Binary Shifts in Systematic Risks

Márcio Estevão Miranda Borges* Alex Luiz Ferreira†

April 6, 2025

Abstract

This paper investigates how shifts in risk parameters influence asset prices, focusing on binary changes in investor beliefs akin to the Peso Problem. We develop a theoretical framework for stochastic discount factors (SDF) under binary regime changes and use the Fama/French European Three Factors dataset (daily data from July 1990 to January 2025) to identify distinct risk regimes empirically. Applying a Gaussian mixture model (GMM), we find that abrupt changes, such as financial crises, lead to significant shifts in risk pricing, indicating that systematic risk is not uniformly priced over time.

Key Words: Regime Shifts, Risk Premia, Asset Pricing.
JEL Codes: G12, F31, D84.

*Universidade de São Paulo, Departamento de Economia FEA-RP, Ribeirão Preto, 14040-900, Brazil. E-mail: marcio_borges@usp.br.

†Universidade de São Paulo, Departamento de Economia FEA-RP, Ribeirão Preto, 14040-900, Brazil. E-mail: alexferreira@usp.br.

1 Introduction

Because asset prices result from forward-looking behaviour, they reflect investors' expectations about future economic conditions and the likelihood of events influencing them. Most economic models assume a single probability distribution for future states of the world. The seminal Arrow-Debreu model, for instance, assigns a unique probability to each distinct payoff, which is assumed to be known and agreed upon by all agents. This framework implies that market participants form expectations based on a single set of possible outcomes and their respective probabilities, which are incorporated into asset prices via the stochastic discount factor.

The Peso Problem, introduced by Rogoff (1977) and Rogoff (1980), describes situations in which anticipated shifts in the probability distribution of economic fundamentals affect asset prices (see, for instance, Evans, 1996). Some authors, such as Barro and Liao (2021), interpret it as the influence of rare disasters or extreme events with disproportionate payoffs.[1] Venturi et al. (2023), for instance, adopts the former view and refers to cases in which agents assign distinct binary probabilities to each potential outcome in every state of nature as "binary uncertainty". This paper adopts the latter interpretation.

Shifts in the probability distribution of fundamentals should be reflected in asset prices. Investors adjust excess demand based on their perception of these probabilities, effectively embedding them into market values (see, for instance, Almeida and Freire, 2022). Certain binary events can substantially reshape these distributions and significantly influence market dynamics. Examples include tight elections (especially second rounds or in bipartisan systems), central bank decisions on when to begin a new cycle of interest rate hikes or cuts — such as those involving the Federal Reserve or the European Central Bank — political or economic referendums, and crucial legislative votes. This study examines how systematic risk exposure evolves during such episodes of market uncertainty.

The related literature has explored the idea that risk perceptions may shift across distinct regimes. Several studies, including Burnside et al. (2011), Ferreira et al. (2022) and Venturi et al. (2023) examine asset price variations driven by changes in stochastic discount factors under shifting "Peso" probability regimes. This paper contributes to that literature by addressing a different question: how do anticipated shifts — particularly binary ones — affect systematic risk exposure? More specifically, given that portfolio betas aggregate linearly under standard asset pricing models,

[1]Explicitly or implicitly referring to the Peso Problem, many studies — including Backus et al. (2011), Gao et al. (2019), and Cortes et al. (2022) — examine the effects of rare disasters on asset prices.

it investigates how changes in market expectations alter the pricing of systematic risk — that is, how they affect risk premia and expected returns — and, consequently, the dynamics of asset prices.

Understanding how changes in probability distributions affect systematic risk is essential for the financial sector. It allows practitioners to anticipate fluctuations in excess returns and manage the uncertainty surrounding asset price volatility. Shifts in risk perceptions can also be both a response to anticipated events and a driver of financial cycles. Hence, understanding these effects enables policymakers to make more informed decisions, essential to economic and financial stability.

This paper examines how changes in risk parameters influence asset pricing, particularly when binary shifts in investor beliefs occur, similar to the Peso Problem. To address this question, we develop a theoretical model based on the Arrow-Debreu framework, capturing the dynamics of stochastic discount factors under regime shifts. For empirical implementation, we use the Fama/French European Three Factors dataset, covering daily data from July 1990 to January 2025. We apply a Gaussian mixture model (GMM) to identify distinct risk regimes for the HML and SMB factors, revealing that abrupt and often unanticipated changes, such as financial crises, can lead to significant shifts in risk pricing. These results suggest that systematic risk exposure evolves dynamically, particularly during periods of heightened economic uncertainty.

The rest of the paper is structured as follows: Section 2 develops a theoretical framework within a general equilibrium model incorporating foreign exchange markets to illustrate how changes in risk parameter distributions affect asset prices and excess returns. Section 3 outlines the empirical methodology. We start with estimating the systematic risk parameters and then specifying a mixture model to capture the dynamics of risk sub-distributions. Section 4 presents the results. Finally, Section 5 provides concluding remarks, summarising the study's key findings and their broader implications.

2 The Model

This section introduces a model that builds on the Consumption Capital Asset Pricing Model (CCAPM) framework to examine the interaction between risky assets and exchange rates internationally. Our approach is motivated by the objective of developing a more general version of the CCAPM that incorporates equity returns and exchange rate dynamics, capturing the broader structure of international risk premia. This extension is consistent with the concept of uncovered equity parity (UEP) as proposed by Hau and Rey (2006). Unlike the uncovered interest parity (UIP), which focuses solely on bond returns, the UEP generalises the relationship by including equity returns and exchange rate movements. Therefore, it is essential to adequately model risky assets in both the H and F economies alongside the exchange rate to capture the dynamics of cross-border risk premia.

To this end, we present a simple Arrow–Debreu model that demonstrates how stochastic discount factors are influenced by the probability of anticipated binary events, as in the Peso Problem. These changes affect asset prices, including equities, exchange rates, bond yields, and associated excess returns and risks.

We begin by outlining the basic structure of the market in a world economy with two representative agents: a domestic agent, H, and a foreign agent, F. For empirical implementation later, the H agent is assumed to be a US resident, while the F agent resides in the Euro area. Both agents live for two periods, t and $t+1$. The world output in period t, denoted by y_t^W, consists of two perishable, exogenously given, country-specific homogeneous goods, and is defined as:

$$y_t^W = y_t + Q_t\, y_t^*,$$

where y_t and y_t^* are the domestic and foreign real endowments, respectively, and Q_t is the real exchange rate (i.e., the price of the foreign good in domestic units). We assume that $y_t,\, y_t^* \in \mathbb{R}_{++}$.

In period t, each agent receives its respective endowment. Uncertainty arises solely from the realization of future states. In period $t+1$, if state $j \in \{1, 2, \ldots, J\}$ occurs, the domestic and foreign agents obtain state-contingent endowments[2] $y_{t+1}(j)$ and $y_{t+1}^*(j)$, respectively, where $y_{t+1}(j),\, y_{t+1}^*(j) \in \mathbb{R}_{++}$, and the corresponding world output in state j is given by

$$y_{t+1}^W(j) = y_{t+1}(j) + Q_{t+1}\, y_{t+1}^*(j).$$

In what follows, we focus on the H agent, as the model is symmetric, and the same applies to the F agent.

[2]We focus on real uncertainty arising from endowment shocks, abstracting from investment and production decisions. While one could imagine a setting in which technological shocks affect the production process — viewing technology as part of the productive capacity that determines endowments — we deliberately leave out such mechanisms and any monetary considerations. This allows us to isolate the impact of the Peso Problem on the stochastic discount factor and maintain a simple, purely real model. Naturally, a more comprehensive framework could incorporate both real and monetary shocks.

Preferences. The period utility function $u : \mathbb{R}^2_{++} \to \mathbb{R}$, is strictly increasing and strictly concave in both arguments. Here, c_t and c_t^* denote domestic and foreign consumption in period t, respectively, and $c_{t+1}(j)$ and $c_{t+1}^*(j)$ represent consumption in period $t+1$, contingent on state j. Lifetime utility is assumed to be additively separable over time and is given by

$$U = u\Big(c_t, c_t^*\Big) + \beta \sum_{j=1}^{J} \pi(j)\, u\Big(c_{t+1}(j), c_{t+1}^*(j)\Big),$$

where $\beta \in (0, 1)$ is the discount factor, and $\{\pi(j)\}_{j=1}^J$ represents the probability distribution over future states, satisfying

$$\pi(j) \geq 0, \quad \forall j \in \{1, \dots, J\}, \quad \text{and} \quad \sum_{j=1}^{J} \pi(j) = 1.$$

Budget Constraints. The domestic agent chooses current consumption, represented by the bundle (c_t, c_t^*) (with c_t measured in domestic units and c_t^* in foreign units), and finances state-contingent future consumption by purchasing Arrow–Debreu (AD) securities. Let Q_t denote the real exchange rate (the price of the foreign good in domestic units), so that the current consumption bundle expressed in domestic units is $c_t + Q_t\, c_t^*$.

Let $p_t(j)$ denote the price at time t (in domestic consumption units) of an AD security that pays one unit of domestic currency in state j. Denote by $b(j)$ the net quantity of AD securities purchased for state j. The domestic agent's saving decision must then satisfy

$$\sum_{j=1}^{J} p_t(j)\, b(j) \;\; \leq \;\; y_t \; - \; \Big(c_t + Q_t\, c_t^*\Big). \tag{1}$$

In each state j at time $t+1$, the agent's resources (in domestic units) consist of the state-contingent endowment $y_{t+1}(j)$ plus any returns from the AD securities, $b(j)$. These resources are allocated between the consumption of domestic and foreign goods. When consumption of foreign goods is converted into domestic units using the state-contingent real exchange rate $Q_{t+1}(j)$, the resource constraint in state j is

$$c_{t+1}(j) + Q_{t+1}(j)\, c_{t+1}^*(j) \;\; \leq \;\; y_{t+1}(j) \; + \; b(j), \quad \forall\, j \in \{1, \dots, J\}. \tag{2}$$

Under the usual assumption of strictly increasing utility, the inequalities (1) and (2) bind at the optimum. Consequently, substituting these conditions into the intertemporal framework yields, in equilibrium, the consolidated budget constraint expressed in domestic consumption units:

$$c_t + Q_t\, c_t^* \;+\; \sum_{j=1}^{J} p_t(j) \Big[c_{t+1}(j) + Q_{t+1}(j)\, c_{t+1}^*(j) \Big] \;=\; y_t \;+\; \sum_{j=1}^{J} p_t(j)\, y_{t+1}(j). \tag{3}$$

Optimization Problem and First-Order Conditions. The domestic agent maximizes expected lifetime utility by choosing her current consumption bundle (c_t, c_t^*) and state-contingent future consumption $\{(c_{t+1}(j), c_{t+1}^*(j))\}_{j=1}^J$, subject to the consolidated intertemporal budget constraint, (3). Necessary first-order conditions (FOCs) imply

$$\frac{u_2(c_t, c_t^*)}{u_1(c_t, c_t^*)} = Q_t, \tag{4}$$

so that the marginal rate of substitution between foreign and domestic consumption equals the real exchange rate. In addition, FOCs yields, for each state j,

$$p_t(j) = \beta\, \pi(j)\, \frac{u_1(c_{t+1}(j), c_{t+1}^*(j))}{u_1(c_t, c_t^*)}, \tag{5}$$

and

$$p_t(j)\, Q_{t+1}(j) = \beta\, \pi(j)\, \frac{Q_t\, u_2(c_{t+1}(j), c_{t+1}^*(j))}{u_2(c_t, c_t^*)}, \quad \forall\, j. \tag{6}$$

Together, equations (5) and (6) characterise the intertemporal allocation for the domestic agent. By symmetry, the foreign agent's optimality conditions take analogous forms. Denote her current consumption bundle by (c_t^F, c_t^{F*}) and her state-contingent future consumption by $\{(c_{t+1}^F(j), c_{t+1}^{F*}(j))\}_{j=1}^J$. Her intratemporal condition is

$$\frac{u_2(c_t^F, c_t^{F*})}{u_1(c_t^F, c_t^{F*})} = \frac{1}{Q_t},$$

and her intertemporal Euler equations are, for each j,

$$p_t^*(j) = \beta\, \pi(j)\, \frac{u_1(c_{t+1}^F(j), c_{t+1}^{F*}(j))}{u_1(c_t^F, c_t^{F*})},$$

$$p_t^*(j)\, Q_{t+1}^{-1}(j) = \beta\, \pi(j)\, \frac{Q_t^{-1}\, u_2(c_{t+1}^F(j), c_{t+1}^{F*}(j))}{u_2(c_t^F, c_t^{F*})}, \quad \forall\, j.$$

These conditions for the foreign agent mirror those of the domestic agent given in (4)–(6).

The law of one price ensures the absence of arbitrage, imposing the condition $p_t(j) = Q_t\, p_t^*(j)$ for any state j.

Market clearing requires that total consumption equals total output in each period. In period t, this condition is $c_t + c_t^* + c_t^F + c_t^{F*} = y_t + y_t^*$.

In each state j at $t+1$, market clearing implies

$$c_{t+1}(j) + c_{t+1}^*(j) + c_{t+1}^F(j) + c_{t+1}^{F*}(j) = y_{t+1}(j) + y_{t+1}^*(j).$$

The foreign agent satisfies a similar budget constraint,

$$c_t^F + Q_t^{-1} c_t^{F*} + \sum_{j=1}^{J} p_t^*(j) \Big[c_{t+1}^F(j) + Q_{t+1}^{-1}(j) c_{t+1}^{F*}(j) \Big] = y_t^* + \sum_{j=1}^{J} p_t^*(j) y_{t+1}^*(j).$$

Together, these conditions determine the equilibrium allocation of consumption, the pricing of contingent claims, and the real exchange rate dynamics.

Assets and Bonds Pricing. Asset pricing stems from each agent's stochastic discount factor (SDF) in this setting. For the domestic agent, the first-order conditions imply that the real price of an Arrow–Debreu security paying one unit of domestic consumption in state j is

$$p(j) = \beta \pi(j) \frac{u'\big(c_{t+1}(j)\big)}{u'(c_t)}.$$

Define the domestic state-specific SDF as

$$M_{t+1}(j) \equiv \beta \, \frac{u'\big(c_{t+1}(j)\big)}{u'(c_t)},$$

so that

$$p(j) = \pi(j) \, M_{t+1}(j). \tag{7}$$

Definition 1 (Risky Asset)**.** *Asset i delivers a payoff $x_i(j)$ in each future state $j \in \{1, \ldots, J\}$. It is classified as* risky *if its payoff varies across states; that is, there exist at least two states $j_1 \neq j_2$ such that $x_i(j_1) \neq x_i(j_2)$.*

Total Asset Price. Let p_i denote the current total price of acquiring all future payoffs $\{x_i(j)\}_{j=1}^{J}$. Since Arrow–Debreu securities are priced at $p(j)$ per unit of payoff in state j, the cost of obtaining these payoffs is

$$p_i \; = \; \sum_{j=1}^{J} p(j) \, x_i(j). \tag{8}$$

We define the gross return on this asset in state j as

$$\tilde{R}^i_{t+1}(j) \;\equiv\; \frac{x_i(j)}{p_i}.$$

Substituting $x_i(j) = p_i\,\tilde{R}^i_{t+1}(j)$ into (8) and applying the expectation operator to this relation yields the standard asset pricing equation:

$$\mathbb{E}_t\Big[M_{t+1}\,\tilde{R}^i_{t+1}\Big] \;=\; 1. \tag{9}$$

Domestic Risk-Free Bond. Consider a domestic risk-free bond that pays exactly 1 unit of domestic consumption at time $t+1$, regardless of the state. Let P^B_t denote its price at time t. By construction, this constant payoff can be replicated via a portfolio of Arrow–Debreu securities. Hence, no-arbitrage implies:

$$P^B_t \;=\; \sum_{j=1}^{J} p(j),$$

where $p(j)$ is the Arrow–Debreu price of one unit of domestic consumption in state j. From (7), we know:

$$P^B_t \;=\; \sum_{j=1}^{J} \pi(j)\,M_{t+1}(j). \tag{10}$$

If the bond pays one unit in every state, its gross return is $R_{t+1} \equiv \frac{1}{P^B_t}$. Combining this definition with (10) yields

$$\mathbb{E}_t\Big[M_{t+1}\,R_{t+1}\Big] \;=\; 1. \tag{11}$$

Since R_{t+1} is risk-free, we have $\mathbb{E}_t\Big[M_{t+1}\Big] \;=\; \frac{1}{R_{t+1}} = \frac{1}{1+r_{t+1}}$, where r_{t+1} is the net risk free return in the H economy.

Excess Return and the SDF. Let r^i_{t+1} denote the net return of asset i in the H economy. From the perspective of the F agent, the gross return on this asset, when converted into F goods, is given by:

$$\tilde{R}^i_{t+1}(j) = (1 + r^i_{t+1}(j))\frac{Q_t}{Q_{t+1}(j)}.$$

Or, equivalently,

$$\tilde{R}^{i}_{t+1}(j) = \frac{(1 + r^{i}_{t+1}(j))}{(1 + \Delta q_{t+1}(j))}, \tag{12}$$

where $\Delta q_{t+1}(j) \equiv \frac{\Delta Q_{t+1}(j)}{Q_t}$. The excess return on the risky i asset in H, relative to the F risk-free bond, is then

$$\tilde{R}^{e,i}_{t+1}(j) \equiv \tilde{R}^{i}_{t+1}(j) - R^{*}_{t+1}, \tag{13}$$

which can also be written as:

$$\tilde{R}^{e,i}_{t+1}(j) = \frac{(1 + r^{i}_{t+1}(j))}{(1 + \Delta q_{t+1}(j))} - \left(1 + r^{*}_{t+1}\right),$$

or, approximately, $\tilde{R}^{e,i}_{t+1}(j) \approx r^{i}_{t+1}(j) - r^{*}_{t+1} - \Delta q_{t+1}(j)$. From the general pricing condition (9) and (10), it follows that[3]:

$$\mathbb{E}_t\left[M_{t+1}\, \tilde{R}^{e,i}_{t+1}\right] = 0, \tag{14}$$

which implies:

$$\mathbb{E}_t\left[\tilde{R}^{e,i}_{t+1}\right] = -\frac{\mathrm{Cov}_t\left(M_{t+1},\, \tilde{R}^{e,i}_{t+1}\right)}{\mathbb{E}_t\left[M_{t+1}\right]}. \tag{15}$$

Since the stochastic discount factor M_{t+1} reflects agents' intertemporal marginal rates of substitution, a negative covariance between $\tilde{R}^{e,i}_{t+1}$ and M_{t+1} raises the expected return required to hold the H asset. Thus, all risky assets with state-dependent payoffs are priced relative to the risk-free benchmark (11) via the stochastic discount factor.

Binary Probability Structure. We assume that the probability of state j follows a mixture model,

$$\pi_t(j) = (1 - P_t)\phi_N(j) + P_t\phi_P(j),$$

where $\phi_N(j)$ represents the probability mass function under the normal regime, $\phi_P(j)$ represents the probability mass function under the "Peso" regime, and $P_t \in [0, 1]$ denotes the probability weight assigned to the "Peso"

[3]The state index j vanishes because we aggregate over all states when computing the expectation, resulting in a pricing relation that applies to the entire excess return rather than individual states.

regime in the mixture distribution. Characterized by a different probability distribution, the “Peso” regime affects expectations accordingly. Under this probability structure, the expectation of the stochastic discount factor is given by

$$\mathbb{E}_t[M_{t+1}] = (1 - P_t)\mathbb{E}_t^N[M_{t+1}] + P_t\mathbb{E}_t^P[M_{t+1}],$$

where

$$\mathbb{E}_t^N[M_{t+1}] = \sum_{j=1}^{J} \phi_N(j)M_{t+1}(j), \quad \mathbb{E}_t^P[M_{t+1}] = \sum_{j=1}^{J} \phi_P(j)M_{t+1}(j).$$

Risk Decomposition. Using (15), we can write

$$\mathrm{Cov}_t\Big(M_{t+1},\, \tilde{R}^{e,i}_{t+1}\Big) = -\Big\{(1 - P_t)\mathbb{E}_t^N[M_{t+1}] + P_t\mathbb{E}_t^P[M_{t+1}]\Big\}\,\mathbb{E}_t[\tilde{R}^{e,i}_{t+1}].$$

Dividing both sides by $Var(M_{t+1})$ yields

$$\beta_{i,t} = -\frac{\Big\{(1 - P_t)\mathbb{E}_t^N[M_{t+1}] + P_t\mathbb{E}_t^P[M_{t+1}]\Big\}\,\mathbb{E}_t[\tilde{R}^{e,i}_{t+1}]}{Var(M_{t+1})}.$$

where $\beta_{i,t} = -\frac{\mathrm{Cov}_t\left(M_{t+1},\tilde{R}^{e,i}_{t+1}\right)}{Var(M_{t+1})}$ is the risk parameter.

3 Empirical Implementation

To capture the idea that agents may assign different probabilities to alternative regimes, we assume that the unobserved risk parameter $\beta_{i,t}$ follows a mixture of two normal distributions. This specification reflects the possibility that market participants anticipate a binary shift in risk conditions — for instance, due to expected policy decisions or institutional events. Formally, its distribution is given by:

$$f(\beta_{i,t};\theta) = (1 - P_t)\,\phi\Big(\beta_{i,t};\mu_N,\sigma_N^2\Big) + P_t\,\phi\Big(\beta_{i,t};\mu_P,\sigma_P^2\Big),$$

where $\phi(\cdot,;\mu,\sigma^2)$ denotes the normal density function with mean μ and variance σ^2. The time-varying parameter P_t captures the probability of being in the Peso regime and serves as an empirical analogue to the shift in beliefs discussed in the theoretical model. The parameter vector $\theta = (\mu_N, \sigma_N^2, \mu_P, \sigma_P^2, P_t)$ is estimated by maximum likelihood, via the Expectation-Maximization (EM) algorithm, using observed estimates $\hat{\beta}_{i,t}$ as inputs.

This approach assigns to each observation a probability of being drawn from either regime. When P_t is low, the data are more consistent with the normal regime. As P_t increases, the likelihood of a shift in expectations rises. This represents a tractable empirical method to detect and interpret regime-dependent deviations from the excess return pricing condition in equation (14).

The Linear Factor Model. We first estimate a time series of risk parameters to apply the mixture model. For this purpose, we use the "Fama/French European Three Factors" dataset, which includes the excess return (defined as the return on a European value-weighted market portfolio minus the United States one-month Treasury bill rate), the HML_t factor (the return spread between portfolios of high and low book-to-market stocks), and the SMB_t factor (the return spread between portfolios of small and large stocks).

We adopt the Fama/French European three-factor model as our baseline specification because it captures the main sources of systematic risk in equity markets: market, size, and value effects. The model is widely used in empirical asset pricing, offering a parsimonious framework that allows for estimating time-varying risk parameters in international markets. Its simplicity and interpretability make it especially well-suited for identifying shifts in systematic risk regimes.

Within this framework, we specify the following linear model for U.S. real excess returns, $\tilde{R}_t^{e,i}$, from the perspective of a domestic investor holding European stocks:

$$\tilde{R}_t^{e,i} = \alpha_t + \beta_{hml,t} \cdot HML_t + \beta_{smb,t} \cdot SMB_t + \epsilon_t, \tag{16}$$

where α_t is the intercept, $\beta_{hml,t}$ and $\beta_{smb,t}$ are the time-varying risk parameters associated with the value and size factors, respectively, and ϵ_t, is an idiosyncratic error term with variance σ_n^2.

The Mixture Model Specification. We thus model the risk parameters $\beta_{hml,t}$ and $\beta_{smb,t}$ as draws from a two-component Gaussian mixture. At each point in time, an observation is assumed to originate from one of two normal distributions: the "Peso regime" with probability P_t, or the "normal regime" with probability $1 - P_t$. After estimating the sub-distributions, it is essential to determine which component corresponds to the Peso regime and to identify the time periods in which the transition between regimes occurs. Because the parameters of the sub-distributions are unknown *ex ante*, they are estimated from the data using the following general specification for each risk parameter:

$$f(\beta_i;\theta) = P_t \cdot \mathcal{N}(\beta_i;\mu_1,\sigma_1^2) + (1-P_t) \cdot \mathcal{N}(\beta_i;\mu_2,\sigma_2^2), \qquad i = \{hml; smb\}, \tag{17}$$

where $f(\beta_i;\theta)$ represents the density of β_i with parameter θ, β_i denotes the variable to be modelled by the mixture model and $\mathcal{N}(\beta_i;\mu,\sigma^2)$ is the normal distribution with mean μ and variance σ^2. As previously mentioned, we apply the mixture model to the parameters risk: $\beta_{hml,t}$ and $\beta_{smb,t}$.

Expectation-Maximization (EM) algorithm. To estimate the mixture model parameters, $\theta = (P, \mu_1, \sigma_1^2, \mu_2, \sigma_2^2)$, we maximise the following likelihood function, using the EM algorithm:

$$\mathcal{L}(\theta) = \prod_{t=1}^{\infty} [P_t \cdot \mathcal{N}(\beta_i;\mu_1,\sigma_1^2) + (1 - P_t) \cdot \mathcal{N}(\beta_i;\mu_2,\sigma_2^2)].$$

After starting with an initial guess for the parameters, the EM algorithm alternates between two steps. In the Expectation Step (E-Step), the expected value of the log-likelihood function is computed, given the observed data and the current parameter estimates. This step incorporates the latent variable structure of the model. In the Maximisation Step (M-Step), the parameter estimates are updated by maximising the expected log-likelihood obtained in the E-Step. These steps are repeated iteratively until convergence, which is typically determined when changes in parameter estimates fall below a predefined threshold or when the likelihood function stabilises.

The next section reports the empirical results. We begin by describing the dataset used to estimate the time series of risk parameters $\beta_{hml,t}$ and $\beta_{smb,t}$, based on rolling regressions. We then present the results from applying the Gaussian mixture model to these series, along with the corresponding descriptive statistics and interpretation.

4 Results

This section provides an overview of the data and presents the main empirical findings. We start by describing the construction of the dataset, including sources and sample characteristics. We then estimate the time-varying risk parameters using rolling regressions and analyse their dynamics. Finally, we report the results of the mixture model applied to these parameters, highlighting the temporal evolution of regime shifts in systematic risk.

The Fama-French Dataset. As mentioned earlier, the data on nominal Treasury bill rates, excess returns, and the HML and SMB factors are drawn from the "Fama/French European Three Factors" dataset. These variables were constructed using European stock return data from Bloomberg. All returns are denominated in U.S. dollars and include both dividends and capital gains. Daily excess returns are computed as the return on a European value-weighted market portfolio minus the one-month U.S. Treasury bill rate.

The European diversified portfolios were constructed using stock data from 16 developed European markets: Austria, Belgium, Denmark, Finland, France, Germany, Greece, Ireland, Italy, the Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, and the United Kingdom. Stocks were sorted based on size (market capitalisation) and the ratio of book equity to market equity.

The construction of the HML and SMB factors involved sorting European stocks into two market capitalisation groups: large stocks (the top 90% of June market capitalisation) and small stocks (the bottom 10%). Three book-to-market (B/M) equity groups were then established at the end of each June using breakpoints at the 30th and 70th percentiles of B/M for the large stocks.

The sample period extends from July 1990 to January 2025. Descriptive statistics for the key variables are presented in Table 1.

As can be visually inferred from Figure 1, the excess return and factor series display stationary behaviour, fluctuating around near-zero means throughout the sample. However, episodes of heightened volatility are visible and tend to coincide with major global events. All three series—excess returns, HML, and SMB—exhibit marked spikes during economic or financial distress periods, such as the early 2000s recession, the 2008 global financial crisis, and the COVID-19 pandemic. These periods are highlighted in the graphs by the shadows representing NBER-dated recessions. Furthermore, in contrast to the results of Fama and French (2012), value premiums are

Table 1: **Descriptive Statistics of Returns and Factors Series.**

Statistic	Mean	St. Dev.	Min	Max
Excess Return: $\tilde{R}_t^{e,i}$	0.02	1.13	-12.00	10.72
SMB	-0.005	0.57	-5.18	3.30
HML	0.01	0.47	-4.14	4.38
Treasury Bill Rate: R_t^F	0.01	0.01	0.00	0.03

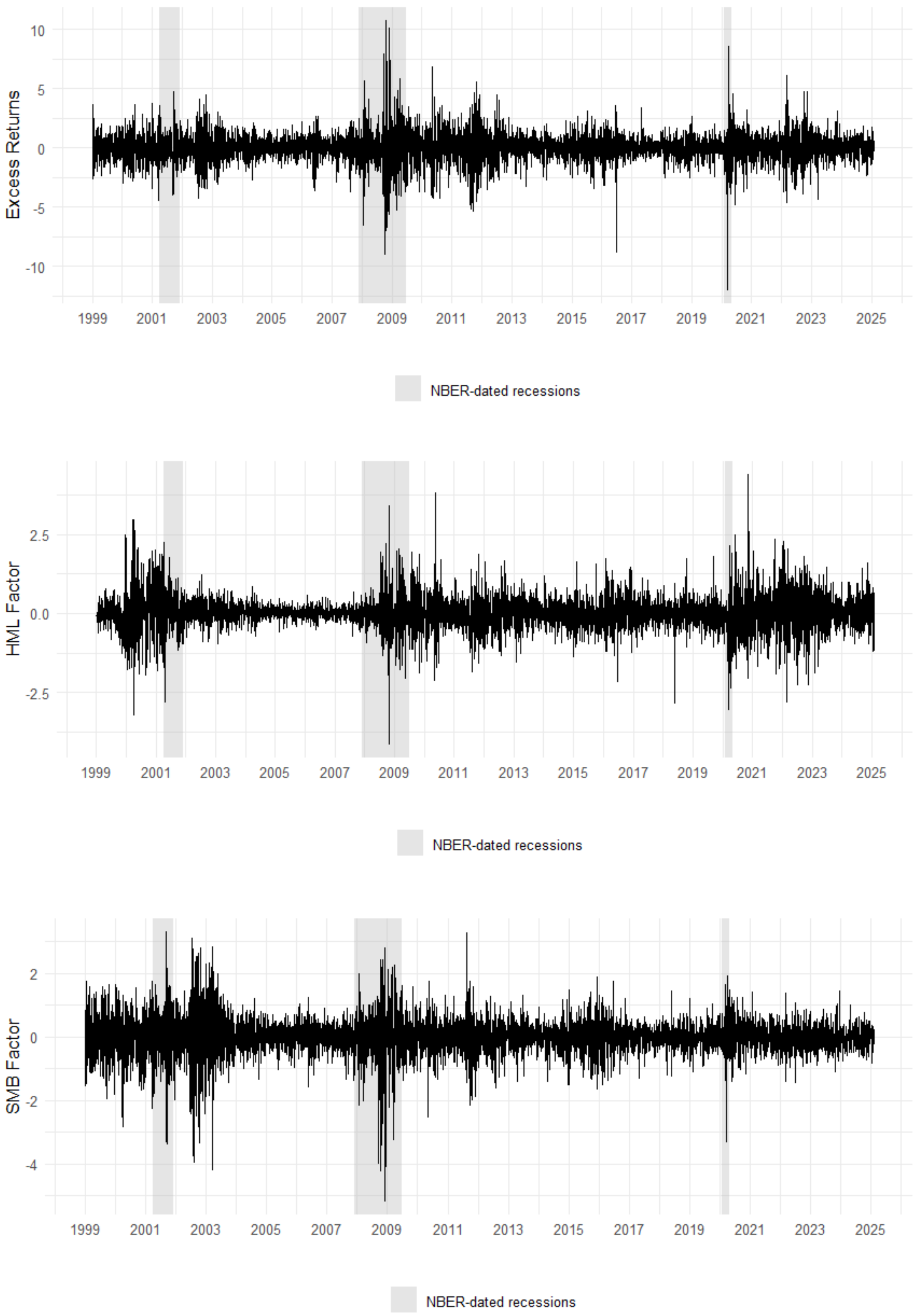
Notes: The table presents descriptive statistics (Mean, St. Dev., Minimum, and Maximum) for the data from the "Fama/French European 3 Factors" dataset—Excess Returns (European value-weighted market portfolio minus the U.S. one-month Treasury Bill rate), SMB, HML, and the Treasury Bill rate—used to estimate the risk parameters $\beta_{hml,t}$ and $\beta_{smb,t}$ via the rolling regression specified in equation (16). The data are daily, denominated in U.S. dollars, and cover the period from July 1990 to January 2025.

larger for big stocks than for small stocks.

These volatility spikes may reflect changing market perceptions of systematic risk. Although many of these events—such as the 2008 financial crisis or the COVID-19 pandemic—were largely unanticipated, they demonstrate how abrupt shifts in market conditions can reshape the pricing of systematic factors, creating a form of uncertainty that resembles the Peso Problem or binary regime changes. This will be later investigated.

Figure 1: **Daily Series: Excess Returns, HML, and SMB Factors**

.



Notes: The figure displays, from top to bottom, the daily series of Excess Returns, HML, and SMB factors from the "Fama/French European 3 Factors" dataset. The excess return corresponds to the return on a European value-weighted market portfolio minus the U.S. one-month Treasury bill rate. The HML factor reflects the return spread between portfolios of high and low book-to-market European stocks. The SMB factor captures the return difference between small and large-cap European portfolios. All returns are expressed in U.S. dollars. The graphs present the period from January 1999 to January 2025. Shading denotes NBER-dated recessions.

Risk Estimation. To estimate the series of risk parameters ($\beta_{hml,t}$ and $\beta_{smb,t}$), we employed OLS regressions using a rolling window of 504 observations (equivalent to approximately two years of trading days). On average, a unit change in HML explains a change of 0.20 in excess return, whereas a unit change in SMB explains a -1.04 variation in excess return (Table 2).

Table 2: **Descriptive Statistics of the Risk Parameters**

.

Statistic	Mean	St. Dev.	Min	Max
$\beta_{hml,t}$	0.20	0.49	−0.83	1.15
$\beta_{smb,t}$	−1.04	0.34	−1.81	−0.34

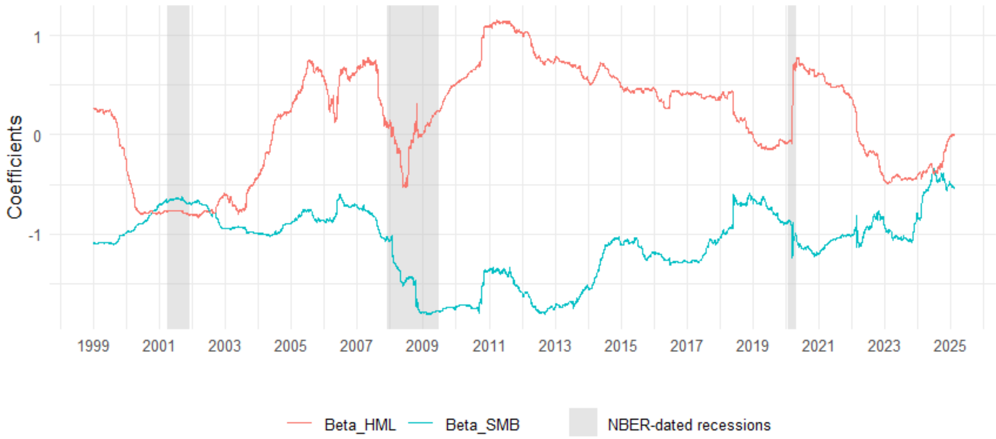
Notes: The Table presents descriptive statistics (Mean, St. Dev., Maximum and Minimum) for the risk parameters estimated, $\beta_{hml,t}$ and $\beta_{smb,t}$, by OLS Rolling Regression specified in equation (16). The rolling regression was employed with a rolling window of 504 observations (equivalent to approximately two years of trading days). The data of these parameters are daily and the sample period extends from July 1992 to January 2025.

Similarly, Fama and French (2012) report a positive β_{hml}; however, in contrast to our findings, they do not observe a negative β_{smb}. This divergence may stem from their focus on global average returns, where SMB spreads favour small-cap stocks, based on the global three-factor model and data from November 1989 to March 2011. VERDELHAN (2018) employs a carry trade HML factor, constructed from country portfolios sorted by interest rates, to explain monthly exchange rate fluctuations, finding predominantly positive β_{hml} values. Schulte et al. (2011) and Lang and Scholz (2015) examine the European real estate equity market using Fama–French factors. Schulte et al. (2011) identifies a generally negative impact of SMB and a positive (in up-markets) or negative (in down-markets) impact of HML, whereas Lang and Scholz (2015) provides further evidence of a positive β_{hml} and negative β_{smb}.

The contrasting dynamics between $\beta_{hml,t}$ and $\beta_{smb,t}$ in Figure 2 reflect the cross-border risk premia from a US investor's perspective. Typically, $\beta_{hml,t}$ is positive, indicating a systematic risk component associated with value stocks. In contrast, $\beta_{smb,t}$ is consistently negative, suggesting a hedging role linked to small-cap stocks.

During the 2001 recession, both betas remained relatively stable, indicating that perceived cross-border risk did not change significantly. However, before the 2008 financial crisis, $\beta_{hml,t}$ decreased considerably, reflecting a reduced risk premium for value stocks, but then rose abruptly in the middle of the recession, indicating heightened risk. In contrast, $\beta_{smb,t}$ became even more negative throughout the recession, suggesting

Figure 2: $\beta_{hml,t}$ **and** $\beta_{smb,t}$ **time series**.



Notes: The figure shows the series of the risk parameters estimated, β_{hml} and β_{smb} by OLS Rolling Regression specified in equation (16). The rolling regression was employed with a rolling window of 504 observations (equivalent to approximately two years of trading days). The data of these parameters are daily and the graphs period extends from January 1999 to January 2025. Shading denotes NBER-dated recessions.

that small-cap stocks increasingly acted as a hedge. A similar divergence appears during the COVID-19 pandemic: $\beta_{hml,t}$ increased sharply, while $\beta_{smb,t}$ became more negative.

This pattern reflects the joint impact of changes in European stock prices and movements in the USD/EUR exchange rate. During US recessions, the dollar often appreciates, which can either amplify or offset the decline in European asset returns from a US investor's perspective. The consistently negative $\beta_{smb,t}$ indicates that small-cap stocks tend to act as a hedge against these risks, while the positive $\beta_{hml,t}$ shows that value stocks carry a more systematic risk component, which becomes pronounced during periods of economic uncertainty.

Mixture Model Estimation. According to equation (17), the risk parameters $\beta_{hml,t}$ and $\beta_{smb,t}$ are modelled over time as a weighted mixture of two distinct sub-distributions from the general population. The parameters of equation (17) were empirically estimated using the EM algorithm and their significance was tested.

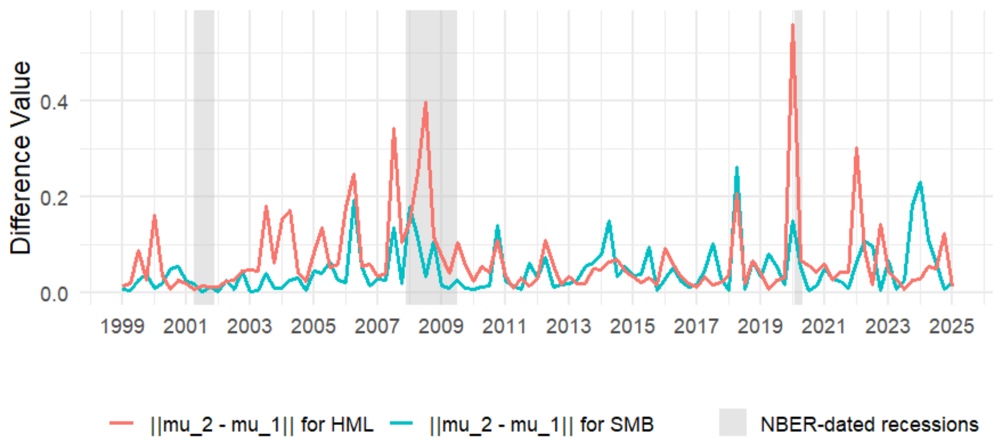
Furthermore, the observations were divided into quarterly periods, and the two distributions were estimated for each quarter to analyse the behaviour of the model parameters over time.

Estimation of quarterly averages: $\hat{\mu}_1$ and $\hat{\mu}_2$. An analysis of the mixture model applied to $\beta_{hml,t}$ reveals the existence of two regimes or sub-distributions that reflect distinct market risk perceptions. The mean series of the two regimes exhibit similar dynamics, although a consistent numerical difference between them is observable. Turning to the mixture model applied to $\beta_{smb,t}$, a similar two-regime pattern emerges, although the dynamics tend to oppose those of $\beta_{hml,t}$ in most quarters. Both for $\beta_{hml,t}$ and $\beta_{smb,t}$, the mean series of the two regimes exhibit similar behaviour, albeit with a modest but consistent numerical difference between them.

Absolute difference ($|\hat{\mu}_1 - \hat{\mu}_2|$). While the mean series for $\beta_{hml,t}$ and $\beta_{smb,t}$ generally exhibit opposing trends, the absolute differences between μ_1 and μ_2 for $\beta_{hml,t}$ and $\beta_{smb,t}$ tend to increase during periods of market turbulence, reflecting heightened market uncertainty. This absolute difference begins to rise slightly before the onset of recessions and intensifies further during these periods, particularly for β_{hml}. This finding reinforces the notion that market stress leads to greater divergence in risk perceptions, regardless of whether the factors move in opposite directions. (Figure 3). The absolute difference between the mean series of the two sub-distributions for $\beta_{hml,t}$ increases substantially. For instance, the most pronounced divergence in the mean series for $\beta_{hml,t}$ occurs at the onset of the COVID-19 pandemic, whereas the largest discrepancy for $\beta_{smb,t}$ is observed during the 2008 financial crisis, highlighting the market's heightened sensitivity to risk during these periods. Furthermore, the absolute difference for $\beta_{hml,t}$ remains consistently greater throughout the analysis period. It becomes even more pronounced during times of crisis, suggesting that the HML factor is associated with greater market uncertainty.

Figure 3: **Absolute Difference series** ($|\hat{\mu}_{1,t} - \hat{\mu}_{2,t}|$) for $\beta_{hml,t}$ and $\beta_{smb,t}$

.

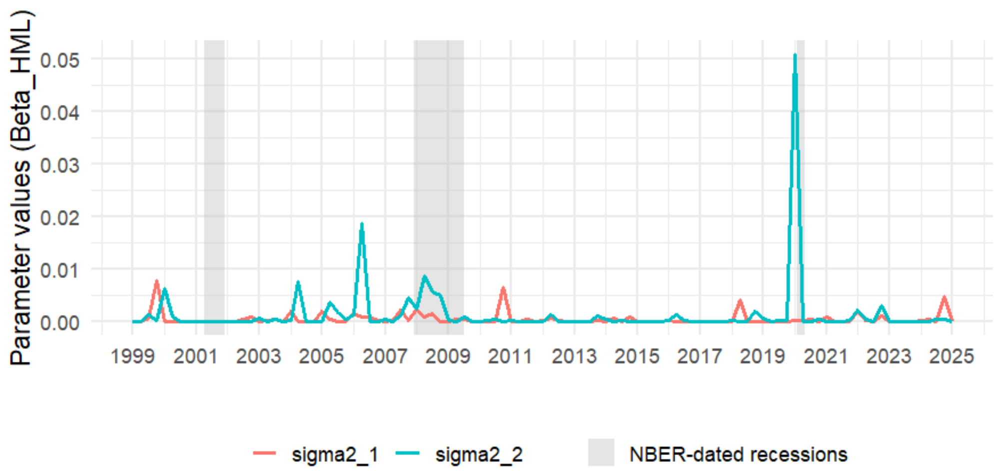


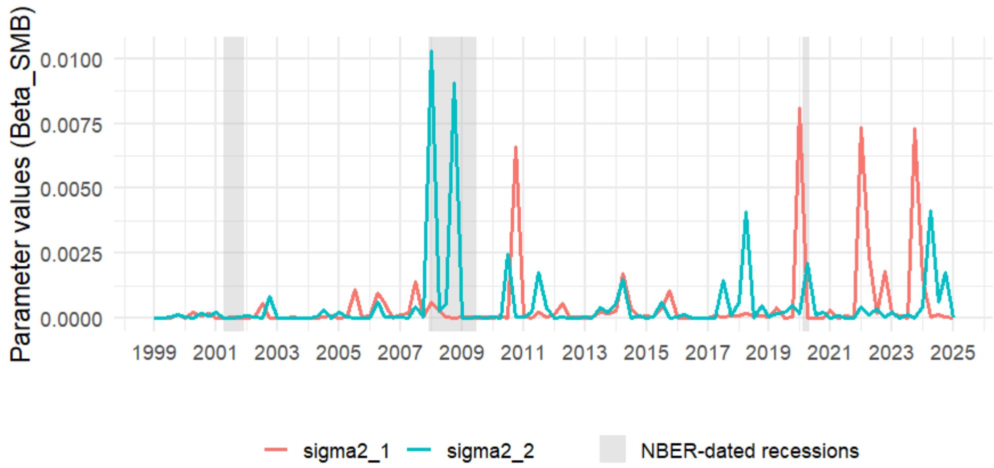
Notes: The figure shows the behaviour of the absolute difference between the estimated parameters $\hat{\mu}_1$ and $\hat{\mu}_2$ from the mixture model specified in equation (17), considering two sub-distributions of β_{hml} and β_{smb} per quarter over the period from 1999:Q1 to 2025:Q1.

Estimation of quarterly variances: $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$. Periods of increasing divergence in the mean series tend to coincide with heightened variance in $\beta_{hml,t}$ and $\beta_{smb,t}$ in at least one of the regimes. For instance, during the 2008 financial crisis, the variance of the first regime of $\beta_{hml,t}$ increased substantially, whereas, during the COVID-19 pandemic, the variance of the second regime became more pronounced. In the case of $\beta_{smb,t}$, the variance of the second sub-distribution rose sharply during the 2008 financial crisis, while the variance of the first sub-distribution increased significantly during the COVID-19 pandemic. This pattern suggests that market uncertainty tends to manifest as heightened volatility in one of the sub-distributions, depending on the nature of the crisis. Moreover, this contrasting pattern further supports the hypothesis that distinct market regimes influence risk perceptions differently over time (Figure 4).

Figure 4: **Series of Estimated Parameters: $\hat{\sigma}^2_{1,t}$ and $\hat{\sigma}^2_{2,t}$ for $\beta_{hml,t}$ and $\beta_{smb,t}$**

.



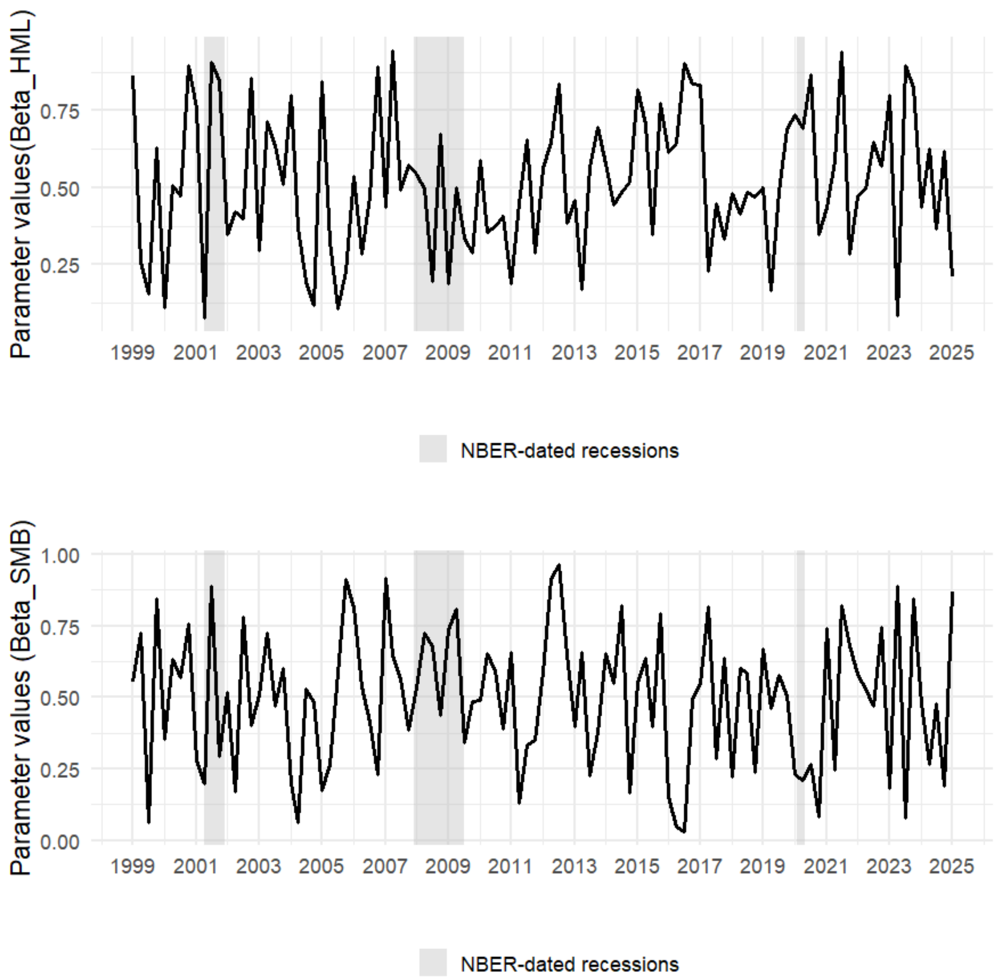


Notes: The figure illustrates the behaviour of the estimated parameters $\hat{\sigma}_1$ and $\hat{\sigma}_2$ from the mixture model specified in equation (17), considering two sub-distributions of β_{hml} and β_{smb} per quarter over the period from 1999:Q1 to 2025:Q1.

Estimation of the regime weights. Certain periods are characterised by the predominance of one of the regimes, which causes the overall distribution to align more closely with the dominant regime during specific years or quarters. This predominance is captured by the estimated probability distribution (Figure 5), which reflects the probability associated with the first regime (P_t). The probability associated with the second regime is given by $(1 - P_t)$. This dynamic implies that the overall risk parameter $\beta_{i,t}$ reflects a weighted mixture of two distinct sub-distributions, whose influence varies over time based on market conditions.

Figure 5: **Series of Estimated Parameters:** $\hat{P}_t$ **for** $\beta_{hml,t}$ **and** $\beta_{smb,t}$

.



Notes: The figure shows the behaviour of the estimated parameter $\hat{P}$ from the mixture model specified in equation (17), considering two sub-distributions of β_{hml} and β_{smb} per quarter over the period from 1999:Q1 to 2025:Q1.

5 Concluding Remarks

This paper investigates how shifts in the distribution of risk parameters influence asset pricing, focusing on binary changes in investor beliefs akin to the Peso Problem. While the Peso Problem typically addresses anticipated changes that affect risk pricing, our findings suggest that abrupt and unanticipated changes—such as financial crises or political disruptions—can also give rise to perceptions of distinct risk regimes. This dynamic highlights how unexpected events can create binary shifts in systematic risk exposure, complementing the conventional understanding of the Peso Problem as strictly related to expected changes.

To examine this phenomenon, we develop a theoretical framework based on a simple Arrow–Debreu model that captures the dynamics of stochastic discount factors under regime shifts. The empirical analysis leverages the Fama/French European Three Factors dataset, which covers daily data from July 1990 to January 2025. The dataset includes excess returns on a European value-weighted market portfolio (in USD) and the HML and SMB factors constructed from 16 developed European markets. Using a rolling regression approach with a window of approximately two years, we estimate time-varying risk parameters, which are then analysed through Gaussian mixture models to detect shifts in perceived risk regimes.

Our findings indicate that systematic risk is not priced uniformly over time. Instead, changes in the probability assigned to different regimes affect the mean and volatility of the estimated betas. The results reveal two distinct risk regimes for HML and SMB factors, reflecting heterogeneous market responses to changes in perceived risk. These shifts become more pronounced before and during economic downturns, suggesting that market stress amplifies divergences in risk perceptions. The empirical evidence indicates that abrupt, often unanticipated changes—such as the 2008 financial crisis and the COVID-19 pandemic—lead to significant shifts in the structure of risk pricing.

The contrasting behaviour of the HML and SMB betas suggests that value stocks (HML) tend to carry a systematic risk component, particularly during crises, while small-cap stocks (SMB) consistently act as a hedge. This pattern reflects the combined effect of fluctuations in European stock prices and movements in the USD/EUR exchange rate from a US investor's perspective. As the dollar typically appreciates during US recessions, it can amplify or mitigate the decline in European asset returns, influencing the risk premia regime.

By linking binary expectation shifts to variations in risk exposure, this study complements the traditional interpretation of the Peso Problem by highlighting that not only anticipated changes but also unexpected, abrupt events can lead to more pronounced regime perceptions. Future research could expand this framework by exploring other asset classes and risk factors.

References

Almeida, C. and Freire, G. (2022). Pricing of index options in incomplete markets. *Journal of Financial Economics*, 144(1):174–205.

Backus, D., Chernov, M., and Martin, I. (2011). Disasters implied by equity index options. *The Journal of Finance*, 66(6):1969–2012.

Barro, R. J. and Liao, G. Y. (2021). Rare disaster probability and options pricing. *Journal of Financial Economics*, 139(3):750–769.

Burnside, C., Eichenbaum, M., and Rebelo, S. (2011). Carry trade and momentum in currency markets. *Annual Review of Financial Economics*, 3(Volume 3, 2011):511–535.

Cortes, G. S., Gao, G. P., Silva, F. B., and Song, Z. (2022). Unconventional monetary policy and disaster risk: Evidence from the subprime and covid–19 crises. *Journal of International Money and Finance*, 122:102543.

Evans, M. D. (1996). 21 peso problems: Their theoretical and empirical implications. In *Statistical Methods in Finance*, volume 14 of *Handbook of Statistics*, pages 613–646. Elsevier.

Fama, E. F. and French, K. R. (2012). Size, value, and momentum in international stock returns. *Journal of Financial Economics*, 105(3):457–472.

Ferreira, A. L., Gong, Y., and Gozluklu, A. E. (2022). Risk-corrected probabilities of a binary event. *SSRN*.

Gao, G. P., Lu, X., and Song, Z. (2019). Tail risk concerns everywhere. *Management Science*, 65(7):3111–3130.

Hau, H. and Rey, H. (2006). Exchange rates, equity prices, and capital flows. *The Review of Financial Studies*, 19(1):273–317.

Lang, S. and Scholz, A. (2015). The diverging role of the systematic risk factors: evidence from real estate stock markets. *Journal of Property Investment & Finance*, 33(1):81–106.

Rogoff, K. (1977). Rational expectations in the foreign exchange market revisited. *Working paper*.

Rogoff, K. S. (1980). *Essays on expectations and exchange rate volatility*. PhD thesis, Massachusetts Institute of Technology.

Schulte, K.-M., Dechant, T., and Schaefers, W. (2011). Systematic risk factors in european real estate equities. *Journal of European Real Estate Research*, 4(3):185–224.

Venturi, P., Ferreira, A., Gozluklu, A., and Gong, Y. (2023). Exchange rates and binary political events. *Oxford Economic Papers*, 76(3):797–822.

VERDELHAN, A. (2018). The share of systematic variation in bilateral exchange rates. *The Journal of Finance*, 73(1):375–418.