On telling apart users and dealers in illicit markets*

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Abstract

We employ an economic model for illicit markets as a strategy to analyze how drug dealing activity is expected to change after a cutoff $\bar{q}$ in drug possession is established for telling apart users and dealers. The analysis shows that increasing $\bar{q}$ from its initial null value (when every possession can be used to qualify the owner as a dealer) is expected to change the correspondent illicit market on ways that are not trivially deduced from a model-free perspective. The model we develop, however, enables us to describe how the intuitive (*ceteris paribus*) impact on drug dealing profitability (increasing it) leads to non-trivial changes on violence and drug dealing activity. The modeling approach we take qualifies our analysis as applicable to a wider range of illicit markets than those for illegal drugs: *the essence of our modeling approach is* given by the illegality of trading and the (implied) intrinsic presence of violence among dealers as the main feature determining the relevant property rights.

Keywords: drug-dealing game, cutoff drug possession, violence, property rights

JEL codes: K42, D78, D74, D23

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1 Introduction

In 2024, the Supreme Court in Brazil (Supremo Tribunal Federal – STF) established an objective criterion for telling apart users and dealers in the market for a specific illegal drug: marijuana. It has been decided that someone caught in possession of no more than 40g in drug cannot be arrested as a drug dealer.

At first, this can be seen as a reasonable and promising way to define sanctions according to the role played by each person participating in this illegal market. Presumably, drug dealers would be subject to much more sanctions than drug users. But, opponents to this criterion argue police job in fighting drug dealing has become much more difficult, since drug dealers now is able to pretend to be a drug user by selling an amount below the established cutoff $\bar{q} = 40$g.

In this paper we employ an economic model for illicit markets as a strategy to analyze how this illegal activity is expected to change after a cutoff in drug possession is established for telling apart users and dealers. Building on Bertolai and Scorzafave (2021)'s model, we are able to provide predictions for the expected changes no only in prices and trading volume but also in the number of dealers acting in this illegal market and in the level violence employed by dealers to ensure their *"property rights"* over territories.

In order to accomplish such a comprehensive analysis, we generalize Bertolai and Scorzafave (2021) model in two critical dimensions. First, we allow the number of dealers to be determined endogenously, allowing free entry. Also, in order to study the distinction criterion between users and dealers, we assume that police is not able to arrest drug dealers as long as their scale of operation remains below some predetermined cutoff $\bar{q} \geq 0$.

Our choice of starting point to the modeling approach in this paper, the Bertolai and Scorzafave (2021)'s model, is a natural one as it should become clear. Bertolai and Scorzafave (2021) builds on the standard model of rational choice between legal and illegal activities proposed by Becker (1968) as the approach to model dealers' choice on the level of drug trading. In order to study the possibility of emergence of a *contractual relationship* between drug dealers acting outside the prisonal system and a group of prisoners with capacity to control other prisoners' welfare,[1] the authors amend this standard model with a previous *turf-war stage*,[2] in which violence is chosen as a strategy to ensure monopoly rights over a share of the territory (which in the model is *tantamount* to the market demand). In advocating for this amendment, the authors argue that violence is a natural feature to observe in illicit markets. Because of their illegal nature, property rights on these markets are not enforced by the State. As a result, agents operating in these markets (dealing drugs) most probably will resort to violence as a strategy to defend what they deem to be their rights and to be important for profitability. Thus, in that sense, an economic model for drug dealing should necessarily take into account interactions between

[1]On this matter, see also Skarbek (2011) and Lessing (2017).

[2]This amendment is inspired in Burrus (1999) and follows the tradition of the "Economics of Conflict" literature (see Garfinkel and Skaperdas (2007)). The main modelling ingredient in this respect is the "Contest Succes Function" $t_i(v)$ discussed below (see Tullock (1980) and Hirshleifer (1989)).

drug dealing and violence.

From this point of view, the choice of starting point to the modeling approach qualifies our analysis as applicable to a wider range of illicit markets than those in which illegal drugs are sold. *The essence of our modeling approach is* given by the illegality of trading and the (implied) intrinsic presence of violence among dealers as the main feature determining the relevant property rights. Whatever the good or service traded in this market, as long as it is an illegal one, our model presents itself as a promising tool for analyzing how a cutoff policy in quantities intended to tell apart users (buyers) and dealers (sellers) would change the corresponding illicit market.

As an overview on the results we found, our analysis shows that increasing $\bar{q}$ from its initial null value is expected to change the correspondent illicit market on ways that are not trivially deduced from a model-free perspective. Our model, however, enables us to describe how the intuitive (*ceteris paribus*) impact on drug dealing profitability (increasing it) leads to non-trivial changes on violence and drug dealing activity. As one would expect, nothing happens in this market if $\bar{q}$ is kept sufficiently near to its initial null level (stays below a low cutoff $w_H^{-1}(w)$). Most surprisingly, aggregate violence and drug dealing can be made higher or lower than their initial level if new $\bar{q}$ is high enough. They can be reduced if the new value of $\bar{q}$ is tailored to be higher but approximate to an intermediate cutoff $w_M^{-1}(w)$. If $\bar{q}$ approximates a high cutoff $w_L^{-1}(w)$, however, they exceed initial levels.

As a conclusion, applying our model to the case that motivated this paper, predictions are determined by how empirically the established cutoff $\bar{q} = 40$g compares itself to cutoff levels $w_L^{-1}(w)$, $w_M^{-1}(w)$ and $w_H^{-1}(w)$. For this hypothetical empirical study, besides the explicitly recognized relevance of wages w, it would be taken into account that functions $w_L(\cdot)$, $w_M(\cdot)$ and $w_H(\cdot)$ (to be described inside the paper) result from drug demand properties (parameters of market size A and price sensibility B) and the policy against drug dealing (parameters of police effort ρ and punishment d).

This paper is organized in five sections, in addition to this introductory discussion. In the next section we present the model for illicit markets we employ for analyzing the cutoff criterion for telling apart users and dealers. Even though we advocate for our model's generality, it is described as a model for illicit drug markets. Section 3, the larger one, defines the equilibrium concept we employ in our analysis and describes in details how it is computed. In section 4, we discuss the insights our model provides for understanding how the cutoff criterion for telling apart users and dealers imply changes in the corresponding illicit market. We conclude our paper in section 5 by presenting some final remarks on future research we believe our contribution *potentially* spurs. All proofs for our lemmata and the main proposition are relegated to the appendix.

2 The Model

As anticipated in section 1, our model is a generalization of Bertolai and Scorzafave (2021)'s model. While in the original model it is assumed that only *two* drug dealers are able to operate in the market, here the number of dealers are endogenously determined by *free entry*. As long as expected profitability is higher than the wage $w > 0$ a person would receive in formal labor market[3] (his or her opportunity cost), he or she enters the market (get involved in drug dealing activities). Because more drug dealers in the market drives profitability down, a finite number of drug dealers emerges in equilibrium when profitability equals the opportunity cost.

The second dimension of Bertolai and Scorzafave (2021)'s model we generalize is the way police effort to fight drug dealing shapes the expected profitability in this activity. Because the establishment of the cutoff criterion $\bar{q} > 0$ intended to tell apart users and dealers makes room for drug dealers to pretend to be drug users, as a strategy to avoid punishment for trading drugs, our model must allow for drug dealers "hide themselves" below the cutoff level $\bar{q}$. Specifically, while in the original model selling $q \geq 0$ units of drug exposes dealers to expected punishment $\rho dq \geq 0$ no matter the value of q, here this punishment is expected only for $q > \bar{q}$. As long as scale of operation q does not exceed $\bar{q}$, no punishment is expected.[4]

2.1 The drug dealing game

Because our model is a generalization of Bertolai and Scorzafave (2021)'s model, in describing it here we closely follows their exposition. There is a homogeneous drug, whose trading is illegal and for which individual demand function is $Q(p) = X - Zp$, where $X > 0$ and $Z > 0$. There is a continuum of consumers (of measure 1) and a large number $N \in \mathbb{N}$ of individuals (players) considering to operate as dealers in this market. Competition instrument is the violence to conquer from other dealers monopoly rights over consumers/territory. There is a police force to arrest those not complying with drug prohibition and/or committing violence acts.

The timing

As illustrated in Figure 1, the game is composed of three stages in which players move, followed by an arrestment stage in which police force acts. First stage is referred as *entry stage* and it can actually be seen as a pre-game stage in which individuals decide whether or not to get involved in drug dealing. The second and third stages are denominated, respectively, *turf war stage* and *trading stage*.

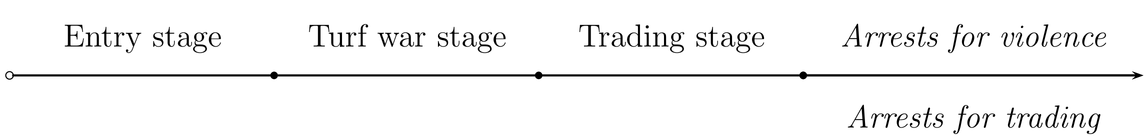

Figure 1: Timing of the two-stage game

Police force acts in the end of the game in a two-fold mission. It arrests criminals for violence acts according to probability $\nu \in (0, 1)$, and it arrests criminals for selling q units of drug according to the

[3]For simplicity, we suppose there is no informal labor market.

[4]Eventual sanctions/costs imposed on drugs users would define this alternative lower punishment. Here, we normalize such eventual sanctions to zero, for simplicity.

probability function $\tilde{\rho} : \mathbb{R}_+ \to [0,1]$ defined as

$$\tilde{\rho}(q) = \begin{cases} \rho \in (0,1) & \text{if } q > \bar{q} \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

Probability distributions ν and $\tilde{\rho}(\cdot)$ are iid among dealers.

The entry stage:

In the first stage, the N individuals (players) *simultaneously* choose between two economic activities to get involved with: a legal one and an illegal one. By choosing going legal, a person acts in labor markets earning wage $w > 0$. For simplicity, all people face the same remuneration w in labor markets. The illegal activity is drug dealing and generates an expected profitability $\Pi(\bar{q})$ when the cutoff level in drug selling is $\bar{q}$. The outcome in this stage is summarized by the number $n \in I \equiv \{1, 2, \cdots, N\}$ of individuals who get involved in drug dealing. The set of such individuals (the players in the subgame that follows) is denoted by $I_n \equiv \{1, 2, \cdots, n\} \subseteq I$, where people's name are always adequately redefined.

The turf war stage:

In the second stage, each dealer chooses how much violence to implement in order to defend himself and to conquer monopoly power over consumers and territories. The fight (outcome) is summarized by a vector of violence amounts $v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n_+$, where v_i stands for the violence amount chosen by the individual referred as dealer $i \in I_n$. Actions (violence) in this stage are simultaneously chosen and determine the distribution of monopoly power over consumers/territory: dealer i gets monopoly power over a proportion $t_i(v)$ of the aggregate demand for drugs. As expected $1 = \sum_{i \in I_n} t_i(v)$.

With probability ν, in the end of the game, dealer $i \in I$ is arrested and suffers punishment h_i for each unit of violence acts he or she has performed. This is the present value of costs associated to being in the jail.[5] For simplicity, we assume the cost h_i is common to all prisoners and equals $h > 0$.[6]

The trading stage:

Suppose turf war has taken place under violence profile v. Dealer $i \in I_n$ trades drugs under monopoly in his or her territory: i decides how much drug to sell, $q_i \geq 0$, in order to fulfill its market demand $t_i(v)Q(p_i)$,[7] so that $q_i = t_i(v)Q(p_i)$ and $p_i \geq 0$ is the price in dealer i's region. We follow Bertolai and

[5] As Bertolai and Scorzafave (2021) discuss, it *"includes the traditional opportunity cost in form of wages he could earn during his time in prison, but also legal expenses during his trial and family expenses to visit him at the prison. Also, it captures monetary value of expected welfare losses of being inside prison. For example, individuals inside prison might face higher probability of getting sick or being victim of aggression, robbery, extortion or murder"*.

[6] This is one of the two dimensions in which we specialize Bertolai and Scorzafave (2021)'s model, and we do so for simplicity purposes. In their model, $h_i = \beta_i h$ where $\beta_i > 0$ is a specific punishment tailored to dealer i. This is in fact one of the two channels of influence the prison gang has over outside prison system criminals. In their study, the authors assume β_i can be controlled to some extent by the prison gang. Because in this paper we are not taking into account this eventual channel of influence, for simplicity, we assume $\beta_i = 1$ for all $i \in I_n$.

[7] Because individual demand for drugs is given by $Q(p)$ and i has monopoly power over $t_i(v)$ consumers, we assume its market demand is $t_i(v)Q(p)$.

Scorzafave (2021) in assuming that there is no cost in producing and supplying drugs and, therefore, profits equal revenues.

Dealer $i \in I$ who has sold q units of drug in the trading stage expects to be arrested for drug dealing in the end of the game under probability $\tilde{\rho}(q) \geq 0$ and to suffer punishment d_i for each unit of drug sold. This is again the present value of costs associated to being in the jail and it is assumed to equal a common value $d > 0$ to all prisoners.[8]

Expected payoffs and the Contest Success Function

After turf war $v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n_+$ has taken place and the distribution of consumers/territories $t(v) \equiv (t_1(v), t_2(v) \cdots, t_n(v))$ has been established, the expected payoff a monopolist dealer gets after selling $q_i \geq 0$ units of drug is the difference between revenues $P\left(\frac{q_i}{t_i(v)}\right) q_i$ and the expected punishment $(1-\nu)\tilde{\rho}(q_i) d q_i + \nu\{[1-\tilde{\rho}(q_i)]hv_i + \tilde{\rho}(q_i)(hv_i + dq_i)\}$, i.e., it equals

$$\pi_i(q_i|n, v) \quad = \quad P\left(\frac{q_i}{t_i(v)}\right) q_i - \tilde{\rho}(q_i) d q_i - \nu h v_i, \tag{2}$$

where $P(q) \equiv Q^{-1}(q) = A - Bq$ denotes the aggregate (inverse) demand function, in which $A = X/Z > 0$ and $B = 1/Z > 0$. Accordingly, the individual (inverse) demand function market share $t_i(v)$ provides to dealer i is given by $P_i(q) = P(q/t_i(v))$. Figure 2 illustrates, for convenience, the typical behavior function $\pi_i(\cdot|n, v)$ exhibits.

The relation between market shares $(t_1, t_2, \cdots, t_n)$ and violence efforts $(v_1, v_2, \cdots, v_n)$ is based in Contest Success Function (CSF) literature that followed Tullock (1980)'s rent-seeking model. In the formulation we assume here (the simplest one), relative success $t_i/(\sum_{j\neq i} t_j)$ of dealer $i \in I_n$ is stated as a function of the ratio of i's resource commitment (violence effort v_i) to that chosen by other players (total violence $|v_{-i}| \equiv \sum_{j\neq i} v_j$ other dealers choose):

$$\frac{t_i}{\sum_{j\neq i} t_j} = \frac{v_i}{|v_{-i}|}, \qquad \text{if } |v_{-i}| > 0. \tag{3}$$

If $|v_{-i}| = 0$, however, the relative sucess of dealer i is *complete* (i.e., $t_i = 1$) when $v_i > 0$ and *symmetric* (i.e., $t_i = 1/n$) if $v_i = 0$. It follows from (3) and $1 = \sum_{j\in I} t_i$ that the contest success function $t : I_n \times \mathbb{R}^n_+ \to [0, 1]$ is defined as

$$t_i(v) = \begin{cases} v_i/|v|, & \text{if } v \neq \mathbf{0} \\ 1/n, & \text{otherwise} \end{cases}, \tag{4}$$

where $\mathbf{0} \equiv (0, 0, \cdots, 0) \in \mathbb{R}^n$ denotes the null vector and $|v| \equiv \sum_{j\in I} v_j$.

[8] This is the second of the two dimensions in which we specialize Bertolai and Scorzafave (2021)'s model. In their model, $d_i = \alpha_i d$ where $\alpha_i > 0$ is a punishment specific to dealer i. This is the second channel of influence the prison gang has over outside prison system criminals: the authors assume α_i can be controlled to some extent by the prison gang. Because in this paper we are not taking into account this eventual channel of influence, for simplicity, we assume $\alpha_i = 1$ for all $i \in I_n$.

3 The equilibrium

The model we have just described is a dynamic game with complete information and $N \in \mathbb{N}$ players. It begins in the entry stage with individuals choice between get involved in drug dealing and going legal. For each outcome $n \in \mathbb{N}$ for this stage, a subgame starts with a simultaneous competition (through violence) among dealers in the turf war stage. Then, after each outcome $(n, v) \in I_n \times \mathbb{R}^n_+$ for the first two stages, another subgame (in some sense, a trivial one) starts with dealers acting as monopolists in the trading stage.

A (pure) strategy for each player is a triple $s = (e, \tilde{v}, \tilde{q})$ such that $e \in \{0, 1\}$ is the action in the entry stage ($e = 1$ meaning get involved with drug dealing), function $\tilde{v} : I \to \mathbb{R}_+$ describes the violence choice $\tilde{v}(n)$ in the subgame starting after outcome n in the entry stage, and function $\tilde{q} : I \times \mathbb{R}^n_+ \to \mathbb{R}_+$ describes the monopolistic choice of drug dealing level in the subgame starting after the outcomes n in the entry stage and v in the turf war stage.

Definitions of pure *Nash equilibrium* (NE) and pure *Subgame Perfect Nash equilibrium* (SPNE) for this game are standard. A (pure) Nash equilibrium is a profile of strategies $(s^*_k)^N_{k=1} = (e^*_k, \tilde{v}^*_k, \tilde{q}^*_k)^N_{k=1}$ such that for each player $k \in I$ the strategy $s^*_k = (e^*_k, \tilde{v}^*_k, \tilde{q}^*_k)$ is a best response to the conjecture $s^*_{-k} = (s^*_j)_{j \neq k}$ about the behavior of the remaining players. A (pure) SPNE is, of course, a (pure) NE that defines (pure) NE in all subgames. In our analysis, we are interested on (pure) SPNE's that are *symmetric* in the sense that violence levels chosen in the turf war stage are the same for each one of the $n \in I$ players that enters the illicit market. Formally, for each outcome $n \in I$ in the entry stage and each dealer $i \in I_n$, equilibrium violence level in a symmetric SPNE is $\tilde{v}^*_i(n) = z_n$ for some $z_n \geq 0$ that does not depends on i.

In order to compute *symmetric subgame-perfect Nash equilibrium* (SPNE) for our game, we must impose that players' strategies constitute a Nash equilibrium in every subgame and that optimal violence levels in the turf war stage are the same for each dealer. We do this computing player's strategies in a backwards induction fashion, computing stage outcomes from the last stage to the first one.

The trading stage equilibrium:
We begin the analysis by discussing the monopolist optimal level of drug dealing for dealer i when his or her market share is $t_i(v)$. In other words, we study the solution to the problem $\max_{q_i \geq 0} \pi_i(q_i|n, v)$, where the objective function has been defined in (2). Optimal levels of drug dealing in each subgame $(n, v) \in I \times \mathbb{R}^n_+$ and the implied optimal expected payoff are reported in Lemma 1.

In solving dealers' problem at the subgame (n, v), *some caution is necessary*, since the objective function (2) is not a continuous one. Because function $\tilde{\rho}(q)$, as defined in (1), is not continuous at $q = \bar{q}$, function $\pi_i(\cdot|n, v)$ defined in (2) is not continuous at the same point. Figure 2 illustrates, for convenience, the typical discontinuous behavior function $\pi_i(\cdot|n, v)$ exhibits. In all four graphs, the solid curve represents the function $\pi_i(\cdot|n, v)$ and its point of discontinuity is identified as $\bar{q}$. Two parabolic curves are used to construct curve $\pi_i(\cdot|n, v)$. They intercept vertical axis at $\pi_i(0|n, v) = -\nu h v_i < 0$ and are invariant to $\bar{q}$. The lower quadratic curve reaches its maximum value at $q = \hat{q}_L$ and the upper one peaks at $q = \hat{q}_H$. The fourth point in each graph, $\hat{q}$, is defined as $q \in [0, \hat{q}_L]$ in which the upper curve reaches the maximum value of the lower curve.

Top left graph is the case $\bar{q} = 0$ and corresponds to the case Bertolai and Scorzafave (2021) has studied. In this case, optimal level of drug dealing clearly equals $\hat{q}_L$. The top right graph illustrates the case $0 < \bar{q} < \hat{q}$. Since $q < \bar{q}$ implies $\tilde{\rho}(q) = 0$, function $\pi_i(\cdot|n, v)$ is given by the upper curve for all $q < \bar{q}$. On the other hand, $\tilde{\rho}(q) = \rho$ for all $q > \bar{q}$ and, therefore, expected profits $\pi_i(\cdot|n, v)$ is given by the lower parabolic curve to the right of $\bar{q}$. Thus, in the top right graph the maximum value of $\pi_i(\cdot|v)$ to the left of $\bar{q}$ is achieved at the point $\bar{q}$. To the right of $\bar{q}$, the maximum is located at $\hat{q}_L$. As suggested in the figure, the optimal level of drug dealing is again $\hat{q}_L$, since $\pi_i(\hat{q}_L|n, v) > \pi_i(\bar{q}|n, v)$.

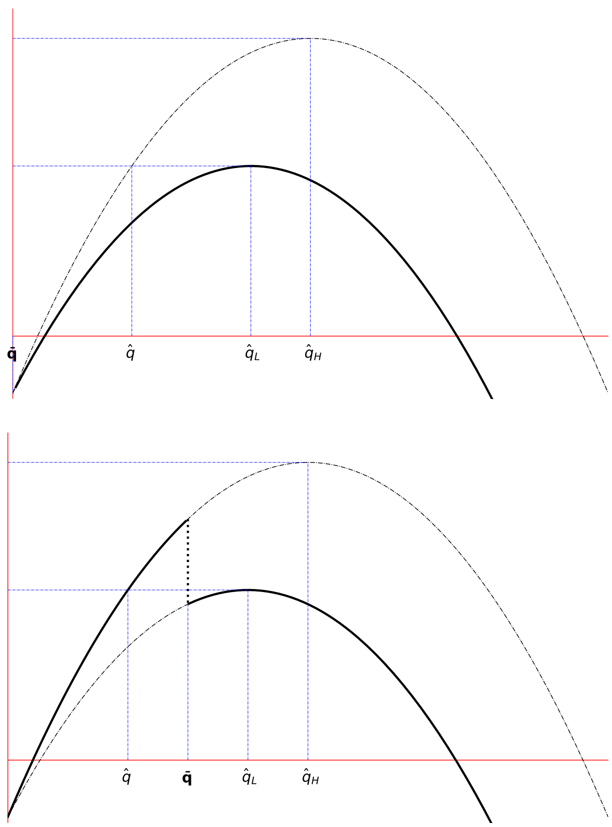

Figure 2: Cutoff $\bar{q} > 0$ implies discontinuous expected payoff $\pi_i(q|n, v)$

Now, consider the case $\hat{q}_L < \bar{q} < \hat{q}_H$ illustrated in the bottom left graph of Figure 2. The discontinuous behavior follows the same pattern identified in the top right graph: function $\pi_i(\cdot|n, v)$ is given by the upper curve to the left of $\bar{q}$ and by the lower curve to the right of $\bar{q}$. Local maxima also follows the same pattern of the last case: there is a local maximum at $\bar{q}$ and another one at $\hat{q}_L$. But now, global maximum is reached at $\bar{q}$. The final case illustrated in Figure 2 is the bottom right graph, in which $\bar{q} > \hat{q}_H$. The pattern of discontinuity remains the same one observed in the last two cases we discussed. The nature of local maxima, on the other hand, changes: to the left of $\bar{q}$ the local maximum happens now at $\hat{q}_H$ and to the right of $\bar{q}$ it is located at $\bar{q}$. Global maximum take place in this case at $\hat{q}_H$.

The dependence of optimal level of drug dealing on parameters $(\bar{q}, \hat{q}, \hat{q}_L, \hat{q}_H)$ illustrated in analyzing Figure 2 is actually quite general, as established in Lemma 1. It becomes clear from this lemma that the lower curve is maximized at $\hat{q}_L \equiv t_i(v)q_L$, the upper curve reaches maximum value at $\hat{q}_H \equiv t_i(v)q_H$, and that the point $\hat{q}$ is always given by $\hat{q} \equiv t_i(v)\tilde{q}$.

Lemma 1. *Let $q_L \equiv (A - \rho d)/2B$, $q_H \equiv A/2B$, and $\tilde{q} \equiv q_H - \sqrt{{q_H}^2 - {q_L}^2}$. Given both an outcome in the entry stage $n \in I$ and a subsequent outcome in turf-war stage $v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n_+$, the optimal choice for dealer i acting as a monopolist in the trading stage is*

$$q_i^*(n, v|\bar{q}) = \begin{cases} t_i(v)q_L & \text{if} \quad 0 \leq \bar{q} \leq t_i(v)\tilde{q} \\ \bar{q} & \text{if} \quad t_i(v)\tilde{q} \leq \bar{q} \leq t_i(v)q_H \\ t_i(v)q_H & \text{if} \quad t_i(v)q_H \leq \bar{q} \end{cases}. \tag{5}$$

Thus, optimal payoff in the trading stage for dealer i is

$$\pi_i^*(n, v|\bar{q}) = \begin{cases} t_i(v)B{q_L}^2 - \nu h v_i & \text{if} \quad 0 \leq \bar{q} \leq t_i(v)\tilde{q} \\ A\bar{q} - B\bar{q}^2/t_i(v) - \nu h v_i & \text{if} \quad t_i(v)\tilde{q} \leq \bar{q} \leq t_i(v)q_H \\ t_i(v)B{q_H}^2 - \nu h v_i & \text{if} \quad t_i(v)q_H \leq \bar{q} \end{cases} \tag{6}$$

Proof. See appendix A. □

As illustrated in the first graph of Figure 3, as a function of $\bar{q}$, the optimal level of drug trading for dealer i under market share $t_i(v)$ presented in (5) is piecewise linear with a discontinuity at $\hat{q} = t_i(v)\tilde{q}$. Then, the matching in Figure 2 between the image of the upper curve at $\hat{q}$ and the maximum of the lower curve translated itself on a discontinuity on optimal choice of drug dealing $q_i^*(n, v|\bar{q})$. The second graph in Figure 3 shows that the optimal expected payoff $\pi_i^*(n, v|\bar{q})$ is continuous in $\bar{q}$. Also, it is differentiable in all $\bar{q}$ such that $0 \leq \bar{q} \neq \hat{q}$: the matching in Figure 2 at $\hat{q}$ translated itself on a kink on optimal expected payoff $\pi_i^*(n, v|\bar{q})$.

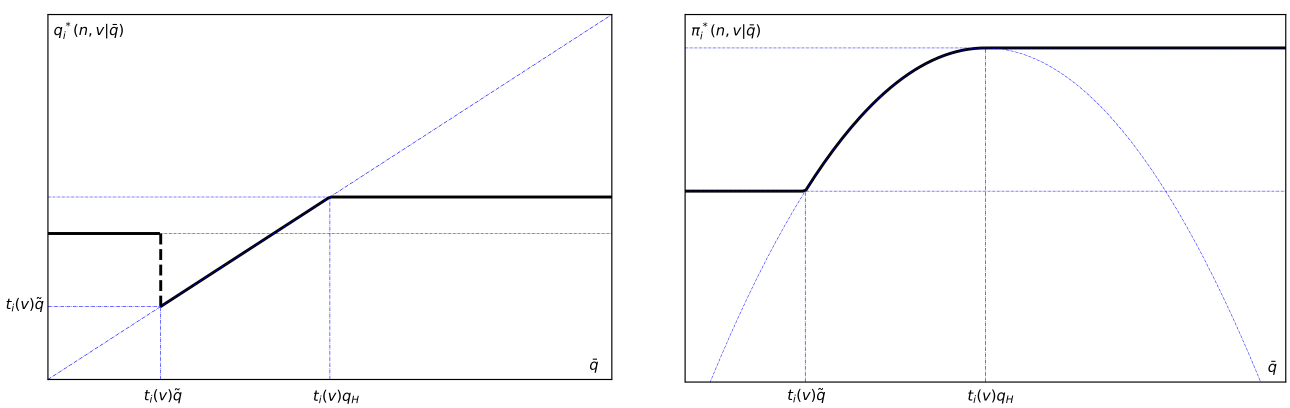

Figure 3: Optimal drug trading and expected payoff of dealer i as a function of $\bar{q}$

The turf war equilibrium:
We now turn analysis to the choice of violence levels in the turf war stage, for each given outcome $n \in I$ in the entry stage and anticipating the optimal expected payoff ($\pi_i^*(n, v|\bar{q})$ defined in (6)) for each possible subsequent trading stage. In particular, we are interested in computing the best (violence) response of each dealer i in the turf war stage for each conjecture $v_{-i} \equiv (v_j)_{j\neq i}$ he or she can form about the behavior of other dealers. In other words, we study the solution to the problem $\max_{v_i\geq 0} \pi_i^*[n, (v_i, v_{-i})|\bar{q}]$, whose objective has been defined in (6). Best (violence) response function in each subgame $n \in I$ is reported in Lemma 2, as described in (9).

In solving dealers' problem at the beginning of subgame n, *much more cautiousness is necessary.* The dependence of the objective function (6) on the choice variable v_i takes place through the linear term $-\nu h v_i$ and the nonlinear function $t_i(v_i+|v_{-i}|)$. This last term, in particular, appears in partitioning function's domain in three intervals: $[0, \hat{q}]$, $[\hat{q}, \hat{q}_H]$, and $[\hat{q}_H, \infty)$. In order to circumvent such a difficult, using definition (4), we rewrite objective (6) alternatively as

$$\pi_i^*(n, v|\bar{q}) = \begin{cases} t^0(v_i)B{q_L}^2 - \nu h v_i & \text{if} \quad 0 \leq \bar{q} \leq t^0(v_i)\tilde{q} \\ A\bar{q} - B\bar{q}^2/t^0(v_i) - \nu h v_i & \text{if} \quad t^0(v_i)\tilde{q} \leq \bar{q} \leq t^0(v_i)q_H \\ t^0(v_i)B{q_H}^2 - \nu h v_i & \text{if} \quad t^0(v_i)q_H \leq \bar{q} \end{cases}, \quad \text{when } |v_{-i}| = 0, \tag{7}$$

and

$$\pi_i^*(n, v|\bar{q}) = \begin{cases} \frac{v_i}{v_i+|v_{-i}|}B{q_H}^2 - \nu h v_i & \text{if} \quad 0 \leq v_i \leq \alpha(\bar{q})|v_{-i}| \\ A\bar{q} - B\bar{q}^2\frac{v_i+|v_{-i}|}{v_i} - \nu h v_i & \text{if} \quad \alpha(\bar{q})|v_{-i}| \leq v_i \leq \beta(\bar{q})|v_{-i}| \\ \frac{v_i}{v_i+|v_{-i}|}B{q_L}^2 - \nu h v_i & \text{if} \quad \beta(\bar{q})|v_{-i}| \leq v_i \end{cases}, \quad \text{when } |v_{-i}| > 0. \tag{8}$$

where $t^0(x) = 1/n$ if $x = 0$ and $t^0(x) = 1$ when $x > 0$. Also, functions $\alpha(\cdot)$ and $\beta(\cdot)$ are such that $\alpha(\bar{q}) \equiv \bar{q}/(q_H - \bar{q})$ if $\bar{q} < q_H$ and $\beta(\bar{q}) \equiv \bar{q}/(\tilde{q} - \bar{q})$ if $\bar{q} < \tilde{q}$. As an abuse of notation, we define $\alpha(\bar{q}) \equiv \infty$ if $\bar{q} \geq q_H$ and $\beta(\bar{q}) \equiv \infty$ when $\bar{q} \geq \tilde{q}$.

Objective function (7) in not well behaved near $v_i = 0$. Clearly, no $v_i > 0$ maximizes the strictly decreasing function (7). Whether $v_i = 0$ is optimal when $|v_{-i}| = 0$ or no optimal solution exists in this case depends on parameters determining function (7)'s value at $v_i = 0$ comparatively to its values in the neighborhood of $v_i = 0$. This is so because function $t^0(x)$ is not continuous at $x = 0$. If the implied discontinuity in $\pi_i^*(n, v|\bar{q})$ is upward as $v_i \to 0^+$, then $v_i = 0$ is optimal. If such a discontinuity is downward as $v_i \to 0^+$, then no $v_i \geq 0$ is optimal.

When $|v_{-i}| > 0$, on the other hand, the objective is the well behaved function (8). As verified in proving Lemma 2, it is continuous everywhere, differentiable at all point $v_i \neq \beta(\bar{q})|v_{-i}|$, and piecewise (strictly) concave. Figure 4 illustrates, for convenience, this typical behavior function $\pi_i^*(n, v|\bar{q})$ exhibits when $|v_{-i}| > 0$ and $\bar{q} < \tilde{q}$. Each graph presents as dashed lines the three curves defining $\pi_i^*(n, v|\bar{q})$ in (8). Horizontal axes is normalized to $v_i/|v_{-i}|$ so that cutoffs in the normalized domain are always $\alpha(\bar{q})$ and $\beta(\bar{q})$. Such cutoffs are presented in each graph as dashed vertical lines. Function $\pi_i^*(n, v|\bar{q})$ is represented in the graphs as the solid curve. All six graphs exhibits the already discussed pattern on continuity, differentiability and concavity.

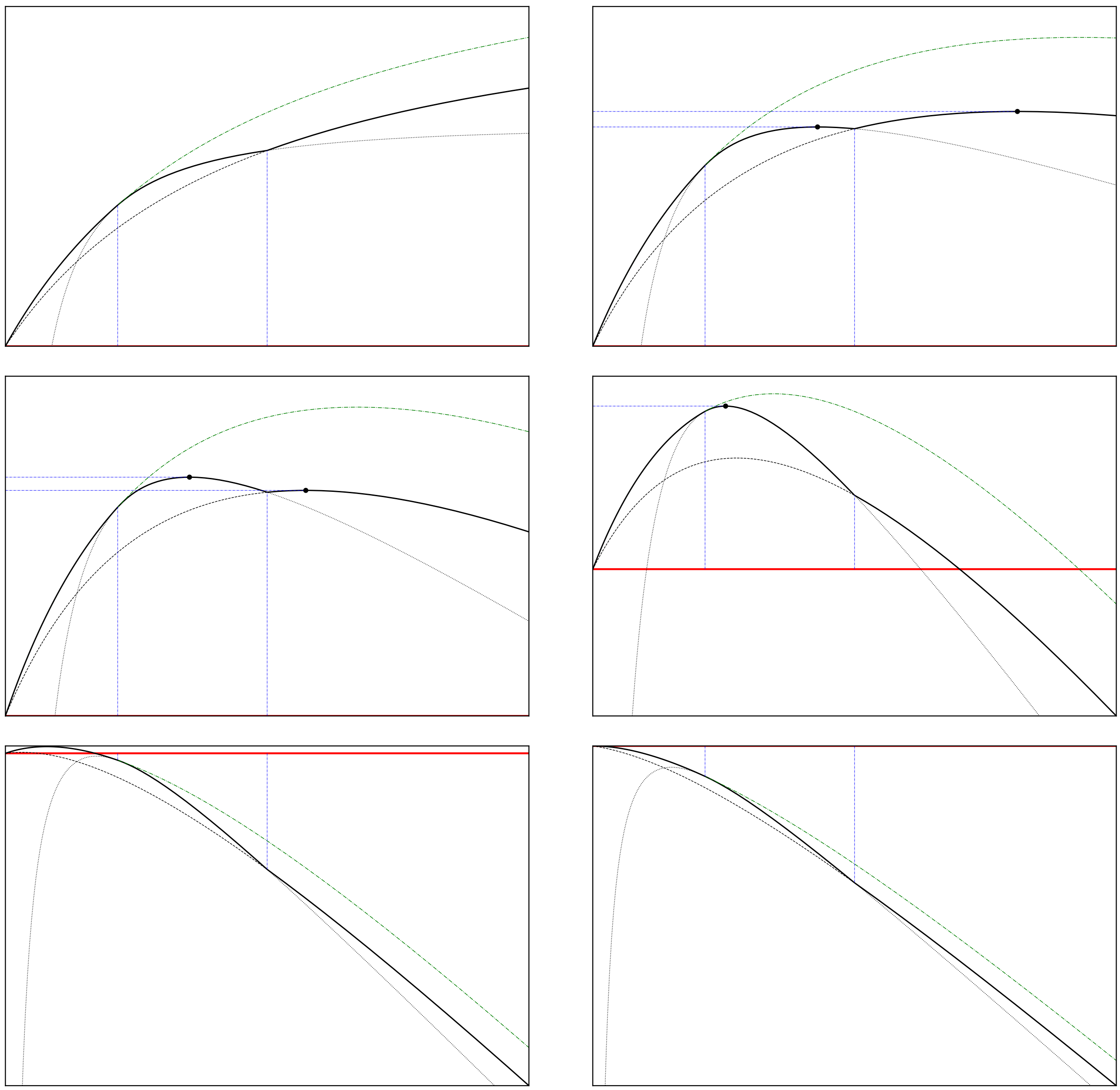

Figure 4: Typical behavior of $\pi_i^*(n, v|\bar{q})$ when $|v_{-i}| > 0$ and $\bar{q} < \tilde{q}$

The bottom right graph shows the typical behavior when $|v_{-i}| \geq v_H^2$ for $v_H \equiv q_H\sqrt{B/\nu h}$. The objective function (the solid curve) in this case is strictly decreasing and, therefore, optimal choice in the horizontal axes is $v_i/|v_{-i}| = 0$. The bottom left graph, on the other hand, presents the typical objective function when $z_H \leq |v_{-i}| \leq v_H^2$ with $z_H \equiv ([v_H - v_{\bar{q}}]^+)^2$ and $v_{\bar{q}} \equiv \bar{q}\sqrt{B/\nu h}$. In this case, the objective is strictly decreasing above $\alpha(\bar{q})$ and exhibits an interior maximum below $\alpha\bar{q}$. Thus, optimal choice maximize the highest curve among the three in the figure: it satisfies $v_i = v_H\sqrt{|v_{-i}|} - |v_{-i}|$.

The middle right graph illustrates the case with $|v_{-i}|$ such that $(v_L/[1+\beta(\bar{q})])^2 \leq |v_{-i}| \leq z_H$ in which $v_L \equiv q_L\sqrt{B/\nu h}$. The objective is strictly increasing below $\alpha(\bar{q})$, it is strictly decreasing above $\beta(\bar{q})$ and peaks at a point interior to $[\alpha(\bar{q}), \beta(\bar{q})]$. As a result, optimal choice maximize the middle curve among the three in the figure and is given by $v_i = v_{\bar{q}}\sqrt{|v_{-i}|}$. Now, the top left graph shows the typical behavior when $|v_{-i}| \leq (v_{\bar{q}}/\beta(\bar{q}))^2$. In this case the objective is strictly increasing below $\beta(\bar{q})$ and peaks after $\beta(\bar{q})$.[9] As a consequence, optimal choice maximize the lower curve among the three in the figure and is given by $v_i = v_L\sqrt{|v_{-i}|} - |v_{-i}|$.

The last two graphs illustrate more sophisticated cases, in which the objective function peaks at two interior points (for convenience, horizontal dashed lines indicate the values objective function achieves at these two local maxima). For $\mu_{\bar{q}} \equiv \sqrt{2v_{\bar{q}}(v_H - v_L)}$ and $z_s \equiv \left([v_L - v_{\bar{q}} - \mu_{\bar{q}}]^+\right)^2$, the top right one represents the case $(v_{\bar{q}}/\beta(\bar{q}))^2 \leq |v_{-i}| \leq z_s$ and the middle left graph the typical behavior of the objective function when $z_s \leq |v_{-i}| \leq (v_L/[1+\beta(\bar{q})])^2$. In the former case, global maximum takes place in the local maximum to the right of $\beta(\bar{q})$. Optimal choice in this case maximize the lower curve and equals $v_i = v_L\sqrt{|v_{-i}|} - |v_{-i}|$. In the latter case (the middle left graph), global maximum is the local maximum at the left of $\beta(\bar{q})$. As a result, optimal choice maximize the middle curve and equals $v_i = v_{\bar{q}}\sqrt{|v_{-i}|}$.

The dependence of the optimal level of violence on parameters $(v_L, v_H, v_{\bar{q}})$ illustrated in analyzing Figure 4 is actually quite general, as established in Lemma 2. It becomes clear from this lemma that only *total* violence $|v_{-i}|$ is relevant from conjecture v_{-i}: how conjectured total violence is distributed among other dealers is not important for dealer i's optimal choice on violence.

Lemma 2. *Let* $v_L \equiv q_L\sqrt{B/\nu h}$, $v_H \equiv q_H\sqrt{B/\nu h}$, *and for* $v_{\bar{q}} \equiv \bar{q}\sqrt{B/\nu h}$ *define* $\mu_{\bar{q}} \equiv \sqrt{2v_{\bar{q}}(v_H - v_L)}$. *Given both a number of dealers* $n \in \mathbb{N}$ *such that* $n > 1$ *and a conjecture* $v_{-i} \in \mathbb{R}_+^{n-1}$ *on the violence levels other dealers has chosen, the optimal choice for dealer* i *in the turf-war stage is*

$$v_i^*(v_{-i}, n|\bar{q}) = \begin{cases} v_L\sqrt{|v_{-i}|} - |v_{-i}| & \text{if} \quad 0 \leq |v_{-i}| \leq \left([v_L - v_{\bar{q}} - \mu_{\bar{q}}]^+\right)^2 \\ v_{\bar{q}}\sqrt{|v_{-i}|} & \text{if} \quad \left([v_L - v_{\bar{q}} - \mu_{\bar{q}}]^+\right)^2 \leq |v_{-i}| \leq \left([v_H - v_{\bar{q}}]^+\right)^2 \\ v_H\sqrt{|v_{-i}|} - |v_{-i}| & \text{if} \quad \left([v_H - v_{\bar{q}}]^+\right)^2 \leq |v_{-i}| \leq v_H^2 \\ 0 & \text{if} \quad v_H^2 \leq |v_{-i}| \end{cases} \tag{9}$$

when some violence is expected from other dealers, $|v_{-i}| > 0$. *As usual for the CSF specification (4), there is no optimal choice for dealer* i *in the turf war stage when* $|v_{-i}| = 0$ *and* $n > 1$. *If* $n = 1$ *and* $|v_{-i}| = 0$ *optimal choice trivially equals* ${v_i}^* = 0$.

Proof. See appendix A. □

As illustrated in Figure 5, as a function of $|v_{-i}|$, the best response of dealer i in the turf war stage is piecewise concave with discontinuity *only* at $z_s = \left([v_L - v_{\bar{q}} - \mu_{\bar{q}}]^+\right)^2$. Differentiability is present in all

[9] Although local maximum above $\beta(\bar{q})$ is not visible in the graph, we know that after increasing in the neighborhood above $\beta(\bar{q})$ the lower curve becomes strictly decreasing. This is so because all three curves are continuous and converge to $-\infty$ as $v_i \to \infty$.

domain, except at two points: the discontinuity point z_s and the intersection point $z_H \equiv \left([v_H - v_{\bar{q}}]^+\right)^2$ between the upper curve ($y = v_H\sqrt{x} - x$) and the middle (increasing) curve ($y = v_{\bar{q}}\sqrt{x}$). The intersection point between the lower curve ($y = v_L\sqrt{x} - x$) and the middle (increasing) one is identified in the figure as $z_L \equiv \left([v_L - v_{\bar{q}}]^+\right)^2$.

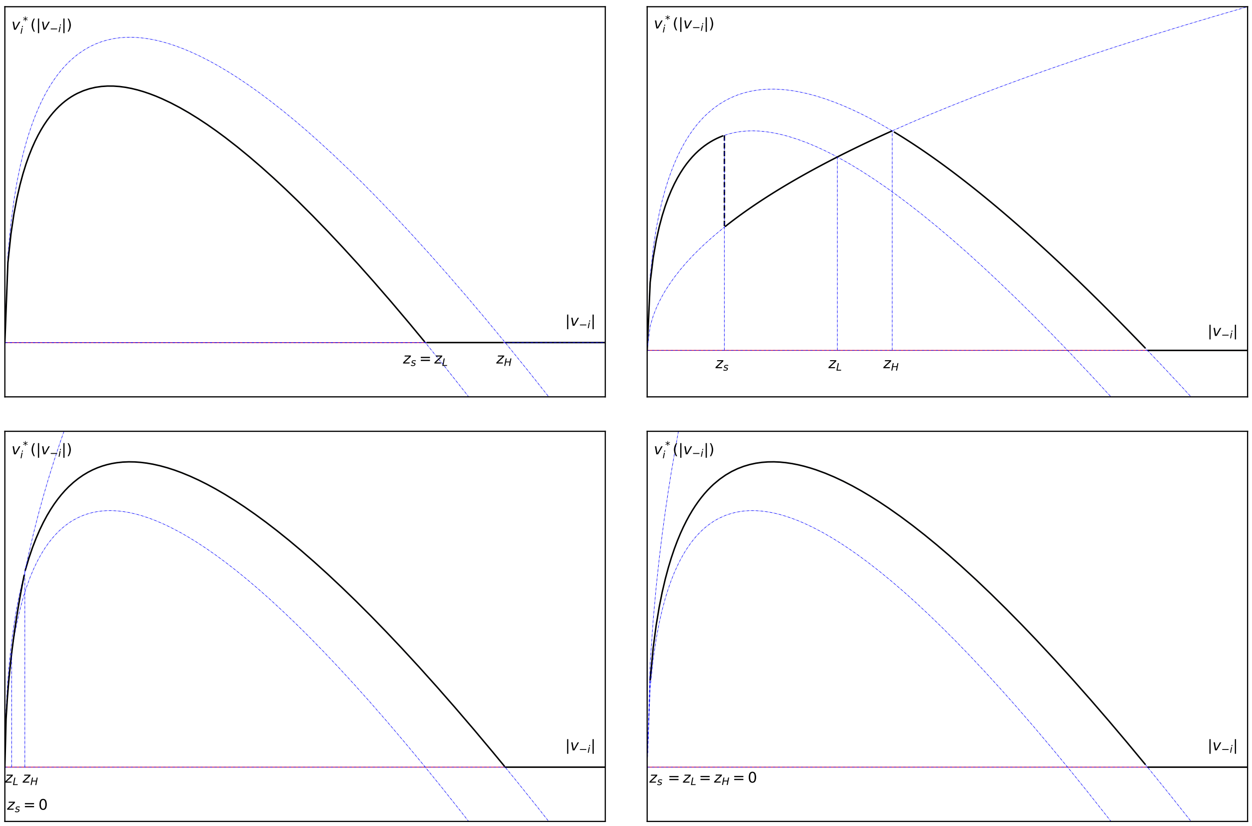

Figure 5: The best response violence

The discontinuity point z_s is given by the value of $|v_{-i}|$ under which middle and lower curves in Figure 4 achieve local maxima at the same value. In that sense, top right and middle left graphs in Figure 4 were constructed under $|v_{-i}| \approx z_s$. The other point in which v_i^* is not differentiable, z_H, is given by the value of $|v_{-i}|$ under which function $\pi_i^*(n, v|\bar{q})$ illustrated in Figure 4 peaks at $v_i = \alpha(\bar{q})|v_{-i}|$. In that sense, middle right and bottom left graphs in Figure 4 were constructed under $|v_{-i}| \approx z_H$.

The next step in computing the turf war equilibrium (meaning the violence equilibrium in subgame $n \in I$ taking optimal payoff in trading equilibrium as given) is demanding that each dealer $i \in I_n$ chooses a level of violence v_i^* that is a best response for $|v_{-i}| = \sum_{j \neq i} v_j^*$. Because we are interested in symmetric equilibria, in this step we actually search for a real number $s \in \mathbb{R}_+$ such that each dealer i chooses $v_i^* = s$ that is a best response for $|v_{-i}| = \sum_{j \neq i} v_j^* = (n-1)s$ since we must impose $v_j^* = s$ for all $j \in I_n$. Our next result, Lemma 3, presents for each subgame $n \in I$ the equilibrium level of violence and the associated equilibrium expected payoff. Figure 6 builds on Figure 5 to illustrate how symmetric equilibria described in Lemma 3 manifest itself in the plane $(|v_{-i}|, v_i^*)$.

In addition to the upper ($y = v_H\sqrt{x} - x$), lower ($y = v_L\sqrt{x} - x$) and middle ($y = v_{\bar{q}}\sqrt{x}$) curves already presented in Figure 5, it is presented in Figure 6 the so called *symmetry line* $y = x/(n-1)$, the solid line from the origin. Over it there are markers identifying its intersections with the original three curves at positive levels of $|v_{-i}|$. The diamond marker $\diamond$ is the intersection with the upper curve and its height is $s_H \equiv (n-1)(v_H/n)^2$. The circle marker $\circ$ is the intersection between the lower curve and the symmetry line and its height is $s_L \equiv (n-1)(v_L/n)^2$. Finally, the triangle marker $\triangle$ identifies the point in which the symmetry line intercepts the middle curve and its height is $s_M \equiv (n-1)v_{\bar{q}}{}^2$.

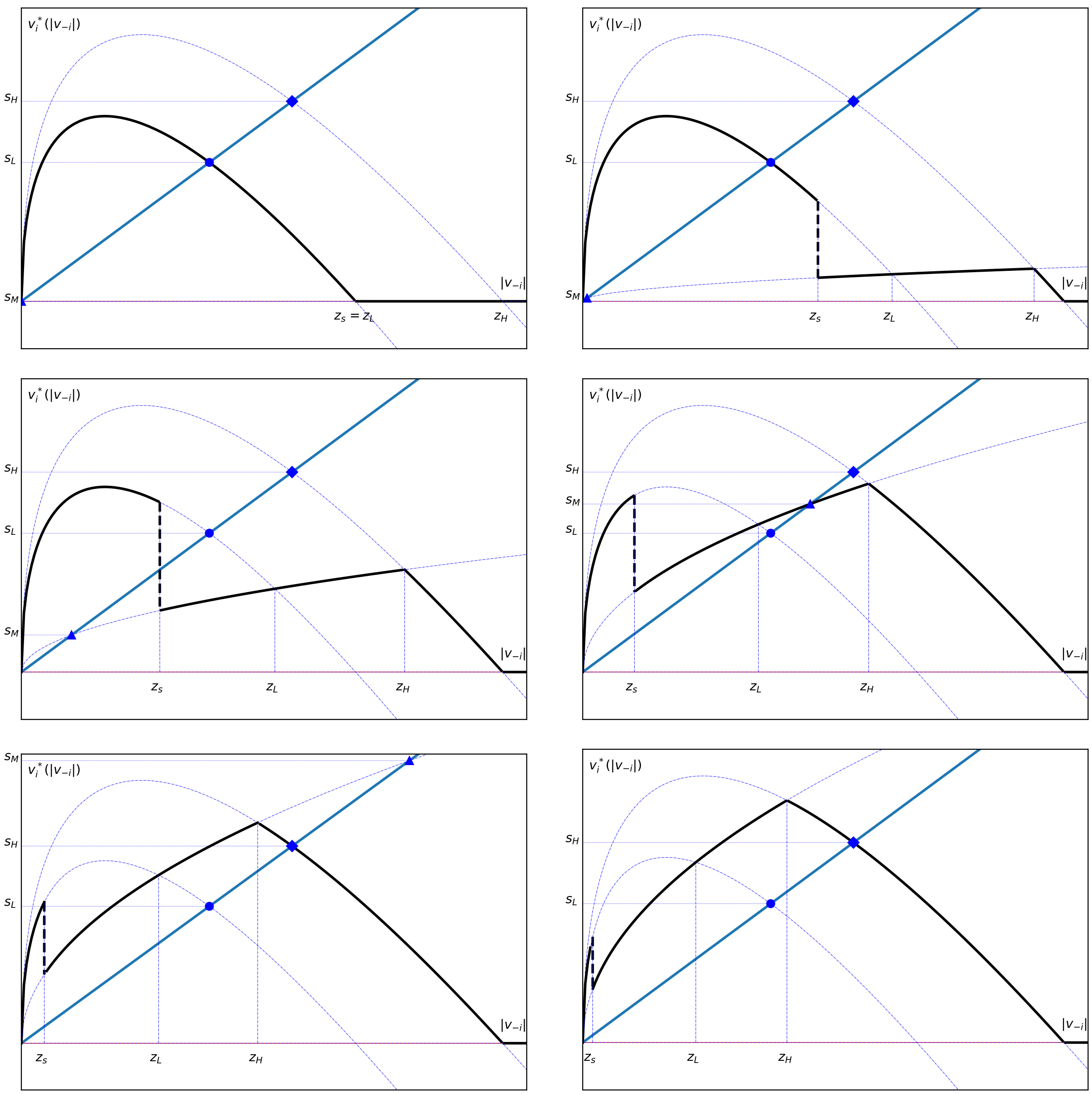

Figure 6: Turf war equilibrium determination when $0 < \bar{q} < \tilde{q}$

We are interested in these intersections because symmetric equilibria correspond to points in the intersection between the symmetry line and the best response curve $v_i^*(n, v_{-i}|\bar{q})$ defined in (9) and plotted as the solid nonlinear curve in the graphs. In effect, for any such point $(|v_{-i}|, v_i^*(|v_{-i}|)) = (x, y)$ we have $v_i^*(x) = y$ from the fact (x, y) is in the best response curve and $y = (n-1)x$ from (x, y) being a point in the symmetry line. As all cases illustrated in Figure 6 suggest, equilibrium multiplicity in turf war stage is not an issue in our model: when a marker meets the best response curve, it is unique. Also, there can be no symmetric equilibria in the turf war stage: for a range of values for $\bar{q}$ no marker meets the best response curve.[10]

From the top left graph to the bottom right one, $\bar{q}$ is increased from $\bar{q} = 0$ to $\bar{q} = (q_H/n + \tilde{q})/2 < \tilde{q}$.[11] It can be seen that equilibrium violence level equals $s_L = (n-1)(v_L/n)^2$ for low values of $\bar{q}$. This case is illustrated in the top left graph (for $\bar{q} = 0$) and in the top right one (for $0 < \bar{q} \leq \delta_L(n)$).[12] When $\bar{q}$ is high enough ($\bar{q} \geq q_H/n$), equilibrium violence level is given by $s_H = (n-1)(v_H/n)^2$, as can be seen in the two bottom graphs. For intermediate values of $\bar{q}$, turf war equilibrium entails violence level $s_M = (n-1){v_{\bar{q}}}^2$ when $\delta_M(n) \leq \bar{q} \leq q_H/n$ (the case in the middle right graph) and disappears when $\delta_L(n) \leq \bar{q} \leq \delta_M(n)$ (the case illustrated in the middle left graph).

The scenario for turf war equilibrium dependence on $\bar{q}$ suggested in Figure 6 is quite general, as established in Lemma 3. Although the analysis for deducing equilibrium violence levels (10) is quite elaborated, as the diversity of graphs in figure 6 suggests, the resulting values are quite simple ones.

Lemma 3. *Given a number of dealers $n > 2$, the profile of violence levels $v^* = (v_1^*, v_2^*, \cdots, v_n^*) \in \mathbb{R}_+^n$ is a symmetric Nash equilibrium in the turf-war stage if for all dealers i*

$$v_i^*(n) = \begin{cases} (n-1)(v_L/n)^2 & \text{if} \quad 0 \leq \bar{q} \leq \delta_L(n) \\ (n-1)v_{\bar{q}}^2 & \text{if} \quad \delta_M(n) \leq \bar{q} \leq (1/n)q_H \\ (n-1)(v_H/n)^2 & \text{if} \quad (1/n)q_H \leq \bar{q} \end{cases} \tag{10}$$

where $\delta_L(n) \equiv \frac{1}{2}\left[\sqrt{q_H - q_L\left(\frac{n-2}{n}\right)} - \sqrt{q_H - q_L}\right]^2$ and $\delta_M(n) \equiv \frac{1}{2n^2}\left[\sqrt{q_H + q_L(2n-1)} - \sqrt{q_H - q_L}\right]^2$. There is no symmetric equilibrium when $\delta_L(n) \leq \bar{q} \leq \delta_M(n)$. Thus, equilibrium payoff for each dealer i

[10] There are standard approaches that could be employed to amend our model in order to deal with this nonexistence result. One approach is to allow for mixed strategy equilibria. Another one is to relax symmetry restriction we impose on our equilibrium concept, allowing for two behaviors in the stage game: a group of dealers performing high level of violence (the image of the lower curve at z_s) and the other group choosing a low level (the value of the middle curve at z_s). Presumably, employing the former would restore equilibrium existence by making the *step* between the lower and middle curve at z_s a vertical solid line. The implied intersection with the new version of the (now *weak*) symmetry line would determine equilibrium probabilities on mixed strategies. The latter approach would restore existence by choosing groups' size as an equilibrium object. Because we believe that dealing with this nonexistence would make our model much more complex, with no obvious implication for the insights our model provides, we refrain from this route. Our solution is restricting our analysis to regions of parameters space in which nonexistence is not an issue.

[11] In terms of equilibria determination, the cases with $\bar{q} \geq \tilde{q}$, which were presented in third and fourth graphs of Figure 5, are very similar to the cases explored in the last two graphs in Figure 6.

[12] Cutoffs $\delta_L(n)$ and $\delta_H(n)$ are defined in Lemma 3's statement.

in a turf war with n dealers is

$$\pi_i^e(n|\bar{q}) = \begin{cases} B\left(q_L/n\right)^2 & \text{if} \quad 0 \leq \bar{q} \leq \delta_L(n) \\ A\bar{q} - B\bar{q}^2(2n-1) & \text{if} \quad \delta_M(n) \leq \bar{q} \leq (1/n)q_H \\ B\left(q_H/n\right)^2 & \text{if} \quad (1/n)q_H \leq \bar{q} \end{cases} . \tag{11}$$

For $n = 1$, turf war equilibrium entails $v_1^* = 0$ and $\pi_i^e(n|\bar{q}) = B{q_L}^2$ as implied by (7).

Proof. See appendix A ☐

As illustrated in the first graph of Figure 7, as a function of $\bar{q}$, the equilibrium level of violence presented in (10) is constant for low and for high values of $\bar{q}$. For intermediate high $\bar{q}$ it is a convex function. As expected, v_i^* is not defined for intermediate low values of $\bar{q}$. Analogously, equilibrium expected payoff defined in (11) is also not defined for intermediate low $\bar{q}$, as illustrated in the second graph of Figure 7. It is invariant to $\bar{q}$ to the left of $\delta_L(n)$ and also to the right of q_H/n. In between $\delta_M(n)$ and q_H/n, equilibrium expected payoff is quadratic concave.

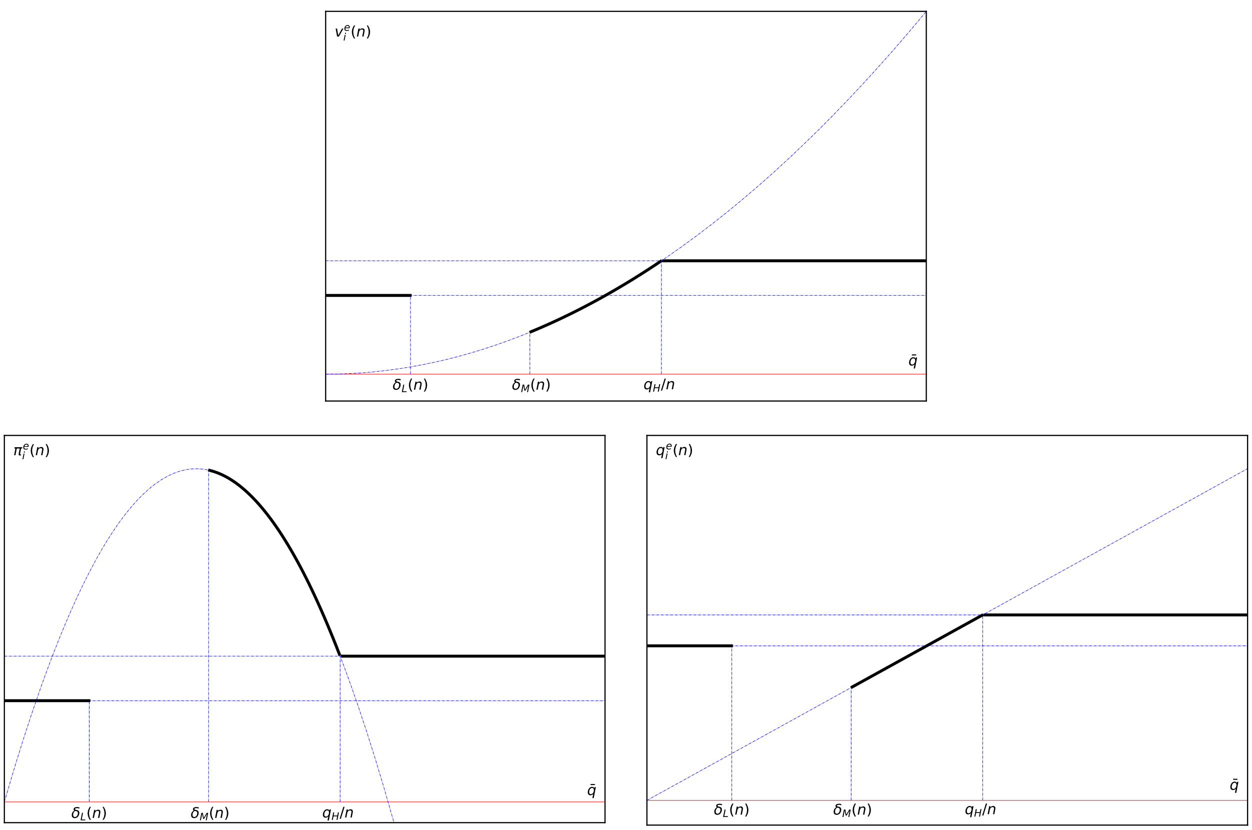

Figure 7: Turf war equilibrium as a function of $\bar{q}$

The entry stage equilibrium:

We finally turn our equilibrium analysis to the entry stage anticipating equilibrium expected payoff

$\pi_i^e(n|\bar{q})$ defined in (11). In particular, we are interested in computing the best (entry) response of each player $i \in I$ in the entry stage for each conjecture $(n-1)$ he or she can form about the number of other individuals choosing to get involved in drug dealing. The option individual i has beyond drug dealing is to develop a legal activity and earn income w. As a consequence, we must solve the problem $\max\{w, \pi_i^e(n|\bar{q})\}$.[13] Solution in this case is trivial for each admissible conjecture $n-1$: individual i gets involved with drug dealing when $\pi_i^e(n|\bar{q}) > w$, chooses a legal economic activity when $\pi_i^e(n|\bar{q}) < w$, while both options are equally attractive for $\pi_i^e(n|\bar{q}) = w$. Because we look for equilibria with some dealer operating in the illegal market and N is assumed sufficiently high to make $\pi_i^e(N|\bar{q})$ below w, the relevant solution for us entails $\pi_i^e(n|\bar{q}) = w$.

Inspecting function $\pi_i^e(n|\bar{q})$ in (11), we know that it is strictly decreasing in the number of drug dealers n in each the three intervals that defines it. A difficult emerges, though, from the fact that cutoff values on $\bar{q}$ that define such intervals are affected by n. In order to circumvent such difficult, using definitions of $\delta_L(n)$ and $\delta_M(n)$, we rewrite function $\pi_i^e(n|\bar{q})$ alternatively as

$$\pi_i^e(n|\bar{q}) = \begin{cases} B\left(q_L/n\right)^2 & \text{if} \quad 0 \leq \; n \; \leq \frac{1}{2}\frac{v_L}{v_{\bar{q}}+\mu_{\bar{q}}} \\ A\bar{q} - B\bar{q}^2(2n-1) & \text{if} \quad {\delta_M}^{-1}(\bar{q}) \leq \; n \; \leq v_H/v_{\bar{q}} \\ B\left(q_H/n\right)^2 & \text{if} \quad v_H/v_{\bar{q}} \leq \; n \end{cases} . \tag{12}$$

where it is assured that $\delta_M(n)$ has a decreasing inverse ${\delta_M}^{-1}(\bar{q})$ defined on $(0, q_L/(q_H - q_L)]$ and such that ${\delta_M}^{-1}(\bar{q})$ converges to ∞ as $\bar{q} \to 0$ and converges to 0 as $\bar{q} \to q_L/(q_H - q_L)$. It now becomes clear that $\pi_i^e(n|\bar{q})$ is continuous for all $n \geq {\delta_M}^{-1}(\bar{q})$.[14] Also, as can be verified ${\delta_M}^{-1}(\bar{q}) \geq \frac{1}{2}\frac{v_L}{v_{\bar{q}}+\mu_{\bar{q}}}$ for all $\bar{q} \geq 0$ so that function $\pi_i^e(n|\bar{q})$ is not defined for n such that $\frac{1}{2}\frac{v_L}{v_{\bar{q}}+\mu_{\bar{q}}} \leq n \leq {\delta_M}^{-1}(\bar{q})$. Then, for $w_H \equiv 4B(\bar{q} + \sqrt{2\bar{q}(q_H - q_L)})^2$ and $w_M \equiv B\bar{q}\left[\bar{q} + 2\left(q_H - \bar{q}{\delta_M}^{-1}(\bar{q})\right)\right]$, equilibrium condition $\pi_i^e(n|\bar{q}) = w$ is satisfied if and only if $w \geq w_H$ or $w \leq w_M$. In particular, equilibrium uniqueness follows from $\pi_i^e(n|\bar{q})$ being strictly decreasing in n.

The first graph in Figure 8 illustrates the discussed typical behavior of $\pi_i^e(n|\bar{q})$ as a function of n. In addition to the cutoffs w_H and w_M already mentioned, a third cutoff $w_L \equiv B\bar{q}^2$ is identified in this graph. It determines how much low w (equivalently, how much high $\bar{q}$) must be for equilibria entail expected payoff $B(q_H/n)^2$. The second graph presents cutoff values w_L, w_M, and w_H as functions of $\bar{q}$ in order to illustrate equilibrium existence in parameters space. Specifically, for each pair $(\bar{q}, w)$ in the colored region of this graph, equilibrium exists and is unique.

From the first graph, we know that pairs above the w_H curve in the second graph (those satisfying $w \geq w_H(\bar{q})$) generates equilibrium with low n and over the lower curve of the first graph, $y = B(q_L/x)^2$. Then, in this case, the equilibrium number of dealers n^e satisfies $w = B(q_L/n^e)^2$ and, therefore, $n^e = n_L$ for $n_L \equiv q_L\sqrt{B/w}$. For pairs $(\bar{q}, w)$ below the w_L curve in the second graph (those satisfying $w \leq w_L(\bar{q})$), the value of n^e is found over the upper curve $y = B(q_H/x)^2$, as the first graph suggests. Thus, it equals

[13] Observe that the illegal payoff is evaluated at n, not at $n-1$. This is so because it must be computed supposing dealer i has joined in drug dealing the conjectured $(n-1)$ other individuals.

[14] At his point, we are studying n as a real number variable.

$n^e = n_H$ for $n_H \equiv q_H\sqrt{B/w}$.

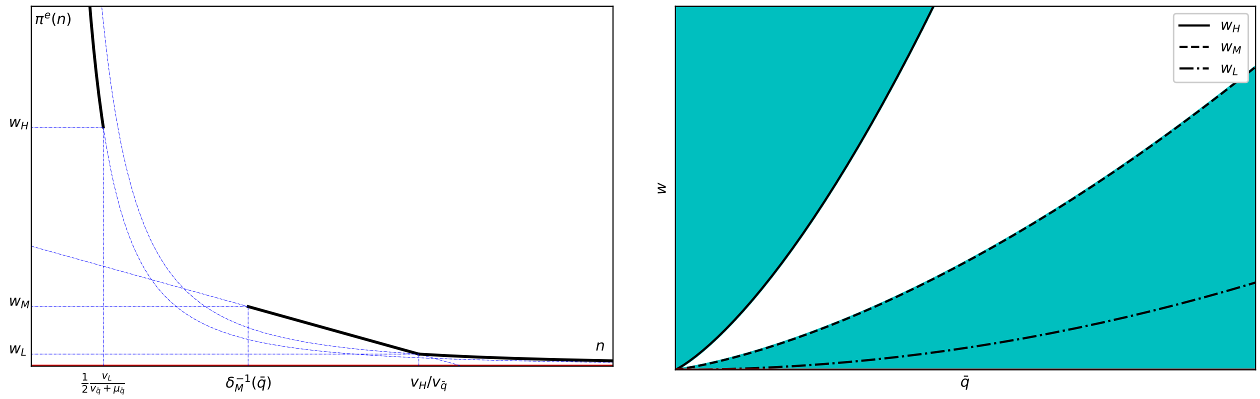

Figure 8: Equilibrium expected payoff as a function of n

The last region of parameters space in which equilibria exist is defined by the pairs $(\bar{q}, w)$ in between the w_L and w_M curves of the second graph of Figure 8, i.e., those satisfying $w_L(\bar{q}) \leq w \leq w_M(\bar{q})$. In this case, as the first graph in the same figure suggests, the equilibrium number of dealers n^e must be found over the curve $y = A\bar{q} - B\bar{q}^2(2x-1)$. Thus, n_e satisfies $w = A\bar{q} - B\bar{q}^2(2n^e - 1)$ and, therefore, is given by $n^e = [{n_{\bar{q}}}^2 + 2n_H n_{\bar{q}} - 1]/2{n_{\bar{q}}}^2$ for $n_{\bar{q}} \equiv \bar{q}\sqrt{B/w}$.

We have then presented our approach to compute equilibria while dealing with existence issues. This is actually the main argument in the proof of Proposition 1 when establishing (13). Computation of equilibrium violence v^e presented in (14) is as simple as plugging (13) back into (10) assuming $n = n^e$. Analogously, the individual volume of drug dealing q^e presented in (15) follows from plugging (13) back into (5) assuming $t_i(v) = 1/n^e$.

Proposition 1. *Define $n_L \equiv q_L\sqrt{B/w}$, $n_H \equiv q_H\sqrt{B/w}$, and $n_{\bar{q}} \equiv \bar{q}\sqrt{B/w}$. For a given cutoff $\bar{q} \geq 0$, suppose that $(n^e, v^e, q^e) \in I \times \mathbb{R}_+^2$ is the symmetric SPNE outcome in which n^e individuals get involved in drug dealing, each of them conquers monopoly rights over a share $1/n^e$ of total demand (territory) by employing violence level v^e and operate under monopoly quantity q^e in this territory. Then, the number of dealers is*

$$n^e = \begin{cases} n_H & \text{if} \quad 0 \leq w \leq w_L(\bar{q}) \\ \frac{2n_H n_{\bar{q}} + {n_{\bar{q}}}^2 - 1}{2{n_{\bar{q}}}^2} & \text{if} \quad w_L(\bar{q}) \leq w \leq w_M(\bar{q}) \\ n_L & \text{if} \quad w_H(\bar{q}) \leq w \end{cases}, \tag{13}$$

the violence level is

$$v^e = \begin{cases} \frac{w}{\nu h}(n_H - 1) & \text{if} \quad 0 \leq w \leq w_L(\bar{q}) \\ \frac{w}{\nu h}\frac{({n_H}^2 - 1) - (n_H - n_{\bar{q}})^2}{2} & \text{if} \quad w_L(\bar{q}) \leq w \leq w_M(\bar{q}) \\ \frac{w}{\nu h}(n_L - 1) & \text{if} \quad w_H(\bar{q}) \leq w \end{cases} \tag{14}$$

and the scale of operation is

$$q^e = \begin{cases} \sqrt{w/B} & \text{if} \quad 0 \leq w \leq w_L(\bar{q}) \\ n_{\bar{q}}\sqrt{w/B} & \text{if} \quad w_L(\bar{q}) \leq w \leq w_M(\bar{q}) \\ \sqrt{w/B} & \text{if} \quad w_H(\bar{q}) \leq w \end{cases} \tag{15}$$

where cutoff functions $w_H(\bar{q}) \equiv 4B\left(\bar{q} + \sqrt{2\bar{q}(q_H - q_L)}\right)^2$, $w_M(\bar{q}) \equiv B\bar{q}\left[\bar{q} + 2\left(q_H - \bar{q}{\delta_M}^{-1}(\bar{q})\right)\right]$, *and* $w_L(\bar{q}) \equiv B\bar{q}^2$, *are stricly increasing and convex. The outcome for the case in which* $w_M(\bar{q}) \leq w \leq w_H(\bar{q})$ *is not reported here because no symmetric equilibrium exists in turf war stage for such* $\bar{q}$.

Proof. See appendix A. □

For convenience, we illustrate in Figure 9 as a function of $\bar{q}$ the typical behavior of equilibrium variables in symmetric SPNE outcomes. The equilibrium number of dealers n^e defined in (13) is presented in the bottom left graph. The equilibrium level of individual violence v^e defined in (14) is presented in the top left graph. Finally, the equilibrium level of individual drug dealing q^e defined in (15) is presented in the middle left graph. The remaining three graphs (on the right) present aggregate variables in equilibrium. Aggregate level of violence is presented in the top right graph, aggregate level of drug dealing can be found in the middle right one, and bottom right graph presents the price under which the illegal drug is sold. Because drug demand is assumed linear, market shares are determined on a linear fashion, and symmetry in equilibrium concept implies the same market share for each drug dealers, price is the same over all territory.

Because equilibrium value for variable n_e must actually be a natural number, results presented in Proposition 1 and, in particular, those illustrated in Figure 9 should be seen as approximate results in the following sense. Inspecting bottom left graph, it can be seen that a value n_L near 2.75 is predicted as the equilibrium number of individuals getting involved with drug dealing when $\bar{q} \approx 0$. Strictly, this is the number n that makes expected payoff from drug dealing $\pi_i^e(n|\bar{q})$ equal to w. Because $\pi_i^e(n|\bar{q})$ is strictly decreasing in n, we know that $\pi_i^e(2|\bar{q}) > w$ and $\pi_i^e(3|\bar{q}) < w$. As a result, the value 2.75 should be seen as the expected number of people involved with drug dealing.[15]

4 Model's insights

The main lessons emerging from our analysis can be discussed using Figure 9 as an illustrative tool. It is worth remembering at this point that our motivating case, Brazil's Supreme Court decision on marijuana possession, have established an objective criterion for telling apart users and dealers in this illegal market: a cutoff value of 40g on the volume of drug possession has been established. Below such

[15] Allowing for mixed strategies in the entry stage is a simple way to formalize this interpretation. Another one is to allow for dispersion of people in different values for w. We do not pursue these formalizations because we believe little would change in our predictions (provided that we interpret them as approximate results of a model with probabilistic entry or heterogeneous w) and additional tractability costs would be paid.

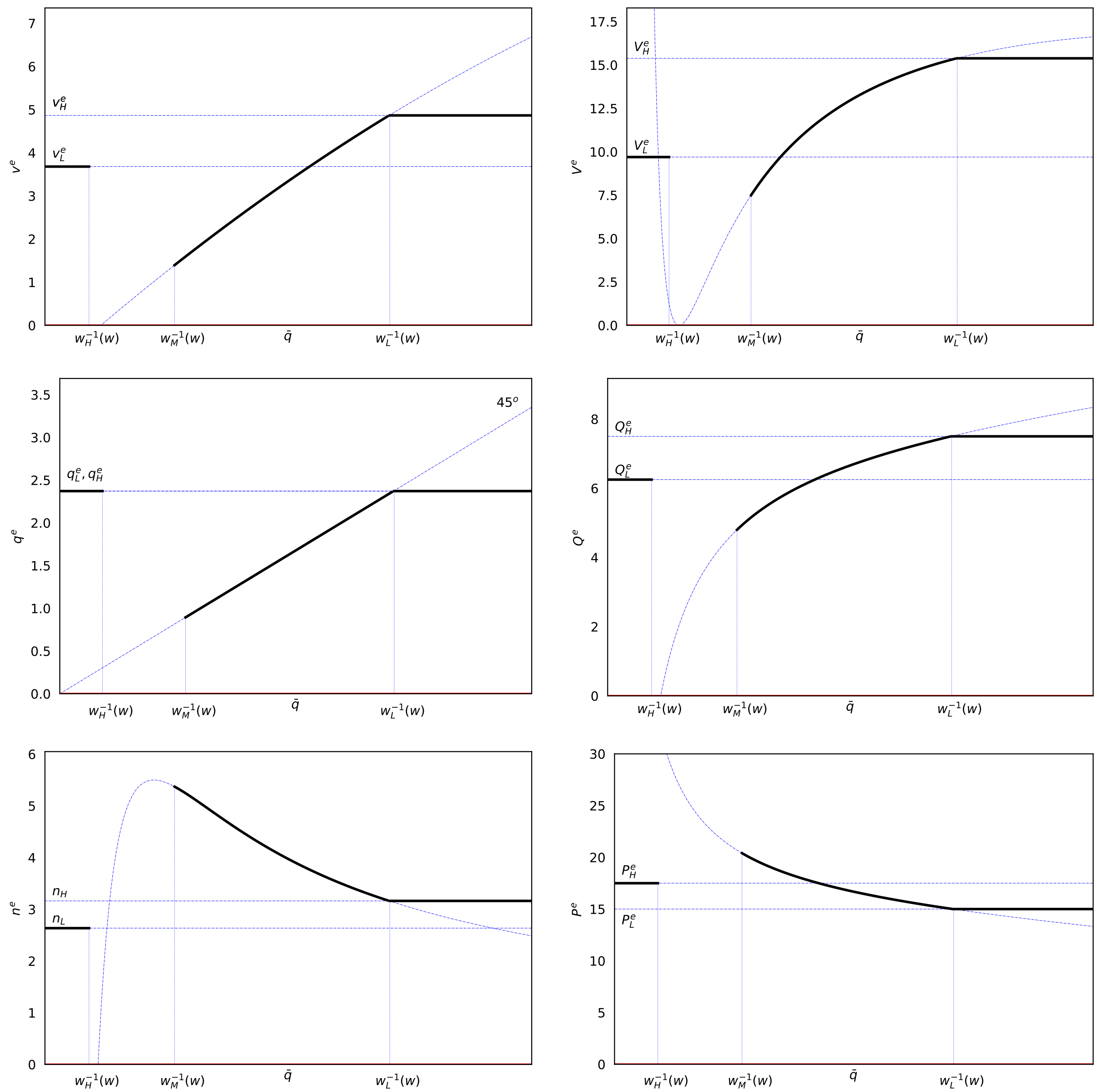

Figure 9: A symmetric SPNE outcome

value, possession allows police force to qualify the drug owner only as an user. Above 40g, possession is evidence of drug dealing. Before such decision, all possession of marijuana could be used to arrest the owner as a drug dealer.

In our model, the creation of this criterion can be seen as an increase in $\bar{q}$ from its initial *null* value to a strictly positive one, $\bar{q} > 0$. Making reference to Proposition 1 and Figure 9 (for illustration), the situation before $\bar{q}$ has been increased entails aggregate level of violence $V_L^e \equiv n^e \times v^e = n_L(n_L - 1)w/\nu h$, aggregate volume of drug dealing $Q_L^e \equiv n_e \times q^e = n_L\sqrt{w/B}$, and price $P_H^e \equiv P(Q_L^e) = A - BQ_L^e$. By construction, expected profitability from drug dealing $\pi_i^e(n|\bar{q})$ defined in (11) always equal wage w under equilibrium number of dealers $n = n^e$. At the individual level, violence is $v_L^e \equiv (n_L - 1)w/\nu h$ and drug dealing equals $q_L^e \equiv \sqrt{w/B}$.

Model's prediction for how this illicit market changes after $\bar{q}$ increases, as illustrated in Figure 9, *depends on how large was such an increase.* According to the model, as long as the new level for $\bar{q}$ remains below $w_H^{-1}(w)$ no change is expected in this illicit market at all. On the other hand, if the new level for $\bar{q}$ is large enough to exceed $w_M^{-1}(w)$, then our model predicts a lot changes for this illicit market.

As can seen from the middle left graph in Figure 9, police work is ineffective when $\bar{q} > w_M^{-1}(w)$ since each drug dealer sells no more than $\bar{q}$ units of drugs. Nobody would be arrested for dealing drugs in our model, although dealers can still be arrested for violence acts. The graph shows that dealers would sell exactly $\bar{q}$ when this new level is not high enough: namely, $\bar{q} < w_L^{-1}(w) = q_L^e$. In other words, for intermediate values ($w_H^{-1}(w) \leq \bar{q} < w_L^{-1}(w) = q_L^e$) drug dealers *decreases* its individual drug dealing as a strategy to protect themselves from police scrutiny. This reasoning has already been discussed in solving dealer's problem at the trading stage (see first graph in Figure 3).

Because intermediate $\bar{q}$ also motivate more people to get involved with drug dealing (see bottom left graph in Figure 9), it is not a direct consequence that such a decrease in *individual* drug dealing results on a decrease on *aggregate* drug dealing. On this issue, the middle graph of Figure 9 shows that aggregate drug dealing decreases as a result lower individual drug dealing only if the intermediate value of $\bar{q}$ is not high enough. For intermediate values of $\bar{q}$ high enough (near to $w_L^{-1}(w) = q_L^e$), the effect of more drug dealers dominates and aggregate drug dealing increases. As expected, price moves in the opposite direction aggregate drug dealing does (see the bottom right graph).

For sufficiently high $\bar{q}$ (namely, $\bar{q} > w_L^{-1}(w) = q_L^e$), individual drug dealing gets back to its original level, as illustrated in the middle left graph, since in this case $q_H^e = \sqrt{w/B} = q_L^e$, as established in (15). This shows the increase in individual drug dealing (from q_L/n to q_H/n) predicted in Lemma 1 (and illustrated in Figure 3) for a sufficiently large increase in $\bar{q}$ does not take place in equilibrium. In effect, such a prediction is constructed for a fixed number of dealers n and a fixed vector of violence v: it is a prediction for a fixed subgame. Because n increases to its new level n_H, all predicted increasing in q_i^* vanishes. In game theory language, the trading-stage subgame in the equilibrium path changes from that associated with $n = n_L$ to another one, that associated with $n = n_H$. The aggregate drug dealing, on the other hand, increases (see the middle right graph) to $Q_H^e \equiv n^e \times q^e = n_H\sqrt{w/B} > Q_L^e$ after $\bar{q}$ increases beyond $w_L^{-1}(w)$, as completely implied by the higher number of drug dealers n_H.

This discussion highlights the relevance in recognizing the number of dealers as an equilibrium variable in modeling illicit drug markets. In particular, a version of our model with a exogenous n would produce quite different predictions for q^e and Q^e. *That n increases* from the initial levels n_L to a new level that is never lower than n_H (with $n_H > n_L$) *follows from the fact that higher levels for $\bar{q}$ make drug dealing more attractive*, as established in (11) and illustrated in Figure 7. This is so because the higher $\bar{q}$, the higher is the range of values for drug dealing under which is possible to "hide" from police scrutiny. Such dependence of drug dealing profitability on $\bar{q}$ for fixed (n, v) has been established in (6) and illustrated in the second graph in Figure 3, in which profitability continuously increases from $(t_i(v)Bq_L{}^2 - \nu h v_i)$ to $(t_i(v)Bq_H{}^2 - \nu h v_i)$.

On the other hand, after taking into account that this higher profitability changes incentives for violence (see Lemma 2), the monotone *increasing* pattern illustrated in the second graph of Figure 3 is replaced by a monotone *decreasing* pattern in the second graph of Figure 7, which takes place after an pronounced increase as $\bar{q}$ changes from $\delta_L(n)$ to $\delta_M(n)$. In words, the higher drug dealing profitability from "hiding below" $\bar{q}$ is increasing in $\bar{q}$ (as illustrated in Figure 3) for fixed v, but it promotes more violence, as first graph in Figure 7 illustrates. This induced higher level of violence, in turn, makes dealing drugs less attractive from ex-ante perspective. The net result is a predominance of the violence effect and profits becomes decreasing in $\bar{q}$. As illustrated in Figure 7, though, such worsening in profitability is not enough to bring equilibrium expected payoff back to its initial level: it stabilizes at $B(q_H/n)^2$ after q_H/n, a level higher than the initial one $B(q_L/n)^2$. Then, from the entry stage perspective, expected profitability increases when $\bar{q}$ gets higher. It is exactly this increase in profits (after violence and drug dealing has already reacted to the new $\bar{q}$) what motivates more people to get involved with drug dealing. And this entry takes place until all increase in profits dissipates: expected payoff should get back to w as free entry equilibrium condition commands.

It is worth remarking at this point that this capacity to make predictions for violence levels is another of the main virtues of our modeling approach (presumably, the best one). Because we follow Bertolai and Scorzafave (2021) in recognizing violence as an intrinsic feature of drug dealing *modeling*,[16] our model is able to predict how violence associated to drug dealing changes after $\bar{q}$ increases to a positive level and alters profit possibilities from drug dealing. Specifically, as first graphs in figures 7 and 9 illustrate, dealers' reduction on drug dealing motivated by intermediate low $\bar{q}$ (see third graph in figure 7 and the middle left graph in figure 9) is accompanied by a reduction in the individual violence level. Looking now to the top right graph in Figure 9, it can be seen that this decreasing is large enough to compensate the higher number of drug dealers (see bottom left graph in the same figure) in reducing the aggregate violence to a level below V_L^e, as long as $\bar{q} \approx w_M^{-1}(w)$. As $\bar{q}$ get large enough to intermediate high level, the already higher number of dealers is combined with a now higher level of individual violence (see the top left graph) to produce a pronounced increase in the aggregate violence (see top right graph). Eventually, for large enough $\bar{q}$ (namely, $\bar{q} > w_L^{-1}(w) = q_L^e$), violence stabilizes on a higher level: $v_H^e = (n_H - 1)w/\nu h$ for individuals and $V_H^e \equiv n_H \times v_H^e$ for the market as a whole.

Our analysis shows that increasing $\bar{q}$ from its initial null value is expected to change the correspondent illicit market on ways that are not trivially deduced from a model-free perspective. The model we develop, however, enables us to describe how the intuitive (*ceteris paribus*) impact on drug dealing profitability (increasing it) leads to non-trivial changes on violence and drug dealing activity. As one would expect, nothing happens in this market if $\bar{q}$ is kept sufficiently near to its initial null level (stays below $w_H^{-1}(w)$). Most surprisingly, aggregate violence and drug dealing can be made higher or lower than

[16] Again, it is a decentralized way to ensure property rights relevant to economic activity that is deemed illicit by society. Although we recognize violence as an essential ingredient in drug dealing *modeling*, we know that violence is not an essential ingredient in drug dealing *equilibrium*. It can vanish as an equilibrium phenomenon, as Bertolai and Scorzafave (2021) shows in their analysis.

their initial level. They can be reduced if the new value of $\bar{q}$ is tailored to be higher but approximate to $w_M^{-1}(w)$. On the other hand, if $\bar{q}$ approximates the initial value of q^e (namely, $w_L^{-1}(w) = q_L^e$), however, they exceed initial levels.

As a conclusion, applying our model to the case that motivated this paper, predictions are determined by how empirically the established cutoff $\bar{q} = 40$g compares itself to cutoff levels $w_L^{-1}(w)$, $w_M^{-1}(w)$ and $w_H^{-1}(w)$. For this hypothetical empirical study, besides the explicitly recognized relevance of w, it would be taken into account that functions $w_L(\cdot)$, $w_M(\cdot)$ and $w_H(\cdot)$ defined inside Proposition 1 result from drug demand properties (parameters of market size A and price sensibility B) and the policy against drug dealing (parameters of police effort ρ and punishment d).

5 Final Remarks

An important remark to make is that, in order to preserve the tractability of Bertolai and Scorzafave (2021)'s model, the analysis presented here makes use of at least two simplifying assumptions that could be *interestingly challenged* by future research on this topic. First, it is natural to expect the demand for illicit drugs to depend on the distinction between users and dealers, but here the distinction cutoff $\bar{q}$ has no effect on demand function $Q(p)$. Second, police effort to restrain drug dealing in reality is clearly not invariant to criminals' behavior, but here the distinction cutoff $\bar{q}$ has no equilibrium effect on police efforts ν and ρ.

For the first limitation, it would be interesting to see in which sense (if any) our conclusions result from the Bertolai and Scorzafave (2021)'s strategy to model the demand side of the market using an exogenous demand curve $Q(p)$. A promising approach would be to recognize *illicit drug users* as players in the (game) model and derive their drug demand from a rational choice that explicitly takes into account the risk to be arrested as a drug dealer when buying more than $\bar{q}$ units of drug. The natural conjecture is that such analysis would produce a drug demand increasing in $\bar{q}$. The overall consequences for equilibrium quantities n, v and q are not so clear, but a preliminary guess could be made using our model. Specifically, the combination of higher $\bar{q}$ with the conjectured increasing drug demand can be *tentatively* viewed here as an increase in parameter A. This makes drug dealing more profitable in each subgame, by increasing both q_L and q_H, which motivates more individual violence and a higher number of drug dealers that dissipates all excess profitability and, therefore, makes individual scale of operation unaffected.[17]

The assumption the police force does not react to criminals behavior (the second limitation) follows the standard approach of rational choice between legal and illegal activities, in the tradition of Becker (1968). It would be interesting to see in which sense (if any) our conclusions result from the Bertolai

[17] By Lemma 1, higher levels for q_L and q_H increase drug dealing and profits in each subgame defined by (v, n). Because trading becomes more profitable, violence and *interim* profits increases in each subgame defined by n, as established in Lemma 3. Proposition 1, however, shows that all excess in profits are dissipated by a higher number of drug dealers. In effect, $q_L^e = q_H^e = \sqrt{w/B}$ is invariant to A.

and Scorzafave (2021)'s strategy to model the society's effort to restrain drug dealing and violence using exogenous arrestment probabilities ν and ρ. A promising approach would be to recognize *police force* as a player in the (game) model and derive their effort from a rational choice that explicitly takes into account the cost society incurs in mobilizing resources to enforce the law.[18] The natural conjecture is that such analysis would conclude police effort to restrain drug dealing decreases as a response for scale of operation falling below $\bar{q}$ and in order to save society's resources. The overall consequences for equilibrium quantities n, v and q are not so clear, but (again) a preliminary guess could be made using our model. Specifically, the conjectured decrease in police effort against drug dealing can be viewed here as an decrease in parameter ρ. This makes the effect of $\bar{q}$ on drug dealing profitability lower, by increasing q_L and n_L. In the limit $\rho = 0$, the cutoff distinction $\bar{q}$ has no effect at all, since $n_H = n_L$ and $q_L = q_H$ in this case. Of course, police force would react to the reestablishment of original scale of operation making ρ strictly positive. The equilibrium value for ρ would balance these incentives and determine all remaining quantities. Comparatively, our guess so constructed is less assertive on how equilibrium changes in the presence of a *police player* than it was possible to conjecture about the presence of a *user player*.

Future research could also *interestingly* explored how the no-violence equilibrium deduced in Bertolai and Scorzafave (2021) analysis would change after $\bar{q}$ increases from its initial null value. Although their analysis assume an exogenous number of dealers, a natural conjecture about allowing for endogenous n in their model is that the prison gang (group of criminals inside prison system with control upon prisoners welfare) would forbid free entry as a feature of the contract proposed to outside prison criminals. This would be so because it avoids profit dissipation and, therefore, increases outside criminals' capacity to pay taxes to the prison gang: the same reason motivates prison gang to suggest no violence among outside prison dealers. In this sense, *such a future research initiative has potential to change our conclusions on a much relevant fashion, since free entry revealed itself a major driving force in our analysis.* The predictions from this hypothetical future research, would be restricted to very few situations, though.[19] As Bertolai and Scorzafave (2021) analysis makes clear, those are situations in which a prison gang has managed to get enough control over inmates' welfare and makes use of it for extorting outside prison dealers. In this sense, our analysis provides predictions for the *standard situation* and this hypothetical future research would provide predictions for a *rare* but much *relevant situation*.

As a final remark, we reemphasizes that our analysis is not applicable only for *drug* dealing illicit markets. Although we have written all model using a language applicable to such markets, the essence here is the *illegal* nature of trade. Whatever the good or service traded in this market, as long as it is an illegal one, our model presents itself as a promising tool for analyzing how a cutoff policy in quantities

[18] More details on the such attractiveness can be found, for example, in the Bjørnskau and Elvik (1992)'s discussion on the superiority of Game Theory approach over the standard rational choice approach of Becker (1968) when studying enforcement of Traffic Law. On this matter, see also modeling approaches of Bertolai et al. (2021), Costa (2023) and Bertolai et al. (2024).

[19] Saying that such situations are rare does not mean their are not relevant. As Bertolai and Scorzafave (2021) analysis shows, it is possible that all violence associated with drug dealing vanishes. If this kind of violence represents a large share of the total violence on an economy, there would be observed in data a pronounced drop in violent crime statistics.

intended to tell apart users (buyers) and dealers (sellers) would change the corresponding illicit market.

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A Proofs

Lemma 1. *Let* $q_L \equiv (A - \rho d)/2B$*,* $q_H \equiv A/2B$ *, and* $\tilde{q} \equiv q_H - \sqrt{{q_H}^2 - {q_L}^2}$*. Given both an outcome in the entry stage* $n \in I$ *and a subsequent outcome in turf-war stage* $v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n_+$*, the optimal choice for dealer* i *acting as a monopolist in the trading stage is*

$$q_i^*(n, v|\bar{q}) = \begin{cases} t_i(v)q_L & \textit{if} \quad 0 \leq \bar{q} \leq t_i(v)\tilde{q} \\ \bar{q} & \textit{if} \quad t_i(v)\tilde{q} \leq \bar{q} \leq t_i(v)q_H \\ t_i(v)q_H & \textit{if} \quad t_i(v)q_H \leq \bar{q} \end{cases}. \tag{5}$$

Thus, optimal payoff in the trading stage for dealer i *is*

$$\pi_i^*(n, v|\bar{q}) = \begin{cases} t_i(v)B{q_L}^2 - \nu h v_i & \textit{if} \quad 0 \leq \bar{q} \leq t_i(v)\tilde{q} \\ A\bar{q} - B\bar{q}^2/t_i(v) - \nu h v_i & \textit{if} \quad t_i(v)\tilde{q} \leq \bar{q} \leq t_i(v)q_H \\ t_i(v)B{q_H}^2 - \nu h v_i & \textit{if} \quad t_i(v)q_H \leq \bar{q} \end{cases} \tag{6}$$

Proof. Let $H(q_i) = P(q_i/t_i(v))q_i - \nu h v_i$ and $L(q_i) = P(q_i/t_i(v))q_i - \rho d q_i - \nu h v_i$. The function $H(q_i)$ is the expected payoff of dealer i when $q_i \leq \bar{q}$, while the function $L(q_i)$ is the expected payoff of dealer i when $q_i > \bar{q}$. Moreover, note that $H(q_i)$ and $L(q_i)$ are concaves and fist order condition shows that they reach their maxima at $q_i = t_i(v)q_H$ and $q_i = t_i(v)q_L$, respectively.

Suppose $0 \leq \bar{q} \leq t_i(v)\tilde{q}$. One can see $t_i(v)q_H > t_i(v)\tilde{q}$. Using that $H(q_i)$ is concave, for $q_i \leq \bar{q}$, dealer i maximizes his payoff by choosing $q_i = \bar{q}$. Hence, for $q_i \leq \bar{q}$, expected payoff is less or equal to $H(t_i(v)\tilde{q})$. Furthermore, $\tilde{q} = q_H - \sqrt{{q_H}^2 - {q_L}^2} = q_L - q_H - q_L - \sqrt{{q_H}^2 - {q_L}^2} < q_L$. Then, $t_i(v)\tilde{q} \leq t_i(v)q_L$. Since $\bar{q} < t_i v(\tilde{q})$, for $q_i > \bar{q}$, dealer i maximizes his payoff by choosing $q_i = t_i(v)q_L$ and his maximum payoff is given by $L(t_i(v)q_L)$. Note that $L(t_i(v)q_L) = (A - Bq_L)t_i(v)q_L - \rho d t_i(v)q_L - \nu h v_i = (A - \rho d)t_i(v)q_L - Bt_i(v){q_L}^2 - \nu h v_i = t_i(v)B{q_L}^2 - \nu h v_i$. By the ther hand, using simple algebra, one has $H(t_i(v)\tilde{q}) = (A - B\tilde{q})t_i(v)\tilde{q} - \nu h v_i = At_i(v)\tilde{q} - B\left(2{q_H}^2 - 2q_H\sqrt{{q_H}^2 - {q_L}^2} - {q_L}^2\right)t_i(v) - \nu h v_i = At_i(v)\tilde{q} + \left(-Aq_H + A\sqrt{{q_H}^2 - {q_L}^2} + B{q_L}^2\right)t_i(v) - \nu h v_i = L(t_i(v)q_L$. Therefore, as $H(q_i) \leq H(t_i(v)\tilde{q}) = L(t_i(v)q_L)$ and $q_i = t_i(v)q_L$ maximizes $L(q_i)$, the optimal choice for dealer i acting as a monopolist in the trading stage, when $0 \leq \bar{q} \leq t_i(v)\tilde{q}$, is ${q_i}^*(v, n|\bar{q}) = t_i(v)q_L$ and his optimal payoff in the trading stage is ${\pi_i}^*(v, n|\bar{q}) = L(t_i(v)q_L) = t_i(v)B{q_L}^2 - \nu h v_i$.

Suppose $t_i(v)\tilde{q} \leq \bar{q} \leq t_i(v)q_H$. Using concavity of $H(x)$, we know that for $\bar{q} \leq t_i(v)q_H$ and $q_i \leq \bar{q}$ dealer i maximizes his payoff choosing $q_i = \bar{q}$. Moreover, we already know that $H(t_i(v)\tilde{q})$ and $L(t_i(v)q_L)$ assume the same value. Using again concavity of $H(q_i)$, we know that $H(q_i)$ is increasing for $q_i < t_i(v)q_H$. Since $t_i(v)q_L$ maximizes $L(q_i)$, then $H(\bar{q}) \geq L(q_i)$ for $t_i(v)\tilde{q} \leq \bar{q} \leq t_i(v)q_H$. Therefore, one has that the optimal choice for dealer i acting as a monopolist in the trading stage, when $t_i(v)\tilde{q} \leq \bar{q} \leq t_i(v)q_H$, is ${q_i}^*(v, n|\bar{q}) = \bar{q}$ and his optimal payoff in the trading stage is ${\pi_i}^*(v, n|\bar{q}) = H(\bar{q}) = A\bar{q} - B\bar{q}^2 - \nu h v_i$.

Suppose now $t_i(v)q_H \leq \bar{q}$. Since $t_i(v)q_H$ maximizes $H(q_i)$, $q_i = t_i(v)q_H$ is the optimal choice for $q_i \leq \bar{q}$. As well, using again that $t_i(v)q_H$ maximizes $H(q_i)$ and $t_i(v)\tilde{q} \leq t_i(v)q_H$, we obtain $H(t_i(v)q_H) \geq$

$H(t_i(v)\tilde{q})$. Since $H(t_i(v)\tilde{q}) = L(t_i(v)q_L)$ and $t_i(v)q_L$ maximizes $L(q_i)$, the optimal choice for dealer i is to choose $q_i = t_i(v)q_H$. Therefore, the optimal choice for dealer i acting as a monopolist in the trading stage, when $t_i(v)q_H \leq \bar{q}$, is $q_i{}^*(v, n|\bar{q}) = t_i(v)q_H$ and his optimal payoff in the trading stage is $\pi_i{}^*(v, n|\bar{q}) = H(t_i(v)q_H) = t_i(v)B{q_H}^2 - \nu h v_i$. □

Lemma 2. *Let* $v_L \equiv q_L\sqrt{B/\nu h}$, $v_H \equiv q_H\sqrt{B/\nu h}$, *and for* $v_{\bar{q}} \equiv \bar{q}\sqrt{B/\nu h}$ *define* $\mu_{\bar{q}} \equiv \sqrt{2v_{\bar{q}}(v_H - v_L)}$. *Given both a number of dealers* $n \in \mathbb{N}$ *such that* $n > 1$ *and a conjecture* $v_{-i} \in \mathbb{R}_+^{n-1}$ *on the violence levels other dealers has chosen, the optimal choice for dealer* i *in the turf-war stage is*

$$v_i^*(v_{-i}, n|\bar{q}) = \begin{cases} v_L\sqrt{|v_{-i}|} - |v_{-i}| & \text{if} \quad 0 \leq |v_{-i}| \leq \left([v_L - v_{\bar{q}} - \mu_{\bar{q}}]^+\right)^2 \\ v_{\bar{q}}\sqrt{|v_{-i}|} & \text{if} \quad \left([v_L - v_{\bar{q}} - \mu_{\bar{q}}]^+\right)^2 \leq |v_{-i}| \leq \left([v_H - v_{\bar{q}}]^+\right)^2 \\ v_H\sqrt{|v_{-i}|} - |v_{-i}| & \text{if} \quad \left([v_H - v_{\bar{q}}]^+\right)^2 \leq |v_{-i}| \leq v_H^2 \\ 0 & \text{if} \quad v_H^2 \leq |v_{-i}| \end{cases} \tag{9}$$

when some violence is expected from other dealers, $|v_{-i}| > 0$. *As usual for the CSF specification (4), there is no optimal choice for dealer* i *in the turf war stage when* $|v_{-i}| = 0$ *and* $n > 1$. *If* $n = 1$ *and* $|v_{-i}| = 0$ *optimal choice trivially equals* $v_i{}^* = 0$.

Proof. All the reasoning for this proof was presented inside the main text using Figure 4. For convenience and completeness, we provide the details of such reasoning in this formal proof.

In order to show continuity and concavity of $\pi_i{}^*(n, v|\bar{q})$ when $|v_{-i}| > 0$ note that the second derivatives of equation (8) in respect to v_i is given by $-2B{q_H}^2|v_{-i}|/(v_i + |v_i|)^3$ if $0 \leq v_i \leq \alpha(\bar{q})|v_{-i}|$, by $-2B\bar{q}^2|v_{-i}|/{v_i}^3$ if $\alpha(\bar{q}) \leq v_i \leq \beta(\bar{q})|v{-}i|$ and by $-2B{q_L}^2|v_{-i}|/(v_i + |v_i|)^3$ if $\beta(\bar{q})|v_{-i}| \leq v_i$. One can see that all these second derivatives are strictly less than 0 in their respectives intervals and, as a consequence, $\pi_i{}^*(n, v|\bar{q})$ is piecewise (strictly) concave.

To show continuity, consider first the case when $\bar{q} \geq q_H$. Since $q_H \geq \tilde{q}$, we have, as an abuse of notation, $\alpha(\bar{q}) = \infty$ and $\beta(\bar{q}) = \infty$. Hence $\pi_i{}^*(n, v|\bar{q}) = v_iB{q_H}^2/(v_i + |v_{-i}|) - \nu h v_i$ for all $v_i \geq 0$, which is always continuous. Suppose now $q_H > \bar{q} \geq \tilde{q}$. Then, $\beta(\bar{q}) = \infty$ and, consequently, $\pi_i{}^*(n, v|\bar{q})$ is given by the first two cases of equation (8). One can see that $\pi_i{}^*(n, v|\bar{q})$ is continuous for $0 < \alpha(\bar{q})|v_{-i}|$ and for $v_i > \alpha(\bar{q})|v_{-i}|$. Moreover, straightfoward algebra can show that the limit of $\pi_i{}^*(n, v|\bar{q})$ as v_i approaches $\alpha(\bar{q})|v_{-i}|$ by the left is the same limit of $\pi_i{}^*(n, v|\bar{q})$ as v_i approaches $\alpha(\bar{q})|v_{-i}|$ by the right, namely $\pi_i{}^*(n, \alpha(\bar{q})|v_{-i}|\bar{q}) = (A/2)\bar{q} - \nu h\alpha(\bar{q})|v_{-i}|$. Hence $\pi_i{}^*(n, v|\bar{q})$ is continuous at $\alpha(\bar{q})|v_{-i}|$ and, therefore, continuous for all $v_i \geq 0$. Finally, suppose $\bar{q} < \tilde{q}$. So, continuous for $0 \leq v_i < \beta(\bar{q})|v_{-i}|$ has been proved in the previous case. For $v_i > \beta(\bar{q})|v_{-i}|$, $\pi_i{}^*(n, v|\bar{q})$ is a continuous function. It remains to show that $\pi_i{}^*(n, v|\bar{q})$ is continuous at $\beta(\bar{q})|v_{-i}|$. Straightforward algebra can show that the limit of $\pi_i{}^*(n, v|\bar{q})$ as v_i approaches $\beta(\bar{q})|v_{-i}|$ by the left is the same limit of $\pi_i{}^*(n, v|\bar{q})$ as v_i approaches $\beta(\bar{q})|v_{-i}|$ by the right, namely $\pi_i{}^*(n, \alpha(\bar{q})|v_{-i}|\bar{q}) = (A/2)\bar{q} - \nu h\alpha(\bar{q})|v_{-i}|$. Hence $\pi_i{}^*(n, v|\bar{q})$ is continuous at $\alpha(\bar{q})|v_{-i}|$ and, therefore, continuous for all $v_i \geq 0$.

For the purpose of conclude that the objective is differentiable at all point $v_i \neq \beta(\bar{q})|v_{-i}|$ consider the first derivative of ${\pi_i}^*(n, v|\bar{q})$ with respect to v_i given by $B{q_H}^2|v_{-i}|/(v_i+|v_{-i}|)^2-\nu h$ if $0 \leq v_i \leq \alpha(\bar{q})|v_{-i}|$, by $B\bar{q}^2|v_{-i}|/{v_i}^2-\nu h$ if $\alpha(\bar{q})|v_{-i}| \leq v_i \leq \beta(\bar{q})|v_{-i}|$ and by $B{q_L}^2|v_{-i}|/(v_i+|v_{-i}|)^2-\nu h$ if $\beta(\bar{q})|v_{-i} \leq v_i$. Straightforward algebra can show that the first derivative is continuous for $0 \leq v_i < \beta(\bar{q})|v_{-i}|$ and for $v_i > \beta(\bar{q})|v_{-i}|$, but is not continuous at $v_i = \beta(\bar{q})|v_{-i}|$ and, therefore, not differentiable at that point.

At this moment, the goal is to show the optimal choice of dealer i in the turf-war stage ${v_i}^*(v_{-i}, n|\bar{q})$.

Suppose ${v_H}^2 \leq |v_{-i}|$. If $0 \leq v_i \leq \alpha(\bar{q})|v_{-i}|$, then $\frac{\partial {\pi_i}^*(n,v|\bar{q})}{\partial v_i} = B{q_H}^2|v_{-i}|/(v_i+|v_{-i}|)^2-\nu h \leq B{q_H}^2/|v_{-i}|-\nu h \leq B{q_H}^2/{v_H}^2-\nu h = 0$. Additionally, if $\alpha(\bar{q})|v_{-i}| \leq v_i \leq \beta(\bar{q})|v_{-i}|$, then $\frac{\partial {\pi_i}^*(n,v|\bar{q})}{\partial v_i} = B\bar{q}^2|v_{-i}|/{v_i}^2-\nu h \leq B\bar{q}^2/\alpha(\bar{q})^2|v_i|-\nu h = B\left(q_H-\bar{q}\right)^2/|v_{-i}|-\nu h \leq B\left(q_H-\bar{q}\right)^2/{v_H}^2-\nu h \leq 0$. Moreover, suppose $\beta(\bar{q})|v_{-i}| \leq v_i$. In this case, $\frac{\partial {\pi_i}^*(n,v|\bar{q})}{\partial v_i} = B{q_L}^2|v_{-i}|/(v_i+|v_{-i}|)^2-\nu h < B{q_L}^2/|v_{-i}|-\nu h \leq B{q_L}^2/{v_H}^2-\nu h \leq 0$. Thereby, using continuity, the objective is always decreasing when ${v_H}^2 \leq |v_{-i}|$ and, therefore, the optimal choice in this case is ${v_i}^*(v_{-i}, n|\bar{q}) = 0$.

The proof for the remaining cases must be written. □

Lemma 3. *Given a number of dealers $n > 2$, the profile of violence levels $v^* = (v_1^*, v_2^*, \cdots, v_n^*) \in \mathbb{R}_+^n$ is a symmetric Nash equilibrium in the turf-war stage if for all dealers i*

$$v_i^*(n) = \begin{cases} (n-1)(v_L/n)^2 & \text{if} \quad 0 \leq \bar{q} \leq \delta_L(n) \\ (n-1)v_{\bar{q}}^2 & \text{if} \quad \delta_M(n) \leq \bar{q} \leq (1/n)q_H \\ (n-1)(v_H/n)^2 & \text{if} \quad (1/n)q_H \leq \bar{q} \end{cases} \tag{10}$$

where $\delta_L(n) \equiv \frac{1}{2}\left[\sqrt{q_H - q_L\left(\frac{n-2}{n}\right)} - \sqrt{q_H - q_L}\right]^2$ and $\delta_M(n) \equiv \frac{1}{2n^2}\left[\sqrt{q_H + q_L(2n-1)} - \sqrt{q_H - q_L}\right]^2$. There is no symmetric equilibrium when $\delta_L(n) \leq \bar{q} \leq \delta_M(n)$. Thus, equilibrium payoff for each dealer i in a turf war with n dealers is

$$\pi_i^e(n|\bar{q}) = \begin{cases} B\left(q_L/n\right)^2 & \text{if} \quad 0 \leq \bar{q} \leq \delta_L(n) \\ A\bar{q} - B\bar{q}^2(2n-1) & \text{if} \quad \delta_M(n) \leq \bar{q} \leq (1/n)q_H \\ B\left(q_H/n\right)^2 & \text{if} \quad (1/n)q_H \leq \bar{q} \end{cases}. \tag{11}$$

For $n = 1$, turf war equilibrium entails $v_1^* = 0$ and $\pi_i^e(n|\bar{q}) = B{q_L}^2$ as implied by (7).

Proof. Suppose $n \geq 2$, $v_L > v_{\bar{q}}$ and $v_H > v_{\bar{q}}$ and $v_l \geq v_{\bar{q}} + \mu_{\bar{q}}$. Let $z_s = \left(v_L - v_{\bar{q}} - \mu_{\bar{q}}\right)^2$, $z_L = \left(v_L - v_{\bar{q}}\right)^2$ and $z_H = \left(v_H - v_{\bar{q}}\right)^2$. Moreover, denote $H(x) = v_H\sqrt{x} - x$, $L(x) = v_L\sqrt{x} - x$ and $M(x) = v_{\bar{q}}\sqrt{x}$. Remind that z_L is the intersection, for $x > 0$, between $L(x)$ and $M(x)$. While z_H is the intersection, for $x > 0$ between $M(x)$ and $H(x)$.

Consider also the symmetry line $S(x) = x/(n-1)$. Because the slope of $L(x)$, $M(x)$ and $H(x)$ tends to infinity as x approaches 0 and $S(x)$ has constant slope, $S(x)$ starts from 0 below all other curves. However, one can see that the slope of $L(x)$, $M(x)$ and $H(x)$ is decreasing. This implies that the intersection between $S(x)$ and the other curves occurs always at 0 and at just another one more point greater or equal to 0 (if $\bar{q} = 0$, then $M(x) = 0$). These points of intersection are given by $e_L \equiv [v_L(n-1)/n]^2$ for $L(X)$, $e_M \equiv [v_{\bar{q}}(n-1)]^2$ for $M(x)$ and $e_H \equiv [v_H(n-1)/n]^2$ for $H(x)$. By (2), we know that the best response function of the individual i is formed using the curves $L(x)$, $M(x)$ and $H(x)$. Thereby, if the equilibrium exists, it must be one of the points e_L, e_M or e_H.

That said, suppose $0 \leq \bar{q} \leq \delta_L(n)$. It can be show $z_s \geq e_L$. In fact, $z_s \geq e_L$ if, and only if, $v_{\bar{q}} + \sqrt{2\bar{q}(q_H - q_L)} - q_L/n \leq 0$, i.e., for $\bar{q} \geq 0$, if, and only if, $\bar{q} \leq \delta_L(n)$.

By lemma 2, we know that the reaction curve of player i is given by $L(x)$ on the left of z_s. Since $z_s \geq e_L$, it can be concluded that e_L is sNE in this case. To show that neither e_M nor e_H is sNE note that $e_M \leq z_s$ if, and only if, $n\bar{q} \leq q_L - \sqrt{2\bar{q}(q_H - q_L)}$, i.e., for $\bar{q} \geq 0$, if, and only if, $q_L \leq \delta_M(n)$. For that reason, since the reaction curve is given by $L(x)$ for $x \leq z_s$ and we have $e_M \leq z_s$, we obtain that e_M cannot be a sNE in this case. Furthermore, $M(x)$ intercepts $H(x)$ only at 0 and z_H. Moreover, $H(x)$ is decreasing for $x > {v_H}^2/4$. Since, $\bar{q} \leq \delta_L(n) \leq (1/n)q_H$, we obtain $z_H \geq e_H > {v_H}^2/4$. Thus, $H(x)$ is decreasing for $x \leq z_H$ and, therefore, the reaction curve is always below $M(x)$ on the right of z_s. Remind that $S(x)$ intercepts $M(x)$ at only two points, 0 and e_M. Noticing that $0 \leq e_M \leq z_s$, for $x > z_s$ the curve $M(x)$ is always below $S(x)$. Therefore, since the reaction curve is always below $M(x)$ for $x > z_s$, e_H is not a sNE as well.

Suppose now $\delta_M(n) \leq \bar{q} \leq (1/n)q_H$. Given that $\bar{q} \geq \delta_M(n) \geq \delta_L(n)$, then $z_s < e_L$ and $z_s < e_M$. Since $L(x)$ composes the reaction function only for $x \leq z_s$, the intersection point between $L(x)$ and $S(x)$, e_L, is not a sNE. One can show $e_M \leq z_H$. Indeed, $e_M = [v_{\bar{q}}(n-1)]^2 \leq \left[(1/n)q_H\sqrt{B/\nu h}(n-1)\right]^2 = [v_H(n-1)/n]^2 = e_H$. But it was already showed that $e_H \leq z_H$ for $\bar{q} \leq (1/n)q_H$. Thus, $e_M \leq z_H$. Hence, since $z_s \leq e_M \leq z_H$ and the reaction curve is given by $M(x)$ in the interval $(z_s, z_H]$, e_M must be a sNE. Again, using the arguments that for $\bar{q} \leq (1/n)q_H$ the reaction curve is always below $M(x)$ when $x > z_s$ and $M(x)$ intercepts $S(x)$ at only two points, one can see that e_H is not in the reaction curve and, therefore, can not be a sNE.

Finally, suppose $\bar{q} \geq (1/n)q_H$. We already know that $\bar{q} \geq (1/n)q_H \geq \delta_M(n) \geq \delta_L(n)$ requires $z_s \leq e_L$ and $e_M > z_s$. Knowing that $L(x)$ composes the reaction curve only for $x \leq z_s$, one has that e_L is not in the reaction curve and, therefore, e_L can not be a sNE. Moreover, it can be show $e_M \geq z_H$. In fact, $e_M = [v_{\bar{q}}(n-1)]^2 = [v_{\bar{q}}n - v_{\bar{q}}]^2 \geq [v_H - v_{\bar{q}}]^2 = z_H$. Using again lemma 2, we know that for $x \geq z_H$ the reaction curve is given by $H(x)$. Hence, e_M is not in the reaction curve and, therefore, can not be a sNE. At last, $e_H = [v_H(n-1)/n]^2 = [v_H - v_H/n]^2 \geq [v_H - v_{\bar{q}}]^2 = z_H$. It follows from lemma 2 that e_H is a sNE.

It remains to show that for $\delta_L(n) < \bar{q} < \delta_M(n)$ there is no sNE. Because $\bar{q} > \delta_L(n)$, we know that $z_s < e_L$. Since $L(x)$ composes the reaction curve only when $x \leq z_s$, it follows that e_L is not a point of the reaction curve. Consequently, e_L can not be a sNE. Also, because $\bar{q} < \delta_M(n)$, it has already been shown that $e_M < z_s$. Again by lemma 2, $M(x)$ composes the reaction curve only for $x \geq z_s$, it follows that e_M is not a sNE. Moreover, since $\bar{q} < (1/n)q_H$, one has $z_H \geq {v_H}^2/4$. Therefore, $H(x)$ is decreasing for $x \geq z_H$. Using the argument of $M(x)$ intercepting $H(x)$ only at 0 and z_H, one can conclude that, for $x \geq z_s$, the reaction curve is always below $M(x)$. Remind that $M(x)$ intercepts $S(x)$ only at two points as well, 0 and e_M. Since $e_M < z_s$, one has that the reaction curve does not intercept $S(x)$ for $x \geq z_s$ and, therefore, e_H can not be a sNE.

Having established (10), straightforward algebra can be used to get (11) by plugging (10) in (6).

Finally, equilibrium with $n = 1$ is trivial. We already know from Lemma (2) that no violence is optimal under $n = 1$. Because there is no strategic interaction with only one dealer, optimal behavior already implies equilibrium.

□

Proposition 1. *Define $n_L \equiv q_L\sqrt{B/w}$, $n_H \equiv q_H\sqrt{B/w}$, and $n_{\bar{q}} \equiv \bar{q}\sqrt{B/w}$. For a given cutoff $\bar{q} \geq 0$, suppose that $(n^e, v^e, q^e) \in I \times \mathbb{R}^2_+$ is the symmetric SPNE outcome in which n^e individuals get involved in drug dealing, each of them conquers monopoly rights over a share $1/n^e$ of total demand (territory) by employing violence level v^e and operate under monopoly quantity q^e in this territory. Then, the number of dealers is*

$$n^e = \begin{cases} n_H & \text{if} \quad 0 \leq \ w \ \leq w_L(\bar{q}) \\ \frac{2n_H n_{\bar{q}} + {n_{\bar{q}}}^2 - 1}{2{n_{\bar{q}}}^2} & \text{if} \quad w_L(\bar{q}) \leq \ w \ \leq w_M(\bar{q}) \\ n_L & \text{if} \quad w_H(\bar{q}) \leq \ w \end{cases}, \tag{13}$$

the violence level is

$$v^e = \begin{cases} \frac{w}{\nu h}(n_H - 1) & \text{if} \quad 0 \leq \ w \ \leq w_L(\bar{q}) \\ \frac{w}{\nu h}\frac{({n_H}^2 - 1) - (n_H - n_{\bar{q}})^2}{2} & \text{if} \quad w_L(\bar{q}) \leq \ w \ \leq w_M(\bar{q}) \\ \frac{w}{\nu h}(n_L - 1) & \text{if} \quad w_H(\bar{q}) \leq \ w \end{cases} \tag{14}$$

and the scale of operation is

$$q^e = \begin{cases} \sqrt{w/B} & \text{if} \quad 0 \leq \ w \ \leq w_L(\bar{q}) \\ n_{\bar{q}}\sqrt{w/B} & \text{if} \quad w_L(\bar{q}) \leq \ w \ \leq w_M(\bar{q}) \\ \sqrt{w/B} & \text{if} \quad w_H(\bar{q}) \leq \ w \end{cases} \tag{15}$$

where cutoff functions $w_H(\bar{q}) \equiv 4B\left(\bar{q} + \sqrt{2\bar{q}(q_H - q_L)}\right)^2$, $w_M(\bar{q}) \equiv B\bar{q}\left[\bar{q} + 2\left(q_H - \bar{q}{\delta_M}^{-1}(\bar{q})\right)\right]$, and $w_L(\bar{q}) \equiv B\bar{q}^2$, are stricly increasing and convex. The outcome for the case in which $w_M(\bar{q}) \leq w \leq w_H(\bar{q})$ is not reported here because no symmetric equilibrium exists in turf war stage for such $\bar{q}$.

Proof. All the reasoning for this proof was presented inside the main text using Figure 9. The details of such reasoning must be written. □