Estimating time-varying factors' variance in the string term structure model with stochastic volatility

Thiago Ramos Almeida[a,*,]

[a]*Department of Economics, Universidade Federal Fluminense (UFF), Niterói, RJ, Brazil*

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ABSTRACT

This paper presents a new string term structured model with stochastic volatility calibrated with futures and options market data. I use interest rate options to build a time series of the variance of the factors that drive the interest rates (level, slope, and curvature). In addition, the variance of the factors is governed by a mean-reverting stochastic volatility process, whose model's parameters are estimated within a new framework through the minimization of the distance between the moments of the gamma distribution. I show the empirical application of the model in Brazil's derivative market, where I present empirical evidence that the Interbank Deposit (DI) future market provides good predictive power for interest rates considering short- and mid-term maturities. However, for long-term maturities, the DI future market does not capture the posterior level of interest rates, suggesting an upward bias due to the hypothetical risk premium embedded in this market.

1. Introduction

In this paper, I develop a multi-factor term structure model assuming that overnight interest rates evolve according to the string-market model of Longstaff, Santa-Clara and Schwartz (2001).[1] I introduce stochastic volatility in the string-market model and apply a new framework to estimate the model's parameters, assuming that the stochastic volatility factors follow the gamma distribution. I minimize the distance between the historical distribution's moments of the variance of the principal components that drive the interest rate and the gamma distribution's moments (mean, variance, skewness, and kurtosis) to find the model's parameters.

Using a few factors, the model can capture the volatility smile in the DI interest rate options market and hedge hundreds of instruments. When the implied volatility increases, the model changes the interest rate's distribution, increasing the probability of left and right tail events, coinciding with the rise of option prices, especially for those that offer protection against extreme events (for illustration, out-of-the-money (OTM) options).

I show the empirical application of the model in the Brazil's derivative market. Brazil Stock Exchange (B3) traded more than 500 interest rate options daily with different maturities. Interestingly, two related interbank deposit (DI) instruments are traded in the Brazilian stock market: (i) the option on IDI, and (ii) the option on DI futures.[2]

As the two options have the same underlying variable (the DI rate), the options theory of Merton et al. (1973) suggest that the relationship between them may be driven primarily by the correlation structure of the forward rates. Consequently, a no-arbitrage relationship may exist.[3]

The second contribution relates to a new method that uses options data to obtain the market expectations of interest rates. Empirical evidence shows that the expectations provided by the OTM put option had an impressive forecasting power for the DI rate considering short-term maturities, while for mid-term maturities, futures market rates showed

*Corresponding author
thiagora2007@gmail.com (T.R. Almeida)
ORCID(s):

[1]This model combines the market-model framework of Brace, Gatarek and Musiela (1997) with the string-shock framework of Goldstein (2000), Santa-Clara and Sornette (2001), and Longstaff et al. (2001). The market-model framework arises from the fact that a set of forward rates that are directly observable in the futures market are used to calibrate the model. On the other hand, in the string-market model, each point of the forward curve has proper dynamics.

[2]DI and IDI, from the acronyms in Portuguese used by the B3, are the average one-day rate of interbank lending and the average one-day interbank deposit rate index, respectively. More details are provided in Section 2.

[3]According to the Futures International Association (FIA), global trading of interest rate products is dominated by exchanges in Europe and North America, which, in 2019, accounted for 79% of worldwide volume in this category. However, in the 2019 ranking of the most traded interest rate products, DI futures contracts (476 million contracts) were second only to Eurodollar futures contracts (687 million contracts). Meanwhile, IDI options (also called IDI index options) appeared in the sixth position (214 million contracts).

higher prediction power.[4] However, for long-term maturities, the options and the DI future market does not captured the level of future interest rates, suggesting a upward bias due to hypothetical risk premium embed in the future market.

Finally, based on the market expectation and volatility series, this paper empirically shows that the term premia rise when Brazil's volatility soars. In contrast, the literature indicates that the term premia fall when volatility increases in advanced economies. For example, Cieslak and Povala (2016) found that interest rates in the U.S. become progressively volatile in anticipation of recessions, and consequently, term premia fall. On the other hand, my findings are consistent with Neumeyer and Perri (2005) and Horvath (2018), who argue that interest rates in emerging economies are countercyclical and lead the business cycle.

Brazil had had one of the most prolonged episodes of hyperinflation in the twenty century, only stabilized in 1994 (see Dornbusch and Cline (1997)). Therefore, investors might be worried about inflation returns due to significant budget deficits and governments with weak fiscal control. Conversely, the central bank's lack of credibility could be one of the reasons for the term premia increases in moments of high volatility.

The rest of this work is organized as follows. Section 2 presents the financial instruments and the payoffs used in the empirical analysis. Section 3 describes the data used. Section 4 describes the option price model. Section 5 describes the methodology to recover the market expectation. Finally, Section 6 concludes the paper.

2. The Interbank Deposit Rates Derivatives Market

This section introduces the options traded on the DI interest rate, i.e., the options on DI futures and IDI[5].

2.1. Option on DI Futures

The option on DI futures has underlying it the DI futures contract. On the exercise day, if the option is in-the-money (ITM), the owner receives a position in a DI futures contract at the agreed interest rate (i.e., the strike). Consequently, the product traded is the interest rate expectation between the option's maturity and the maturity of the underlying DI futures contract. In other words, the forward interest rate lies between those two maturities. The futures contract has daily liquidity, so quitting the position immediately after exercising the option is possible.

Below I present the payoff of the derivative. Let T_1 be the option's exercise date, and let T_2 be the maturity date of the futures contract. The option payoff can be expressed in T_1 as follows:

$$\text{payoff}\left[T_1\right] = Q \times \max\left(cp\left[\frac{100,000}{\exp\left(k\left(\frac{T_2-T_1}{252}\right)\right)} - FUT_{T_1,T_2}\right], 0\right), \tag{1}$$

where Q is the contract quantity, cp is the variable that defines if the option is a call or put option, and FUT_{T_1,T_2} is the notional value of the futures contract at time T_1 with maturity in T_2. The futures contract will have a notional value of R$ 100,000 at maturity (T_2).

After some algebraic manipulations, it is possible to transform the payoff in Eq. (1) to the payoff of a swap option (swaption) and so reduce the computational burden (see Carreira and Brostowicz Jr (2016)),

$$\begin{aligned}&\text{payoff}\left[T_2\right] = \\ &Q \times \frac{100,000}{\exp\left(k\left(\frac{T_2-T_1}{252}\right)\right)} \times \left(\frac{T_2-T_1}{252}\right) \times \max\left(cp\left[R_t^{\text{linear}} - K^{\text{linear}}\right], 0\right),\end{aligned} \tag{2}$$

where the interest rate R_t^{linear} represents the implicit linear DI rate between T_1 and T_2. K_t^{linear} is the strike of the option transformed to the linear rate[6].

[4] For interest rates forecasts in Brazil, see Lima, Luduvice, Tabak et al. (2006), Pinheiro, Almeida and Vicente (2007), Barbedo and de Melo (2012), and Baghestani and Marchon (2012).

[5] In the Appendix, I provide a more detailed explanation about the interbank deposit rate and the DI future contract.

[6] I transform the continuous exponential rates into linear rates as follows, where k_t and r_t are the strikes and the DI rate in continuous time capitalization, respectively:

$$1 + K_t^{\text{linear}}\left(\frac{T_2-T_1}{252}\right) = \exp\left(k_t\left(\frac{T_2-T_1}{252}\right)\right), \tag{3}$$

2.2. Option on IDI

The option on IDI trades the average DI interest rate between the trade date and the tenor of the option. Underlying it is the DI rate index. This index launched on 1 February 2009 with a value of R$ 100,000.00 and is updated daily by the DI interest rate (using the bank's annual calendar of 252 business days). To obtain the index for any date T, I use the expression below, following Carreira and Brostowicz Jr (2016):

$$IDI_T = IDI_{\text{basedate}} \prod_{t_i=\text{basedate}}^{T} (1 + DI_{t_i})^{\frac{1}{252}}. \tag{5}$$

The option on IDI is also European's type, and the strike is expressed in R$. The option is exercised at maturity if the DI index remains above the strike for the call option, i.e., the buyer receives payment if the interest rate exceeds the agreed implicit rate in the strike price. In contrast, the put option is exercised if the DI index remains below the strike.

This instrument resembles the caps and floors traded in international markets, mainly used to protect buyers against fluctuations in interest rates. Indeed, options on IDI can offer a similar hedge mechanism for different periods, primarily associated with monetary policy uncertainty.

The option on IDI payoff at T maturity date is given by,

$$\text{payoff}_{T+1} = Q \times \max\left(cp\left[IDI_T - K\right], 0\right), \tag{6}$$

where Q is the number of contracts, K is the option strike, and cp is the variable that defines whether the option is call or put. Note that the DI rate on any given day is released at the end of the same day, which means that the index price at T can be known only in $T+1$.

The payoff can also be expressed with a continuous-time interest rate,

$$\text{payoff}_{T+1} = Q \times \max\left(cp\left[IDI_t \times \exp\left(r_t\left(\frac{T-t}{252}\right)\right) - K\right], 0\right). \tag{7}$$

3. Data

Table 1 summarizes the main features of the IDI options data, which covers October 2015 to January 2019. In Panel A, the options are sorted by the tenor in bank calendar days. The average price of put options exceeds call options by a substantial margin (nearly six-fold). The average daily number of options shows that most contracts have a tenor below 300 business days (approximately one calendar year). In Panel B, ATM options are sorted by moneyness, defined as the ratio of the strike to the forward price. The average price and number of ATM put options per trading day are much larger than for ATM call options. The sample size comprises 154,190 options, and all data come from the B3.[7]

Table 2 summarizes the main features of options on DI futures data covering October 2015 to January 2019. Panel A sorts the instruments by the bank calendar tenor (considering only business days). The average price of put options exceeds call options by a considerable margin (almost two-fold). The average daily number of options shows that most contracts have a tenor below 200 business days. In Panel B, ATM options are sorted by moneyness, defined as the ratio of the strike to the forward price. The average price and number of ATM put options per trading day are much larger than for ATM call options. The sample size comprises 38,930 options, and the data is from the B3 website[8],

Table 3 shows descriptive statistics for the DI futures contract data covering 9 October 2015 to 18 January 2019. The standard deviation for long tenors is less than for short maturities, reflecting notable monetary policy changes in the period (short-term rates reached a peak of 13.82% and an annual minimum of 5.96%). The negative skewness of the shortest tenors shows that rates stay above average for most of the sample, while the reverse occurs for long tenors. Kurtosis increases with the tenor, showing heavy tails for the medium- and long-term rates. The data is from the B3 website.

$$1 + R_t^{\text{linear}}\left(\frac{T_2 - T_1}{252}\right) = \exp\left(r_t\left(\frac{T_2 - T_1}{252}\right)\right). \tag{4}$$

[7] To avoid market microstructure effects, I use options with maturities above eight business days for the IDI options and DI futures.

[8] Accessed on February 19, 2021, <http://www.b3.com.br/pt_br/market-data-e-indices/servicos-de-dados/market-data/historico/boletins-diarios/pesquisa-por-pregao/pesquisa-por-pregao/>.

Table 1
IDI option data.

Panel A: Sorted by time-to-maturity*	[8, 150)	[150, 300)	[300, 450)	⩾ 450	All
Call option					
Average number per trading day	430.26	336.83	165.94	113.55	1,048.42
Average price	464.34	1,360.69	1,387.42	4,283.75	1,311.69
Average maturity	75.60	218.30	367.57	563.62	220.65
Put option					
Average number per trading day	538.00	420.29	196.86	164.97	1,323.72
Average price	4,562.05	4,909.93	8,993.25	13,269.41	6,418.18
Average maturity	76.20	217.54	367.54	632.09	233.88

Panel B: Sorted by moneyness K/F	ATM call < 0.99	[0.99, 1)	ATM put [1, 1.04)	⩾ 1.08	All
Average number per trading day	120.47	204.63	642.56	280.33	1248.01
Average price	8052.89	1262.71	3444.05	21798.01	7654.10
Average maturity	339.38	184.31	188.66	300.48	227.61

Note: The table summarizes the IDI options data sorted by maturity and moneyness. The statistics are from sample data covering 9 October 2015 to 18 January 2019. Panel A reports the average number of options available per trading day. The average implied volatility is the average annual percentage of the Black-Scholes (Black (1976)) implied volatility for each moneyness level (the ratio of the strike price K to the forward price F). Panel B reports the same three measures for the ATM options sorted by moneyness. The weekly data for options represent Friday closing rates. Data from the B3 website. *Business days using the bank calendar of 252 annual workdays.

Table 2
DI futures options data.

Panel A: Sorted by time-to-maturity*	[8, 50)	[50, 200)	[200, 400)	⩾ 400	All
Call option					
Average number per trading day	66.88	141.72	61.22	3.00	270.23
Average price	369.95	332.74	870.90	1,196.06	464.90
Average maturity	29.35	115.62	285.64	422.48	133.31
Put option					
Average number per trading day	78.12	171.31	75.05	4.20	328.68
Average price	1,217.43	1,159.96	1,076.87	748.90	1,149.40
Average maturity	28.62	116.21	282.98	418.43	137.33

Panel B: Sorted by moneyness K/F	ATM call < 0.90	[0.90, 1)	ATM put [1, 1.4)	⩾ 1.4	All
Average number per trading day	32.69	35.11	168.14	77.43	313.37
Average price	1,909.68	828.40	876.41	2,888.24	1,475.94
Average maturity	169.70	147.55	133.43	113.74	133.93

Note: The table summarizes the options on DI futures data sorted by tenor and moneyness. The statistics are from sample data covering 9 October 2015 to 18 January 2019. Panel A reports the average number of options available per trading day. The average implied volatility is the average annual percentage of the BSIV for each moneyness level (the ratio of the strike price K to the forward price F). Panel B reports the same three measures for the ATM options sorted by moneyness. The weekly data for options represent Friday closing rates. Data from the B3 website. * Business days using the bank calendar of 252 annual workdays.

Figure 1 shows the time series of the DI forward rate. I use Friday closing rates, and the sample contains data for 164 days (9 October 2015 to 31 January 2019), with approximately 40 maturity dates for each day. I obtain the forward rates by bootstrapping the interpolated par curve. The interpolation technique is commonly used in the literature by practitioners, namely, the piecewise polynomial interpolation (cubic spline).

Table 3
Summary statistics for DI futures contracts.

Business day*	100	400	800	1200	1600	2000
Sample size	164	164	164	164	164	164
Mean	9.46%	9.64%	10.33%	10.69%	10.90%	10.99%
Std. Dev.	0.030	0.025	0.021	0.018	0.017	0.016
Minimum	5.96%	6.54%	7.70%	8.30%	8.61%	8.82%
25%	6.43%	7.44%	8.80%	9.34%	9.64%	9.75%
50%	8.97%	8.83%	9.63%	10.19%	10.41%	10.57%
75%	12.88%	11.78%	11.40%	11.39%	11.57%	11.62%
Maximum	13.82%	15.08%	15.39%	15.38%	15.46%	15.53%
Skewness	-1.73	-0.78	0.21	0.49	0.51	1.10
Kurtosis	0.18	0.67	1.07	1.15	1.16	1.29

Note: The table summarizes the DI futures contracts. The data are sorted by the tenor in business days (using the bank calendar of 252 annual workdays) from 9 October 2015 to 18 January 2019. The weekly data for interest rates represent Friday closing rates in continuous-time capitalization. Data from the B3 website.

Table 3 shows that the standard deviation of the DI futures contract with maturity in 100 business days is almost twice that of the contract with maturity in 2,000 business days. The sample comprises a period when the monetary policy target rate changed considerably. Since the Central Bank controls the short-term section of the curve through its monetary policy, this could explain the high standard deviation of the short-term rates.

In 2015, Brazil entered a deep recession, with an accumulated contraction of 7.5% in real GDP and a dramatic accumulated fall in per capita income of 9.2% in just two years (2015-2016) (Nassif, Feijó and Araújo, 2020). After reaching a high output gap and unemployment, the consumer inflation index converged to below-targeted levels in 2017 and 2018. Consequently, the weakened economy opened up the possibility of reducing the SELIC benchmark interest rate. Note that turnarounds in monetary policy credibility during this period (see (Ferreira, Goes and Arruda, 2018)) also may have contributed to the interest rate fall.

4. Option Pricing Model

To capture the volatility smile, Goldstein (2000), and Santa-Clara and Sornette (2001) modeled the evolution of the interest rate term structure as a *stochastic string*. In this approach, which is a generalization of the Heath, Jarrow and Morton (1992) model, each forward rate $f(t, T_i, T_j)$ has a proper dynamic, but the shocks that affect any forward rate are correlated to the others.

To model the dynamics of the DI interest rates term structure, I applied a multi-factor *string market* model. When a single factor arbitrage-free specification is applied in a multi-factor economically motivated framework, it can generate suboptimal hedging strategies with arbitrage opportunities (Backus, Foresi and Zin (1998)). Besides that, single-factor models, such as the widely used Black (1976) model, cannot capture the implied volatility smile.

Following Longstaff et al. (2001), the evolution of the forward rate on the risk-neutral measure is,

$$dF_i = \alpha_i F_i dt + \sigma_i F_i dZ_i, \tag{8}$$

where α_i is an unknown *drift*, σ_i is a deterministic volatility function of the forward price, and dZ_i is a Brownian motion process. The forward rate with continuous-time capitalization is,

$$F_i = \log\left(\frac{P(t, T_i)}{P(t, T_i + \tau)}\right)\frac{252}{\tau} \tag{9}$$

In the *string market* model, the evolution of the bonds' prices is based on a stochastic differential equation with two terms, the *drift* extracted from the spot interest rates, and the *diffusion* composed by the multiplication of (i) a Jacobian matrix of the sensitivities of the forward rates to each pair correlations, (ii) a matrix of the implied covariance, (iii) a vector of forward rates.

Although the model is specified in terms of the forward rates, following Longstaff et al. (2001), I consider a more efficient implementation using the vector of the bond prices. Applying Itô's rule to the vector P of the bond prices, I obtain its evolution in the risk-neutral measure,

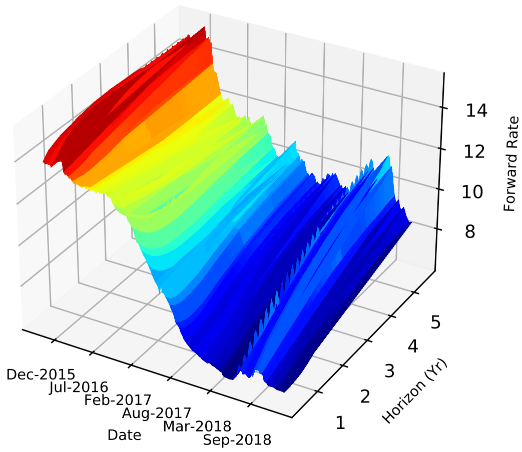

Figure 1: DI-over future contract. The data set consists of weekly observations from October 2015 to January 2019, using Friday closing rates. The sample contains 164 days, each with approximately 40 maturity dates. The forward rates are computed from 100 to 1,450 business days up to maturity. I use a cubic spline to interpolate the DI future contract rates' par curve and bootstrap the forward curve.

$$dP = rPdt + J^{-1}\Sigma_t F dB, \tag{10}$$

where,

$$J = \begin{bmatrix} -\frac{P(1)}{P^2(2)} & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{P(3)} & -\frac{P(2)}{P^2(3)} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{P(4)} & -\frac{P(3)}{P^2(4)} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{P(N-1)} & -\frac{P(N-2)}{P^2(N-1)} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{P(N)} & -\frac{P(N-1)}{P^2(N)} \end{bmatrix},$$

$$\Sigma_t = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,N-1} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N-1,1} & \sigma_{N-1,2} & \cdots & \sigma_{N-1,N-1}^2 \end{bmatrix},$$

where $r(t)$ is the spot interest rate, $\Sigma_t F dZ$ is a vector defined by the second terms in the Eq.((8)), and J is a Jacobian matrix obtained through the derivation of the forward vector $(F_1, F_2, \dots, F_{N-1})$ concerning the discount bond price $(P_1, P_2 ... P_N)$. The Brownian motions B are correlated through $d\langle B_i, B_j\rangle = \rho_{ij}$, where ρ_{ij} is the historical correlation between the forward rates that have the shocks Z_i and Z_j.

The model's drift term is calibrated with the rates traded in the DI futures market. The *string market* model may be interpreted as a collection of DI rates with a different dynamic depending on the forward rates spanning maturities. However, the shocks are correlated between them. Accordingly, I start the evolution of the bond prices in equation 10, with the rate $r(t)$ traded in the DI futures market.

In the calibration of the diffusion term in equation (10), P, J, and F are also estimated from the DI futures market. However, the estimation of Σ_t, the instantaneous variance-covariance matrix, is done through the implied volatility of the option prices.

The main contribution of this paper is the introduction of stochastic volatility in the Longstaff et al. (2001) model. I consider that $\Sigma_t = U_t \Psi_t U_t'$ (see Eq. (13)) and the variance of the factors (Ψ_t) is represented by the following matrix,

$$\Psi_t = \begin{pmatrix} \left(a_1\sqrt{v_{1,t}} + \frac{b_1}{\sqrt{v_{1,t}}}\right)^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \left(a_2\sqrt{v_{2,t}} + \frac{b_2}{\sqrt{v_{2,t}}}\right)^2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \left(a_3\sqrt{v_{3,t}} + \frac{b_3}{\sqrt{v_{3,t}}}\right)^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \left(a_{30}\sqrt{v_{30,t}} + \frac{b_{30}}{\sqrt{v_{30,t}}}\right)^2 \end{pmatrix},$$

where $v_{i,t}$ evolves stochastically under the risk neutral measure according to,

$$dv_{i,t} = \alpha(\theta - v_{i,t})dt + \xi\sqrt{v_{i,t}}dZ_t \qquad i = 1,2...30 \tag{11}$$

Grasselli (2017) join the Heston-like and the 3/2-like diffusion terms in a unified framework, forming a two-factor model. The 4/2 (i.e., 1/2+3/2) has instantaneous volatility of the form $(a\sqrt{v_t} + b/\sqrt{v_t})$, where a and b give the weight of each model in the volatility. For Detemple and Kitapbayev (2018), the 4/2 model has important features as the variance is bounded away from zero (see Gatheral (2008) for this stylized fact) and fits the shapes of the implied volatility surface. Lin, Li, Luo and Chern (2017) investigate whether the 4/2 model pricing performance of options is better than others (3/2, Heston and Black Scholes model). According to the authors, the 4/2 model had the best overall performance in pricing VIX's future and its options,

The instantaneous variance is mean-reverting and is driven by v_t whose parameters are α, the speed of mean reversion, θ, the mean reversion level, and ξ, the volatility of volatility. Hence, the variance is a combination of a 1/2 and 3/2 processes,

$$V_t^2 = \left(a^2 v_t + \frac{b^2}{v_t} + 2ab\right) \tag{12}$$

The Brownian motions Z, B are correlated through $d\langle B, Z\rangle_t = \rho dt$, where ρ is the correlation factor between underlying asset returns and the volatility. Each of $\alpha, \theta, \xi, \rho$ are constants, and I assume that the Feller condition holds, $2\alpha\theta > \xi^2$ for each v_i. With $b = 0$, I can recover the Heston (1993) model, and with $a = 0$, the 3/2 [9] model of Platen (1998).

4.0.1. Implied Covariance Matrix

This section builds a time-varying variance-covariance matrix of the DI rates using the DI future options and the DI future contracts. This matrix, $\hat{\Sigma}$ (which is symmetrical and positive semi-definite), is used in my multi-factor term structure model, as shown in Eq.(10).

Firstly, I calculate the historical sample correlation H (my sample covers five years) of the DI future market contracts. Matrix H can be decomposed as:

$$H = U\Lambda_0 U', \tag{13}$$

[9] The name 3/2 is due to the stochastic differential equation (SDE) in which there is the power of 3/2 in the diffusion term.

where Λ_0 is a diagonal matrix of the eigenvalues of H (non-negative), and U corresponds to the N eigenvectors. I generate $\hat{\Sigma}_t$, assuming that it shares the same eigenvectors with the historic correlations matrix H. $\hat{\Sigma}_t$ is decomposed as follows:

$$\hat{\Sigma} = U\Psi_t U', \tag{14}$$

where U is a constant matrix which contain the eigenvectors, conforming Eq.((13)). Ψ_t is a time-varying diagonal matrix of eigenvalues with non-negative entries. The i-th diagonal element of Ψ_t can be interpreted as the instantaneous variance of the i-th factor that drives the interest rate evolution.

To estimate Ψ_t, I build a new matrix Ω_t using the implied volatility of the options market. Initially, I estimate an implied volatility term structure for each sample day using the DI futures options. For each tenor, I take four calls and four put options with strikes closest to the underlying price and find unique volatility that minimizes the error between the Black (1976) model and the market prices of a set of options. To obtain the market-implied volatility for each tenor, I solve the following optimization problem,

$$\begin{aligned} &\underset{\sigma\in\Re^+}{\text{minimize}} \quad \hat{F}(\sigma)_t = \\ &= \frac{1}{N^c}\left(\sum_{i=1}^{N^c} C_{t,i}(T_j, K_i^{\text{linear}}, R^{\text{linear}}, \sigma_t) - y_i\right)^2 + \frac{1}{N^p}\left(\sum_{i=1}^{N^p} P_{t,i}(T_j, K_i^{\text{linear}}, R^{\text{linear}}, \sigma_t) - y_i\right)^2, \end{aligned} \tag{15}$$

where $C_{t,i}(T_j, K^{\text{linear}}, R^{\text{linear}}, \sigma_t)$ and $P_{t,i}(T_j, K^{\text{linear}}, R^{\text{linear}}, \sigma_t)$ are the model prices at time t, with maturity in T_j, strike K^{linear}, volatility σ_t, and linear interest rate R^{linear}, and where N^c and N^p are the total number of call and put options, respectively.

For each t, I then interpolate the set of volatilities with a cubic spline to build the implied volatility term structure. In sequence, I allocate the volatilities in the diagonal of Ω_t and square them to obtain the variance. To obtain the covariance, I use the historical correlation matrix and the known expression that $cov(x, y) = \rho_{xy}\sigma_x\sigma_y$, to fill the remaining entries of Ω_t.

After getting Ω_t, I apply singular value decomposition-SVD on it,

$$\Omega_t = U_t S_t V_t^T, \tag{16}$$

where S (diagonal matrix) contains the singular values. I consider that $\Psi_t = S_t$.[10] Finally,

$$\hat{\Sigma}_t = U S_t U' = U\Psi_t U' \tag{17}$$

In Figure 2, I depict, in four subplots, the time series for the variance of each of the four main factors. Between September 2018 and October 2018, the variance increased, reaching a peak around the election period (where the first round was on October 7, 2018, and the second round was on October 28, 2018).

4.1. Change of Measure

As in the 3/2 model, the risk-neutral paradigm's potential failure may also occur in the 4/2 model. Lewis (2000) extended the results of Sin (1998) furthermore, showing that the probability of an explosion under ubiquitous stochastic volatility models is above zero ($P(V_{t,T} \to \infty) > 0$). As a result, the process under the physical measure and the Girsanov transformation to the risk-neutral measure may not be equivalent. There is zero probability of an explosion in the volatility in the physical measure. Consequently, the discounted asset price is a strict local martingale under the (putative) risk-neutral probability measure.

For the 3/2 stochastic volatility model, Drimus (2012) showed that two conditions must be satisfied to have a well-behaved model. Using the Feller's boundary condition, the Heston process cannot reach the zero boundaries if,

[10] The eigenvectors of $\Omega_t\Omega_t'$ are in the columns of V and the eigenvectors of $\Omega_t'\Omega_t$ are in the columns of U. But the singular values of S represent the square root of the eigenvalues of $\Omega_t\Omega_t'$ and $\Omega_t'\Omega_t$.

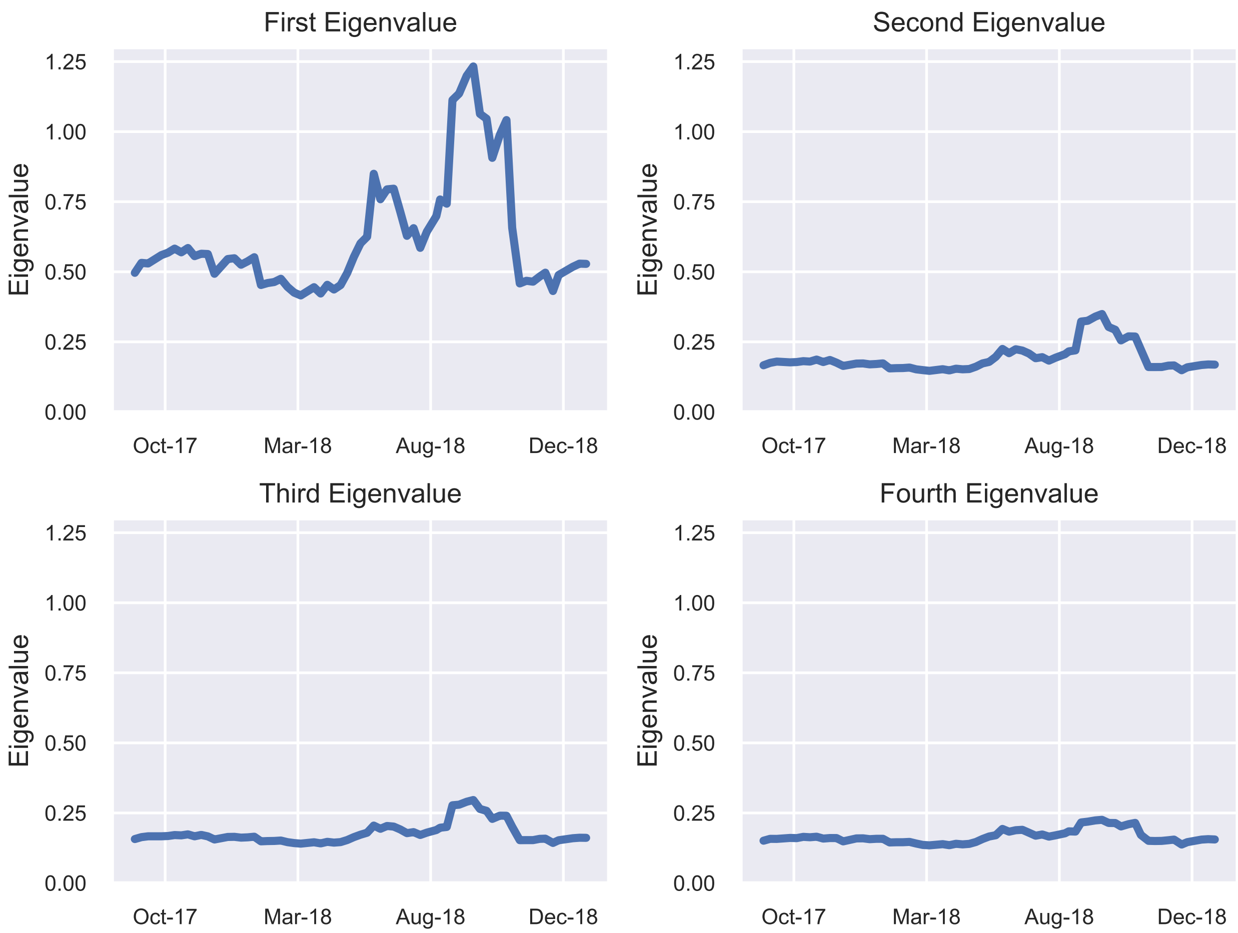

Figure 2: Time series of eigenvalues. The four subplots show the eigenvalues computed weekly from September 29, 2017 to January 18, 2019. I obtain the values for each day by minimizing the errors between the model price (Black (1976)) and market price using a set of options (four call options and four put options with the strike closest to the underlying price). Interpolating the set of implied volatilities that minimize each tenor error, I obtain the implied volatility for other tenors of the DI futures contract and build a covariance matrix with the historical correlation matrix. Finally, I obtain this matrix's eigenvalues as the factors' variances.

$$2\alpha\theta \geqslant \xi^2, \tag{18}$$

which I denote as a condition I. If v_t is a Heston process then $1/v_t$ is a 3/2 process (see Heston (1997), Platen (1998)). A non-zero probability of $1/v_t$ reaching infinity implies that a non-zero probability of v_t reaches zero. Consequently, the condition in Eq. (18) must also be satisfied for the 3/2 model.

The second condition requires that the non-explosion test for $1/v_t$ also must be satisfied under the measure, which takes the asset price as a numeraire. To test the martingale property for the discount asset, Grasselli (2017) made the following change in the Brownian motion for the volatility process:

$$d\tilde{Z}_t = dZ_t - \rho\left(a\sqrt{v_t} + \frac{b}{\sqrt{v_t}}\right)dt$$

Under the historical measure ($\tilde{Z}$), v solves,

$$dv_t = (\alpha\theta + \rho\xi b - (\alpha - \rho\xi a)v_t)dt + \xi\sqrt{v_t}d\tilde{Z}_t$$

As indicated above, in the risk-neutral probability measure, $1/v_t$ does not reach infinity if the Feller condition is satisfied, i.e., Eq (18). On the other hand, in the historical measure, the process $1/v_t$ has zero probability of explosion if the corresponding Feller condition is satisfied (condition II),

$$\alpha\theta + 2\rho\xi b \geqslant \xi^2 \tag{19}$$

The correlation parameter ρ plays a vital role in determining if the Feller condition will be satisfied in Eq. (19). Usually, the behavior of prices and volatility display a negative correlation. For example, volatility is typically very high in the stock market at moments of highly negative returns. This stylized fact is also known as the leverage effect. However, the middle- and long-term interest rates are crescent with volatility in the DI futures market. Table 4 elucidates that the correlation between term premia and volatility is highly positive, easing the model's implementation, given that this facilitates the Feller condition requirement.

4.2. Model's Parameter Estimation

I model the observed volatility according to Eq. (11). To avoid the identifiability problem, I set $a = 1$, as in Escobar-Anel and Gong (2020). Consequently, the model can be interpreted as a Heston plus a fraction of the 3/2 model.

To estimate b I use the fact that V_t^2 reach the infimum when v_t is equal to b, where $V_t^2 = v_t + b^2/v_t + 2b$, is the variance of the factors. Then, the estimator for b is,

$$\hat{b} = \frac{\min V^2(t_i)}{4}, \quad \text{i=0,...,T.}$$

I assume, following Cox, Ingersoll Jr and Ross (2005), that as $t \to \infty$, v_t is asymptotically gamma distributed,

$$f(v) = \frac{\omega^\eta}{\Gamma(\eta)} v^{\eta-1} e^{-\omega v}, \tag{20}$$

where the distribution parameters are $\eta = 2\alpha\theta/\xi^2$ (shape parameter) and $\omega = 2\alpha/\xi^2$ (scale parameter). Consequently, $1/v_t$ is asymptotically inverse gamma distributed, and the parameters are the same in the two distributions, [11]

To estimate the parameters, I use the following moments of the distribution: mean ($\Lambda_{1,i}$), variance ($\Lambda_{2,i}$), skewness ($\Lambda_{3,i}$) and kurtosis ($\Lambda_{4,i}$) of the gamma and inverse gamma distribution,

[11] To see if the parameters are the same, I apply the transformation technique, where $g(X) = 1/X = Y$,

$$\begin{aligned} f_Y(y) &= f_Y(g^{-1}(y)) \left|\frac{dx}{dy}\right| \\ &= \frac{\omega^\eta}{\Gamma(\eta)}(1/y)^{\eta-1} \mathrm{e}^{\frac{-\omega}{y}} \left|\frac{1}{y^2}\right| \\ &= \frac{\omega^\eta}{\Gamma(\eta)}(y)^{-\eta-1} \mathrm{e}^{\frac{-\omega}{y}} \end{aligned}$$

The inverse gamma distribution's parametrization does not change due to the transformation; hence, the parameters are the same.

$$
\begin{aligned}
\Lambda_{1,i} &= \mathbb{E}\left(v_t + \frac{\hat{b}^2}{v_t}\right) = \frac{\eta}{\omega} + \hat{b^2}.\frac{\omega}{\eta-1} \\
&= \theta + \hat{b^2}.\frac{2\alpha}{2\alpha\theta-\xi^2}
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{2,i} &= \mathbb{E}\left[\left(v_t + \frac{\hat{b}^2}{v_t}\right)^2\right] - \mathbb{E}\left[\left(v_t + \frac{\hat{b}^2}{v_t}\right)\right]^2 = \frac{\eta}{\omega^2} + \hat{b^4}.\frac{\omega^2}{(\eta-1)^2(\eta-2)} \\
&= \frac{2\alpha\theta(2\alpha\theta+\xi^2)}{4\alpha^2} + 2\hat{b^2} - \frac{4\hat{b^2}\alpha\theta}{(2\alpha\theta-\xi^2)} - \theta^2
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{3,i} &= \frac{\mathbb{E}\left[\left(\left(v_t + \frac{\hat{b}^2}{v_t}\right) - \mathbb{E}\left(v_t + \frac{\hat{b}^2}{v_t}\right)\right)^3\right]}{\left(\mathbb{E}\left[\left(v_t + \frac{\hat{b}^2}{v_t}\right) - \mathbb{E}\left(v_t + \frac{\hat{b}^2}{v_t}\right)^2\right]\right)^{3/2}} \approx \frac{2}{\sqrt{\eta}} + \hat{b}.\frac{4\sqrt{\eta}-2}{\eta-3} \\
&= \frac{2}{\sqrt{2\alpha\theta/\xi^2}} + \hat{b}.\frac{4\sqrt{2\alpha\theta/\xi^2}-2}{2\alpha\theta/\xi^2-3}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\Lambda_{4,i} &= \frac{\mathbb{E}\left[\left(\left(v_t + \frac{\hat{b}^2}{v_t}\right) - \mathbb{E}\left(v_t + \frac{\hat{b}^2}{v_t}\right)\right)^4\right]}{\left(\mathbb{E}\left[\left(v_t + \frac{\hat{b}^2}{v_t}\right) - \mathbb{E}\left(v_t + \frac{\hat{b}^2}{v_t}\right)^2\right]\right)^{2}} \approx \frac{6}{\eta} + \hat{b}.\frac{6(5\eta-11)}{(\eta-3)(\eta-4)} \\
&= \frac{6}{2\alpha\theta/\xi^2} + \hat{b}.\frac{6(10\alpha\theta/\xi^2-11)}{(2\alpha\theta/\xi^2-3)(2\alpha\theta/\xi^2-4)}
\end{aligned}
$$

To obtain the parameters, I minimize the objective function for each factor,

$$
\begin{aligned}
F_i = &\left(\Lambda_{1,i} - \frac{1}{n}\sum_{j=0}^{n}(V_{t,i}^2 - 2\hat{b})\right)^2 + \left(\Lambda_{2,i} - \frac{1}{n}\sum_{j=0}^{n}(V_{t,i}^2 - \mu_i)^2\right)^2 \\
&+ \left(\Lambda_{3,i} - \frac{\frac{1}{n}\sum_{j=0}^{n}\left(V_{t,i}^2 - \mu_i\right)^3}{\left[\frac{1}{n}\sum_{j=0}^{n}\left(V_{t,i}^2 - \mu_i\right)^2\right]^{3/2}}\right)^2 \\
&+ \left(\Lambda_{4,i} - \frac{\frac{1}{n}\sum_{j=0}^{n}\left(V_{t,i}^2 - \mu_i\right)^4}{\left[\frac{1}{n}\sum_{j=0}^{n}\left(V_{t,i}^2 - \mu_i\right)^2\right]^{2}}\right)^2,
\end{aligned}
\tag{22}
$$

where μ_i is the arithmetic mean of V_i^2. To overcome the fact that Feller condition is often violated (see Da Fonseca and Grasselli (2011) and Lin et al. (2017)), I estimate the parameters under the restrictions (conditions I and II), forcing the algorithm only to search the estimators in the space outside the restrictions.

Table 4
Diebold and Mariano (2002) and likelihood ratio statistics.

Model pairs	DM stats	Likelihood ratio stats
V1–V2	-1.03	**3,748.87**
V2–V3	**-2.28**	**205.04**

Note: The total number of cross-sectional and time series observations (27,923 over the entire sample) times the differences between the SSE logarithms for the two models asymptotically follow a χ^2_{65} distribution, and the 1% critical values are 94. The DM statistics, calculated according to Eq. (25), follow an asymptotic standard normal distribution under the null hypothesis of an equal pricing error. A negative statistic means that the more sophisticated model has more minor pricing errors. **Bold** indicates significance at the 5% level for the DM statistics and 1% level for the likelihood ratio test. The sample is composed of ATM IDI options.

4.3. Stochastic Volatility Model Empirical Results

I first used an incremental likelihood ratio test to estimate the number of factors to price IDI options. I compared nested models with one to three factors to verify statistical differences. The test is as follows: for a model with N factors, I compute the squared errors of the differences between the market and model price and then use the exact computation for a model with $N + 1$ factors. The error has a chi-square distribution with 65 degrees of freedom (χ^2_{65}) in the one-tail test (The χ^2_{65} 1% critical value is 94).

Table 4 shows the likelihood ratio test results. The model with two factors is statistically different from the model with only one; the likelihood ratio test is 3,748.87, well above the 1% critical value (94). The model with three factors also is statistically different from the model with two factors (the likelihood ratio test is 205.04).

I also adopt the Diebold-Mariano (DM) approach (Diebold and Mariano (2002)) to test if the two forecasting models are statistically equal. I first define the forecast error for each t and instrument i as $\epsilon_{i,t} = (\hat{p}_{i,t} - p_{i,t})/p_{i,t}$. I take a function $g(.)$ as a loss function, considering $g(\epsilon_{i,t})$ as $g(\epsilon_{i,t}) = |\epsilon_{i,t}| \times 100$. The loss differential between the two models is $d_{i,t} = g(\epsilon^1_{i,t}) - g(\epsilon^2_{i,t})$. I take the weekly sum-of-squared-errors (SSE) as $v_t = \sum_{i=1}^{N} |\epsilon_i| \times 100$. Defining the loss differential between the two forecasts as $d_t = v^1_t - v^2_t$ and taking the forecasts as having equal accuracy if and only if the loss differential has zero expectations for all t, then the null hypothesis is that both forecasts are equally accurate: $H_0 \,:\, E(d_t) = 0\ \forall t$. If $\{d_t\}_{t=1}^{T}$ is covariance stationary with short memory, then:

$$\sqrt{T}(\bar{d} - \mu) \sim N(0, 2\pi f_d(0)), \tag{23}$$

where $\bar{d} = \frac{1}{T}\sum_{t=1}^{T} d_t$, is the sample mean of the loss differential and $\mu = E(d_t)$ is the population mean of the loss differential. The spectral density of the loss differential at frequency 0 is as follows:

$$f_d(0) = \frac{1}{2\pi}\left(\sum_{k=-\infty}^{\infty} \gamma_d(k)\right), \tag{24}$$

where $\hat{\gamma}_d(k) = \frac{1}{T}\sum_{t=|k|+1}^{T}(d_t - \bar{d})(d_{t-|k|} - \bar{d})$. In practice I consider $M = T^{\frac{1}{3}}$ and a good estimator is as follows,

$$DM = \frac{\bar{d}}{\sqrt{\frac{\sum_{k=-M}^{M}\hat{\gamma}_d}{T}}}, \tag{25}$$

Table 5
Parameter Estimates of Stochastic Volatility Models

	SV1	SV2	SV3
Parameter	Estimate	Estimate	Estimate
α_1	0.7467	0.7467	0.7467
α_2		0.2027	0.2027
α_3			0.6659
θ_1	0.3858	0.3858	0.3858
θ_2		0.0841	0.0841
θ_3			0.0838
ξ_1	0.2461	0.2461	0.2461
ξ_2		0.1360	0.1360
ξ_3			0.1364
b_1	0.1038	0.1038	0.1038
b_2		0.0366	0.0366
b_3			0.0351
$\rho 1$	0.4259	0.4259	0.4259
$\rho 2$		0.3731	0.3731
$\rho 3$			0.3842
Objective function	0.1303	0.1214	0.0982

Note: This table reports parameter estimates of the one-, two-, and three-factor stochastic volatility models (SV1, SV2 and SV3, respectively). I obtain the estimates by minimizing Eq. (22) using the eigenvalues of the covariance matrix observed on a weekly frequency from September 29, 2017, to January 19, 2019. The objective functions reported in the table are rescaled SSEs over the entire sample at the estimated model parameters and are equal to the RMSE of different IDI options.

As the magnitude of the IDI option price varies considerably, I adapt the DM test (Diebold and Mariano (2002) to award the same weight to all instruments. Jarrow, Li and Zhao (2007) calculate the loss function as the square of pricing errors. I, therefore, take a loss function defined as the absolute value of the difference between the market and model prices divided by the market price.

Comparing the two- and three-factor models, Table 4 shows that the three-factor model is better at reducing the pricing error, and the models are statically different from the model with two factors with a 95% level of confidence. According to Table 4, the DM statistic shows that the two-factor model reduces the pricing error compared to the one-factor model. However, the difference is not statically significant at the 95% level of confidence.

Table 4.3 shows the coefficients of the stochastic volatility model for the first factor (associated with the level of the yield curve), second factor (associated with slope), and third factor (associated with curvature). Table 4.3 also elucidates the objective function results (root mean squared error (RMSE) between the model and market price) and shows that the three-factor model has produced more minor pricing errors.

5. Recovering Market Expectation with IDI Options

This section extracts different market expectations from different subsets of options. Carlson, Craig and Melick (2005) used options on federal fund futures to obtain market expectations for possible outcomes of Federal Open Market Committee (FOMC) meetings in the U.S., i.e., the market consensus view of monetary policy pathways. Sihvonen and Vähämaa (2014), using data from the U.K. (3-month sterling LIBOR-referenced futures options), examined whether expectations implied by options prices were consistent with Taylor-type monetary policy rules. The authors found that interest rate expectations implied by options prices are sensitive to the expected output gap and economic uncertainty.

To evaluate if market expectations for interest rates differ for alternative subsets of options, such as ATM and OTM put options, I construct a measure of market expectations using the IDI option. This instrument has the most liquidity in the interest rate portfolio of options traded in the B3. Eq. (26) illustrates the optimization problem to find market expectations:

$$\min_{r^* \in \Re^+} \quad \hat{G}(r^*) = \sum_{i=1}^{N^c} \left(\left[IDI_t \times \exp\left(r_t^* \left(\frac{T-t}{252} \right) \right) - K_i^c \right] \times D_{t,T}) - y_i \right) + \sum_{i=1}^{N^p} \left(\left[K_i^p - IDI_t \times \exp\left(r_t^* \left(\frac{T-t}{252} \right) \right) \right] \times D_{t,T}) - y_i \right), \quad (26)$$

where N^c and N^p are the number of call and put options, respectively, traded on the day t and with maturity in T. IDI_t is the interbank deposit rate index on date t, K^c is the strike of the call, and K^p is the strike of the put. Segregation between call and put options is necessary because the payoffs are different. Moreover:

$$D_{t,T} = \frac{1}{\exp\left(r_t^* \frac{T-t}{252} \right)}$$

For each sample day and maturity, I minimize the expression in Eq. (26) to find the expected interest rate for each tenor. In sequence, I use a cubic spline for each traded date to interpolate the expected rates.

In my estimation of the market expectations, I regard the probability measure as the physical measure, given that the stochastic discount factor (SDF) as used in Eq. (26) ($D_{t,T}$) is not the risk-free rate. Instead, I construct the SDF with the option-implied interest rate.

To evaluate the forecasting performance of the implied market expectations regarding interest rates based on ATM (call and put), OTM (only put), and the futures market data, I apply ordinary least squares (OLS) regression as follows,

$$y_{t,t+\tau} = \beta_0 + \beta_1 x_{t,t+\tau} + \epsilon_{t,t+\tau}, \quad (27)$$

where $y_{t,t+\tau}$ is the realized (ex-post) DI average rate between t and $t+\tau$, and $x_{t,t+\tau}$ is the implied market DI rate expectation (ex-ante) for the same period. The expectation contained in $x_{t,t+\tau}$ were formed through the IDI options and futures market rates. The error term has the following specification,

$$E_t(\epsilon_{t,t+\tau}) = 0 \quad (28)$$

Table 6 presents the results of the regression represented by Eq(27), where τ is 200, 300, 400, 500, 700, and 900 business day (bd). The t-statistic (t(NW)) was obtained with the parameters' standard errors heteroskedasticity and autocorrelation corrected (HAC), following Newey, West et al. (1987). I use a lag length equal to 2h (to account for the overlap), following the approach of Bollerslev, Todorov and Xu (2015).

Table 6 shows that all coefficients of the market expectation are statistically significant at the 1% level. Panel A and B of Table 6 show that the prediction for the short-term DI rate (200, 300, and 400 bd) was highly accurate. Using the expectations provided by OTM put, ATM, and futures market, the R^2 were above 0.8 for all regressions. However, as the time horizon increase, the statistical significance diminishes. For example, the prediction for 500 bd had smaller t(NW), especially for the OTM put options: 4.11, and the R^2 was 0.65. Panel C of Table 6 shows the estimation results for 700 and 900 bd. Panel C revealed that the DI futures market had higher forecast power than ATM options for long-term maturity, considering the R^2 and the T(NW) statistics.[12]

The forecasting power can also be inferred from the coefficient β_1 value. When this coefficient is close to one, its prediction power is high. However, for 900 bd, the coefficient is well below one (0.8813). Table 6 shows that β_1 distance from one when the time horizon increase. Consequently, the DI average rate in the DI futures market was above the real DI rate for the sample period.

Figure 3 shows the fitted values (model predictions) of the OLS regression using the model of Eq.(27) and the parameters contained in Table 6. The *ex-ante* variable is the expectation formed through the DI future market. The confidence interval of 95% practically captured the realized DI rates in the two subplots of Figure 3, the left considering

[12]The sample has not OTM put options with 700 and 900 bd until maturity.

700 bd until maturity and the right considering 900 bd.[13]. Interestingly, Figure 3 shows that the expectation of the future market captured the trend of a fall in the interest rates but not it's level. The expectation stayed above the realized DI rate for all days, for 700 and 900 bd.

Low-interest rates have been a phenomenon in advanced economies for several decades (see Mehrotra and Sergeyev (2021), and Whited, Wu and Xiao (2021)). According to Blanchard (2019), since 1980, interest in U.S. government bonds has decreased steadily, and by 2021, their yield was lower than the nominal growth rate. Japan, the UK., and the EU now have a 10-year nominal interest rate below the nominal growth rate. Brazil has had a clear downward trend in interest rates since 1999 (see Baghestani and Marchon (2012)).

Swanson (2006), using U.S. data for 1989 to 2003, describes increased Federal Reserve transparency after the adoption of an inflation-targeting framework for monetary policy[14]. The same author concludes that private sector forecasting errors for interest rates, contrasting with GDP and inflation rates, have fallen considerably, suggesting an improvement in the private sector's ability to forecast interest rates. As mentioned earlier, Federal Reserve transparency is one of the reasons for this significant improvement. The growing transparency of Brazil's Central Bank can be one of the reasons for the highly accurate expectations provided by the derivative data, as seen in the regression results presented in Tables 6.

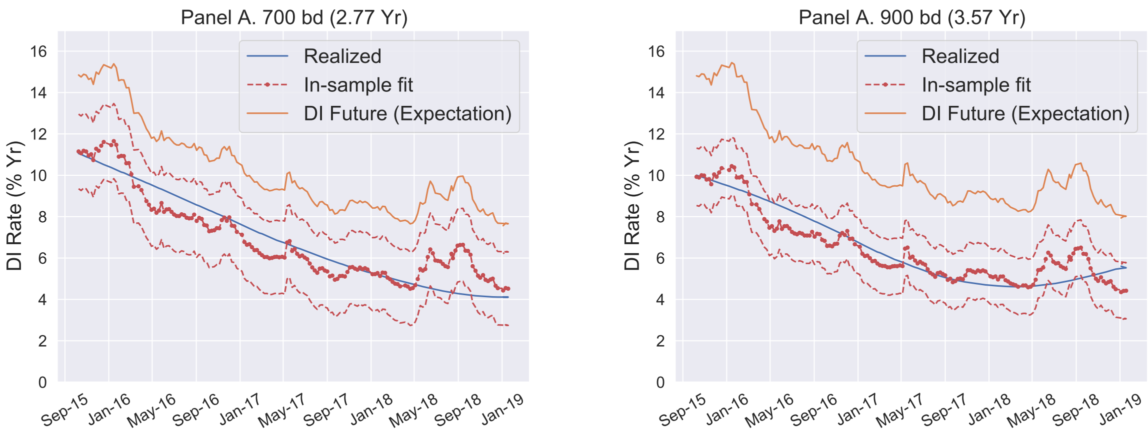

Figure 3: The realized rate (blue line) is the DI average rate computed between the trading date (t) and the tenor ($t+\tau$), is the *ex-post* rate. The fitted values of the regression using the parameters presented in Table 6 are in the red dotted points. The orange line represents the expected DI rate formed through DI future market, which is the *ex-ante* rate. The confidence interval is 95%. Data from B3.

5.1. Correlation between Volatility and Expectation

Figure 4 shows the correlation between volatility, term premia,[15], and the interest rate expectation formed through the OTM put options, ATM options, and the futures contract for 100, 200, 300, 400, and 2000 business days ahead. The high positive correlation between volatility and term premia indicates that moments of financial stress are associated with increases in the interest rate for long-term contracts.

Many authors predicted the U.S. GDP growth from the slope of the yield curve (see Laurent et al. (1988), Harvey (1989), Stock and Watson (1989), Harvey (1991), and Hamilton and Kim (2000)). A steepening curve with significant term premia can predict positive economic growth, while an inverted yield curve can predict recession (Campbell (1995), and Ang, Piazzesi and Wei (2006)). On the other hand, financial market volatility can predict recession (Baker, Bloom and Davis (2016), and Gulen and Ion (2016)). However, my findings suggest that volatility is associated with

[13] The expectation is the rate traded in the DI futures market. The realized DI rate is the real average rate between the trade date and the maturity, computed in business days.

[14] Including regularly scheduled meetings, the announcement of changes in federal fund rate targets, the rationale for policy actions, the release of votes of individual committee members and the preferred policy choices of any dissenters, and the announcement of policy tilts indicating most likely future interest rate actions

[15] Equivalent to the differences between the future contract with maturity in 2,000 business days and 100 business days.

a steep curve in Brazil since term premia rise with volatility, even considering that my sample period coincided with a monetary easing.

My results also contrast with Cieslak and Povala (2016), who found that interest rates (in the U.S.) become progressively volatile, and term premia fall, in anticipation of recessions. However, Cieslak and Povala (2016) also argue that the correlation between term premia and expectations is very volatile and can often switch signs. My findings are also consistent with Neumeyer and Perri (2005) and Horvath (2018), who argue that real interest rates in emerging economies are countercyclical and lead the business cycle.

Some factors might explain Brazil's high correlation between term premia and volatility. Brazil had one of the most extended episodes of hyperinflation in the 20th century, which only stabilized in 1994 (see Dornbusch and Cline (1997)). Therefore, long-memory investors might be worried about unanchoring inflation expectations due to significant budget deficits and weak fiscal control, possibly resulting in high-interest rates to combat inflation. On the other hand, the Central Bank's lack of credibility could explain the increases in expected long-term rates in moments of high volatility.

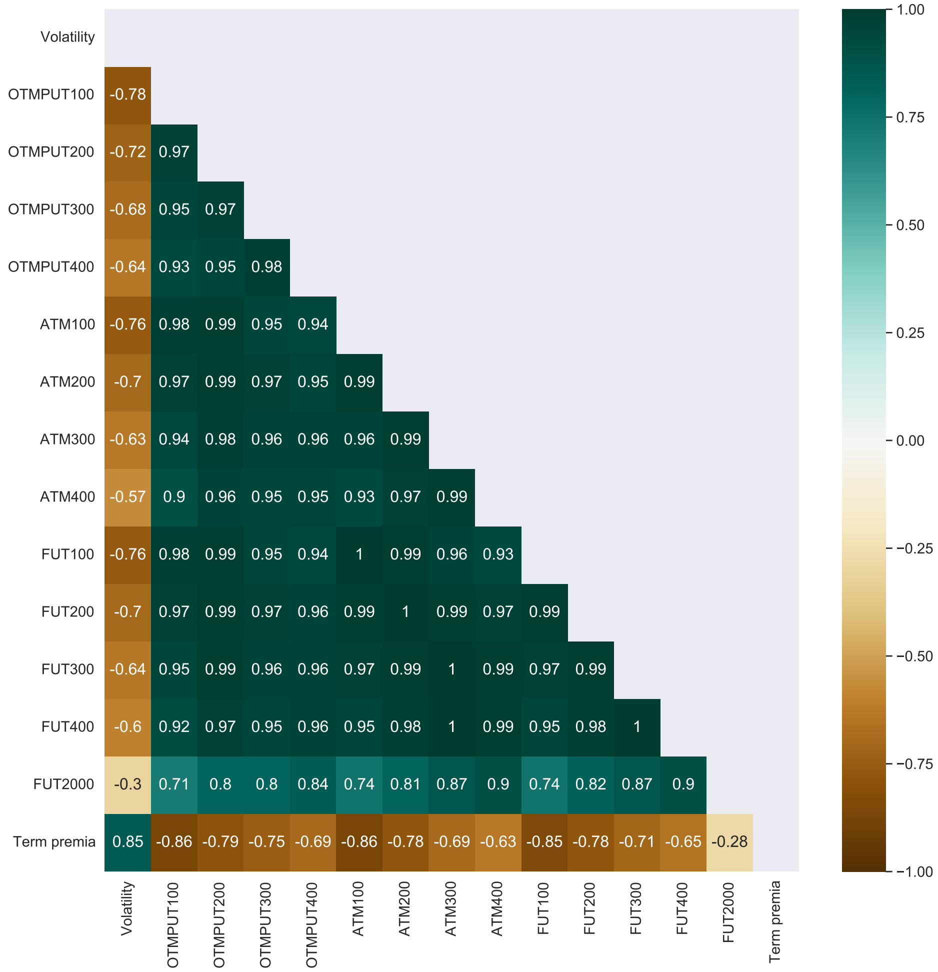

Figure 4: Correlation between volatility, term premia, and expected interest rates. "OTMPUT" expectation formed through OTM put options; "ATM" expectation formed through ATM options; "FUT" expectation formed through futures contract; and 100, 200, 300, 400, and 2000 business days.

6. Conclusions

This paper presents a new multi-factor term structure model with stochastic volatility. The new term structure model can price bonds and derivatives and forecast macroeconomic variables. To estimate the model's parameters, I show a new method derived from the gamma distribution and the distribution's moments of the historical variance of the factors that drive the interest rates. To illustrate the model's empirical application, I priced options written on DI interest rate (traded in Brazil's stock market, B3). The model with three factors had better pricing power (related to IDI options) than the model with two and one factors.

This study also shows that IDI options and DI future market can provide a robust but straightforward measure of market expectations for possible outcomes of Brazilian Central Bank target interest rates (defined in COPOM meetings; see Appendices), especially for short- and mid-term maturities (200, 300, and 400 bd). I regress the realized DI rate (*ex-post*) between time t until time $t + \tau$ on the market expectation (*ex-ante*) for the same period. However, for long-term maturities (700 and 900 bd), the market expectation captured the fall trend in interest rates, but the level was not captured. The expectation was above the realized DI rate for all sample days by a substantial amount.

The time series of expectations and volatility built in this paper showed some interesting empirical features. For example, term premia rise when volatility soars in Brazil. In contrast, the literature on term structure models shows that the first and second moments of the yield curve have a negative correlation in developed countries. That term premia fall when volatility increases. In Brazil, particular factors might explain this phenomenon, such as episodes of hyperinflation, which historically were associated with a lack of fiscal discipline and weak central bank credibility.

My theoretical and empirical contributions demonstrate that estimating market expectations, volatility, and options prices can be done within a unified framework using the *string-market* model with stochastic volatility presented in this paper. Future research can test the hypothesis that future market rates embed a risk premium for long-term maturities, justifying that the traded rates in the future market were above the realized rates for all days in my sample period.

Table 6: Regression on *ex-post* DI rate with the *ex-ante* DI rate as explanatory variable.

Pane A						
	OTM PUT	ATM	DI Future	OTM PUT	ATM	DI Future
	200 bd (0.8 Yr)	200 bd (0.8 Yr)	200 bd (0.8 Yr)	300 bd (1.19 Yr)	300 bd (1.19 Yr)	300 bd (1.19 Yr)
	Realized	Realized	Realized	Realized	Realized	Realized
Constant	-0.0090	-0.0086	-0.0096	-0.0095	-0.0093	-0.0131
Expectation	1.1639	1.0815	1.0879	1.1640	1.0274	1.0651
t(NW)	(52.72)	(42.15)	(44.84)	(32.54)	(28.23)	(38.19)
R^2	0.980	0.974	0.975	0.920	0.946	0.958
Observations	161	161	164	161	161	164
Panel B						
	OTM PUT	ATM	DI Future	OTM PUT	ATM	DI Future
	400 bd (1.58 Yr)	400 bd (1.58 Yr)	400 bd (1.58 Yr)	500 bd (2 Yr)	500 bd (2 Yr)	500 bd (2 Yr)
	Realized DI rate	Realized DI rate	Realized DI rate	Realized DI rate	Realized DI rate	Realized DI rate
Constant	-0.0096	-0.0109	-0.0158	0.0166	-0.0138	-0.0188
Expectation	1.0909	0.9752	1.0202	0.6955	0.9351	0.9766
t(NW)	(25.31)	(26.77)	(33.81)	(4.11)	(23.79)	(27.99)
R^2	0.89	0.93	0.94	0.65	0.88	0.91
Observations	161	161	164	161	161	164
Panel C						
		ATM	DI Future		ATM	DI Future
		700 bd (2.77 Yr)	700 bd (2.77 Yr)		900 bd (3.57 Yr)	900 bd (3.57 Yr)
		Realized DI rate	Realized DI rate		Realized DI rate	Realized DI rate
Constant		-0.0141	-0.0255		0.0058	-0.0209
Expectation		0.8043	0.9226		0.5337	0.8113
t(NW)		(23.14)	(21.04)		(9.99)	(21.81)
R^2		0.69	0.83		0.50	0.85
Observations		161	164		161	164

Note: This table reports estimates from time series regressions of realized interest rates, implied option market, and future market moments for the real DI rate. I show the NW t-statistics in parentheses. The statistics are generated from sample data for 9 October 2015 to 18 January 2019. The weekly data for options represent Friday closing rates. "bd" means business days (bank calendar with 252 business days per year).

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A. Interbank Deposit Rates

The SELIC target rate is the main instrument of monetary policy in Brazil. It is defined by the Brazilian Monetary Policy Committee (COPOM), as a central bank organ with macroeconomic price stabilization as its primary objective. Therefore, treasury bonds issued by the federal government, which are post-fixed like the SELIC treasury bond (*Letra Financeira do Tesouro*, LFT), follow variations in the SELIC rate. Given that this interest rate remunerates the safest asset (i.e., LFTs) in Brazil, almost all interest rates used in the economy, such as loans or banking capital, are marked by the SELIC rate.

The DI rate is the weighted average rate traded between banks in prefixed operations with a one-day maturity. The primary collateral given by financial institutions in the interbank lending market is federal securities, essentially LFTs. Consequently, the DI rate is directly related to the SELIC rate.

A.1. The DI Futures Contract

In 1991, the B3 launched the DI futures contract to meet the demand for hedging by banks and companies. The interest rate futures market is vital in transferring risk among agents. The main strategies include changing a fixed rate to a floating rate or the reverse. The underlying of the DI futures contract is the DI interest rate. This derivative is usually used to hedge and manage risk exposure on liabilities and debt positions in the Brazilian currency (R$).

The DI futures contract mechanism is similar to a swap contract (where the traded rate at the beginning of the contract is the agreed rate): one party receives a positive cash flow if the agreed rate remains above the market rate (fixed leg). In contrast, the other part receives a positive cash flow if the agreed rate remains below the market rate (float leg). One of the main differences with the swap contract is that the futures contract has daily movement on the margin account based on differences between the market and traded rates.

The DI futures contract is one of the most volatile interest rate contracts globally, attracting much speculation and offering arbitrage opportunities across the entire term structure. Some of the technical aspects of the futures contract are that the notional value at maturity is R$ 100,000.00, the market quotation is expressed as an annual percentage, compounded daily based on each year consisting of 252 business days, and the contract size is expressed in terms of unit price (UP) in R$.

Consider a continuous trading economy with a continuum of default-free bonds traded with different maturities. Assume bonds are traded at dates $t \in [0, T]$. I define $P(t, T)$ as the bond price at time $t \in [0, T]$, with a face value of R$ 1.00 at maturity T. Henceforth, $P(T, T) = 1$ and $P(t, T) > 0$ for all $t \in [0, T]$ (no default-free condition of the bonds).

The UP of the one-day interbank deposit futures contract is defined as:

$$UP(t, T) = \frac{100,000}{\left(1 + k_t\right)^{\frac{(T-t)}{252}}}, \tag{29}$$

where k_t is the agreed interest rate, and where the difference $T - t$ represents business days, i.e., the trading date t, inclusive, and the maturity date of the contract T, exclusive. As mentioned, DI futures contracts have a face value of R$ 100,000.00 at maturity, so the UP represents R$ 100,000.00 discounted by the agreed interest rate. Since the bank calendar in Brazil uses annual periods of 252 business days, the DI rate is computed only for this period each year.

Given two maturity dates, T_i and T_j, I define the forward interest rate $f(t, T_i, T_j)$ as the implicit interest rate between T_i and T_j inferred in t, obtained as follows:

$$f(t, T_i, T_j)^{\text{annual}} = \left(\frac{UP(t, T_i)}{UP(t, T_j)}\right)^{\frac{252}{T_j - T_i}} - 1, \tag{30}$$

$\forall\, T_i, T_j \in [t, T], T_i \leq T_j, t \in [0, T]$, where $UP(t, T_i)$ and $UP(t, T_j)$ are the UPs at time t of contracts with maturity at T_i and T_j, respectively.

To simplify the equations and facilitate numerical implementations of the model, I use all interest rates with continuous-time capitalization. I use the following expressions to obtain the annual forward interest rate with continuous-time capitalization:

$$f(t, T_i, T_j)^{\text{continuous}} = \log\left(1 + f(t, T_i, T_j)^{\text{annual}}\right), \tag{31}$$

$\forall\, T_i, T_j \in [t, T], T_i \leq T_j, t \in [0, T]$. With the DI rates for the diverse maturities of the DI futures contract, $f(t, T_i, T_j)$, I obtain the bond prices that pay R$ 1.00 at maturity,

$$P(t, T_j) = \frac{1}{\exp\left(f(t, t, T_j)\frac{T-t}{252}\right)}, \tag{32}$$

$\forall\, T_j \in [t, T]$, $t \in [0, T]$, where $r(t)$ is the spot rate:

$$r(t) = f(t, t, t), \quad \forall t \in [0, T]. \tag{33}$$

A.2. Black Option Pricing Model

The classical model for valuing interest rate options is the Black (1976) model, where the price is quoted in terms of implied volatility. In my framework, assuming linear interest rates between T_1 and T_2, and assuming that R^{linear} has a log-normal distribution, the pricing model for the option on DI futures is as follows (see Santos and Silva (2017)):

$$C_t = D^{DI}_{t,T_2} \frac{100,000}{\exp\left(k\left(\frac{T_2-T_1}{252}\right)\right)} \frac{(T_2-T_1)}{252}\left(R^{\text{linear}}_{T_1,T_2} N(d1) - K^{\text{linear}}_{T_1,T_2} N(d2)\right), \tag{34}$$

and

$$P_t = D^{DI}_{t,T_2} \frac{100,000}{\exp\left(k\left(\frac{T_2-T_1}{252}\right)\right)} \frac{(T_2-T_1)}{252}\left(K^{\text{linear}}_{T_1,T_2} N(-d2) - R^{\text{linear}}_{T_1,T_2} N(-d1)\right), \tag{35}$$

with

$$d1 = \frac{\ln\left(\frac{R^{\text{linear}}_{T_1,T_2}}{K^{\text{linear}}_{T_1,T_2}}\right) + 0.5\sigma^2\left(\frac{T_1-t}{252}\right)}{\sigma\sqrt{\left(\frac{T_1-t}{252}\right)}},$$

$$d2 = d1 - \sigma\sqrt{\left(\frac{T_1-t}{252}\right)},$$

where C_t and P_t are the call and put option prices, respectively, D^{CDI}_{t,T_2} is the discount factor between the date t and T_2, based on the curve of the DI future,

k is the option strike defined in rate, σ is the annual volatility, $T_2 - T_1$ is the contract maturity considering the bank calendar, $N(.)$ is the cumulative normal distribution function, and $K^{\text{linear}}_{T_1,T_2} = \left(\exp\left(\left(\log(1+k)\frac{(T_2-T_1)}{252}\right)\right) - 1\right) \times \frac{(252)}{T_2-T_1}$ and $R^{\text{linear}}_{T_1,T_2}$ are the linear interest rates between T_1 and T_2.

To estimate $R^{\text{linear}}_{T_1,T_2}$, I need to adjust the bond price present value to T_1:

$$P(T_1,T_2) = P(t,T_2)\exp\left(f(t,t,T_1)\left(\frac{T_2-T_1}{252}\right)\right) \tag{36}$$

Therefore, the implicit interest rate with continuous capitalization between T_1 and T_2 is:

$$r^{\text{continuous}}_{T1,T_2} = \log\left(\frac{1}{P(T_1,T_2)}\right)\frac{252}{(T_2-T_1)}, \tag{37}$$

and the corresponding linear interest rate is:

$$R^{\text{linear}}_{T_1,T_2} = \left(\exp\left(r^{\text{continuous}}_{T1,T_2}\left(\frac{T_2-T_1}{252}\right)\right) - 1\right)\frac{252}{T_2-T_1}. \tag{38}$$

For the option on IDI, the Black (1976) pricing model expression is as follows:

$$C_t = [IDI_T N(d1) - KN(d2)]D^{DI}_{t,T_2}, \tag{39}$$

and

$$P_t = [KN(-d2) - IDI_T N(-d1)]D^{DI}_{t,T_2}, \tag{40}$$

with

$$d1 = \frac{\ln\left(\frac{IDI_T}{K}\right) + 0.5\sigma^2\left(\frac{T_1-t}{252}\right)}{\sigma\sqrt{\left(\frac{T_1-t}{252}\right)}}, \tag{41}$$

$$d2 = d1 - \sigma\sqrt{\left(\frac{T_1-t}{252}\right)}. \tag{42}$$

IDI_T is the DI index updated to date T with the interest rate obtained in the futures market.