On steady-state money distributions and monetary policy: smoothness in a matching model*

Jefferson Bertolai[†1] and Nathan Machado[2]

[1]*FEARP/USP*

School of Economics, Bussiness and Accounting at Ribeirão Preto – University of São Paulo

[2]*FFCLRP/USP*

Faculty of Philosophy, Sciences and Letters at Ribeirão Preto – University of São Paulo

[1,2]*LEMC/USP*

Economic Theory Research Lab – University of São Paulo

July 22, 2024

Abstract

We study smoothness properties of steady-state (*non-degenerate*) money distributions in a matching model of money with individual holdings bounded above and indivisible money. It is shown that money creation through the monetary policy studied in Deviatov and Wallace (2001) drives steady-state money distributions in a continuous (differentiable) fashion. This opens venue for sharper analytical results and simpler numerical approaches in the analysis of optimal monetary policy using matching modelling of money.

keywords: matching model, money distributions, monetary policy
JEL codes: E52, E58

*Work developed in the context of academic activities at the Economic Theory Research Lab (LEMC-FEARP/USP). Jefferson Bertolai and Nathan Machado thank *"Programa Unificado de Bolsas"* (PUB/USP) for the financial support in the development of academic activities within LEMC-FEARP/USP.

†Corresponding author. ✉ jbertolai@fearp.usp.br.

1 An overview

It is a well known fact in monetary theory literature that tractability is a central issue for matching models of money that produces non-degenerate distribution of money as an equilibrium or optimum phenomenon. Frequently, interesting insights in such class of models obtains only by resorting to the numerical approach and by making modelling choices that help to keep the model solvable.[1]

Assuming money is indivisible and restricting individual money holdings below some *ad hoc* level $\bar{m} \in \mathbb{N}$ are two of the most successful modelling choices in this literature. In this case, the distribution of money is a vector $p = (p_0, p_1, \cdots, p_{\bar{m}})$ with non-negative component $p_m \geq 0$ representing the proportion of people whose individual money holding is $m \in \{0, 1, \cdots, \bar{m}\}$. When population is modelled as a continuum of unitary measure, it must be the case that $p_0 + p_1 + \cdots + p_{\bar{m}} = 1$. Also, the total quantity of money coincides with the per capital amount of money $\mu = \sum_{m=0}^{\bar{m}} m p_m$.

In a such model, assuming time is discrete, the current distribution of money $p = (p_0, p_1, \cdots, p_{\bar{m}})$ evolves to a new distribution $p' = (p'_0, p'_1, \cdots, p'_{\bar{m}})$ next period according to how money exchange hands through monetary payments this period. If trade opportunities and monetary payments dictate that people beggining this period with i units of money exit trading with j units of money under probability $t_{ij} \in [0, 1]$, then the proportion of people with j units of money after monetary payments is given by $p'_j = \sum_{i=0}^{\bar{m}} p_i t_{ij}$. Using matrix notation $T = [t_{ij}]$, this law of motion from $p = (p_0, p_1, \cdots, p_{\bar{m}})$ to $p' = (p'_0, p'_1, \cdots, p'_{\bar{m}})$ can be succinctly written as $p' = pT$.

As one would expect, monetary payments does not change the total amount of money in the economy so that if $\mu = \sum_{j=0}^{\bar{m}} j p_j$, then $\mu = \sum_{j=0}^{\bar{m}} j p'_j$. Monetary policy, on the other hand, could be used to alter aggregate money μ to a new level μ'. Consider, as proposed by Deviatov and Wallace (2001), a monetary policy that creates $a \geq 0$ units of money and delivers them to $\alpha(1 - p_{\bar{m}})$ individuals, choosing randomly who gets it among those individuals with money holdings lower than $\bar{m}$ and limiting such monetary gifts to at most one unit. As a consequence, $\alpha(1 - p_{\bar{m}}) = a$ and $\mu' = \mu + a$. Also, each individual with less than $\bar{m}$ units of money gets one new unit of money under probability α and gets no addtion money under probability $1 - \alpha$. Those individuals with money holdings $\bar{m}$ gets no money from monetary policy.

Combining monetary policy with monetary payments in trade opportunities, the proportion of people with j units of money after monetary payments and monetary policy is given by $p'_j = (1 - \alpha) \sum_{i=0}^{\bar{m}} p_i t_{ij} + \alpha \sum_{i=0}^{\bar{m}} p_i t_{i(j-1)}$ if $0 < j < \bar{m}$, it equals $p'_0 = (1 - \alpha) \sum_{i=0}^{\bar{m}} p_i t_{i0}$ if $j = 0$, and is given by $p'_{\bar{m}} = \sum_{i=0}^{\bar{m}} p_i t_{i\bar{m}} + \alpha \sum_{i=0}^{\bar{m}} p_i t_{i(j-1)}$ if $j = \bar{m}$. Using matrix notation $A = [a_{ij}]$, where $a_{\bar{m}\bar{m}} = 1$, $a_{ii} = 1 - \alpha$ and $a_{i(i+1)} = \alpha$ for $i < \bar{m}$ and $a_{ij} = 0$ otherwise, this extended law of motion from $p = (p_0, p_1, \cdots, p_{\bar{m}})$ to $p' = (p'_0, p'_1, \cdots, p'_{\bar{m}})$ can be more transparently described as $p' = pTA$.

In order to be able to study the monetary policy just described in a model in which people are not able to carry more than $\bar{m}$ units of money, Deviatov and Wallace (2001) proposes a probabilistic destruction

[1] See, for example, Kiyotaki and Wright (1989, 1991), Trejos and Wright (1995), Shi (1995), Cavalcanti and Wallace (1999a,b), Cavalcanti et al. (1999), and Molico (2006).

of money that mimics in a indivisble money economy the well known normalization macroeconomic literature use when studying balanced growth paths. Each unit of money (the old ones and the new ones) disappears under a probability $\delta \in [0,1]$. Under this normalization trick, someone with m units of money after monetary payments and monetary policy starts next period with $j \leq m$ units of money under probability $d_{mj} = \binom{m}{j}\delta^{m-j}(1-\delta)^j$. Using matrix notation $D = [d_{ij}]$, where $d_{ij} = 0$ whenever $i < j$, the complete law of motion from $p = (p_0, p_1, \cdots, p_{\bar{m}})$ to $p' = (p'_0, p'_1, \cdots, p'_{\bar{m}})$ is given by

$$p' = pTAD. \tag{1}$$

Stationary distributions of money are objects of great interest for the economy discussed here. They are defined as vectors $p \in \Delta \equiv \{x \in \mathbb{R}_+^{1+\bar{m}} | \sum_{i=0}^{\bar{m}} x_i = 1\}$ that satisfy (1) with $p' = p$. Although law of motion (1) suggests that stationary p can be computed as the eigenvector associated with the unitary eingenvalue of matrix TAD, appropriately normalized to be an element of Δ, this is not the case. In the matching model of money studied here, people meet in pair so that trading opportunities behing probabilities t_{ij} depends on the current p. As precisely descrided in this paper, the dependence of transtion matrix T on current money distribution p is given by $T = I_{\bar{m}} + (1/N)\begin{bmatrix} F_0 p & F_1 p & \cdots & F_{\bar{m}} p \end{bmatrix}$ for appropriately defined matrix F_j that reflects monetary payments. The natural number $N \geq 3$ is related to scarce trade opportunities and $I_{\bar{m}}$ denotes the identity matrix of dimension $1 + \bar{m}$. Acknowledging such dependence in (1), it follows that p is a stationary distribution if, and only if,

$$p = \left(p + \begin{bmatrix} \frac{pF_0p}{N} & \frac{pF_1p}{N} & \cdots & \frac{pF_{\bar{m}}p}{N} \end{bmatrix}\right) AD. \tag{2}$$

In the language we proposed using (2), the analysis Deviatov and Wallace (2001) presents on how the set of stationary money distributions depends on monetary policy can be described as how the set of vectors in Δ satisfying (2) depends on matrices A and D when $\bar{m} = 2$ and each matrix F_j assumes a spefic value F_j^*. Their analysis conclude this set is infinite (a continuum) for $(\alpha, \delta) = (0, 0)$ and has one, and only one, element for $(\alpha, \delta) \in \mathbb{R}_+^2$ in a neighborhood of $(0, 0)$. In terms of smoothness, the correspondence from (α, δ) to the set of stationary money distributions is not continuous at $(\alpha, \delta) = (0, 0)$ when each matrix F_j equals F_j^*. Figure 1 illustrates such discontinuous behaviour in the simplex Δ. The set of stationary distributions at $(\alpha, \delta) = (0, 0)$ is the curve connecting triangle vertices $p = (1, 0, 0)$ and $p = (0, 0, 1)$. The set of stationary distributions for $(\alpha, \delta) \approx (0, 0)$ is illustrated assuming $\delta = (2/3)\alpha$ and each such pair $(\alpha, \delta) = (\alpha, (2/3)\alpha)$ generates one of the points in the other curve in the figure.

This illustration suggests that the correspondence Deviatov and Wallace (2001) studies is not lower hemicontinuous at $(\alpha, \delta) = (0, 0)$ but it is a continuous function in the neighborhood of this point. We verify in our analysis (see appendix A) that their correspondence is indeed not lower hemicontinuous at $(\alpha, \delta) = (0, 0)$.

In this paper, motivated by the relevance of tratactability issues to the class of matching models Deviatov and Wallace (2001) studies, *we further explore smoothness properties* of stationary money

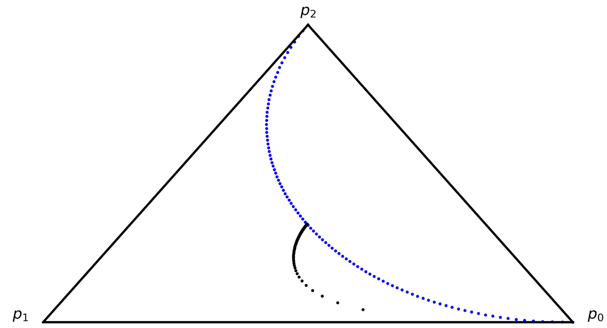

Figure 1: The discontinuous behaviour in Deviatov and Wallace (2001)'s mapping

distributions. However, we do so by taking an approach slightly different from Deviatov and Wallace (2001)'s strategy. Formally, we study how the set of vectors $p \in \Delta$ satisfying (2) depends on monetary payments (defining matrices F_j's) and monetary policy (defining matrices A and D *and the initial money creation* μ). Also, we explicitly acknowledge that normalization parameter δ must destroy money just enough to bring the increased amount of money after money creation $\mu + a$ back to the initially created amount of money μ. Specifically, the amount of money that survives destruction stage $(1-\delta)(\mu + a)$ is assumed to equal μ by choice of δ. Thus, we assume

$$\delta = \frac{a}{\mu + a} = \frac{\alpha(1 - \tilde{p}_{\bar{m}})}{\mu + \alpha(1 - \tilde{p}_{\bar{m}})} \tag{3}$$

where $\tilde{p}_{\bar{m}} = \sum_{i=0}^{\bar{m}} p_i t_{i\bar{m}} + \alpha \sum_{i=0}^{\bar{m}} p_i t_{i(j-1)}$ denotes the proportion of people with $\bar{m}$ units of money after monetary payments (that shape t_{ij}) and money creation (that is driven by α).

Our analysis, documented in the following sections, shows for the case Deviatov and Wallace (2001) studies (i.e., $\bar{m} = 2$) that the set of stationary distributions can be writtens as a *differentiable function* of money creation parameters (μ, α) and monetary payments parameters $\lambda \equiv (\lambda_{ij}^k) \in \Delta^{3\times 3}$ defining matrices F_j's. Formally, we show that if $\mathbb{P}(\mu, \alpha, \lambda) \subseteq \Delta$ is the set of all steady-state money distributions when initial money creation is $\mu \in [0, 2]$, the rate of (*probabilistic*) money creation in each period is driven by $\alpha \in [0, 1]$, and (*probabilistic*) money payments are $\lambda \equiv (\lambda_{ij}^k) \in \Delta^9$, then

correspondence $\mathbb{P} : [0, 2] \times [0, 1] \times \Delta^9 \rightrightarrows \Delta$ *is a differentiable function of* (μ, α, λ) *almost everywhere.*

The set $\mathbb{P}(\mu, \alpha, \lambda)$ is not unitary only in points (μ, α, λ) in which both monetary payments and money creation per period do not affect current distribution (i.e., $F_j = \mathbf{0}$ for all j and $A = I_{\bar{m}}$).[2] This is not surprising, though, since this is a trivial case in which $T = I_{\bar{m}}$ and, therefore, each $p \in \Delta$ with $\mu = \sum_{j=0}^{\bar{m}} j p_j$ qualifies as a stationary money distribution.

In a sense, the implied discontinuity of correspondence $\mathbb{P}(\cdot)$ in such points are not troublesome since monetary payments that imply $F_j = \mathbf{0}$ for all j are tipically not expected as an optimality or

[2]When $\bar{m} = 2$, they are defined as the points that satisfy $0 = \alpha = \lambda_{02}^1 = \lambda_{11}^1$.

an equilibrium phenomenon, since concave value functions and more divisibility on money motivate monetary exchanges that modify current distribuion of money (i.e., $F_j \neq \mathbf{0}$ for some j). Even when such monetary payments are interesting, when combined with $\alpha = 0$, they have no implication for the distribution of trade opportunities (the extensive margin), by definition. The trade-off between extensive and intensive margins would collapse to the choice of the initial amount of money μ (as it is the case when $\bar{m} = 1$) and tractability is an objective much less challenging.

The discontinuity of Deviatov and Wallace (2001)'s mapping discussed above, on the other hand, could reveal itself a critical problem. This is so because their discontinuity takes place at a point in which monetary payments λ behind their F_j^*'s matrices are attractive to society. In effect, they are chosen as the best payment schedule for society when it is assumed $\alpha = 0$. Thus, it could reasonably be the case that the social optimum be found at their discontinuity. A central result in the Deviatov and Wallace (2001)'s analysis establishes that the optimal allocation is not at the discontinuity point (i.e., does not feature $\alpha = 0$) as long as intertemporal discount factor is sufficiently high to eliminate the trade-off between extensive and intensive margins under $\alpha = 0$. What is optimal when this trade-off is a relevant issue (when low intertemporal discount factor, for example) or when $\bar{m} > 2$ was left for future research to deal with.

Deviatov (2006) promotes this research agenda by studying the same economy, but using an alternative notion of implementability[3] and resorting to numerical approach. Several examples in which money creation ($\alpha > 0$) is not optimal is presented, even when the intertemporal discount factor is high. As expected from the Deviatov and Wallace (2001)'s mapping, the numerical approach must deal with the discontinuity at $\alpha = 0$. In justifying the numerical strategy, it is stated that "Another difficult is that the mapping $F(p) = pTAD - p$ is ill-behaved at $\alpha = \delta = 0$ (Deviatov, 2006, p.14-15)" and, then, credited to Deviatov and Wallace (2001) the study of the properties of this mapping.

Still in the same class of matching models of money, but assuming $\bar{m} = 1$, Deviatov and Wallace (2014) builds on Cavalcanti and Wallace (1999a)'s model of inside money to study optimality of inflation when money creation can also take place inside meetings, when monitored people print money to pay non-monitored people for consumption. Again, the normalization trick (change of units of measure) is employed to deal with money creation in an economy with $\bar{m} < \infty$, but money can also be taken out of circulation when monitored people produce to non-monitored people in meetings. As a general result, it is not optimal to create money on an outside fashion, as the probability $\alpha > 0$ above. Optimal allocation features inflation, but (net) money creation is done by monitored people.

Barros Jr. (2017) build on these previous efforts to develop an extensive study, allowing upper bound on individual holdings $\bar{m}$ as large as four units of money, the presence for monitored people in some specifications in order to study inside money, and turning on and off the demand for trade terms to be on the pairwise-core. Again, although some relevant progress is reported, dimensionality remains a central issue constraining the divisibility of money one can study while keeping the model solvable.

[3]While Deviatov and Wallace (2001) demand only ex-post individual rationality, Deviatov (2006) weakens individual rationality to the ex-ante notion but demands trade recomendations to be in the ex ante pairwise core.

The discussion in this detailed introduction to the subject of this paper (the dependence of the set stationary distributions on monetary policy) has made clear how relevant are results on the smoothness of steady state distributions like ours. They open venue for sharper analytical results and simpler numerical approaches, for example, in the analysis of optimal monetary policy using matching modelling of money.

In addition to this unusually extensive introductory section, this paper is organized in three more sections. In section 2, for convenience, we present with more details the model Deviatov and Wallace (2001) study. In subsection 2.1, the environment is concisely presented, and subsection 2.2 discusses monetary policy modelling. Our smoothness result is established in section 3. Section 4 presents some final remarks complementary to the overview discussion in section 1. All proofs are presented in appendix A.

2 The model

The model Deviatov and Wallace (2001) study is a matching model of money in the tradition of Kiyotaki and Wright (1989, 1991). They propose to study monetary policy in the version of that model in which consumption and production are divisible goods, as proposed by Trejos and Wright (1995) and by Shi (1995), and the upper bound on individual money holdings $\bar{m}$ is increased to two units.

2.1 The environment

Time is discrete and the horizon is infinite. In each period there are $N \geq 3$ perishable and perfectly divisible consumption goods. People in this economy live forever and are heterogeneous with respect to preferences and productive capacity. A person is said to be a type-n agent if he or she consumes only the good n and is able to produce only the good $n+1$ (modulo N). For each type $n \in \{1, 2, \cdots, N\}$, there are an unitary measure of type-n people so that total population amounts to N.

People meet at each date in a pairwise fashion. Each agent meets one, and only one, other agents at random. This bilateral meeting is the trading oportunity each person receives in a given period. Because of people's productive and consumption specialization with $N \geq 3$, however, trading opportunities with double coincidence of interest never take place. Only simple coincidence of interest is possible, beyond meetings with no coincidence of wants. In a productive meeting (in which there is simple coincidence of wants) between a producer with i units of money and a consumer with j units of money, production equals consumption is denoted by $y_{ij} \geq 0$ while monetary payments are probabilistic with $\lambda_{ij}^k \in [0, 1]$ standing for the probability the consumer pays k units of money to the producer. Obviously, $1 = \sum_{k=0}^{\bar{m}} \lambda_{ij}^k$.

Each person maximizes expected discounted utility, where $\beta \in (0, 1)$ is the intertemporal discount factor. At each date, an agent of type n who produces $y \geq 0$ (of the $n+1$ good) experiences the utility $-y$, while the person of type $n+1$ who consumes y units (of the $n+1$ consumption good) experiences the utility $u(y)$. It is assumed that the function u is strictly concave and increasing with $u(0) = 0$ and

$u'(0) = \infty$. Also, there exists $\bar{y} > 0$ such that $u(\bar{y}) = \bar{y}$ and we denote $y^* > 0$ the production level that maximizes $u(y) - y$ so that $u'(y^*) = 1$.

It is not possible to store goods from one period to another. The only asset that can be stored across periods is fiat money. Fiat money is indivisible and individual money holdings cannot be larger than $\bar{m} = 2$. The distribution of money is $p = (p_0, p_1, \cdots, p_{\bar{m}}) = (p_0, p_1, p_2) \in \Delta$, where $p_j \geq 0$ is the proportion of people whose indivivual money holdings equal $j \in \{0, 1, 2\}$.

Both individuals' type and money holdings are observable in each meeting. People are not able to commit to future actions and their history is private information.

2.2 Money distribution dynamics and the monetary policy

In this environment, each person participates of a productive meeting in a given period under probability $2/N$: a type-n person becomes a producer (meets someone whose type is $n+1$) with probability $1/N$ and becomes a consumer (meets someone whose type is $n-1$) with probability $1/N$. Also, the person each individual meets has k units of money under probabililty p_k. Finally, after participating with i units of money of a productive meeting with a partner with k units of money, a producer exits with j units of money under probability λ_{ik}^{j-i} while a consumer exits the meeting with j units of money under probability λ_{ki}^{i-j}. Therefore, the ex-ante probability that a person with money holdings i exits meetings with $j \neq i$ units of money is given by

$$t_{ij} = \frac{1}{N} \sum_{k \in M} p_k \left(\lambda_{ik}^{j-i}\right) + \frac{1}{N} \sum_{k=0}^{\bar{m}} p_k \left(\lambda_{ki}^{i-j}\right).$$

Because $1 = \sum_{j=0}^{\bar{m}} t_{ij}$, it is the case that $t_{ii} = 1 - \sum_{j \neq i} t_{ij}$. From this, we know that the proportion of people with j units of money after monetary payments take place is given by $\sum_{i=0}^{\bar{m}} p_i t_{ij}$. Using matrix notation $T = [t_{ij}]$, the distribution of money after monetary payments is given by pT.

Lemma 1. *Let $I_{\bar{m}}$ denote the identity matrix with dimension $1 + \bar{m}$. For each $j \in \{0, 1, 2, \cdots, \bar{m}\}$, define $F_j = \left[f_{kl}^j\right]$ as the matrix that defines flows of people to level j of money holdings. Specifically, its generic entry f_{kl}^j equals the probability a producer in meeting (k, l) exits with j units of money (λ_{kl}^{j-k}) if $k < j$, is given by the probability a consumer in meeting (k, l) exits with j units of money (λ_{lk}^{k-j}) if $k > j$. If $k = j$, f_{kl}^j equals $-\sum_i \left(\lambda_{kl}^{i-k} + \lambda_{lk}^{k-i}\right)$. Then, transition matrix T can be written as*

$$T = I_{\bar{m}} + \frac{1}{N} \begin{bmatrix} F_0 p & F_1 p & \cdots & F_{\bar{m}} p \end{bmatrix}, \tag{4}$$

Also, in the aggregate the (net) flows must cancel (in the sense that $\sum_{j=0}^{\bar{m}} F_j = \mathbf{0}$) and average flow matrix $\bar{F} \equiv \sum_{j \in M} j F_j$ is anti-symmetric (i.e., $\bar{F} = -\bar{F}^t$). In particular, this last result ensures that for all $x \in \mathbb{R}^{1+\bar{m}}$ it must be the case that $x \bar{F} x^t = 0$.

Proof. See appendix A. $\square$

After meetings, money creation takes place. Each person with money holdings not equal to $\bar{m}$ receives one unit of money with probability α and get no additional money under probability $1-\alpha$. People with $\bar{m}$ units of money get no additional money. As already antecipated in section 1, such money creation defines a transition matrix $A = [a_{ij}]$ that assumes the format presented in (5) when $\bar{m} = 2$. In order to study money creation in an economy with finite upper bound on individual money holdings, each unit of money is assumed to disappear under probability $\delta \in [0, 1]$. In matrix notation, as already antecipated in section 1, this normalization is represented through matrix $D = [d_{ij}]$ which assumes the format presented in (5) when $\bar{m} = 2$.

$$A = \begin{pmatrix} 1-\alpha & \alpha & 0 \\ 0 & 1-\alpha & \alpha \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ \delta & 1-\delta & 0 \\ \delta^2 & 2\delta(1-\delta) & (1-\delta)^2 \end{pmatrix} \tag{5}$$

Because normalization is assumed to take place after money creation and money creation is assumed to occur after monetary payments, the law of motion for money distribuion becomes $p' = pTAD$.

As explained by Deviatov and Wallace (2001), this kind of policy is a random version of the standard lump-sum money creation policy. In models with divisible money, the standard policy creates money at a constant rate, with the injections of money being handed out to people on a *lump-sum* fashion. This is the famous helicopter money. In order to study steady state allocations/equilibria with money creation, that policy is usually modelled togheter with a normalization. The injection of money is followed by a reduction in each person's holdings that is proportional to the person's holdings. The proportional reduction is nothing but a normalization. The creation (α part) in (5) is done on a per person basis, while the de desintegration/normalization (δ part) is proportional to money holdings holdings.[4]

Definition 1. *Let $p \in \Delta$ a distribution of money. Given $(\alpha, \delta) \in [0, 1]^2$ and $\lambda \equiv \left(\lambda^l_{mn}\right) \in \Delta^{\bar{m}^2}$, distribution p is stationary $p = pTAD$*

3 The smoothness result

As already anticipated in section 1, because money creation under probability α makes *unitary* monetary gifts only to people with monetary holdings after monetary payments less than $\bar{m}$, the amount of new money each period is given by $a = \alpha(1 - e_{\bar{m}}pT)$, where $e_{\bar{m}} = (0, 0, \cdots, 0, 0, 1) \in \mathbb{R}^{1+\bar{m}}$ is the last how of identity matrix $I_{\bar{m}}$. In order to explicitly acknowledge that normalization parameter δ must destroy money just enough to bring the increased amount of money after money creation $\mu + a$ back to the

[4]Moreover as Deviatov and Wallace (2001) explain, in a model with divisible money and a nondegenerate distributions of money holdings, the standard policy has two effects: it tends to redistribute real money holdings shifting the edges of the money holdings to the mean and it has incentive effects by making money less valuable to acquire. This two effects is also present in our policy. Indeed, as regards incentives, producers can receive a unit of money without working for it (the lump-sum part) or they can loose money for which they have worked (the normalization part). This two events makes money less valuable to acquire. And, for the same reasons, consurmers are more willing to part with money, shifting the edges os the money holdings distribution.

initially created amount of money μ, we choose the value δ that makes the amount of money that survives destruction stage $(1-\delta)(\mu+a)$ equal to μ. Thus, we assume that

$$\delta = \frac{a}{\mu+a} = \frac{\alpha(1-e_{\bar{m}}pT)}{\mu+\alpha(1-e_{\bar{m}}pT)} \tag{6}$$

Lemma 2. *For each distribution $p \in \Delta$, each monetary police $(\mu,\alpha) \in [0,2]\times[0,1]$, and each schedule monetary payments $\lambda = (\lambda_{ij}^k) \in \Delta^{(1+\bar{m})^2}$, let $x = (\mu,\alpha,\lambda)$ and define*

$$f(p;x) = p - pTAD,$$

where matrix D is defined by δ given in (6) as a function of x. Then, p is an stationary money distribution under $x = (\mu,\alpha,\lambda)$ if, and only if, $f(p;x) = 0$. Also, when $\bar{m} = 2$ we have $f(p;x) = [f_0(p;x), f_1(p;x), f_2(p;x)]$, where

$$\begin{aligned}
f_1(p;x) &\equiv \frac{pF_2p}{N}(1-\delta)\Big\{\alpha\delta + (1-\alpha)(1-\delta)\Big\} - p_0\alpha(1-\delta) + p_1(1-\alpha(1-\delta))\delta + \delta^2 p_2 \\
f_2(p;x) &\equiv \frac{pF_2p}{N}(1-\delta)\Big\{\alpha - 2\left[(1-\delta)(1-\alpha)+\alpha\delta\right]\Big\} + p_0\alpha(1-\delta) - p_1\alpha(1-2\delta) \\
&\quad -\delta p_1(1-\alpha(1-2\delta)) + 2\delta(1-\delta)p_2 \\
f_3(p;x) &\equiv \frac{pF_2p}{N}(1-\delta)^2(1-2\alpha) + p_1\alpha(1-\delta)^2 - p_2\delta(2-\delta).
\end{aligned}$$

and $pF_2p = p_1p_1\lambda_{11}^1 - p_0p_2\lambda_{02}^1$.

Proof. Straightforward but tedious algebra can be used to deduce results in this lemma. □

According to Lemma 2, p is a stationary distribution under $x = (\mu,\alpha,\lambda)$ if is a solution for the system of equations $f(p;x) = 0$. Using this result, for each $x = (\mu,\alpha,\lambda) \in [0,\bar{m}]\times[0,1]\times\Delta^{(1+\bar{m})^2}$ define $\mathbb{P}(x) \equiv \{p \in \Delta | f(p;x) = 0\}$ as the set of all stationary money distributions under x. Resorting to the Implicit Function Theorem (IFT), we are able to establish in Proposition 1 in which points $f(p;x) = 0$ implicitly defines p as a differentiable function of $x = (\mu,\alpha,\lambda)$. In this case, correspondence $\mathbb{P}(\cdot)$ is a differentiable function. Even in points $x = (\mu,\alpha,\lambda)$ for which we cannot invoke the IFT, we are able do show that p is a continuous funtion of at x, i.e., $\mathbb{P}(\cdot)$ is a continuous function. Only when x is such that $A = I_{\bar{m}}$ and $F_j = \mathbf{0}$ for all j, we have more than one element in the set $\mathbb{P}(x)$.

Proposition 1. *Suppose $\bar{m} = 2$. For each $x = (\mu,\alpha,\lambda) \in [0,\bar{m}]\times[0,1]\times\Delta^{(1+\bar{m})^2}$, define $g(p,x) = \begin{pmatrix} 1-\sum_i p_i \\ \mu - \sum_i ip_i \\ f_2(p,x) \end{pmatrix}$ and the set of stationary money distribution under x is $\mathbb{P}(x) \equiv \{p \in \Delta | f(p;x) = 0\}$. Then, $\mathbb{P}(x) = \{p \in \mathbb{R}^{1+\bar{m}} | g(p;x) = \mathbf{0}\}$ for all admissible x and*

- *$\mathbb{P}(x)$ equals the set $\{p \in \Delta | \mu = \sum_{j=0}^2 jp_j\}$ if $0 = \alpha = \lambda_{11}^1 = \lambda_{02}^1$;*

- $\mathbb{P}(x)$ *is a differentiable function in every* x *such that* $0 = \alpha = \lambda_{11}^1 = \lambda_{02}^1$ *does not hold. Derivatives are*

$$\mathbb{P}'(x) = -\left[\frac{d}{dp}g(\mathbb{P}(x); x)\right]^{-1}\left[\frac{d}{dx}g(\mathbb{P}(x); x)\right], \tag{7}$$

Proof. See appendix A. □

An interesting consequence of Proposition 1 is that the IFT also provides derivates for $\mathbb{P}(x)$. This could be used to study how social welfare depends on $x = (\mu, \alpha, \lambda)$. When $\bar{m} = 2$ and production and consumption y_{ij}^k is made possible through monetary payments λ_{ij}^k in meetings $(0,1)$, $(0,2)$, $(1,1)$, and $(1,2)$, for example, the social welfare is proportional to

$$w(p, \lambda, y) = \sum_{i=0}^{1}\sum_{j=1}^{2}\sum_{k=1}^{\min\{j,2-i\}} p_i p_j \lambda_{ij}^k \left[u(y_{ij}^k) - y_{ij}^k\right]$$

In principle, derivatives in $\mathbb{P}'(x)$ could be used to maximize $w(p, \lambda, y)$ choosing y and $x = (\mu, \alpha, \lambda)$ subject to $p = pTAD$ and $p = \mathbb{P}(x)$.

In studying $w(p, \lambda, y)$, Deviatov and Wallace (2001) chooses the first-best level for production, $y_{ij}^k = y^*$, in order to maximize the intensive margin. Then, a *constrained* optimal choice is made to maximize the extensive margin $E_\lambda(p) \equiv \sum_{i=0}^{1}\sum_{j=1}^{2}\sum_{k=1}^{\min\{j,2-i\}} p_i p_j \lambda_{ij}^k$ choosing (λ, p) subject to $p = pT$ and $(\alpha, \delta) = (0, 0)$. As a solution, they find that extensive margin $E_\lambda(p)$ is maximized at $p^* = (1/3, 1/3, 1/3)$ with $\lambda_{11}^{1*} = \lambda_{02}^{1*} = 1$. Then, using their derivative of p in α and assuming $\delta = (2/3)\alpha$, they differentiate $E_\lambda(p)$ in α and evaluate the obtained derivative at p^* and λ^*. Their result on optimality of money creation ($\alpha^* > 0$) then follows from the fact that the resulting derivative is strictly positive (and that β is high enough to ensure y^* and λ^* are individually rational in all relevant meetings).

4 Final Remarks

We believe our smoothness result opens venue for sharper analytical results and simpler numerical approaches in the analysis of optimal monetary policy using matching modelling of money. In particular, we have briefly discussed some important advances in this literature that have had to deal with tractability issues implied by the discontinuity Deviatov and Wallace (2001) has reported in their analysis. For example, there are studies with different implementability notions (Deviatov, 2006), other allowing for inside money creation (Deviatov and Wallace, 2014), while other face the challenge of increasing $\bar{m}$ (Barros Jr., 2017).

It would be interesting to see research effort being channeled back to these questions and modelling approaches now that we know the correspondence $\mathbb{P}(x)$ is a differenciable function. In particular, the confidence Deviatov and Wallace (2001) manifest in applying their proof technique[5] on economies with

[5] Namely, computing optimal extensive and intensive margins ignoring the trade-off that eventually exist between these margins and then searching for a high enough β to ensure that such choices are individually rational.

higher $\bar{m}$ when saying "we are confident we could produce a version of proposition 1 for any finite bound of individuals holdings (Deviatov and Wallace, 2001, p.11)" should now be taken even more seriously, once derivatives of $\mathbb{P}(x)$ follows as a subproduct of Implicit Function Theorem.

From the tecnichal point of view, generalizing our analysis for $\bar{m} > 2$ sounds like an interesting and natural mathematical question. Because we rely on the Implicit Function Theorem in our proof and we are able to decompose transition matrix T in (4) for arbitrary $\bar{m}$, the generalization for higher $\bar{m}$ is in principle a promising mathematical task. That said, we restrict our analysis to the case $\bar{m} = 2$ because we believe this is already enough to subsidize new interesting contributions in this research agenda.

References

F. A. d. Barros Jr. *Essays on monetary theory*. PhD thesis, 2017.

R. d. O. Cavalcanti and N. Wallace. Inside and outside money as alternative media of exchange. *Journal of Money, Credit and Banking*, pages 443–457, 1999a.

R. d. O. Cavalcanti and N. Wallace. A model of private bank-note issue. *Review of Economic Dynamics*, 2(1):104–136, 1999b.

R. d. O. Cavalcanti, A. Erosa, and T. Temzelides. Private money and reserve management in a random-matching model. *Journal of Political Economy*, 107(5):929–945, 1999.

A. Deviatov. Money creation in a random matching model. *Topics in Macroeconomics*, 6(3), 2006.

A. Deviatov and N. Wallace. Another example in which lump-sum money creation is beneficial. *Topics in Macroeconomics*, 1(1):153460131001, 2001.

A. Deviatov and N. Wallace. Optimal inflation in a model of inside money. *Review of Economic Dynamics*, 17(2):287–293, 2014.

N. Kiyotaki and R. Wright. On money as a medium of exchange. *Journal of political Economy*, 97(4): 927–954, 1989.

N. Kiyotaki and R. Wright. A contribution to the pure theory of money. *Journal of economic Theory*, 53(2):215–235, 1991.

M. Molico. The distribution of money and prices in search equilibrium. *International Economic Review*, 47(3):701–722, 2006.

S. Shi. Money and prices: a model of search and bargaining. *Journal of Economic Theory*, 67(2): 467–496, 1995.

A. Trejos and R. Wright. Search, bargaining, money, and prices. *Journal of political Economy*, 103(1): 118–141, 1995.

A Proofs

Lemma 1. *Let $I_{\bar{m}}$ denote the identity matrix with dimension $1+\bar{m}$. For each $j \in \{0,1,2,\cdots,\bar{m}\}$, define $F_j = \left[f_{kl}^j\right]$ as the matrix that defines flows of people to level j of money holdings. Specifically, its generic entry f_{kl}^j equals the probability a producer in meeting (k,l) exits with j units of money (λ_{kl}^{j-k}) if $k<j$, is given by the probability a consumer in meeting (k,l) exits with j units of money (λ_{lk}^{k-j}) if $k>j$. If $k=j$, f_{kl}^j equals $-\sum_i \left(\lambda_{kl}^{i-k}+\lambda_{lk}^{k-i}\right)$. Then, transition matrix T can be written as*

$$T = I_{\bar{m}} + \frac{1}{N}\begin{bmatrix} F_0 p & F_1 p & \cdots & F_{\bar{m}} p \end{bmatrix}, \tag{4}$$

Also, in the aggregate the (net) flows must cancel (in the sense that $\sum_{j=0}^{\bar{m}} F_j = \mathbf{0}$) and average flow matrix $\bar{F} \equiv \sum_{j\in M} jF_j$ is anti-symmetric (i.e., $\bar{F} = -\bar{F}^t$). In particular, this last result ensures that for all $x \in \mathbb{R}^{1+\bar{m}}$ it must be the case that $x\bar{F}x^t = 0$.

Proof. Let $M = \{0,1,2,\ldots,\bar{m}\}$. Because each row of T sums to unit, we have diagonal terms $t_{ii} = 1-\sum_{j\neq i} t_{ij}$. Then, T can be written as $T = I + \frac{1}{N}Q$, where $Q = [q_{ij}]$ with $q_{ij} = \sum_k p_k \lambda_{ik}^{j-i}$ when $j>i$, $q_{ij} = \sum_k p_k \lambda_{ki}^{i-j}$ when $j<i$, and $0 = \sum_j q_{ij}$. Then, it is a simple matter to see that the matricial product $F_j p$ produces a column vector whose i-th row equals q_{ij}. This shows (4)

Let $A = \sum_{j\in M} F_j$. Therefore, the general element a_{kl} of A is given by $a_{kl} = e_k A e_l^T = \sum_{j=0}^{\bar{m}} f_{kl}^j = \sum_{j<k} f_{kl}^j + \sum_{j=k} f_{kl}^j + \sum_{j>k} f_{kl}^j = \sum_{j<k} \lambda_{lk}^{k-j} + f_{kk} + \sum_{j>k}\lambda_{kl}^{j-k} = f_{kk} - f_{kk} = 0$.

Moreover, let $B = \sum_{j\in M} jF_j$. Note that $B = -B^T$ if, and oly if, $b_{kl} = -b_{kl}$ for each $(k,l) \in M^2$. Observe that $b_{kl} = e_k B e_l^t = e_k \left(\sum_{j\in M} jF_j\right) e_l^T = \sum_{j\in M}(e_k jF_j e_l^t) = \sum_{j=0}^{\bar{m}} j f_{kl}^j = \sum_{j<k} j\lambda_{lk}^{k-j} + k f_{kl}^k + \sum_{j>k} j\lambda_{kl}^{j-k} = \sum_{j<k} j\lambda_{lk}^{k-j} + \sum_{j>k} j\lambda_{kl}^{j-k} - k\left(\sum_{j<k}\lambda_{lk}^{k-j} + \sum_{j>k}\lambda_{kl}^{j-k}\right) = \sum_{j<k}\lambda_{lk}^{k-j}(j-k) + \sum_{j>k}\lambda_{kl}^{j-k}(j-k)$. Using symmetry, $b_{lk} = \sum_{j<l}\lambda_{kl}^{l-j}(j-l) + \sum_{j>l}\lambda_{lk}^{j-l}(j-l)$. It follows from this that $a_{kl} + a_{lk} = \left[\sum_{j<l}\lambda_{kl}^{l-j}(j-l) + \sum_{j>k}\lambda_{kl}^{j-k}(j-k)\right] + \left[\sum_{j<k}\lambda_{lk}^{k-j}(j-k) + \sum_{j>l}\lambda_{lk}^{j-l}(j-l)\right] =$
$\left[\sum_{j>k}\lambda_{kl}^{j-k}(j-k) - \sum_{j<l}\lambda_{kl}^{l-j}(l-j)\right] + \left[\sum_{j>l}\lambda_{lk}^{j-l}(j-l) - \sum_{j<k}\lambda_{lk}^{k-j}(k-j)\right] =$
$\left[\sum_{m=1}^{\bar{m}-k}\lambda_{kl}^m m - \sum_{m=1}^{l}\lambda_{kl}^m m\right] + \left[\sum_{m=1}^{\bar{m}-l}\lambda_{lk}^m m - \sum_{m=1}^{k}\lambda_{lk}^m m\right]$.

Remind that $\lambda_{ij}^n = 0$ if $n > \min\{j, \bar{m}-i\}$. Therefore, $()s_{lk} \leq \bar{m}-l$ and $s_{lk\leq k})$ and $(s_{kl} \leq \bar{m}-k$ and $s_{kl} \leq l)$. Thus, $b_{kl} + b_{lk} = [\sum_{m=1}^{s_{kl}}\lambda_{kl}^m m - \sum_{m=1}^{s_{kl}}\lambda_{kl}^m m] + \left[\sum_{m=s_{kl}+1}^{\bar{m}-k}\lambda_{kl}^m m - \sum_{m=s_{kl}+1}^{l}\lambda_{kl}^m m\right]$ $+[\sum_{m=1}^{s_{lk}}\lambda_{kl}^m m - \sum_{m=1}^{s_{lk}}\lambda_{kl}^m m] + \left[\sum_{m=s_{lk}+1}^{\bar{m}-l}\lambda_{lk}^m m - \sum_{m=s_{lk}+1}^{k}\lambda_{lk}^m m\right] = [\mathbb{E}(m_{kl}) - \mathbb{E}(m_{kl})] + [\mathbb{E}(m_{lk}) - \mathbb{E}(m_{lk})] = 0$.

Since $(k,l) \in M^2$ was arbitrary, then $\forall (k,l) \in M^2 (b_{kl} = -b_{lk})$, Therefore, $B = -B^T$

It remains to show that $C \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix if, and only if, $\forall x \in R^n (xCx^t = 0)$. Suppose C is a skew-symmetric matrix, that is, $C + C^t = \mathbf{0}$. Let $x \in \mathbb{R}^n$ arbitrary. Note that

$xCx^t = x(-C^t)x^t = -xC^tx^t = (-1)(xC^tx^t)^t = (-1)\left((x^t)^t(C^t)^tx^t\right) = (-1)xCx^t$. So, $2xC^tx^t = 0$. What implies that $xCx^t = 0$.

Suppose now that $\forall x \in \mathbb{R}^n(xCx^t = 0)$. Note that $0 = (ei+e_j)(C+C^t)(e_i+e_j)^t = e_i(C+C^t)e_i^t+e_i(C+C^t)e_j^t+e_j(C+C^t)e_i^t+e_j(C+c^t)e_j^T = e_jCe_i^t+e_jC^te_i^t+e_iCe_j^t+e_iC^te_j^t) = (a_{ji}+a_{ij})+(a_{ij}+a_{ji}) = 2(a_{ij}+a_{ji})$. Therefore, $a_{ij} = -a_{ji}$. □

Proposition 1. *Suppose $\bar{m} = 2$. For each $x = (\mu, \alpha, \lambda) \in [0, \bar{m}] \times [0, 1] \times \Delta^{(1+\bar{m})^2}$, define* $g(p, x) = \begin{pmatrix} 1 - \sum_i p_i \\ \mu - \sum_i ip_i \\ f_2(p, x) \end{pmatrix}$ *and the set of stationary money distribution under x is $\mathbb{P}(x) \equiv \{p \in \Delta | f(p; x) = 0\}$. Then, $\mathbb{P}(x) = \{p \in \mathbb{R}^{1+\bar{m}} | g(p; x) = \mathbf{0}\}$ for all admissible x and*

- *$\mathbb{P}(x)$ equals the set $\{p \in \Delta | \mu = \sum_{j=0}^{2} jp_j\}$ if $0 = \alpha = \lambda_{11}^1 = \lambda_{02}^1$;*
- *$\mathbb{P}(x)$ is a differentiable function in every x such that $0 = \alpha = \lambda_{11}^1 = \lambda_{02}^1$ does not hold. Derivatives are*

$$\mathbb{P}'(x) = -\left[\frac{d}{dp}g(\mathbb{P}(x); x)\right]^{-1}\left[\frac{d}{dx}g(\mathbb{P}(x); x)\right], \tag{7}$$

Proof. Because matrices T, A and D produce $pTAD \in \Delta$, the linear combination $\sum_{j\in M} f_j(p; x) = \sum_{j\in M} p_j - (1, 1, 1)pTAD = 1 - 1 = 0$. Since δ is chosen to keep the amount of money μ unaltered, as presented in (6), we have $(0, 1, 2)pTAD = \mu$. Then, $\sum_{j\in M} jf_j(p; x) = \sum_{j\in M} p_j - (0, 1, 2)pTAD = \mu - \mu = 0$. This shows that IFT cannot be used in the system $f(p; x) = \mathbf{0}$. We actually study the system $g(p; x) = \mathbf{0}$, in which the first two rows have been replaced by constraints $1 = \sum_{i\in M} p_i$ and $\mu = \sum_{i\in M} ip_i$. The jacobian determinant associated to the system $g(p; x) = \mathbf{0}$ is

$$|D| \equiv \frac{\partial}{\partial p}g(p; x) = \begin{vmatrix} -1 & -1 & -1 \\ 0 & -1 & -2 \\ \frac{\partial q(\lambda)}{\partial p_0}\Phi & \frac{\partial q(\lambda)}{\partial p_1}\Phi + \alpha(1-\delta)^2 & \left(\frac{\partial q(\lambda)}{\partial p_2} + 1\right)\Phi - 1 + 2\alpha(1-\delta)^2 \end{vmatrix},$$

where for convenience we denote $\Phi \equiv (1-\delta)\left[(1-\delta)(1-2\alpha) - \frac{\partial f_3}{\partial \delta}\frac{\alpha}{\mu+\alpha\theta}\right]$ and $q(\lambda) = pF_2p/N = ({p_1}^2\lambda_{11}^1 - p_0p_2\lambda_{02}^1)/N$. Observe that we can write $|D| = \Phi\left(1 + \kappa(p, \lambda)\right) - 1$ if $\kappa(p, \lambda) \equiv \frac{\partial q(\lambda)}{\partial p_2} + \frac{\partial q(\lambda)}{\partial p_0} - 2\frac{\partial q(\lambda)}{\partial p_1}$.

<u>Case $\alpha = 0$</u>: For $\alpha = 0$, it follows that $\Phi = 1$. This produces $|D| = \kappa(p, \lambda) = \frac{\partial q(\lambda)}{\partial p_2} + \frac{\partial q(\lambda)}{\partial p_0} - 2\frac{\partial q(\lambda)}{\partial p_1} = -\frac{1}{N}p_0\lambda_{02}^1 - \frac{1}{N}p_2\lambda_{02}^1 - \frac{4}{N}p_1\lambda_{11}^1 = -\frac{\lambda_{02}^1}{N} + \frac{p_1}{N}(\lambda_{02}^1 - 4\lambda_{11}^1)$. Therefore, in this case $|D| = 0$ if, and only if, $p_1(\lambda_{02}^1 - 4\lambda_{11}^1) = \lambda_{02}^1$. This is equivalent to

$$(\mu - 2p_2)(\lambda_{02}^1 - 4\lambda_{11}^1) = \lambda_{02}^1. \tag{8}$$

If $\lambda_{02}^1 - 4\lambda_{11}^1 = 0$, then (8) implies that $\lambda_{02}^1 = 0$. But then $\lambda_{11}^1 = 0$. So, in the case that $\lambda_{02}^1 = 4\lambda_{11}^1$, only $(\lambda_{11}^1, \lambda_{02}^1) = (0, 0)$ generates $|D| = 0$. We then consider the case $\lambda_{02}^1 - 4\lambda_{11}^1 \neq 0$, in which (8) says that the determinant vanishes if, and only if, $p_2 = \frac{1}{2}\left(\mu - \frac{\lambda^1_{02}}{\lambda^1_{02}-4\lambda^1_{11}}\right)$.

If $\lambda^1_{02} = 0 < \lambda^1_{11}$, we have $p_2 = \mu/2$ and, therefore, $p_1 = \mu - 2(\mu/2) = 0$. Thus, $p = (1-\mu/2, 0, \mu/2)$ is the only vector in Δ that makes $|D| = 0$ when $\lambda^1_{02} = 0 < \lambda^1_{11}$. For $\lambda^1_{11} = 0 < \lambda^1_{02}$, we have $p_2 = \frac{1}{2}(\mu - 1)$. Then, $p_2 \geq 0$ demands $\mu \geq 1$. In this case $p_1 = \mu - 2p_2 = 1$, and, therefore, $p = (0, 1, 0)$ is the only vector in Δ that makes $|D| = 0$ when $\lambda^1_{11} = 0 < \lambda^1_{02}$.

Suppose now that $\lambda^1_{11} > 0$ and $\lambda^1_{02} > 0$ and $\lambda^1_{02} \neq 4\lambda^1_{11}$. If $\lambda^1_{02} > 4\lambda^1_{11}$, then, $p_2 = \frac{1}{2}\left(\mu - \frac{\lambda^1_{02}}{\lambda^1_{02}-4\lambda^1_{11}}\right) \leq \frac{\mu}{2}$. Suppose $\mu \geq \frac{\lambda^1_{02}}{\lambda^1_{02}-4\lambda^1_{11}}$. Thereby: $p_1 = \mu - 2p_2 = \mu - 2\left[\frac{1}{2}\left(\mu - \frac{\lambda^1_{02}}{\lambda^1_{02}-4\lambda^1_{11}}\right)\right] = \frac{\lambda^1_{02}}{\lambda^1_{02}-4\lambda^1_{11}}$. Since $\lambda^1_{02} > 4\lambda^1_{11}$, then $p_1 \geq 0$. But, we have $p_1 \leq 1$ if, and only if, $\frac{\lambda^1_{02}}{\lambda^1_{02}-4\lambda^1_{11}} \leq 1$, what occurs when $0 \leq -4\lambda^1_{11}$. Contradiction with $\lambda^1_{11} > 0$.

Suppose $\lambda^1_{02} < 4\lambda^1_{11}$. Hence, $p_2 = \frac{1}{2}\left(\mu - \frac{\lambda^1_{02}}{\lambda^1_{02}-4\lambda^1_{11}}\right) \geq 0$. Furthermore, $p_2 \leq 1$ if, and only if, $\mu \leq 2 + \frac{\lambda^1_{02}}{\lambda^1_{02}-4\lambda^1_{11}}$. Suppose then $\mu \leq 2 + \frac{\lambda^1_{02}}{\lambda^1_{02}-4\lambda^1_{11}}$. Thereby, $p_1 = \mu - 2p_2 = \mu - 2\left[\frac{1}{2}\left(\mu - \frac{\lambda^1_{02}}{\lambda^1_{02}-4\lambda^1_{11}}\right)\right] = \frac{\lambda^1_{02}}{\lambda^1_{02}-4\lambda^1_{11}} < 0$. Contradiction with $p_1 \in [0, 1]$.

Case $\alpha > 0$. The goal is to prove that for $\alpha > 0$, $|D|$ is always nonzero.

Remind that the determinant is zero if, and only if, $\Phi(1 + \kappa(p, \lambda)) = 1$. But this can never happen since the absolute value of Φ is always less than 1 and the absolute value of $(1 + \kappa(p, \lambda))$ is always less or equal to 1. Indeed, using the fact that $\kappa(p, \lambda) \leq 0$, can be obtained $(1 + \kappa(p, \lambda)) \leq 1$. To show $(1+\kappa(p, \lambda)) \geq -1$ it is enough to see that $\kappa(p, \lambda) \geq -\frac{1}{N}(1+3p_1) \geq -\frac{4}{N}$. So, $(1+\kappa(p, \lambda)) \geq 1 - \frac{4}{N} > -1$.

To conclude the proof, we must prove that the absolute value of Φ is less than 1, what is equvalent to show that $-1 < \Phi < 1$. One can show that $\Phi = \frac{(1-\delta)^3}{\mu}\left[(1 - 2\alpha)(\mu + \alpha(2 - \theta)) + 2\mu\alpha^2\right]$. The proof is not trivial and it is divided in cases for $\alpha \leq \frac{1}{2}$ and $\alpha > \frac{1}{2}$.

Suppose $\alpha > \frac{1}{2}$. Observe that $\frac{(1-\delta)^3}{\mu} = \frac{\mu^2}{(\mu+\alpha\theta)^3}$. Furthermore, $(\mu + \alpha\theta)^3 \geq \mu^3 + 3\mu^2\alpha\theta > \mu^2(\mu + \frac{3}{2}\theta)$ Since, for $\mu < 2$, $\theta > 0$, we have that $\mu^2(\mu + \frac{3}{2}\theta) \geq \mu^2$. Hence, $0 < \frac{(1-\delta)^3}{\mu} < 1$. It remains to show that $-1 < (1 - 2\alpha)(\mu + \alpha(2 - \theta)) + 2\mu\alpha^2 < 1$. Taking the deravative of this last term with respect to α we obtain $2\mu(2\alpha - 1) - 4\alpha(2 - \theta) + 2 - \theta$ which is always less than 0. Therefore, evaluating $(1 - 2\alpha)(\mu + \alpha(2 - \theta)) + 2\mu\alpha^2$ in $\alpha = \frac{1}{2}$ and $\alpha = 1$, we obtain $\frac{\mu}{2}$ and $\mu - 2 + \theta$, which both are between -1 and 1.

Suppose $\alpha \leq \frac{1}{2}$. The strategy used here is quite different from the previous one. Note that

$$\frac{\partial\Phi}{\partial\alpha} = (1 - \delta)^3 \left[-(1 - 2\alpha)(4\theta + 6\delta(1 - \theta) + 2(\mu - 1)) - 6(1 - \delta)\alpha^2\theta + 2\alpha(\theta - 2)\right],$$

one can show that $(4\theta + 6\delta(1 - \theta) + 2(\mu - 1)) \geq 0$ for $\alpha \leq \frac{1}{2}$. Therefore, Φ is decreasing in α for $0 \leq \alpha \leq \frac{1}{2}$. Evaluating Φ in $\alpha = 0$ and $\alpha = \frac{1}{2}$ we obtain, respectively, $\Phi = 1$ and $\Phi = \frac{\mu}{2}$. Hence, we conclude that $-1 < \Phi < 1$. Therefore, for $\alpha > 0$, the determinant $|D|$ is always nonzero. □

Proposition 2. *Let $\mathcal{E} : [0, 1]^2 \rightrightarrows \Delta$ such that $\mathcal{E}(\alpha, \delta) \equiv \{p \in \Delta | pTAD = p\}$ denote the correspondence Deviatov and Wallace (2001) use to study stationary money distributions. Then, $\mathcal{E}$ is not lower semi-continuous at $(\alpha, \delta) = (0, 0)$.*

Proof. Let $\mathcal{M} = \Delta$. The goal is to show that $\mathcal{E}(\alpha, \delta)$ is not lower semi-continuous in $(\alpha, \delta) = (0, 0)$, that is, $\exists V \subseteq \mathcal{M}(V \text{ is open in } \mathcal{M}) \wedge \mathcal{E}(0,0) \cap V \neq \emptyset)$ such that $\forall \delta > 0 \exists b \in B_\delta((0,0)) \cap D(\mathcal{E}(b) \cap V = \emptyset)$

Let $\varepsilon_V = \frac{\|(1/3,1/3,1/3)-(1,0,0)\|}{3}$. Seja $V = B_{\varepsilon_V}((1,0,0))$. Note that V is open in $\mathcal{M}$. Furthermore, $(1,0,0) \in \mathcal{E}(0,0)$ and $(1,0,0) \in V$. Therefore, $\mathcal{E}(0,0) \cap V \neq \emptyset$. It remains to show that $\forall \delta > 0 \exists b \in B_\delta((0,0)) \cap D(\mathcal{E}(b) \cap V = \emptyset)$.

Let $\delta > 0$ arbitrary. We want to show that $\exists b \in B_\delta((0,0)) \cap D(\mathcal{E}(b) \cap V = \emptyset)$. It is known that (from Deviatov and Wallace (2001)), for $(\alpha, \delta) = (\alpha_1, \frac{2}{3}\alpha_1)$ and α_1 going to 0, we have that $\mathcal{E}(\alpha_1, \frac{2}{3}\alpha_1)$ goes to $(1/3, 1/3, 1/3)$, that is, $\forall \vartheta_\mathcal{E} > 0 \exists \delta_\mathcal{E} > 0(\alpha \in B_{\delta_\mathcal{E}}(0) \implies \mathcal{E}(\alpha, \frac{2}{3}\alpha) \in B_{\vartheta_\mathcal{E}}((1/3, 1/3, 1/3)))$.

Therefore, from the previous one, in particular for $\vartheta_\mathcal{E} = \varepsilon_V > 0$, it is obtained that $\exists \delta_\mathcal{E} > 0(\alpha \in B_{\delta_\mathcal{E}}(0) \implies \mathcal{E}(\alpha, \frac{2}{3}\alpha) \in B_{\varepsilon_V}((1/3, 1/3, 1/3)))$. Let $\delta_\mathcal{E} = \delta_{\mathcal{E}_0} > 0$. So, $\alpha \in B_{\delta_\mathcal{E}}(0) \implies \mathcal{E}(\alpha, \frac{2}{3}\alpha) \in B_{\varepsilon_V}((1/3, 1/3, 1/3)))$

To show the previous one, take $b = (\alpha_2, \frac{2}{3}\alpha_2)$ such that $0 < \alpha_2 < min\{\delta, \delta_{\mathcal{E}_0}, 1\}$. Thus, $\alpha_2 \in B_{\delta_\mathcal{E}}(0)$. Therefore, we can conclude that $\mathcal{E}(b) \in B_{\varepsilon_V}((1/3, 1/3, 1/3))$.

Thus, $V \cap \mathcal{E}(b) = \left\{x \mid \|x - (1,0,0)\| < \frac{\|(1,0,0)-(1/3,1/3,1/3)\|}{3}\right\} \cap \{x \mid \|x - (1/3, 1/3, 1/3)\| < \|\frac{(1,0,0)-(1/3,1/3,1/3)}{3}\|\Big\} = \emptyset$. Therefore, it was proved that $\mathcal{E}(\alpha, \delta)$ is not lower semi-continuous in $(0, 0)$. □