitions n is fixed. If we make b tend to zero, the partitions become evenly distributed along the state space. As we shall see, having evenly distributed partitions increase welfare.

Lemma 4. *In a partition equilibrium defined by Lemma 3, welfare increases as the number of partitions increase, and as the partitions become more evenly distributed. Therefore, welfare increases as the conflict of interest, b, decreases.*

As this is a simple model with relative few parameters, we present a numerical example, which we will return to throughout the paper

Example. *Let the parameters be $\overline{m} = 28$, $\underline{m} = 0$ and $b = 2$. For these parameters, we have a maximum of 3 partitions since $\overline{m} - \underline{m} = 28 > 2bn\,(n-1) = 24$. The partitions are given by $a^3 = 28$, $a^2 = {}^{32}\!/_3$, and $a^1 = {}^{4}\!/_3$.*

The results from these benchmarks will be important to the equilibria discussed in the following sections.

3 Perfect information transmission

We now look at our first equilibria in a repeated game setting. It seems natural to begin with the most informative equilibrium possible, which we call the Perfect information transmission equilibrium. From Lemma 1, we know such equilibrium it is impossible to reach in a static game due to lack of commitment. But since repeated interaction allows for new possible arrangements, we can verify if such equilibria exists.

Remember from section 2 that stage game is repeated every period, and in every period a new state m_t is draw from an independent distribution. We seek an equilibrium based on trigger strategies that allow players to reach greater perfect communication. We'd like an equilibrium that features perfect communication, so the sender's signal $s_t = m_t$ in every period. In turn, the receiver sets an action that is shifted towards the Sender's preference. We denote this shift by d. For simplicity, we assume the action shift is constant over time and is independent of the message s_t. That means the action set in every period is $y_t = s_t + d$.[2] For obvious reasons, we restrict attention to action shifts that are smaller than the conflict of interest, that is $b \geq d \geq 0$. The mechanics of the proposed equilibrium is relatively simple. The Sender promises to send a precise truthful message in every period while the receiver promises to reward him with a favorable action after receiving the information. as this is

Trigger strategies also require credible punishments from the proposed equilibrium strategies. That is, a description of the off the equilibrium strategies. Off-the-equilibrium strategies should impose the toughest credible punishment for the players. We assume the punishment is to to play a babbling equilibrium forever, stating the next period. Since players observe their payoff at the end of every stage, this strategy is perfectly implementable. Moreover, since babbling is an equilibrium of the static game, it constitutes a credible punishment, so we this strategies satisfy subgame perfection.

Let us now look at the conditions that support this equilibrium. We begin looking at the Sender's incentives. If she complies to the proposed strategy, her instantaneous payoff will be constant over time and equal to $v = -(b - d)$. If the equilibrium is played forever, his payoff is

$$V = -\frac{b-d}{1-\delta}.$$

In case of deviation, the players get their payoff from a babbling equilibrium, which is

$$\underline{v} = -\frac{(\overline{m} - \underline{m})}{4} - \frac{b^2}{(\overline{m} - \underline{m})}.$$

Finally, the Sender can increase her instantaneous payoff if she deviates from the trigger strategy and sends a signal $s_t = m_t + b - d$ in which case she enjoys an instantaneous payoff of zero (because the Receiver implements $y_t = m_t + b$). The perfect information transmission must satisfy

$$-\frac{(b-d)}{1-\delta} \geq 0 - \frac{\delta}{1-\delta}\left[\frac{(\overline{m} - \underline{m})}{4} + \frac{b^2}{(\overline{m} - \underline{m})}\right]. \tag{2}$$

Now, let us look at the incentives for the receiver. If he complies to the proposed strategy, his instantaneous payoff is $u = -d$. When repeated infinitely, this becomes

$$U = \frac{-d}{1-\delta}.$$

[2] We could think of more complex action shifts, as multiplicative shifts, or even making them conditional to the state m_t or the time period. But as a first pass, we focus on linear and constant shift.

In turn, the instantaneous payoff from playing a babbling equilibrium is given by

$$\underline{u} = \frac{\overline{m} - \underline{m}}{4},$$

while the deviation that gives the highest instantaneous payoff is to set the action equal to the signal, that is $y_t = s_t$. This gives him a payoff of zero. So, in order to play this trigger strategy, it must be that

$$-\frac{d}{1-\delta} \geq 0 - \frac{\delta}{1-\delta}\frac{\overline{m} - \underline{m}}{4}. \tag{3}$$

So, from (2) and (3) we have the combined conditions for β and d that compose our first proposition

Proposition 1. *There exists a trigger strategy equilibrium of the repeated cheap-talk game with perfect information transmission with an action shift $d \in [0, b]$ provided players are sufficiently patient, that is $\delta \geq \delta^*$, where*

$$\delta^* = \max\left\{\frac{4\left(\overline{m} - \underline{m}\right)\left(b - d\right)}{\left(\overline{m} - \underline{m}\right)^2 + 4b^2}, \frac{4d}{\left(\overline{m} - \underline{m}\right)}\right\}.$$

Proposition 1 is a variation of a folk theorem, showing that patient players can overcome the conflicts of static games. Notice that the two equilibrium conditions have opposite signs regarding the action shift d. The condition for the Sender is decreasing with respect to d. This means that a greater action shift makes it possible for less patient Senders to comply to a trigger strategy equilibrium. On the other hand, the condition for the Receiver is increasing with respect to d, meaning that a a greater d implies Receivers must be more patient to comply to the trigger strategy.

Working with those conditions, we get the following two corollaries

Corollary 1. *The minimum cutoff δ^* is achieved with a action shift given by*

$$d^* = \frac{\left(\overline{m} - \underline{m}\right)^2 b}{2\left[\left(\overline{m} - \underline{m}\right)^2 + 2b^2\right]}. \tag{4}$$

This expression is found by choosing the action shift d that makes the two conditions within the max operator equal. The corollary shows that the actions shift can be used to ease the conditions and help sustain a perfect information transmission.

The second corollary shows that perfect information transmission can be achieved even with an action shift of zero.

Corollary 2. *There exists a $\check{\delta} \geq \delta^*$ that can sustain perfect information transmission even* *with $d = 0$. It is given by*

$$\check{\delta} = \frac{4\left(\overline{m} - \underline{m}\right)b}{\left(\overline{m} - \underline{m}\right)^2 + 4b^2}. \tag{5}$$

Corollary 2 shows that a positive action shift is not a necessary condition to achieve perfect information transmission. The construction of the trigger strategy equilibrium is such that the babbling equilibrium is the punishment. In such punishment, the Sender has an uncertain payoff. On the other hand, in a perfect information transmission equilibrium, his payoff is constant over time. As he dislikes uncertainty and enjoys no action shift in babbling, he may comply to the "no shift" trigger strategy equilibrium if he is patient enough. However, the space of parameters where perfect information transmission can be achieved is smaller with an action shift of zero.

All these perfect information transmission equilibria are equivalent in terms of welfare, as the sum of players payoffs is always equal to $-b/(1-\delta)$ regardless of the policy bias. If we are to think of this model in the spirit of lobby games, the payoff of the Receiver should have more weight than that of the Sender and we should look for the equilibrium with the lowest action shift d possible.

Example. *With the same parameters of our running example. If we take an action shift of $d = .5$, then the cutoff delta that sustains perfect information transmission is $\delta^* = 0,21$.*

The actions shift that minimizes the cut off delta is $d^* = 784/792 = 0,9898...$, reaching a cutoff delta of $\delta^* = 0,1414....$ Finally, the lowest delta that support perfect information transmission with a action shift of zero ($d = 0$) is $\check{\delta} = 0,28$.

The results from this section present the conditions for achieving a perfect information equilibrium. From Corollary 1, we know that the cutoff δ^* is positive, which shows that for low δ, perfect information transmission will not be achieved. If, however, players are not patient enough, there are still improvements in communication that can be achieved in partition equilibria.

4 Repeated partition equilibrium

In the previous section, we saw that perfect information transmission is possible for patient players. So, why should we take a step back, and study equilibria where less information is transmitted. The fact is, in partition equilibrium, the Sender's short-term gain from misinformation is much smaller in a partition equilibrium. She is cannot induce the Receiver to choose her optimal policy. So, with different payoffs for deviation, a repeated partition equilibrium can be found whenever the static game has a partition equilibrium. Therefore, the repeated partition equilibrium can be sustained in a different range of parameters than the perfect information equilibrium. In particular, we will show the partition equilibrium can be supported with low levels of δ.

In section 2.1, we saw that the partitions are found using a kind of truth telling condition. In a partition, the Sender is indifferent between which partition to report. We combine this logic to that of a trigger strategy, using the continuation payoffs to create a higher benefit of complying to the proposed strategy. As a result, we are able to achieve more informative structure of communication. The trigger strategy proposes a given partition equilibrium to be played in each stage. If both players play according to the proposed equilibrium, the same partition equilibrium will be played in the next period. If one player does not play according to the proposed strategy, the players will play babbling equilibrium forever. The partitions are set according to the expected payoff of the repeated partition equilibrium. In turn, this payoff depends on the partitions. So the equilibrium must be a fixed point of the expected continuation payoff.

In order to ease the exposition, we begin looking at the case only the Sender complies to the trigger strategy. That is, we find partitions that are incentive compatible taking into account the continuation payoff and the threat of punishment. However, the Receiver keeps choosing her optimal static action but he agrees to punish in case the Sender deviates from the trigger strategy. In turn, we look at an equilibrium where the Receiver moves away from her optimal action and favors the Sender with a linear action "shift".

4.1 Sender's trigger strategy

In a static partition equilibrium, remember the signal space is divided into smaller partitions, $\{a^1, a^2, \ldots, a^n\}$, and communication reveals where the true state lies. The partitions are set in such way that the communication is truthful, that is, the sender wishes to inform the correct partition. In the static game, there is no commitment, so the Receiver chooses the best policy for him, given the information he receives. As a first pass, we will assume the receiver still chooses his best policy in every period, given the received information. This allow us to focus on the conditions for information transmission for the Sender.

We compute the partitions from the condition for truthful communication, as in the static game. In a repeated game, this condition is now based on a trigger strategy equilibrium. The trigger strategy is based on a proposed partition equilibrium, a deviation and a punishment.

We will compute partitions presuming we are in a stable equilibrium in the following periods. Therefore, the continuation payoff, Ev^r, is fixed for the definition of the partitions. Of course, future payoffs depend on the equilibrium partitions, which means the partitions are going to be defined recursively.

We begin with a signal $s^j = a^j$ informing the state m_t lies within partition $[a^{j-1}, a^j]$. The receiver then sets action $y_t = {(a^j + a^{j-1})}/{2}$. This partitions sustain communication if

$$EV\left(\left(y_j, y^2\right), \left(s_j, s^2\right)\right) \geq EV\left(\left(y', \underline{y}^2\right), \left(s', \underline{s}^2\right)\right),$$

where y^2 and s^2 are the sequences of expected actions and signals in trigger strategy equilibrium, from tomorrow to infinity and $\underline{y}^2$ and $\underline{s}^2$ are the respective sequences of actions and signals in a babbling equilibrium, from tomorrow to infinity. This condition boils down to

$$\underbrace{v\left(y^j, s^j\right)}_{\text{Equilibrium}} + \underbrace{\frac{\delta}{1-\delta} Ev^r}_{\text{Continuation}} \geq \underbrace{v\left(y', s'\right)}_{\text{Deviation}} + \underbrace{\frac{\delta}{1-\delta} E\underline{v}}_{\text{Punishment}}$$

We will begin looking at a simple type of condition that prevents the Sender from reporting the state lies in the superior adjacent partition (in the proof, of the next proposition, we show that this is indeed the relevant condition). Then the trigger strategy becomes

$$-\left|\frac{a^j + a^{j-1}}{2} - \hat{m} - b\right| + \frac{\delta}{1-\delta} Ev^r \geq -\left|\frac{a^{j+1} + a^j}{2} - \hat{m} - b\right| + \frac{\delta}{1-\delta} E\underline{v}$$

We then approximate $\hat{m}$ to the border of the partition, a^j, this condition boils down to

$$\frac{a^j + a^{j-1}}{2} - a^j - b \geq a^j + b - \frac{a^{j+1} + a^j}{2} - \frac{\delta}{1-\delta}\left[Ev^r - E\underline{v}\right].$$

And, finally, we get:

$$\left(a^{j+1} - a^j\right) \geq \left(a^j - a^{j-1}\right) + 4b - 2\frac{\delta}{1-\delta}\left[Ev^r - E\underline{v}\right] \tag{6}$$

As in the static game, the conditions for information transmission in partition equilibrium boils down to a difference equation. This condition is linear, as (1), but (6) has an additional term, related to δ and to the difference in the continuation payoff from a repeated partition and a babbling equilibrium. This additional term works as a reduction of the conflict of interest between the players. Unlike in the static game, this condition holds in inequality.[3]

[3]We can find another condition, limiting the maximum size of the partitions when we the true state lies within $[a^j, a^{j+1}]$. When $m_j \in [a^j, a^{j+1}]$, then truthful communication imposes

$$-\left|\frac{a^j + a^{j-1}}{2} - \tilde{m} - b\right| + E\underline{v} \leq -\left|\frac{a^{j+1} + a^j}{2} - \tilde{m} - b\right| + Ev^r$$

But such condition limits the size of partitions from above. Since welfare is increasing with respect to the number of partitions, we restrict attention to (6) by assuming it holds with equality.

As mentioned before, (6) only defines the partitions implicitly, since the continuation payoff Ev^r depends on the partitions which, in turn, depend on the continuation payoff. So partition are defined recursively and we must check if the equilibrium indeed exists. The Proposition below shows that is indeed the case.

Proposition 2. *There exists a repeated partition equilibrium of the repeated cheap-talk game. The partitions of a repeated game $n-$partition equilibrium are characterized by*

$$a^j = \frac{j}{n}\overline{m} + \frac{n-j}{n}\underline{m} - 2j\left(n-j\right)b', \tag{7}$$

where

$$b' = \max\left\{b - \frac{\delta}{2\left(1-\delta\right)}\left(v^r - \underline{v}\right), 0\right\},$$

and

$$v^r = -\sum_{j=1}^{n}\int_{a^{j-1}}^{a^j}\left|\frac{a^j + a^{j-1}}{2} - x - b\right| dx$$

Moreover, the maximum number of partitions $\overline{n}$ is highest integer that satisfies

$$\frac{\left(\overline{m} - \underline{m}\right)}{b'} > 2\overline{n}\left(\overline{n} - 1\right) \tag{8}$$

The important aspect of Proposition 2 is that $b \geq b'$. That means the continuation payoff from repeated interaction can be simplified as a static partition equilibrium with a smaller bias. With a smaller conflict of interest, the number of partitions is weakly superior. That is evident since the left hand-side of condition (8) is decreasing with respect to b', potentially allowing for a higher n^*. Additionally, from (7) it is straightforward to check the partitions are more evenly distributed as b' decreases.[4] From Lemma 4, we know that more evenly distributed partitions increases induce higher welfare. So, the repeated interaction improves communication and increases welfare.

Making $\tilde{m}$ approximate a^j from the right, and manipulating the expression, gives us

$$\left(a^{j+1} - a^j\right) \leq \left(a^j - a^{j-1}\right) + 4b + \frac{2\delta}{1-\delta}\left[Ev^r - E\underline{v}\right]$$

The condition above is different from (6), showing that we have some slackness to define the partitions. However, as we focus on informative equilibrium, we focus on condition (6) since it allows the greater number of partitions for the same set of parameters.

[4] To understand this point more clearly, imagine the number of partitions is fixed while $\hat{b}$ tends to zero. In that case, the partitions would be equal and each with size $(\overline{m}+\underline{m})/n$.

Corollary 3. *The $n-$partition equilibrium has a weakly greater number of partitions, which are more evenly distributed than the static equilibrium, for every $\delta > 0$.*

Notice that Proposition does not establishes a minimum parameter of impatience, delta. In fact, it holds even for very small delta, since even the difference between the payoff of any partition equilibrium and the babbling equilibrium is positive. So, even for ϵ positive delta, the positions of the partitions are going to be different from the static partition equilibrium. Therefore, the partition equilibrium can be understood as the most informative equilibrium for deltas below the threshold, of Corollary 2, $\check{\delta}$.

Corollary 4. *There exists a repeated partition equilibrium of the repeated cheap talk game for every non negative $\delta \geq 0$. In particular, if $\delta > 0$, the equilibrium partitions are different from those of the static game.*

To highlight our results, we look at our running example, in a repeat game version.

Example. *Considering the same parameters of our running example, if we take a delta of $\delta = 0.2$, the partition equilibrium has at most 3 partitions, as in the static game. The partitions are approximately given by $a^3 = 28$, $a^2 \approx 12.52$, and $a^1 \approx 3.19$. If, alternatively, $\delta = 0,3$, then there are up to 4 partitions in equilibrium, approximately given by $a^4 = 28$, $a^3 \approx 14.18$, $a^2 \approx 4.91$, and $a^1 \approx 0.18$. Remember that with $\delta > 0.28$, the players can achieve perfect information transmission.*
If, on the other hand, the delta is $\delta = 0.01$, the equilibrium has at most 3 partitions, given by $a^3 = 28$, $a^2 = 10.73$, and $a^1 = 1.4$, slightly different from the static partition equilibrium.

The example highlights the results from Corollary 3, that is, as players become more patient, the communication is improved, the number partitions increase as they become more evenly distributed. Finally, the example also shows the partitions change even if players are impatient (low delta).

4.2 Action shift

In section 4.1 we presumed the receiver always chose his optimal policy, given the information he receives. That is, if he receives signal $s_t = [a^{j-1}, a^j]$, he chooses policy $p_t = {}^{(a^j + a^{j-1})}/_2$, which is the midpoint of the partition. But there are different strategies the receiver can adopt. We will look at trigger strategies where the receiver chooses an action with a constant shift from his optimal policy. That is, given a signal $s_t = [a^{j-1}, a^j]$, the receiver chooses policy $p_t = {}^{(a^j + a^{j-1})}/_2 + d$. This rule specifies that, whatever the partition, the implemented policy is midpoint of the partition plus a shift in favor of the Sender's preferred action.[5] Clearly, the action shift must be greater than zero and smaller than the bias, $0 \leq d \leq b$, if not, the policy would be Pareto inferior.

The logic of looking for strategies with a more favorable policy is that the Sender conditions for truthful communication are eased and information transmission can improve. The reason players cannot make such arrangements in the static game is that receiver lacks commitment to announce a favorable policy in "exchange" for improved communication, because when he receives information, he is no longer bound to offer an action shift that the reduces his payoff. But the strength of the trigger strategy is exactly to allow for players to overcome conflicts of the static game, such as lack of commitment.

On the other hand, the action shift creates the possibility of deviation for the receiver, as he can deviate from the proposed policy by choosing his optimal action in the static game. Therefore, we must also impose a condition for the Receiver to comply to the proposed trigger strategy equilibrium. But let us begin with the conditions for the Sender. Again, we take the continuation payoff, Ev^r as given and look at the partitions that can be sustained in equilibrium. We begin looking at the conditions for truthful communication.

Let a signal $s^j = a^j$ inform the state m_t lies within partition $[a^{j-1}, a^j]$. The receiver then sets action $y = {}^{(a^j + a^{j-1})}/_2 + d$. This partitions sustain communication if

$$EV\left(\left(y_j, y^2\right), \left(s_j, s^2\right)\right) \geq EV\left(\left(y', \underline{y}^2\right), \left(s', \underline{s}^2\right)\right),$$

where y^2 and s^2 are the sequences of expected actions and signals in trigger strategy equilibrium, from tomorrow to infinity while $\underline{y}^2$ and $\underline{s}^2$ are the respective sequences of actions and signals in a babbling equilibrium, from tomorrow to infinity. This condition boils down to

$$\underbrace{v\left(y^j, s^j\right)}_{\text{Equilibrium}} + \underbrace{\frac{\delta}{1-\delta} Ev^r}_{\text{Continuation}} \geq \underbrace{v\left(y', s'\right)}_{\text{Deviation}} + \underbrace{\frac{\delta}{1-\delta} E\underline{v}}_{\text{Babbling}}$$

If the sender do not wish to send a message informing the state is on the adjacent partition, we get

$$-\left|\frac{a^j + a^{j-1}}{2} - \hat{m} - (b-d)\right| + \frac{\delta}{1-\delta} Ev^r \geq -\left|\frac{a^{j+1} + a^j}{2} - \hat{m} - (b-d)\right| + \frac{\delta}{1-\delta} E\underline{v}.$$

[5] We could look at different, more complex, ways for the policy to favor the Sender. For example a multiplicative shift, or a shift conditional on the partition. Given the linear nature of the model, we focus on this linear shift as a first pass.

As we approximate $\hat{m}$ to a^j in regular conditions, this becomes

$$\frac{a^j + a^{j-1}}{2} - a^j - (b - d) \geq a^j + (b - d) - \frac{a^{j+1} + a^j}{2} - \frac{\delta}{1-\delta}\left[Ev^r - E\underline{v}\right].$$

which simplifies to

$$a^{j+1} - a^j = a^j - a^{j-1} + 4(b - d) - \frac{\delta}{1-\delta}\left[Ev^r - E\underline{v}\right]. \tag{9}$$

This is a difference equation that defines a transition rule for the partitions. Again, it shows partitions must be increasing in size. However, the action shift d reduces the increment necessary to achieve a truthful communication. It works as an additional reduction of the conflict of interest beyond that originated from the continuation payoff.

On the other hand, we must also look at the Receiver's condition to comply to the proposed trigger strategy equilibrium. The timing of the stage game is such that he receives the signal $s_t \in [a^{j-1}, a^j]$ and expects it to the truthful. He then updates his beliefs and computes his expected stage game payoffs, given by

$$E(u_t \mid s_t) = -\frac{1}{a^j - a^{j-1}} \int_{a^{j-1}}^{a^j} \left|y_t^j - m - b\right| dm,$$

therefore, when the receiver complies to the equilibrium, the payoff is

$$E(u_t^r \mid s_t) = -\frac{1}{a^j - a^{j-1}} \int_{a^{j-1}}^{a^j} \left|\frac{a^j + a^{j-1}}{2} - (m - d) - b\right| dm,$$

and if he deviates,

$$E\left(u_t^D \mid s_t\right) = -\frac{1}{a^j - a^{j-1}} \int_{a^{j-1}}^{a^j} \left|\frac{a^j + a^{j-1}}{2} - m - b\right| dm,$$

The trigger strategy is sustained if

$$\underbrace{E(u_t^r \mid s_t)}_{\text{Equilibrium}} + \underbrace{\frac{\delta}{1-\delta} Eu^r}_{\text{Continuation}} \geq \underbrace{E\left(u_t^D \mid s_t\right)}_{\text{Deviation}} + \underbrace{\frac{\delta}{1-\delta} E\underline{u}}_{\text{Babbling}} \tag{10}$$

One aspect of the Receiver's condition is that it is signal contingent. That means that, given the given an equilibrium partition, the receiver may be tempted to comply in some states, but not on others. Evidently, in order reach a stable equilibrium, the condition must hold for every partition. Luckily, we can restrict attention to the first partition. If the

Receiver complies in the first partition, he complies in all other partitions. It is summarized in the following lemma.

Lemma 5. *If the receiver does not deviate from the trigger strategy in the first partition, then he does not deviate in other partitions. Therefore, the condition for the receiver to comply to a trigger strategy equilibrium*

$$\underbrace{E\left(u_t^r \mid s_t^1\right)}_{Equilibrium} + \underbrace{\frac{\delta}{1-\delta}Eu^r}_{Continuation} \geq \underbrace{E\left(u_t^D \mid s_t^1\right)}_{Deviation} + \underbrace{\frac{\delta}{1-\delta}E\underline{u}}_{Babbling},$$

where

$$E\left(u_t^r \mid s_1\right) = -\frac{1}{a^1-\underline{m}}\int_{\underline{m}}^{a^1}\left|\frac{a^1-\underline{m}}{2}+d-m-b\right|dm,$$

and

$$E\left(u_t^D \mid s_1\right) = -\frac{1}{a^1-\underline{m}}\int_{\underline{m}}^{a^1}\left|\frac{a^1-\underline{m}}{2}-m-b\right|dm.$$

This condition simplifies to

$$\frac{d^2}{a^1-\underline{m}} \leq \frac{\delta}{1-\delta}\left[Eu^r - E\underline{u}\right]$$

Lemma 5 simplifies the Receiver's conditions that sustain the equilibrium. Notice that when the first partition is very small, the incentives to deviate become too big. When the Receiver was choosing his static best, $d = 0$, and the condition would be met trivially. Plus, new partitions emerged very small in size. If we were to allow such small partition when $d > 0$, would violate his condition. condition may not be met when the first partition is too small. Therefore, this condition works as a minimum size of the first partition.

Now we are in position to present the equilibrium

Proposition 3. *The partitions of a repeated game $n-$partition equilibrium with a policy shift are characterized by*

$$a^j = \frac{j}{n}\overline{m} + \frac{n-j}{n}\underline{m} - 2j\left(n-j\right)\tilde{b},$$

where

$$\tilde{b} = \max\left\{b-d-\frac{\delta}{2\left(1-\delta\right)}\left(v^r-\underline{v}\right),0\right\},$$

and

$$v^r = -\sum_{j=1}^{n}\int_{a^{j-1}}^{a^j}\left|\frac{a^j+a^{j-1}}{2}-x-(b-d)\right|dx$$

Moreover, the maximum number of partitions $\overline{n}$ *is highest integer that satisfies*

$$\frac{(\overline{m} - \underline{m})}{\hat{b}} > 2\overline{n}\,(\overline{n} - 1)$$

and

$$\frac{d^2}{a^1 - \underline{m}} \leq \frac{\delta}{1-\delta}\left[Eu^r - E\underline{u}\right],$$

where

$$Eu^r = -\sum_{j=1}^{n} \int_{a^{j-1}}^{a^j} \left| \frac{a^j + a^{j-1}}{2} + d - x \right| dx$$

Plus, there exists a positive $d < b$ *for which there is a partition equilibrium of the repeated cheap talk game, with a policy shift whenever there is a partition equilibrium (without a policy shift) of the static cheap talk game.*

Proposition 3 shows that the action shift is an addition factor reducing the partitions. It allows greater number and more evenly distributed partitions. Thus, the action shift works as an additional reduction of conflict of interest. It carries the same results from Corollaries 3 and 4. So, the threat of punishment in a babbling equilibrium and an action shift together can improve information transmission.

Example. *Again, let us turn to our running example with* $\overline{m} = 28$*,* $\underline{m} = 0$*,* $b = 2$*, and* $\delta = 0,2$*, plus, with an action shift of* $d = 0,5$*. Then, the equilibrium has 4 partitions, given by* $a^4 = 28$*,* $a^3 \approx 15.35$*,* $a^2 \approx 6.46$*, and* $a^1 \approx 1.35$*, and the trigger strategy for he Receiver is slack. Remember from section 4.1 that with these same parameters (and a* $d = 0$*), the partition equilibrium had only 3 partitions. So, the action shift allowed for an additional partition. Moreover, remember from the example in section 3 that with an action shift of* $d = 0,5$ *and a level of impatience of* $\delta = 0,2$*, there is no equilibrium with perfect information transmission (the minimum delta to sustain perfect information transmission is* $\delta = 0,21$*).*

The action shift also raises new questions regarding welfare. Without the action shift, the Receiver was always maximizing his payoff. The Sender also benefited from repeated interaction since it improves communication. With the action shift, the Sender gains from a more beneficial action and from the improved communication, thus she is better of with the action shift $d \in [0, b]$. The Receiver, on the other hand, faces a trade-off. With a greater shift, the he has an inferior policy but with improved communication. In particular, if the effect of communication is dominates, an action shift can be welfare improving. The following proposition addresses this issue

Proposition 4. *The derivative of the action shift, d, on the Receiver's payoff is given by* *the derivative below*

$$\frac{\partial U}{\partial d} = \begin{cases} \frac{1}{1-\delta}\frac{2n}{\Delta m}\left[-d + \tilde{b}\frac{\left(n^2-1\right)}{3}\right] & \textit{for } d \leq \frac{1}{3}\left[2b - \frac{\delta}{1-\delta}\left(v^r - \underline{v}\right)\right] \\ \frac{1}{1-\delta}\frac{1}{\Delta m}\left[-\left(2n-1\right)d - \left(a_1 - \underline{m}\right) + 2\tilde{b}\frac{\left(n^3-n\right)}{3} + \frac{\Delta m - 2n(n-1)}{2n}\right] & \textit{for } d > \frac{1}{3}\left[2b - \frac{\delta}{1-\delta}\left(v^r - \underline{v}\right)\right] \end{cases}.$$

We notice that $\partial U/\partial d$ *is not continuous, is piecewise negative with respect to* d*, but it is positive for near zero* d*.*

Proposition 4 presents the derivative of Receiver's payoff with respect to the policy bias. Although the derivative is a non monotone and discontinuous function, we find that it is positive near zero. This shows that some policy shift is welfare improving (zero shift is not optimal). Given the expression complexity, hard to identify a closed form solution for an optimal policy shift since, the derivative jumps with the number of partitions in equilibrium. The Proposition also highlights the importance of improving communication, with greater number and more evenly distributed partitions. Even with a policy that directly harms the receiver, the indirect effect of communication compensates, at least for a range of small action shifts.

The results from Proposition 4 are also distinct from what we get in perfect information transmission. In that case, once information is transmitted perfectly, the policy shift only transfers payoff from the Receiver to the Sender. But as all information is being transferred, there's no marginal increase in communication. Aggregate surplus is maximized for any shift above the δ cutoff from Proposition 1.

5 Lobbying

One of the many applications of the cheap talk game is lobbying. Whereas the predominant view of lobbies is that of rent seeking (GH and many others), a branch of the literature views them as information providers. In that alternative view, lobbies naturally have better information than policy makers because of their are directly impacted by the policy. However, conflicts of interest could limit that transmission to the policy maker as seen in Austen-Smith (1993) and Schnakenberg (2016). So we can see the cheap talk model fits the description of this strategic situation.

In particular, evidence from i Vidal et al. (2012) on revolving door politics indicates that long term relationships are also important for lobbying. They show that the market value of lobbying firms decrease if the former (politician) boss of a current employee fails to be reelected. This points out that lobbying influence might not take place as quid pro quo of money in exchange for favorable policy.

In our paper we bring together the two elements of cheap talk communication and repeated interaction. The Sender is the lobby while the receiver is the policy maker. All the results apply almost directly to the example of information lobbying. However, one important difference is that, typically, the policy maker's payoff is the society's welfare. Therefore, Proposition 4 takes the interpretation of the derivative of welfare with respect to a policy shift.

The interpretation is that lobbies end up getting a favorable policy (or capture) in exchange for the information they are providing. However, we show that, at least for low level of influence, the favorable policy is welfare increasing, since it eases communication that ultimately improves decision making. Thus, repeated interaction between lobbyists and policy makers may indeed result in influence, even without money contribution. The catch is that such influence may actually be welfare improving.

6 Conclusion

In this paper we analyzed an infinitely repeated version of Crawford and Sobel (1982) model. We focus on two classes of equilibria based on trigger strategies. We first study a perfect information transmission equilibrium, where the Sender reveals the true state of the world at every period and the Receiver chooses an action "shifted" towards the Sender's preferences. This equilibrium is sustained when players are patient enough. Moreover, the minimum patience level has a V-shaped relationship with the action shift granted by the Receiver.

We also find a class of partition equilibrium in repeated games. The repeated partition equilibrium follows the same structure of the static partition equilibrium, but the continuation payoff works as a reduction of the conflict of interests. Therefore, the repeated interaction allows for richer information transmission, with a weakly greater number of partitions and and better positions partitions. That is, repeated interaction improves the strategic communication. Importantly, this equilibrium exists even for impatient players (even near zero impatience).

The communication also improves with an action shift, in favor of the Sender's preferences. If the Receiver sent shift actions in favor of the Sender, this allows for improved information transmission. This results in a trade-off for the Receiver, between better information and inferior action. The trade-off is complex, as the relationship between action shift and Receiver's payoff is not continuous. Nevertheless, we find that a zero action shift is not optimal. So, the Receiver benefits from choosing an action that favors the Sender, at least for low levels of favoring. The results show that new and richer equilibria can be found in a repeated CS model.

Appendix

Proof of Lemma 1. Lemma 1 states that perfect information transmission is not an equilibrium. By perfect information transmission we mean an equilibrium where the Sender's signal is $s = m$ and the Receiver's action is $y = s = m$.

In order to reach a contradiction, let us assume that the sender offers a signal $s = m$ for any state. Upon receiving a truthful signal, the Receiver chooses $s = m$, his optimal action is to choose action $y = s = m$. However, if the Receiver's action is to choose $y = s$, the Sender's best response is to choose $s = m + b$. Therefore, sending a signal $s = m$ is not a best response. Therefore, perfect information transmission is not an equilibrium of the static game. □

Proof of Lemma 2. The so called Babbling equilibrium features no information transmission, that is, the Sender's signal is $s = \emptyset$ and the Receiver chooses his best action given he does not know any information beyond the prior.

Let us assume the Receiver chooses

$$\max_y \frac{1}{\overline{m} - \underline{m}} \int_{\underline{m}}^{\overline{m}} - |y - m| \, dm$$

or

$$\max_y \frac{1}{\overline{m} - \underline{m}} \left[\int_{\underline{m}}^{y} -(y - m)dm + \int_{y}^{\overline{m}} -(m - y)dm \right],$$

which has $\overline{y} = {(\overline{m} - \underline{m})}/{2}$ as maximum. Since the Receiver chooses $\overline{y}$ regardless of the signal, the Sender is indifferent between any message, and can choose $s = \emptyset$. Therefore, babbling is an equilibrium of the static game, which proves Lemma 2.

Moreover, the players payoffs in a babbling equilibrium are given by

$$\underline{u} = \frac{1}{\overline{m} - \underline{m}} \left[\int_{\underline{m}}^{\frac{\overline{m} - \underline{m}}{2}} -(\frac{\overline{m} - \underline{m}}{2} - m)dm + \int_{\frac{\overline{m} - \underline{m}}{2}}^{\overline{m}} -(m - \frac{\overline{m} - \underline{m}}{2})dm \right]$$

which, after straightforward algebra, simplifies to

$$\underline{u} = -\frac{\overline{m} - \underline{m}}{2}$$

Finally, the Sender's payoff in a babbling equilibrium is given by

$$\underline{u} = \frac{1}{\overline{m} - \underline{m}} \left[\int_{\underline{m}}^{\frac{\overline{m}-\underline{m}}{2}+b} -(\frac{\overline{m} - \underline{m}}{2} + b - m)dm + \int_{\frac{\overline{m}-\underline{m}}{2}+b}^{\overline{m}} -(m - b - \frac{\overline{m} - \underline{m}}{2})dm \right],$$

which, after straightforward algebra, simplifies to

$$\underline{v} = -\frac{\overline{m} - \underline{m}}{2} - \frac{b}{\overline{m} - \underline{m}}$$

□

Proof of Lemma 3. A partition equilibrium features a set of messages that equal the set partitions itself (for simplicity, reporting $s = a^j$ means the Sender is reporting the true state lies between a^{j-1} and a^j) and the Receiver's action rule is to chose the midpoint of the reported partition. Although this proof reproduces results from (Crawford and Sobel (1982)), we will solve it in detail since it offers subsidies for future propositions in our paper.

Let us begin showing that choosing the midpoint of the partition is the optimal action rule provided the Sender chooses partitions truthfully. If the state m lies within partition $[a^{j-1}, a^j]$ then Receiver's best response is given by

$$\max_y E_{a^j} [-|y - m|] = \frac{1}{a^j - a^{j-1}} \int_{a^{j-1}}^{a^j} -|y - m|\, dm$$

or

$$\max_y \frac{1}{a^j - a^{j-1}} \left[\int_{a^{j-1}}^{y} (m - y)dm + \int_{y}^{a^j} (y - m)dm \right]$$

The derivative of this expression is given by

$$(y - y) + \int_{a^{j-1}}^{y} (-1)dm - (y - y) + \int_{y}^{a^j} 1dm = 0$$

which becomes

$$-(y - a^{j-1}) + (a^j - y) = 0.$$

Simple manipulation shows that

$$y = \frac{a^j + a^{j-1}}{2}.$$

The next step is to find the partitions that induce the Sender to communicate partitions truthfully. Let us continue assuming the true state m belongs to partition $[a^{j-1}, a^j]$. For truthful communication, given any m within the partition, the sender wants to reveal the correct partition.

We begin looking at deviations to the adjacent superior partition $[a^j, a^{j+1}]$. If we take two different states $m > m'$ with $m, m' \in [a^{j-1}, a^j]$, let us show that if the Sender reports the true partition with m, then she must also choose the true partition with m' as well. Let us begin showing that for any $m' < {}^{a^j+a^{j-1}}/_2 - b$, the Sender will never deviate to the upwards adjacent partition. The sender does not deviate when

$$-\left|\frac{a^j + a^{j-1}}{2} - m' - b\right| \geq -\left|\frac{a^{j+1} + a^j}{2} - m' - b\right| \tag{11}$$

where the action y was already replaced by the Receiver's action rule. But when $m' < {}^{(a^j+a^{j-1})}/_2 - b$, (11) becomes

$$m' + b - \frac{a^j + a^{j-1}}{2} > m' + b - \frac{a^{j+1} + a^j}{2}, \tag{12}$$

which holds directly since $a^{j+1} > a^{j-1}$. Thus, we do not have to look for any deviation for any such states.

But again, let us consider m and m'. If the Sender reports the true partition with m it must be that

$$-\left|\frac{a^j + a^{j-1}}{2} - m - b\right| \geq -\left|\frac{a^{j+1} + a^j}{2} - m - b\right| \tag{13}$$

We want to show that the above condition implies that (11) also holds.

For simplicity, we will assume that $a^{j+1} - a^j > 2b$ for any j and then check ex-post if this holds. Since this is a module function, we need to consider different cases. Remember that $m, m' \geq {}^{(a^j+a^{j-1})}/_2 - b$. With these two conditions, the first inequality above becomes

$$\frac{a^j + a^{j-1}}{2} - m - b \geq m + b - \frac{a^{j+1} + a^j}{2}$$

and since $m > m'$, we must have

$$\frac{a^j + a^{j-1}}{2} - m' - b > \frac{a^j + a^{j-1}}{2} - m - b \geq m + b - \frac{a^{j+1} + a^j}{2} > m' + b - \frac{a^{j+1} + a^j}{2}$$

therefore, for any $m > m'$, if the Sender reports the correct partition with m she will report the correct partition with m'. Using this same logic, we can show that if the Sender does not wishes to report the upwards adjacent partition, she also does not wishes to falsely report a greater partition. Thus, if she does not deviates to the upwards adjacent partition, she does not deviate to any other partition to the right.

Another implication is that to ensure the Sender will always report the correct partition, we must check if she has the right incentives when for the highest state within the partition, that is, when $m = a^j$.

We will consider a state m that approximates a^j from the left, so that $m \in [a^{j-1}, a^j]$. The Sender will report the correct partition if (13) holds. As m approaches a^j this becomes

$$\frac{a^j + a^{j-1}}{2} - a^j - b \geq a^j + b - \frac{a^{j+1} + a^j}{2}$$

Reorganizing this expression, we get the following transition rule.

$$(a^{j+1} - a^j) \geq (a^j - a^{j-1}) - 4b \tag{14}$$

Now, we must look at deviation to the downwards adjacent partition. We now consider two states $m < m'$, with $m, m' \in [a^j, a^{j+1}]$. We begin noting that, if $m' > {}^{(a^{j+1}+a^j)}/_2$, then the sender will never report the partition is a^j. That is because when such condition holds, the inequality

$$-\left|\frac{a^j + a^{j-1}}{2} - m' - b\right| \leq \left|\frac{a^{j+1} + a^j}{2} - m' - b\right|$$

becomes

$$\frac{a^j + a^{j-1}}{2} - m' - b \leq \frac{a^{j+1} + a^j}{2} - m' - b$$

Therefore, if $m' \geq {}^{(a^{j+1}+a^j)}/_2$, then, she never deviates. If, on the other hand, $m \leq m' \leq {}^{(a^{j+1}+a^j)}/_2$, then if

$$-\left|\frac{a^j + a^{j-1}}{2} - m - b\right| \leq \left|\frac{a^{j+1} + a^j}{2} - m - b\right|$$

Becomes

$$\frac{a^j + a^{j-1}}{2} - m - b \leq -\frac{a^{j+1} + a^j}{2} + m + b \tag{15}$$

Since $m' \geq m$, we get from (15)

$$\frac{a^j + a^{j-1}}{2} - m' - b < \frac{a^j + a^{j-1}}{2} - m - b \leq -\frac{a^{j+1} + a^j}{2} + m + b < -\frac{a^{j+1} + a^j}{2} + m + b.$$

So the Sender does not wish to deviate to a^j.[6] This implies that it is enough to check the

[6] As in the for deviation to the upwards adjacent partition, we can show quite directly that if Sender does not wish deviate to the downward adjacent partition, she also does not wish to deviate to any further downward partition.

incentives for truthful communication on the lowest point in a partition, that is, a^j. We will choose an m' that approaches a^j from the right, so that m' always belong to $[a^j, a^{j+1}]$. If the Sender communicates the correct partition, it must be that

$$-\left|m' - \frac{a^j + a^{j-1}}{2} - b\right| \leq -\left|m' - \frac{a^{j+1} + a^j}{2} - b\right|$$

As m' tends to a^j, this inequality becomes

$$\frac{a^j + a^{j-1}}{2} - b - a^j \leq a^j + b - \frac{a^{j+1} + a^j}{2}, \tag{16}$$

which, after straightforward algebra, becomes

$$(a^{j+1} - a^j) \leq (a^j - a^{j-1}) + 4b. \tag{17}$$

Combining (14) and (17), we get

$$(a^{j+1} - a^j) = (a^j - a^{j-1}) + 4b. \tag{18}$$

This difference equation is the basis of the expression for the partitions. We can write (18) as

$$a^{j+1} - 2a^j + a^{j-1} = 4b \tag{19}$$

This is a second-order, linear difference equation with terminal conditions $a^n = \overline{m}$ and $a^0 = \underline{m}$. We can solve this with a particular integral $a^p = 2bj^2$, the complementary function is given by $a^c = A^1 v^k + v^k j A^2$. Where v^k, with $k = 1, 2$ are the solutions to the auxiliary equation $\lambda^2 - 2\lambda + 1 = 0$, based on (19). That is

$$v = \frac{2 \pm \sqrt{4-4}}{2} = 1.$$

So there is a unique solution. A given partition is the sum of the two solutions, that is $a^j = a^p + a^c$. From the initial and terminal condition we find the constants for the complementary function. Since $a^0 = \underline{m}$, we get

$$\underline{m} = A^1.$$

From the terminal condition, we get

$$\overline{m} = \underline{m} + nA^2 + 2bn^2,$$

rearranging, gives

$$A^2 = \frac{\overline{m} - \underline{m}}{n} - 2nb$$

Therefore, the full expression for any given partition is

$$a^j = \underline{m} + j\frac{\overline{m} - \underline{m}}{n} - 2jnb + 2bj^2$$

which simplifies to

$$a^j = \frac{j}{n}\overline{m} + \frac{n-j}{n}\underline{m} - 2bj(n-j).$$

This proves the first part of the Lemma. The second part of Lemma 3 regards the maximum number of partitions, n^* in a given equilibrium. We can find, n^* by ensuring that a^1 is above $a^0 = \underline{m}$. That is, we find the number of partitions such that

$$\frac{\overline{m}}{n^*} + \frac{n^*-1}{n^*}\underline{m} - 2b(n^*-1) > \underline{m}.$$

This expression can be rearranged to

$$\frac{\overline{m} - \underline{m}}{b} > 2n^*(n^*-1)$$

Which completes the proof. One important feature of this proof is that n^* is the maximum number of partitions. But we can construct equilibria with a smaller number of partitions $n < n^*$. □

Proof of Lemma 4. In order to show how the number and position of the partitions affect welfare we must distinguish two effects. The first is the impact of b on the position of the partitions, and the second is the impact of b on the number of partitions in equilibrium. From Lemma 3, we have the following difference equation (14).

$$(a^{j+1} - a^j) = (a^j - a^{j-1}) + 4b$$

and

$$a^j = \frac{j}{n}\overline{m} + \frac{n-j}{n}\underline{m} - 2bj(n-j).$$

Combining the two, we get

$$a^j - a^{j-1} = \frac{\overline{m} - \underline{m}}{n} - 2b\left(n - 2j + 1\right) \tag{20}$$

So we have an expression for the size of the partitions as a function of b. Now let us look at the expressions for the payers' expected payoff. Let us begin with the receiver. The expected payoff given a set of partitions is

$$E(u(y,m)) = \frac{1}{\overline{m}-\underline{m}} \sum_{j=1}^{n} \int_{a^{j-1}}^{a^j} - \left| \frac{a^j + a^{j-1}}{2} - m \right| dm$$

which can be re-written as

$$E[u(p,m)] = \frac{1}{\overline{m}-\underline{m}} \sum_{j=1}^{n} \left[\int_{a^{j-1}}^{\frac{a^j+a^{j-1}}{2}} \left(m - \frac{a^j + a^{j-1}}{2} \right) dm + \int_{\frac{a^j+a^{j-1}}{2}}^{a^j} \left(\frac{a^j + a^{j-1}}{2} - m \right) dm \right]$$

We solve this integral to get the following expression

$$E(u) = -\sum_{j=1}^{n} \frac{(a^j - a^{j-1})^2}{4(\overline{m}-\underline{m})}.$$

with $(a^j - a^{j-1})$ given by (20)

Let us now compute the Sender's payoff.

$$E(v(p,m)) = \frac{1}{\overline{m}-\underline{m}} \sum_{j=1}^{n} \int_{a^{j-1}}^{a^j} - \left| \frac{a^j + a^{j-1}}{2} - b - m \right| dm$$

which can be expanded to

$$E[v(p,m)] = \frac{1}{\overline{m}-\underline{m}} \sum_{j=1}^{n} \left[\int_{a^{j-1}}^{\frac{a^j+a^{j-1}}{2}-b} \left(m + b - \frac{a^j + a^{j-1}}{2} \right) dm + \int_{\frac{a^j+a^{j-1}}{2}-b}^{a^j} \left(\frac{a^j + a^{j-1}}{2} - b - m \right) dm \right], \tag{21}$$

when $(\overline{m}-\underline{m}) \geq 2bn^2$. That expression can be simplified to

$$E(v) = \sum_{j=1}^{n} \frac{(a^j - a^{j-1})^2}{4(\overline{m}-\underline{m})} - \frac{nb^2}{\overline{m}-\underline{m}}.$$

When $(\overline{m}-\underline{m}) < 2bn^2$, the first partition's size is smaller than $2b$, so $((a^1-a^0)/2 - b) < a^0 = \underline{m}$ and we compute expected payoff from different equation than (21). The payoff is given by

$$E[v(p,m)] = \frac{1}{\overline{m}-\underline{m}} \sum_{j=2}^{n} \left[\int_{a^{j-1}}^{\frac{a^j+a^{j-1}}{2}-b} \left(m + b - \frac{a^j+a^{j-1}}{2} \right) dm + \int_{\frac{a^j+a^{j-1}}{2}-b}^{a^j} \left(\frac{a^j+a^{j-1}}{2} - b - m \right) dm \right], + \frac{1}{\overline{m}-\underline{m}} \left[\int_{\underline{m}}^{a^1} (\frac{a^1-\underline{m}}{2} - b - m) dm. \right] \tag{22}$$

which simplifies to

$$E(v) = \frac{1}{\overline{m} - \underline{m}} \left[\sum_{j=1}^{n} \frac{(a^j - a^{j-1})^2}{4} - nb^2 - b(a^1 - \underline{m}) \right].$$

So, if we consider the case where $(\overline{m} - \underline{m}) \geq 2bn^2$

The derivative of both players' payoff with respect to b is given by

$$\frac{\partial E(u)}{\partial b} = \frac{\partial E(v)}{\partial b} = \frac{1}{\overline{m} - \underline{m}} \sum_{j=1}^{n} \left[\frac{(a^j - a^{j-1})}{2} \times 2j(n-j) \right] > 0.$$

If, on the other hand, $(\overline{m} - \underline{m}) < 2bn^2$, then the expected payoff for the Sender is given by (22) and we have

$$\frac{\partial E(v)}{\partial b} = \frac{1}{\overline{m} - \underline{m}} \left[\sum_{j=2}^{n} \left[\frac{(a^j - a^{j-1})}{2} \times 2j(n-j) \right] + b2j(n-j) \right] > 0$$

Therefore, in every case, an increase in b, keeping the number of partitions n fixed, increases the players payoff.

[Still incomplete!!]

□

Proof of Proposition 1. Proposition 1 argues there exists a trigger strategy equilibrium of the repeated cheap-talk with perfect information transmission with an action shift of $d \leq b$. The result is a combination of the Sender's and receiver's trigger strategy. Manipulating (2), we get the following condition for the impatience parameter δ

$$\delta \geq \frac{(\overline{m} - \underline{m})(b-d)}{(\overline{m} - \underline{m})^2 + 4b^2}$$

Now, manipulating (3), we get

$$\delta \geq \frac{4d}{(\overline{m} - \underline{m})}$$

So the two trigger strategy place restrictions on the minimum value of δ, which must be greater than both cutoffs. Therefore, provided

$$\delta \geq \delta^* = \max \left\{ \frac{(\overline{m} - \underline{m})(b-d)}{(\overline{m} - \underline{m})^2 + 4b^2}, \frac{4d}{(\overline{m} - \underline{m})} \right\}$$

both trigger strategies will be met and the players will sustain a perfect information transmission equilibrium. □

Proof of Corollary 1. The cutoff δ from Proposition 1 is a function of the action shift d. Thus, it is interesting to find the value of d that leads to the lowest δ^*. Since δ^* is the maximum of two cutoffs, one increasing with respect to d and the other decreasing with respect to d. As δ^* must be grater than both cutoffs, the lowest δ^* is that when both cutoffs are equal. Therefore, d^* is found by the equation below

$$\frac{(\overline{m}-\underline{m})(b-d^*)}{(\overline{m}-\underline{m})^2+4b^2}=\frac{4d^*}{(\overline{m}-\underline{m})}$$

Then, straightforward manipulation leads to (4). This completes the proof. □

Proof of Corollary 2. Corollary 2 argues that if players are patient enough, there exists an equilibrium with $d=0$. It is simply to replace $d=0$ on (4) to get

$$\check{\delta}=\frac{4(\overline{m}-\underline{m})b}{(\overline{m}-\underline{m})^2+4b^2},$$

which ends the proof □

Proof of Proposition 2. The proof of Proposition 2 has two main parts. The first one is finding the partitions for a given synthetic conflict of interest parameter $\hat{b}$

$$\hat{b}=\max\left\{0,b-\frac{\delta}{2(1-\delta)}[Ev^r-\underline{v}]\right\}$$

(which is a function of a given v^r). The second part consists of showing that the Sender's expected payoff

$$Ev^r=-\sum_{j=1}^{n}\int_{a^{j-1}}^{a^j}\left|\frac{a^j+a^{j-1}}{2}-x-b\right|dx \tag{23}$$

is bounded and continuous with respect to $\hat{b}$. This conditions ensure that there exist a fixed point in v^r, that is a given v^r that induces a synthetic conflict of interest $\hat{b}$ that leads to the to exact payoff v^r. Let us begin with the first part of the proof.

Much like in the proof of Lemma 3, we will compute the equilibrium partitions from the difference equation (6). However, before we can use such condition, we must show that the Sender will not deviate from partitions that follow such rule.

We focus on deviations in the current period, assuming that the Sender will not deviate in future periods when following the stationary equilibrium strategy (in the spirit of trigger strategy equilibrium). So, for the deviations considered here, we will fix the future strategies and associated payoff sequencies.

It is simple to check that the Receiver's optimal strategy is still to choose the midpoint of the reported partition today, just as we have shown in the proof of Lemma 3. Then, we must look at the Sender's decision to which information to reveal. Again, we are looking for partition equilibrium, so the signal structure are the partitions. We must then show when the partitions induce truth telling for every possible state m.

Let us begin assuming the true state m_t belongs to partition $[a^{j-1}, a^j]$. Let us look at deviations towards the adjacent superior partition, $[a^j, a^{j+1}]$. If we take two different states $m_t > m_t'$, with $m_t, m_t' \in [a^{j-1}, a^j]$, let us show that if the Sender does not wishes to send message $s_t = [a^j, a^{j+1}]$ when the state is m, then she will also not deviate when the state is m'. That is, if

$$-\left|\frac{a^{j-1}+a^j}{2} - m_t - b\right| + \frac{\delta}{1-\delta}Ev^r > -\left|\frac{a^j+a^{j-1}}{2} - m_t - b\right| + \frac{\delta}{1-\delta}\underline{v} \tag{24}$$

Then it must be that

$$-\left|\frac{a^{j-1}+a^j}{2} - m_t' - b\right| + \frac{\delta}{1-\delta}Ev^r > -\left|\frac{a^j+a^{j-1}}{2} - m_t' - b\right| + \frac{\delta}{1-\delta}\underline{v} \tag{25}$$

We can have three distinct relevant cases to consider. $m' < m < {(a^j+a^{j-1})}/2 - b$, $m' < {(a^j+a^{j-1})}/2 - b < m < {(a^{j+1}+a^j)}/2 - b$ and ${(a^j+a^{j-1})}/2 - b < m' < m < {(a^{j+1}+a^j)}/2 - b$.

Let us begin looking at the cases where $m' < {(a^j+a^{j-1})}/2 - b$. In such cases

$$m' + b - \frac{a^j+a^{j-1}}{2} + \frac{\delta}{1-\delta}Ev^r > m' + b - \frac{a^{j+1}+a^j}{2} + \frac{\delta}{1-\delta}\underline{v}$$

A careful inspection of the inequality above reveals that the inequality always holds, provided the partitions are increasing and the payoff from the trigger strategy is greater than the payoff from a babbling equilibrium. Therefore, if $m' < {(a^j+a^{j-1})}/2 - b$, the Sender will never deviate to the upwards adjacent partition. This rules out the first two cases.

We then have to tackle the case where ${(a^j+a^{j-1})}/2 - b < m' < m < {(a^{j+1}+a^j)}/2 - b$. Is such case, inequality (24) becomes

$$\frac{a^j+a^{j-1}}{2} - b - m + \frac{\delta}{1-\delta}Ev^r > m + b - \frac{a^{j+1}+a^j}{2} + \frac{\delta}{1-\delta}\underline{v}$$

But since $m' < m$, we have

$$\begin{aligned}\frac{a^j+a^{j-1}}{2} - b - m' + \frac{\delta}{1-\delta}Ev^r > \frac{a^j+a^{j-1}}{2} - b - m + \frac{\delta}{1-\delta}Ev^r > \\ > m + b - \frac{a^{j+1}+a^j}{2} + \frac{\delta}{1-\delta}\underline{v} > \quad m' + b - \frac{a^{j+1}+a^j}{2} + \frac{\delta}{1-\delta}\underline{v}\end{aligned}$$

which ensures that (25) holds. Therefore, we only have to consider the deviation around the partitions. That is, when $m \to_- a^j$. In that case, we must have

$$\frac{a^j + a^{j-1}}{2} - b - a^j + \frac{\delta}{1-\delta} E v^r \geq a^j + b - \frac{a^{j+1} + a^j}{2} + \frac{\delta}{1-\delta} \underline{v} \tag{26}$$

Manipulating (26) gives us (6). This difference equation (6), can be reorganized to

$$(a^{j+1} - a^j) \geq (a^j - a^{j-1}) - 4b' \tag{27}$$

with

$$b' = \max \left\{ b - \frac{\delta}{2(1-\delta)} [E v^r - \underline{v}], 0 \right\},$$

and this leads to a solution much like the one in Lemma 3, but with a different conflict of interest b'. Therefore, we can compute partitions in a similar expression from that of Lemma 3. However, the partitions depend on the Sender's payoff, which in turn, depend on the partitions. So, there is a circular definition which can be solved by a fixed point.

We will tackle the fixed point for the Sender's payoff function. It is a function of the partitions, which in turn are a function of b', which is a function of the Sender's expected payoff Ev^r. In order to have a fixed point, Ev^r must be a superior hemi-continuous, compact and nonempty correspondence of itself.

We begin defining the set where the correspondence belongs to. The payoff of the babbling equilibrium is a lower bound in terms of payoff for the sender. Any partition equilibrium will yield a greater payoff for the sender. On the other hand, the payoff function is a loss function which is never greater than zero. So the Sender's payoff belongs to the set $\Theta \equiv [\underline{v}, 0]$. Therefore, the Sender's payoff for any partition equilibrium belongs to Θ, which is a compact and closed set.

Now we show that the Sender's payoff function is a continuo, us function of b', which in turn, is a continuous function of Ev^r. The Sender's expected payoff (23) is the sum of many module functions that can take different forms. If in a given partition j' we have ${(a^{j'} - a^{j'-1})}/{2} - b > a^{j'-1}$ the contribution of this partition to the overall payoff is given by

$$\frac{1}{\Delta m} \left[\int_{a^{j'-1}}^{\frac{a^{j'} + a^{j'-1}}{2} - b} \left(-\frac{a^{j'} + a^{j'-1}}{2} + b + m \right) dm + \int_{\frac{a^{j'} + a^{j'-1}}{2} - b}^{a^{j'}} \left(-m - b + \frac{a^{j'} + a^{j'-1}}{2} \right) dm \right] = -\frac{1}{\Delta m} \left[\frac{(a^{j'} - a^{j'-1})^2}{4} + b^2 \right]$$

where $\Delta m = \overline{m} - \underline{m}$ But if ${(a^{j'} - a^{j'-1})}/{2} - b < a^{j'-1}$ then the contribution of this partition to the payoff is given by

$$\frac{1}{\Delta m}\int_{a^{j'-1}}^{a^{j'}}\left(-m-b+\frac{a^{j'}+a^{j'-1}}{2}\right)dm=-\frac{1}{\Delta m}b(a^{j'}-a^{j'-1})$$

Therefore, we must find the partition k that splits the payoff in each of these rules.

So, for a given expected payoff Ev^r, that defines the synthetic b' to we find the number of partitions $\overline{n}$. Then, we apply the difference equation (27) to find the partitions and we find the partitions. Then we find the k such that

$$\begin{aligned}\frac{a^k+a^{k-1}}{2}-b\leq& a^{k-1}\\ \frac{a^{k+1}+a^k}{2}-b\geq& a^k\end{aligned}$$

which becomes

$$\begin{aligned}a^k-a^{k-1}\leq& 2b\\ a^{k+1}-a^k\geq& 2b\end{aligned}$$

Using that

$$a^j-a^{j-1}=\frac{\Delta m}{n}-2b'(n-2j+1)$$

we get that k is the positive integer that satisfies

$$\frac{n-1}{2}+\frac{b}{2b'}-\frac{\Delta m}{4b'n}\leq k\leq\frac{n+1}{2}+\frac{b}{2b'}-\frac{\Delta m}{4b'n} \tag{28}$$

Notice that k can be zero

that defines the Sender's payoff

$$Ev^r=-\frac{1}{\Delta m}\left[\sum_{j=k}^{\overline{n}}\left(\frac{(a^j-a^{j-1})^2}{4}+b^2\right)+b(a^k-\underline{m})\right], \tag{29}$$

where k is the integer that satisfies (28). Since Ev^r is a continuous functions of the partitions, the direct relationship between b' and Ev^r is continuous. However, n and k are integers numbers, implicitly defined by b'. Therefore, we must show that Ev^r is continuous with respect to n and k. Let us begin looking at n. There is a new partition every time

$$\frac{\Delta m}{b'} = 2\overline{n}(\overline{n}-1)$$

which can be rewritten as

$$\frac{\Delta m}{n} = 2b'(n-1) \tag{30}$$

In this case, the first partition is given by

$$a^1 = \frac{\overline{m}}{n} + \frac{n-1}{n}\underline{m} - 2b'(n-1)$$

which, using (30), gives $a^1 = \underline{m}$. Therefore, the first partition has size zero. So the Sender's payoff from this equilibrium with $\overline{n}$ partitions is identical to his payoff from an equilibrium with $\overline{n}-1$ partitions, since the mass of the first partition is zero. Effectively, the partitions are equal between the two equilibria. That is, when $\frac{\Delta m}{b'} = 2\overline{n}(\overline{n}-1)$, then

$$Ev^r(\overline{n}) = \sum_{j=1}^{\overline{n}} \int_{a^{j-1}}^{a^j} \left| \frac{a^j + a^{j-1}}{2} - m - v \right| dm = Ev^r(\overline{n}-1) = \sum_{j=1}^{\overline{n}-1} \int_{a^{j-1}}^{a^j} \left| \frac{a^j + a^{j-1}}{2} - m - v \right| dm$$

Therefore, the Sender's payoff is continuous with respect to n.

Now, let us look at changes at k. Let us look at the case where

$$\begin{aligned}
\frac{a^k + a^{k-1}}{2} - b &< a^{k-1} \\
\frac{a^{k+1} + a^k}{2} - b &= a^k \\
\frac{a^{k+2} + a^{k+1}}{2} - b &> a^{k+1}
\end{aligned}$$

therefore

$$a^{k+1} - a^k = 2b$$

So the payoff from this partition is given by

$$\begin{aligned}
\frac{1}{\Delta m}\left[\int_{a^k}^{a^{k+1}} \left(-m - b + \frac{a^{k+1}+a^k}{2}\right) dm\right] =& \frac{1}{\Delta m}\left[\int_{\frac{a^{k+1}+a^k}{2}-b}^{a^{k+1}} \left(-m - b + \frac{a^{k+1}+a^k}{2}\right) dm\right] \\
&+ \frac{1}{\Delta m}\left[\int_{a^k}^{\frac{a^{k+1}+a^k}{2}-b} \left(-\frac{a^{k+1}+a^k}{2} + b + m\right) dm\right]
\end{aligned}$$

where the right-hand side is the Sender's payoff from an equilibrium computed with $k-1$. Therefore, the welfare function is continuous with respect to k.

Since Ev^r is a continuous function from $\Theta \to \Theta$, there exists a fixed point in Ev^r. Therefore, there exists an equilibrium set of partitions for our repeated cheap talk game. In particular, since $Ev^r > \underline{v}$, there are more partitions in equilibrium and the partitions are better positioned along the state space, since $b > b'$. This completes the Proof of Proposition 2.

□

Proof of Corollary 3. Since the partitions are computed using the effective conflict of interest $b' < b$, it is straightforward to check that there is a weakly greater number of partitions and that the partitions are more evenly distributed.

□

Proof of Corollary 4. From the expression of b', it is straightforward to notice $b' < b$ whenever $\delta > 0$. Therefore, from corollary 3, we have an equilibrium with weakly greater number of partitions, and more evenly distributed partitions. □

Proof of Lemma 5. The Receiver's decision to comply to a trigger strategy is conditional on the message he receives. Assuming the message is truthful, after he receives the message he can update the information about the state m. So, we can compute the expected payoff after receiving a message m_t^j by

$$E\left(u_t^r|s^j\right) = \frac{1}{a^j - a^{j-1}} \int_{a^{j-1}}^{a^j} \left| \frac{a^j + a^{j-1}}{2} + d - m \right| dm$$

which becomes

$$E\left(u_t^r|s^j\right) = \frac{1}{a^j - a^{j-1}} \left[\int_{\frac{a^j + a^{j-1}}{2}+d}^{a^j} \left(\frac{a^j + a^{j-1}}{2} + d - m \right) dm + \int_{a^{j-1}}^{\frac{a^j + a^{j-1}}{2}+d} \left(m - \frac{a^j + a^{j-1}}{2} - d \right) dm \right]$$

which simplifies to

$$E\left(u_t^r|s^j\right) = -\frac{1}{a^j - a^{j-1}} \left[\frac{(a^j - a^{j-1})^2}{4} + d^2 \right] \tag{31}$$

Now, we must compute the payoff from deviating a trigger strategy in a repeated partition equilibrium. In such case, the message is truthful and informs the true partition. So, the best the receiver can do is to choose the midpoint of the partition, just as in section 4.1. Then, his payoff from the deviation is given by

$$E\left(u^D|s_t^j\right) = -\frac{1}{a^j - a^j - 1}\left[\frac{(a^j - a^{j-1})^2}{4}\right] \tag{32}$$

Then, from (10), we know that, given a message m^j, the Receiver complies to the trigger strategy if

$$\frac{d^2}{a^j - a^{j-1}} \leq -E(u^r - \underline{u}),$$

where the right side is positive. This condition must be satisfied for every partition. Thus, the smaller the partition, the greater the receiver's payoff from deviating. Since partitions are increasing in size, it must be that the first partition has the highest payoff from deviation. Then, if the receiver does not wish to deviate in case the state falls into the first partition, he does not wish to deviate in any other partition. Therefore, complience to the trigger strategy mean

$$\frac{d^2}{a^1 - \underline{m}} \leq -E(u^r - \underline{u}), \tag{33}$$

This completes the proof. □

Proof of Proposition 3. This proof has X parts. For the first part, we will define a synthetic equilibrium. The synthetic equilibrium is one where partitions are defined by the Sender's transition rule for truthful communication, given by equation (9). In a synthetic equilibrium, we ignore the receiver conditions to comply to the trigger strategy. from (9) we can define a modified conflict of interest $\tilde{b}$ in the spirit of the proof of Proposition 2. The $\tilde{b}$ is denoted by

$$\tilde{b} = \max\left\{b - d - \frac{\delta}{2(1-\delta)}\left[Ev^r - \underline{v}\right], 0\right\}$$

With this substitution, we are able to compute the partitions of the synthetic equilibrium, just as in the proof of Lemma 2. So, even though a positive d means the Receiver is tilting the policy towards the Sender's preferences, the compliance to the trigger strategy is the same as in an equilibrium where the Sender is not favored, but has a smaller conflict of interest b.

One important aspect of this synthetic equilibrium is that, if we have an equilibrium with $n^* \geq 2$, then, there also exists an equilibrium with $n^* - 1$, since it satisfies all conditions. This will be important for the remainder of this proof.

Given the similarities to the proof of Proposition 2, the synthetic equilibrium satisfies the following conditions

$$a^j = \frac{j}{n}\overline{m} + \frac{n-j}{n}\underline{m} - 2j\left(n-j\right)\tilde{b},$$

where

$$\tilde{b} = \max\left\{b - d - \frac{\delta}{2(1-\delta)}(v^r - \underline{v}), 0\right\},$$

and

$$v^r = -\sum_{j=1}^{n}\int_{a^{j-1}}^{a^j}\left|\frac{a^j + a^{j-1}}{2} - x - (b-d)\right| dx$$

Moreover, the maximum number of partitions $\overline{n}$ is highest integer that satisfies

$$\frac{(\overline{m} - \underline{m})}{\hat{b}} > 2\overline{n}(\overline{n} - 1)$$

The two last conditions are related to the Receiver conditions to comply to the trigger strategy. In particular, condition (33) depends on the Receiver's continuation payoff Eu^r and the size of the first partition a^1.

So, when is a synthetic equilibrium a really an equilibrium? It is a matter of checkin whether the Receiver complies to this trigger strategy. Therefore, if conditions from Lemma 3 are satisfied. That is, if

$$\frac{d^2}{a^1 - \underline{m}} \leq \frac{\delta}{1-\delta}\left[Eu^r - E\underline{u}\right], \tag{34}$$

where

$$Eu^r = -\sum_{j=1}^{n}\int_{a^{j-1}}^{a^j}\left|\frac{a^j + a^{j-1}}{2} + d - x\right| dx$$

If it is satisfied, then, we have an equilibrium. Otherwise, we can reduce the number of partitions, since there is a synthetic equilibrium with fewer partitions. This synthetic equilibrium with fewer partitions has a smaller payoff for the Receiver and a greater size for the first partition, so there are greater chances condition (34) is satisfied. Still, that does not ensure there exists an equilibrium with a smaller number of partitions, since the condition may not be met even with a smaller n. Plus, if the number of equilibrium partitions is two, there is no chance the Receiver will comply to a policy shift in a babbling equilibrium.

So, when can we be sure there is a partition equilibrium with a policy shift in a cheap talk game? One way to answer this question is to look at (34) imagining $d = 0$ in the repeated partition equilibrium. In such case, the receiver's expected payoff Eu^r is greater than the payoff of the babbling equilibrium $\underline{u}$, and the right side of (34) is zero. Therefore, we can find a positive d (possibly near zero) that such that this condition will still hold. Therefore, if there exists a partition equilibrium with no policy shift cheap talk game, there is a repeated partition equilibrium with a positive policy shift d in the repeated cheap talk game. This completes the proof.

□

Proof of Proposition 4. This proof is organized in two parts. The first one is to compute the Receiver's expected payoff as a function of the policy shift d. The second part is to take the derivative of this function. We begin computing the Receiver's expected payoff. There are two possibilities for this expression, the first one is given by

$$Eu^r = -\sum_{j=1}^{n} \int_{a_{j-1}}^{a^j} \left| \frac{a^j + a^{j-1}}{2} + d - x \right| dx$$

which can be written as

$$Eu^r = \sum_{j=1}^{n} \left[\int_{a^{j-1}}^{\frac{a^j+a^{j-1}}{2}+d} \left(x - \frac{a^j + a^{j-1}}{2} - d \right) dx + \int_{\frac{a^j+a^{j-1}}{2}+d}^{a^j} \left(\frac{a^j + a^{j-1}}{2} + d - x \right) dx \right]$$

this expression becomes

$$Eu^r = \sum_{j=1}^{n} \left[\left(\frac{x^2}{2} - x\frac{a^j + a^{j-1}}{2} - xd \right) |_{a^{j-1}}^{\frac{a^j+a^{j-1}}{2}+d} + \left(x\frac{a^j + a^{j-1}}{2} + xd - \frac{x^2}{2} \right) |_{\frac{a^j+a^{j-1}}{2}+d}^{a^j} \right]$$

which becomes

$$Eu^r = \sum_{j=1}^{n} \left[-\frac{1}{2} \left(\frac{a^j + a^{j-1}}{2} + d \right)^2 - \left(\frac{(a^{j-1})^2}{2} - a^{j-1} \left(\frac{a^j + a^{j-1}}{2} + d \right) \right) \right]$$
$$\sum_{j=1}^{n} \left[a^j \left(\frac{a^j + a^{j-1}}{2} + d \right) - \frac{(a^j)^2}{2} - \frac{1}{2} \left(\frac{a^j + a^{j-1}}{2} + d \right)^2 \right]$$

which simplifies to

$$Eu^r = -\sum_{j-1}^{n} \left[\frac{(a^j - a^{j-1})^2}{4} + d^2 \right]$$

will be such whenever $2d < a^1 - \underline{m}$.

Otherwise, the Receiver's welfare will be

$$Eu^r = \sum_{j=2}^{n}\left[\int_{a^{j-1}}^{\frac{a^j+a^{j-1}}{2}+d}\left(x - \frac{a^j + a^{j-1}}{2} - d\right)dx + \int_{\frac{a^j+a^{j-1}}{2}+d}^{a^j}\left(\frac{a^j + a^{j-1}}{2} + d - x\right)dx\right]$$
$$\int_{\underline{m}}^{\frac{a^j+a^{j-1}}{2}+d}\left(x - \frac{a^1 + \underline{m}}{2} - d\right)dx$$

This expression simplifies to

$$Eu^r = -\sum_{j=2}^{n}\left[\frac{(a^j - a^{j-1})^2}{4} + d^2\right] - \frac{1}{2}\left(\frac{a^1 - \underline{m}}{2} + d\right)^2$$

Now, we can use the fact that

$$a^j - a^{j-1} = \frac{\Delta m}{n} - 2\tilde{b}(n - 2j + 1)$$

and $a^1 = \overline{m}/n + (n-1)\underline{m}/n - 2(n-1)\tilde{b}$ to find the derivative of these expressions with respect to d.

When $2d < a^1 - \underline{m}$, we have

$$\frac{\partial Eu^r}{\partial d} = -\sum_{j=1}^{n}\left[\frac{(a^j - a^{j-1})}{2}2(n - 2j + 1)\right] - 2nd$$

which simplifies to

$$\frac{\partial Eu^r}{\partial d} = \frac{2\tilde{b}n}{3}\left(n^2 - 1\right) - 2nd$$

while, if $2d > a^1 - \underline{m}$, we have

$$\frac{\partial Eu^r}{\partial d} = \frac{2\tilde{b}n^2}{3}(n-1) - \left(\frac{\Delta m}{n} - 2\tilde{b}(n-1) + d\right)(2n-1)$$

which completes the proof.

□

References

Aumann, R. and Hart, S. (2003). Long cheap talk. *Econometrica*, 71(6):1619–1660.

Aumann, R. and Maschler, M. (1995). *Repeated Games with Incomplete Information*. MIT Press.

Austen-Smith, D. (1993). Information and influence: Lobbying for agendas and votes. *American Journal of Political Science*, 37(3):799–833.

Crawford, V. P. and Sobel, J. (1982). Strategic information transmission. *Econometrica*, 50(6):1431–1451.

Forges, F. and Koessler, F. (2008). Long persuation games. *Journal of Economic Theory*, 143(1):1–35.

Golosov, M., Skreta, V., Tsyvinski, A., and Wilson, A. (2014). Dynamic strategic information transmission. *Journal of Economic Theory*, 151:304–341.

i Vidal, J. B., Draca, M., and Fons-Rosen, C. (2012). Revolving door lobbyists. *American Economic Review*, 102(7):3731–3738.

Renault, J., Solan, E., and Vieille, N. (2013). Dynamic sender-receiver games. *Journal of Economic Theory*, 148(2):502–534.

Renault, J., Solan, E., and Vieille, N. (2017). Optimal dynamic information provision. *Games and Economic Behavior*, 104:329–349.

Schnakenberg, K. (2016). Informational lobbying and legislative voting. *American Journal of Political Science*, 61(1):129–145.