Recursive General Equilibrium in Multi-Sector Economies: an Explicit Solution

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Abstract

This paper presents an explicit recursive solution of a multi-sector growth model with global heterogeneity. The recursive equilibrium is Lipschitz with a minimal state space and can be computed through iterations of a contraction on a Banach space. Additionally, we present a stead state equilibrium with a closed form solution. We conclude that aggregate capital endowment could be viewed as a minimal state space. Although some results in the literature point out that the lack of equilibrium is related to its multiplicity, this example makes it clear that the existence of equilibrium crucially depends on the shape of the aggregate capital transition. Intuitively, when we obtain conditions in the exogenous parameters which make the capital transition non-degenerate, we then obtain the existence of equilibrium with the minimal state space. The main fact that supports this result is based on the idea that the capital transition parameterizes the sequential equilibrium, generating it in a recursive way.

Keywords: Multi-sector economy, recursive equilibrium, growth models, minimal state space.

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1 Introduction

Since the work of Lucas and Prescott (1971) and Prescott and Mehra (1980), recursive equilibrium has been a key focal point of both applied and theoretical work in characterizing sequential equilibrium for dynamic general equilibrium models in such fields as macroeconomics, international trade, growth theory, industrial organization, financial economies, and monetary theory. Specifically, in general dynamic models with infinitely lived agents economists have focused on so-called minimal state space recursive equilibrium, i.e. a pair of stationary transition and policy functions that relate the endogenous variables in any two consecutive periods, defined on the natural state space. Apart from its simplicity, (minimal state space) recursive equilibrium is also widely used in applied or computational works, as powerful recursive methods provide algorithms to compute it efficiently. Results regarding equilibrium existence are necessary prerequisites for a theoretical and computational analysis, however.

Unfortunately, there are well known examples where recursive equilibria (in specific function spaces) in dynamic economies are non existent (see Santos (2002) for economies with taxes, Kubler and Schmedders (2002) for economies with incomplete asset markets or Krebs (2004) for economies with large borrowing limits). Some recent attempts that address the question of minimal state space recursive equilibrium existence and its approximation, include contributions of Datta et al. (2002) and Datta et al. (2018) for models with homogeneous agents, who propose a monotone maps method applied on the equilibrium version of the household first order conditions and prove equilibrium existence along with its comparative statics, using versions of Tarski fixed point theorem. Unfortunately, there are no known results on how to extend these techniques to models with heterogeneous agents and multiple assets. Next, Brumm et al. (2017) apply some powerful results from stochastic games literature and by adding sufficient shocks prove existence of a recursive equilibrium using operators defined on households first order conditions and applying Kakutani-Fan-Gliksberg fixed point theorem on the operator defined on the Walrasian auctioneer problem. The underlying topology is weak-star and the obtained recursive equilibrium a measurable map on the state space. The measure theoretical results together with recent contributions in stochastic games allow to prove minimal state space recursive equilibrium existence without sunspots or public coordination devices.

More specifically, one of the canonical equilibrium models analyzed in the literature that significantly influenced the fields of financial economics, macroeconomics, monetary theory, optimal taxation and econometrics, was developed by Lucas Jr (1978). However, despite the model's wide application, typical assumptions involve a representative agent. In fact, presence of infinitely lived *heterogeneous* agents can be the key to explain several peculiarities of market frictions from the perspective of models with rational expectations. Apart from mentioned Brumm et al. (2017) contribution, there are only few known results concerning recursive equilibrium existence in the Lucas three model with heterogeneous agents. These include Raad (2016), who show the existence of a possibly non-continuous recursive equilibrium with a minimal state space, however, the model assumes that agents have exogenous beliefs on portfolio transitions.[1] Kubler and Schmedders (2002) present an example of an infinite-horizon economy with Markovian fundamentals, where the recursive competitive equilibrium (defined on a state space of equilibrium asset holdings and exogenous shocks) does not exists. In their example, there must exist two different nodes of a tree such that along the equilibrium path the value of the equilibrium asset holdings is the same but such that there exist more than one equilibrium for both of the continuation economies. Although they claim that a slight perturbation in individual endowments will restore the existence of a weakly recursive equilibrium, we detail the set of conditions that

[1] Agents make mistakes directly or indirectly on prices by inaccurate anticipation of transition portfolios and an equilibria with rational expectations and perfect foresight can *not* be implemented in this environment. Therefore, we cannot apply Raad's result in this paper. In fact, he shows that an equilibrium allocation for an economy with agents making large enough errors on price expectations cannot be a Radner equilibrium, assuming quite general conditions on the primitives. The author also presents an example elucidating this fact even if agents make errors only on the portfolio transitions.

rules out Kubler and Schmedders (2002) example from the model analyzed in our paper. Finally, Raad and Woźny (2019) show the existence of recursive equilibrium in a model with heterogeneous agents with minimal state space but for the case of only one asset and no production.

In this paper, we present a simple form equilibrium solution of a multi-sector growth model with global heterogeneity. The obtained sequential solution is implemented by a Lipschitz recursive equilibrium with a minimal state space. Furthermore, it allows us to conjecture the possible non-existence of an equilibrium with capital performing the role of an asset and of an input for production. We conclude that positive aggregate profits could restore the existence of equilibrium by decoupling capital and stock markets. In Section 4 we compute the stead state equilibrium using the software Julia. The explicit equations presented here can be used for inferences around capital stock boundaries that precludes equilibrium implementation (Cao, 2020). Furthermore, the recursive equilibrium could be useful in calibration of macroeconomic n-sector growth models.

2 The Model

In this paper we follow the notation given in Mas-Colell et al. (1995). Suppose that there exists a finite set of agents types denoted by $I = \{1, \cdots, \mathrm{I}\}$ and a finite set of firms $J = \{1, \cdots, \mathrm{J}\}$ shared into three sectors[2] (capital production, capital rental and consumption) indexed by the sets J_k, J_κ and J_c with cardinality $(\mathrm{J}_k, \mathrm{J}_\kappa, \mathrm{J}_c)$ respectively. Assume that firms of the same sector are identical.

Regarding capital sector, assume that there exists a capital which starts to be produced at the beginning of each period using two primary inputs: a raw material with amount denoted by $m^j \in M^j := \mathbb{R}_+$ and a high skilled labor with amount denoted by $l^j \in L^j := \mathbb{R}_+$. Write the amount of produced capital by $k^j \in K^j$ and suppose that it is available at the end of each period and has a depreciation $1 - \gamma$ for $\gamma < 1$ for $j \in J_k$.

[2] This model could be extended to several sectors each one containing an intermediary capital as an asset.

Regarding consumption sector, assume that there is one consumption good produced by renting an amount of capital input from the previous period denoted by $\kappa^j \in K^j := \mathbb{R}_+$ and by employing low skilled labor with amount denoted by $\ell^j \in L^j$. Denote the produced consumption amount by $c^j \in C^j = \mathbb{R}_+$ for each firm $j \in J_c$.

Regarding capital rental sector, assume that firms employ an amount of previous period capital to rent denoted by $\kappa^j \in K^j := \mathbb{R}_-$ and produces only capital depreciation with amount denoted by $k^j \in \mathbb{R}_-$ for $j \in J_\kappa$. Assume to simplify that firms do not employ labor in the rental trade. The input of capital denoted as $\kappa^j \in K^j$ represents a contract between the two capital sectors specifying the endowment of capital given in the previous period and available to rent, that is, $\kappa^j < 0$ for $j \in J_\kappa$ and to be rented, that is, $\kappa^j \geq 0$ for $j \in J_k$ at the beginning of the period. Assume that the total amount of physical capital employed in consumption sector and the total amount of physical capital given at the previous period coincide.

Capital is endowed by consumers indirectly through the allocation of an asset available in the capital rental market. Let $K^i = \mathbb{R}_+$ be[3] the set where capital choices are defined, consider $C^i = \mathbb{R}_+$ the set where agent i's consumption is chosen.[4] There are J assets or equities with amounts[5] denoted by the portfolio $a^i \in A^i = \mathbb{R}_+^{\mathrm{J}}$ and traded only by consumers. Assume that physical capital and equity one are identified, that is, the first equity represents the asset endowment in the capital rental sector. More precisely, we have $a_1^i = k^i$ for each agent i, that is, $a^i = (k^i, a_2^i, \cdots, a_{\mathrm{J}}^i)$ for all $a^i \in A^i$. Define $A^i = K^i \times A_2^i \times \cdots \times A_{\mathrm{J}}^i$ for each $i \in I$ and denote by $\epsilon^i = (a_2^i, \cdots, a_{\mathrm{J}}^i)$ the equity allocation for all $i \in I$. Equities $j = 2, \cdots, \mathrm{J}$ have unitary net supply, that is, $\sum_{i \in I} \epsilon^i = 1$.

The set of equity prices is denoted by[6] $Q = \mathbb{R}^{\mathrm{J}}$ with a typical element given

[3] The boundary on K^i will be exogenously defined later.

[4] We assume that durable capital can be stored, in contrast to any type of raw material which is totally consumed in the transformation.

[5] Each equity also share profits through a *pro-rata* rule.

[6] Since we consider the case of zero profits at steady state equilibrium, equities may have negative prices outside the steady state.

as a row matrix[7] $q = [q_1, \cdots, q_{\text{J}}]$. Therefore the equity one and capital price coincide, that is, $q_1 = p_k$. Note that the equity price in the capital production and consumption sectors represents the total value of a firm while in the capital rental sector it represents the value of a unit of capital. Denote by $p_\epsilon = [q_2, \cdots, q_{\text{J}}]$ the vector of equity prices and define $P_\epsilon = \mathbb{R}^{\text{J}-1}$ as the set of such prices p_ϵ. Define $p_e = [p_\kappa, p_\ell, p_l, p_m, p_c, p_k]$ as the vector of intermediary and final good endowment unitary prices. Write $P_e = \mathbb{R}^6_{++}$ as the set of such prices p_e. Additionally, assume that good price is normalized to one, that is, $p_c := \bar{p}_c = 1$. The set of all prices is defined as $P = \mathbb{R}^{5+\text{J}}_{++}$ with a typical element given as a row matrix $p = [p_\kappa, p_\ell, p_l, p_m, p_c, p_k, q_2, \cdots, q_{\text{J}}]$ representing prices of capital rental, labor, raw material, consumption good, capital, and equities respectively. Define shortly $p = [p_e, p_\epsilon]$ for all $p \in P$. This configuration defined on prices is necessary because capital plays the role of asset, input and produced good simultaneously. Write $X^i = C^i \times A^i$ for all $i \in I$ and define the set of firms choices $Y^j = C^j \times K^j$ for all $j \in J$. A typical element of X^i is given by a column matrix $x^i = (c^i, a^i)$ and a typical element of Y^j is given by $y^j = (c^j, k^j)$. Write $\Phi = \mathbb{R}^I$ as an auxiliary set in which will be defined the residual capital demand.

Let $A = \prod_{i \in I} A^i$ be the state space[8] with a typical element denoted by the $\text{J} \times \text{I}$ matrix $\bar{a} = [\bar{a}^1_-, \cdots, \bar{a}^{\text{I}}_-]$ representing previous period asset distribution. Denote the set of all continuous functions[9] $\hat{p} : A \to P$ by $\widehat{P}$. Moreover, consider $\widehat{C}$ as the space of all continuous functions $\hat{c} : A \to C$ representing the transition of optimal consumption choices and $\widehat{A}$ as the space of all continuous functions $\hat{a} : A \to A$ representing the transition of asset distribution.

Define the symbol without upper index as the Cartesian product over all agents for any allocation variable. For instance, write $C = \prod_{\iota \in I \cup J} C^\iota$ with a typical element as a row matrix $c = [c^\iota]_{\iota \in I \cup J}$. Write $x^I = [x^i]_{i \in I}$ as a $2 \times \text{I}$ matrix. The

[7] We denote a row matrix as $[\cdots]$ and a column matrix as $(\cdots)$.

[8] This set contains the minimal state space. We exhibit the minimal state space further ahead.

[9] Note that we are using the "$\hat{}$" and "$\widehat{}$" symbols to denote the space of functions from A to the specified set or an operator is this space. Moreover we use the "$\tilde{}$" symbol to denote functions defined outside the state space A.

other variables have an analogous interpretation. Define analogously the symbol without upper index for functions. Given $\{\nu, \nu'\} \subset \mathbb{R}^N$ write $\nu^{N+} = \sum_{n \in N} \nu^n$ and $\nu\nu' = \sum_{n \in N} \nu_n \nu'_n$. We state that $\nu \geq \nu'$ when $\nu_n \geq \nu'_n$ for all $n \in N$. Finally, we denote by 0 as the vector $(0, 0, \cdots, 0) \in \mathbb{R}^{\mathrm{N}}$.

2.1 Firms

Heterogeneous firms produce the consumption and capital goods in two sectors. Each sector has a large number of identical firms. Write the set of all input allocations as $\mathcal{I}^j = L^j \times M^j \times K^j \times L^j$ with a typical element written as $\iota^j = (l^j, m^j, \kappa^j, \ell^j) \in \mathcal{I}$ for all $j \in J$. We consider production functions described by a log Cobb-Douglas function. Technology in each sector $j \in J$ is represented by the production function $f^j : \mathcal{I}^j \to C^j \times K^j$ and the production set of feasible allocations $Y^j_{\mathrm{F}} \subset Y^j$ as

$$Y^j_{\mathrm{F}} = \{y^j \in Y^j : \exists (l^j, m^j, \kappa^j, \ell^j) \in \mathcal{I}^j \text{ with } f^j(l^j, m^j, \kappa^j, \ell^j) \geq y^j\}$$

and firms' optimal costs[10] for each $j \in J$ are then given for each $p_e \in P_e$ and each $y^j = (c^j, k^j) \in Y^j_{\mathrm{F}}$ by[11]

$$\tilde{\varsigma}^j(y^j, p_e, \bar{k}^{I+}_{-}) = \min\{p_l l^j + p_m m^j + p_\kappa \kappa^j + p_\ell \ell^j : f^j(\iota^j) \geq y^j,\ \kappa^j \geq -\bar{k}^{I+}_{-}/\mathrm{J}_\kappa\}.$$

for each $\bar{a}_{-} \in A$. Write $\tilde{\varsigma}^j = (\tilde{l}^j, \tilde{m}^j, \tilde{\kappa}^j, \tilde{\ell}^j)$. Firms' optimal profits are then given for each $p_e \in P_e$ by[12]

$$\tilde{\pi}^j(p_e, \bar{k}^{I+}_{-}) = \max\{p_c c^j + p_k k^j - \tilde{\varsigma}^j(c^j, k^j, p_e, \bar{k}^{I+}_{-}) : (c^j, k^j) \in Y^j_{\mathrm{F}}\} \text{ for all } j \in J. \quad (1)$$

Write $\tilde{\pi}(p_e, \bar{k}^{I+}_{-})$ as the row vector $[\tilde{\pi}^j(p_e, \bar{k}^{I+}_{-})]_{j \in J}$. Firms' optimal production are then given for each $p_e \in P_e$ and each $j \in J$ by

$$\tilde{y}^j(p_e, \bar{k}^{I+}_{-}) = \operatorname{argmax}\{p_c c^j + p_k k^j - \tilde{\varsigma}^j(c^j, k^j, p_e, \bar{k}^{I+}_{-}) : (c^j, k^j) \in Y^j_{\mathrm{F}}\}. \quad (2)$$

[10] We adopt the convention that $\min\{\emptyset\} = \infty$, that is, the cost of producing an unfeasible allocation is arbitrarily large.

[11] We assume that each firm in consumption sector has the same capital input constraint, that is, $\kappa^j \geq -\bar{k}^{I+}_{-}/\mathrm{J}_\kappa$.

[12] Recall that $\bar{a}_{-} = [(\bar{k}^i_{-}, \bar{\epsilon}^i) : i \in I]$

We assume that a typical firm in the capital rental market provides a certain amount of capital at the beginning of the period and rent it to the firms in consumption sector. Suppose to simplify that capital rental sector use labor with that same level of skills such as in the physical capital sector and hence leisure has the same price in the whole capital sector.[13] Finally we consider that rental sector produces only capital depreciation and hence $K^j \subset \mathbb{R}_-$ for $j \in J_\kappa$.

For a given vector of technical coefficients $(\sigma^j_m, \sigma^j_l, \sigma^j_\kappa, \sigma^j_\ell)$ and such that[14] $(\sigma^j_m, \sigma^j_l) = (0,0)$ for $j \in J_c$, $(\sigma^j_\kappa, \sigma^j_l, \sigma^j_m) = (1-\gamma, 0, 0)$ for $j \in J_\kappa$ and $(\sigma^j_\kappa, \sigma^j_\ell) = (0,0)$ for $j \in J_k$, capital and good sectors have a technology represented by the following production functions[15] $f^j_\kappa : K^j \to \mathbb{R}_-$, $f^j_k : L^j \times M^j \to \mathbb{R}_+$ and $f^j_c : K^j \times L^j \to \mathbb{R}_+$ defined respectively by

$$\begin{aligned} f^j_k(\kappa^j) &= \sigma^j_\kappa \kappa^j \text{ for } j \in J_\kappa \qquad (3)\\ f^j_k(l^j, m^j) &= \sigma^j_l \log(l^j) + \sigma^j_m \log(m^j) \text{ for } j \in J_k \\ f^j_c(\kappa^j, \ell^j) &= \sigma^j_\kappa \log(\kappa^j) + \sigma^j_\ell \log(\ell^j) \text{ for } j \in J_c \end{aligned}$$

where (l^j, m^j) is the amount of primary labor and capital inputs employed on the capital sector and (κ^j, ℓ^j) is the amount of capital and labor inputs employed on the consumption goods sector. Note that f^j_k and f^j_c and the Cobb-Douglas functions have the same isoquants. Define $f^j = (f^j_c, f^j_k)$ for $j \in J$.

2.2 Agents' features

Consider a model with one good and agents with instantaneous utility function defined by $\hat{u}^i(c^i) = \log(c^i)$ for all $c^i \in C^i$ and $i \in I$. Assume agents have the same discount rate, that is $\beta^i = \bar{\beta}$ for all $i \in I$. There is no disutility of labor nor utility from other (non-consumption) goods.

Consumption, labor (or leisure) and primary capital endowments are given by $e^i_\kappa \in \mathbb{R}_+$, $e^i_\ell \in \mathbb{R}_+$, $e^i_l \in \mathbb{R}_+$, $e^i_m \in \mathbb{R}_+$, $e^i_k \in \mathbb{R}_+$, and $e^i_c \in \mathbb{R}_+$ for all $i \in I$.

[13] That is, workers have the same wage.

[14] Assume to simplify there is no labor employment in capital rental sector.

[15] Assume that there is no production when some input is employed below one unit. This implies that all production functions lead to a non-negative amount of production.

Write $e^i = (e^i_\kappa, e^i_\ell, e^i_l, e^i_m, e^i_c, e^i_k)$, $e = [e^i : i \in I] \in \mathbb{R}^{6\times\mathrm{I}}_+$ and recall that $p_e = [p_\kappa, p_\ell, p_l, p_m, p_c, p_k]$. We consider as a convention that agents have null capital endowments, that is, $e^i_\kappa = 0$ and $e^i_k = 0$. The consumers budget sets are given for each $i \in I$ by

$$\tilde{b}^i(a^i_-, \bar{k}^{I+}_-, p) = \{x^i \in X^i : p_c c^i + q a^i \leq (q + \tilde{\pi}(p_e, \bar{k}^{I+}_-))a^i_- + p_e e^i\}.$$

3 Recursive Equilibrium

For each $i \in I$, consider $\widehat{V}^i$ as the set of all continuous functions $\hat{v}^i : A^i \times A \to \mathbb{R}$ and for each $j \in J$, consider $\widehat{V}^j$ as the set of all continuous functions $\hat{v}^j : A \to \mathbb{R}$. Write $\widehat{V} = \prod\{V^\iota : \iota \in I \cup J\}$. For each transition of prices and savings variables $(\hat{p}, \hat{a})$ we have the following definition.

Definition 3.1. Define for each $i \in I$ the map $\hat{\mu}^i_v : \widehat{V} \times \widehat{P} \times \widehat{A} \to \widehat{V}^i$ as

$$\hat{\mu}^i_v(\hat{v}, \hat{p}, \hat{a})(a^i_-, \bar{a}_-) = \max\left\{\hat{u}^i(c^i) + \bar{\beta}\hat{v}^i(a^i, \hat{a}(\bar{a}_-)) : x^i \in \tilde{b}^i(a^i_-, \bar{k}^{I+}_-, \hat{p}(\bar{a}_-))\right\}$$

for all $(a^i_-, \bar{a}_-) \in A^i \times A$ and the map $\hat{\mu}^i_x : \widehat{V} \times \widehat{P} \times \widehat{A} \to \widehat{X}^i$ as

$$\hat{\mu}^i_x(\hat{v}, \hat{p}, \hat{a})(\bar{a}_-) = \mathrm{argmax}\left\{\hat{u}^i(c^i) + \bar{\beta}\hat{v}^i(a^i, \hat{a}(\bar{a}_-)) : x^i \in \tilde{b}^i(\bar{a}^i_-, \bar{k}^{I+}_-, \hat{p}(\bar{a}_-))\right\}$$

for all $(a^i_-, \bar{a}_-) \in A^i \times A$.

Define for each $j \in J$ the maps $\hat{\mu}^j_v : \widehat{P} \to \widehat{V}^j$ and $\hat{\mu}^j_y : \widehat{P} \to \widehat{Y}^j$ as

$$\hat{\mu}^j_v(\hat{p})(\bar{a}_-) = \tilde{\pi}(\hat{p}_e(\bar{a}_-), \bar{a}_-) \text{ and } \hat{\mu}^j_y(\hat{p})(\bar{a}_-) = \tilde{y}^j(\hat{p}_e(\bar{a}_-), \bar{a}_-) \text{ for all } \bar{a}_- \in A.$$

Definition 3.2. The transition vector $(\hat{x}, \hat{y}, \hat{p}, \hat{v}) \in \widehat{X} \times \widehat{Y} \times \widehat{P} \times \widehat{V}$ is a recursive equilibrium if it satisfies for each $\bar{a}_- = [\bar{k}_-, \bar{\epsilon}_-] \in A$

1. $\hat{v}^i = \hat{\mu}^i_v(\hat{v}, \hat{p}, \hat{a})$ for all $i \in I$ and $\hat{v}^j = \hat{\mu}^j_v(\hat{p})$ for all $j \in J$;
2. $\hat{x}^i \in \hat{\mu}^i_x(\hat{v}, \hat{p}, \hat{a})$ for all $i \in I$ and $\hat{y}^j \in \hat{\mu}^j_y(\hat{p})$ for all $j \in J$;
3. $\sum_{i\in I} \hat{k}^i(\bar{a}_-) = \sum_{j\in J} \hat{k}^j(\bar{a}_-) + \sum_{i\in I} \gamma\bar{k}^i_-$ for all $\bar{a}_- \in A$;
4. $\sum_{i\in I} \hat{a}^i_j(\bar{a}_-) = 1$ for all $j \in \{2, \cdots, \mathrm{J}\}$;

5. $\sum_{i\in J} \hat{c}^i(\bar{a}_-) = \sum_{j\in J} \hat{c}^j(\bar{a}_-) + \sum_{i\in I} e_c^i$ for all $\bar{a}_- \in A$;

6. $\sum_{j\in J} \hat{l}^j(\bar{a}_-) = \sum_{i\in I} e_l^i$;

7. $\sum_{j\in J} \hat{m}^j(\bar{a}_-) = \sum_{i\in I} e_m^i$;

8. $\sum_{j\in J_\kappa} -\hat{\kappa}^j(\bar{a}_-) = \sum_{j\in J_k} \hat{\kappa}^j(\bar{a}_-) = \sum_{i\in I} \bar{k}_-^i$;

9. $\sum_{j\in J} \hat{\ell}^j(\bar{a}_-) = \sum_{i\in I} e_\ell^i$.

4 The explicit solution

Suppose hereafter that an arbitrary price transition $\hat{p}$, savings transition $\hat{a}$, and input-output transition $\hat{y}$ are taken as given for consumers. Equity prices $(\hat{q}_2, \cdots, \hat{q}_{\mathrm{J}})$ are chosen in such a way that for each agent $i \in I$ the portfolio $\bar{\epsilon}_- = (\bar{a}_2^i, \cdots, \bar{a}_{\mathrm{J}}^i)$ is an optimal steady state choice. We exhibit a solution in which optimal equity choices are constant and equal to $\bar{\epsilon}_-$. Therefore, the state space is given as $A = \prod_{i\in I} A^i$ where

$$A^i = \{(\bar{k}^i, \bar{\epsilon}^i) : \bar{k}^i \in K^i\} \text{ for all } i \in I.$$

To ensure that optimal equity choices are constant, we consider an equation relying the equity prices such that $\bar{\epsilon}_-$ is actually an arbitrage-free portfolio. Moreover, we will show that the minimal state space will be given by a certain subset of $\sum_i K^i$.

4.1 Firms' optimal choices

Proposition 4.1. *The optimal input firms' choices evaluated at any transition price $\hat{p}$ are given as a solution of the following functional equations for all $\bar{a}_- =$*

$[\bar{k}_{\text{-}}, \bar{\epsilon}_{\text{-}}] \in A$ *by*

$$\hat{l}^j(\bar{a}_{\text{-}}) = \frac{\sigma_l^j \hat{p}_k(\bar{a}_{\text{-}})}{\hat{p}_l(\bar{a}_{\text{-}})} \textit{ and } \hat{m}^j(\bar{a}_{\text{-}}) = \frac{\sigma_m^j \hat{p}_k(\bar{a}_{\text{-}})}{\hat{p}_m(\bar{a}_{\text{-}})} \textit{ for } j \in J_k \tag{4}$$

$$\hat{\kappa}^j(\bar{a}_{\text{-}}) = \frac{\sigma_\kappa^j \hat{p}_c(\bar{a}_{\text{-}})}{\hat{p}_\kappa(\bar{a}_{\text{-}})} \textit{ and } \hat{\ell}^j(\bar{a}_{\text{-}}) = \frac{\sigma_\ell^j \hat{p}_c(\bar{a}_{\text{-}})}{\hat{p}_\ell(\bar{a}_{\text{-}})} \textit{ for } j \in J_c \tag{5}$$

$$\hat{k}^j(\bar{a}_{\text{-}}) = f_k^j\big(\hat{l}^j(\bar{a}_{\text{-}}), \hat{m}^j(\bar{a}_{\text{-}})\big) \textit{ for } j \in J_k \tag{6}$$

$$\hat{c}^j(\bar{a}_{\text{-}}) = f_c^j\big(\hat{\kappa}^j(\bar{a}_{\text{-}}), \hat{\ell}^j(\bar{a}_{\text{-}})\big) \textit{ for } j \in J_c \tag{7}$$

$$(\hat{k}^j(\bar{a}_{\text{-}}), \hat{\kappa}^j(\bar{a}_{\text{-}})) = \begin{cases} \big(\sigma_\kappa k_{\text{-}}^{I+} \mathrm{J}_\kappa^{-1}, k_{\text{-}}^{I+} \mathrm{J}_\kappa^{-1}\big) \textit{ if } \hat{p}_\kappa(\bar{a}_{\text{-}}) \geq \sigma_\kappa \hat{p}_k(\bar{a}_{\text{-}}) \\ (0,0) \textit{ if } \hat{p}_\kappa(\bar{a}_{\text{-}}) < \sigma_\kappa \hat{p}_k(\bar{a}_{\text{-}}) \end{cases} \quad \textit{for } j \in J_\kappa \tag{8}$$

$$\hat{c}^j(\bar{a}_{\text{-}}) = 0 \textit{ and } \hat{l}^j(\bar{a}_{\text{-}}) = 0 \textit{ for } j \in J_\kappa \tag{9}$$

Proof: See Section 6.1 in the appendix. Notice that since all objetive funtions are concave then solving the first order equations is a sufficient procedure for finding the global optimum. We exhibit explicitly solutions of the first order equations. □

Remark 4.1. Equations (1) and (4) imply that optimal firm j's profit for $j \in J_k \cup J_c$ is given for all $\bar{a}_{\text{-}} \in A$ at the optimum by

$$\tilde{\pi}^j(p_e, \bar{k}_{\text{-}}^{I+}) = p_c \hat{c}^j(\bar{a}_{\text{-}}) + p_k \hat{k}^j(\bar{a}_{\text{-}}) - p_l \hat{l}^j(\bar{a}_{\text{-}}) - p_m \hat{m}^j(\bar{a}_{\text{-}}) - p_\kappa \hat{\kappa}^j(\bar{a}_{\text{-}}) - p_\ell \hat{\ell}^j(\bar{a}_{\text{-}})$$

and hence

$$\tilde{\pi}^j(\hat{p}_e(\bar{a}_{\text{-}}), \bar{k}_{\text{-}}^{I+}) = \begin{cases} \hat{p}_k(\bar{a}_{\text{-}})(\hat{k}^j(\bar{a}_{\text{-}}) - \sigma_m^j - \sigma_l^j) \text{ if } j \in J_k \\ \hat{p}_c(\bar{a}_{\text{-}})(\hat{c}^j(\bar{a}_{\text{-}}) - \sigma_\kappa^j - \sigma_\ell^j) \text{ if } j \in J_c \end{cases}$$

for each price transition $\hat{p} \in \widehat{P}$. In addition, We assume that the profit function of the capital rental sector is given per unit of capital.[16] The reason for this convention is the fact that we will consider the share of these firms given in

[16] Note that capital price and capital rental price are given per units of capital.

rented capital unit of account. Therefore, for $j \in J_\kappa$ and $k^j \geq -\sigma_\kappa \bar{k}_-^{I+}/\mathrm{J}_\kappa$ we have by (3)

$$\begin{aligned}\tilde{\varsigma}^j(c^j, k^j, \hat{p}_e(\bar{a}_-), \bar{k}_-^{I+}) &= \min\left\{\hat{p}_\kappa(\bar{a}_-)\kappa^j : \kappa^j \geq \max\{k^j/\sigma_\kappa, -\bar{k}_-^{I+}/\mathrm{J}_\kappa\}\right\} \\ &= \hat{p}_\kappa(\bar{a}_-)k^j/\sigma_\kappa\end{aligned}$$

for all $\bar{a}_- \in A$ since $K^j \subset \mathbb{R}_-$ and hence

$$\tilde{\kappa}^j(y^j, \hat{p}_e(\bar{a}_-), \bar{a}_-) = k^j/\sigma_\kappa \geq -\bar{k}_-^{I+}/\mathrm{J}_\kappa. \tag{10}$$

Thus capital rental firm j's profit[17] is given by

$$\begin{aligned}\tilde{\pi}^j(\hat{p}_e(\bar{a}_-), \bar{k}_-^{I+}) &= \max\{\bar{p}_c c^j + \hat{p}_k(\bar{a}_-)k^j - \tilde{\varsigma}^j(c^j, k^j, \hat{p}_e(\bar{a}_-), \bar{k}_-^{I+}) : (c^j, k^j) \in \mathrm{Y}_\mathrm{F}^j\} \\ &= \max\{\big(\hat{p}_k(\bar{a}_-) - \hat{p}_\kappa(\bar{a}_-)/\sigma_\kappa\big)k^j : k^j \in K^j\} \\ &= \max\{\big(\hat{p}_\kappa(\bar{a}_-) - \sigma_\kappa \hat{p}_k(\bar{a}_-)\big)|k^j|/\sigma_\kappa : k^j \in K^j\}.\end{aligned}$$

Thus for $k^j \geq -\sigma_\kappa \bar{k}_-^{I+}/\mathrm{J}_\kappa$ we get

$$\tilde{\pi}^j(\hat{p}_e(\bar{a}_-), \bar{k}_-^{I+}) = \max\{0, \big(\hat{p}_\kappa(\bar{a}_-) - (1-\gamma)\hat{p}_k(\bar{a}_-)\big)\bar{k}_-^{I+}/\mathrm{J}_\kappa\}.$$

Finally, we will see next that at the recursive equilibrium

$$\tilde{\pi}^j(\hat{p}_e(\bar{a}_-), \bar{k}_-^{I+}) = \big(\hat{p}_\kappa(\bar{a}_-) - (1-\gamma)\hat{p}_k(\bar{a}_-)\big)\bar{k}_-^{I+}/\mathrm{J}_\kappa \text{ for all } \bar{a}_- \in A.$$

In this model we will consider an example of recursive equilibrium assuming identical firms in each sector and hence symmetric choices of inputs and outputs. Therefore we have the following definition

Definition 4.1. Define the optimal input choices for each firm j as

$$\bar{l}^j = e_l^{I+}/\mathrm{J}_k; \quad \bar{m}^j = e_m^{I+}/\mathrm{J}_k; \quad \hat{\kappa}^j(k_-^{I+}) = k_-^{I+}/\mathrm{J}_c; \quad \bar{\ell}^j = e_\ell^{I+}/\mathrm{J}_c.$$

Define the optimal capital produced $\bar{k}^J$ as

$$\bar{k}^j = f_k^j(\bar{l}^j, \bar{m}^j) \text{ for all } j \in J_k$$

[17] Recall that $\sigma_\kappa = 1-\gamma$ and $(c^j, k^j) \in \mathrm{Y}_\mathrm{F}^j$ and $j \in J_\kappa$ implies $c^j = 0$.

and the consumption optimal produced $\hat{c}^J : K^+ \to \mathbb{R}^J_+$ as

$$\hat{c}^j(k^{I+}_-) = f^j_c(\hat{\kappa}^j(k^{I+}_-), \bar{\ell}^j) \text{ for all } j \in J_c$$

moreover, define for $j \in J_k \cup J_c$ the profit $\hat{\pi}^j_\epsilon : \widehat{P} \times A \to \mathbb{R}_+$ as

$$\hat{\pi}^j_\epsilon(\hat{p}, \bar{a}_-) = \tilde{\pi}^j(\hat{p}_e(\bar{a}_-), \bar{k}^{I+}_-) \text{ for all } \bar{a}_- \in A.$$

We consider profits in capital rental sector defined per units of capital rented. Since $\hat{k}^j(\bar{a}_-) = -\sigma_\kappa \bar{k}^{I+}_- / \mathrm{J}_\kappa$ then by (10)

$$\hat{\kappa}^j(\bar{a}_-) = \hat{k}^j(\bar{a}_-)/\sigma_\kappa = -\bar{k}^{I+}_-/\mathrm{J}_\kappa$$

Therefore define $\hat{\pi}^j : \widehat{P}_e \times A \to \mathbb{R}_+$ for $j \in J_\kappa$ as

$$\begin{aligned}\hat{\pi}^j(\hat{p}_e, \bar{a}_-) &= \tilde{\pi}^j(\hat{p}_e(\bar{a}_-), \bar{k}^{I+}_-)/|\hat{\kappa}^j(\bar{a}_-)| \\ &= \max\{0, \hat{p}_\kappa(\bar{a}_-) - (1-\gamma)\hat{p}_k(\bar{a}_-)\}\end{aligned}$$

and write $\hat{\pi}(\hat{p}_e, \bar{a}_-) = [\hat{\pi}^j(\hat{p}_e, \bar{a}_-) : j \in J]$ for all $\bar{a}_- \in A$.

4.2 Agents' optimal choices

The agents' demand are given by the following result. First consider the definition of an arbitrage-free price.

Definition 4.2. We say that the transition vector $(\hat{a}, \hat{p}, \hat{\pi}) \in \widehat{A} \times \widehat{P} \times \widehat{Q}$ satisfies the arbitrage-free property if there exist transitions $\hat{\delta} : A \to (0,1)$, $\hat{\varphi} : A \to \mathbb{R}^I$ and $\hat{\nu} : A \to \mathbb{R}^I$ such that

1. $\hat{q}(\bar{a}_-) = \hat{\delta}(\bar{a}_-)\hat{q}(\hat{a}(\bar{a}_-)) + \hat{\delta}(\bar{a}_-)\hat{\pi}(\hat{p}_e, \hat{a}(\bar{a}_-))$ for all $\bar{a}_- \in A$
2. $\hat{\varphi}^i(\bar{a}_-) = \hat{\delta}(\bar{a}_-)\hat{p}_e(\hat{a}(\bar{a}_-))e^i + \hat{\delta}(\bar{a}_-)\hat{\varphi}^i(\hat{a}(\bar{a}_-)) - \bar{\beta}\hat{p}_e(\bar{a}_-)e^i$ for all $\bar{a}_- \in A$
3. $\hat{\nu}^i(\bar{a}_-) = \bar{\beta}\log(\bar{\beta}/\hat{\delta}(\bar{a}_-))(1-\bar{\beta})^{-1} + \bar{\beta}\hat{\nu}^i(\hat{a}(\bar{a}_-))$ for all $\bar{a}_- \in A$

Proposition 4.2. *Consider the transition vector* $(\hat{a}, \hat{p}, \hat{\pi}) \in \widehat{A} \times \widehat{P} \times \widehat{Q}$ *satisfying the arbitrage-free property. Define for each* $(a^i_-, a^i, \bar{a}_-) \in A^i \times A^i \times A$

$$\tilde{c}^i(a^i_-, a^i, \bar{a}_-) = (\hat{q}(\bar{a}_-) + \hat{\pi}(\hat{p}_e, \bar{a}_-))a^i_- - \hat{q}(\bar{a}_-)a^i + \hat{p}_e(\bar{a}_-)e^i. \tag{11}$$

Suppose that $\tilde{a}^i : A^i \times A \to A^i$ *and* $\tilde{v}^i : A^i \times A \to \mathbb{R}$ *satisfy*

1. $\hat{q}(\bar{a}_-)\tilde{a}^i(a^i_-, \bar{a}_-) = \bar{\beta}(\hat{q}(\bar{a}_-) + \hat{\pi}(\hat{p}_e, \bar{a}_-))a^i_- - \hat{\varphi}^i(\bar{a}_-)$ *for all* $(a^i_-, \bar{a}_-) \in A^i \times A$;
2. $\tilde{v}^i(a^i_-, \bar{a}_-) = \log(\tilde{c}^i(a^i_-, \tilde{a}^i(a^i_-, \bar{a}_-), \bar{a}_-))(1-\bar{\beta})^{-1} + \hat{\nu}^i(\bar{a}_-)$ *for all* $(a^i_-, \bar{a}_-) \in A^i \times A$.

Then for each $(a^i_-, \bar{a}_-) \in A^i \times A$

$$\tilde{a}^i(a^i_-, \bar{a}_-) = \operatorname{argmax}\{\log(\tilde{c}^i(a^i_-, a^i, \bar{a}_-)) + \bar{\beta}\tilde{v}^i(a^i, \hat{a}(\bar{a}_-)) : a^i \in A^i\}$$
$$\tilde{v}^i(a^i_-, \bar{a}_-) = \max\{\log(\tilde{c}^i(a^i_-, a^i, \bar{a}_-)) + \bar{\beta}\tilde{v}^i(a^i, \hat{a}(\bar{a}_-)) : a^i \in A^i\}.$$

Proof: See Section 6.2 in the appendix. □

4.3 Recursive equilibrium

In this section we will show the recursive relations relying equilibrium prices, based on previous sections optimal choices. The recursive equilibrium existence is characterized by the following Theorem. First we exhibit a definition with auxiliary objects. Suppose hereafter that K^ι is bounded for all $\iota \in I \cup J$.

Definition 4.3. Let $Z^{I+} \subset \mathbb{R}_+$ a bounded set. We say that a function $g : Z \to \mathbb{R}$ is locally Lp_g-Lipschitz at $\bar{z} \in Z$ when

$$|g(z) - g(\bar{z})| \leq \text{Lp}_g\,|z - \bar{z}| \text{ for all } z \in Z.$$

In the notation below we will define an auxiliary set that will later represent the domain of the aggregate capital variable.

Notation 4.1. Given a fixed $\bar{k}_{-}^{I+} \in \mathbb{R}_{++}$ and $\Delta_{\mathrm{K}} < \bar{k}_{-}^{I+}$ define[18]

$$K^{+} = \{k_{-}^{I+} \in \mathbb{R}_{++} : |k_{-}^{I+} - \bar{k}_{-}^{I+}| \leq \Delta_{\mathrm{K}}\}.$$

The next definition establishes the sets of the residual capital demand and equity transition functions.

Definition 4.4. Given a fixed $\bar{k}_{-}^{I+} \in K^{+}$ define:

1. $\widehat{\Phi}^{I+}$ as the set of all transitions $\hat{\varphi}^{I+} : K^{+} \to \Phi^{I+}$ continuous locally Lp_{φ}-Lipschitz at $\bar{k}_{-}^{I}$ and such that $\hat{\varphi}^{I+}(\bar{k}_{-}^{I+}) = 0$.

2. $\widehat{P}_{\epsilon}^{J+}$ as the set of all transitions $\hat{p}_{\epsilon}^{J+} : K^{+} \to P_{\epsilon}^{J+}$ continuous locally $\mathrm{Lp}_{p\epsilon}$-Lipschitz at $\bar{k}_{-}^{I}$ and such that $\hat{p}_{\epsilon}^{J+}(\bar{k}_{-}^{I+}) = 0$

3. $\widehat{\Phi}$ as the set of all transitions $\hat{\varphi} : K^{+} \to \Phi$ such that $\hat{\varphi}^{I+} \in \widehat{\Phi}^{I+}$.

4. $\widehat{P}_{\epsilon}$ as the set of all transitions $\hat{p}_{\epsilon} : K^{+} \to P_{\epsilon}$ such that $\hat{p}_{\epsilon}^{J+} \in \widehat{P}_{\epsilon}^{J+}$

Define the metric on $\widehat{P}_{\epsilon}^{J+} \times \widehat{\Phi}^{I+}$ by[19]

$$||(\hat{p}_{\epsilon}^{J+}, \hat{\varphi}^{I+})|| = \sup\{\max\{|\hat{p}_{\epsilon}^{J+}(k_{-}^{I+})|, |\hat{\varphi}^{I+}(k_{-}^{I+})|\} : k_{-}^{I+} \in K^{+}\}.$$

and the metric on $\widehat{P}_{\epsilon} \times \widehat{\Phi}$ by[20]

$$||(\hat{p}_{\epsilon}, \hat{\varphi})|| = \sup\{\max\{||\hat{p}_{\epsilon}(k_{-}^{I+})||, ||\hat{\varphi}(k_{-}^{I+})||\} : k_{-}^{I+} \in K^{I}\}.$$

Remark 4.2. Note that $\widehat{P}_{\epsilon}^{J+} \times \widehat{\Phi}^{I+}$ and $\widehat{P}_{\epsilon} \times \widehat{\Phi}$ endowed with the metric $||\cdot||$ are complete metric spaces. Write $\xi^{+} = (\hat{p}_{\epsilon}^{J+}, \hat{\varphi}^{I+})$ as a typical element of $\widehat{P}_{\epsilon}^{J+} \times \widehat{\Phi}^{I+}$ and $\xi = (\hat{p}_{\epsilon}, \hat{\varphi})$ as a typical element of $\widehat{P}_{\epsilon} \times \widehat{\Phi}$.

[18] We do not consider the dependence of K^{+} on $\bar{k}_{-}^{I+}$ and Δ_{K} for the sake of simplicity.
[19] Consider $||\cdot||$ as the max norm.
[20] Consider $||\cdot||$ as the max norm.

Definition 4.5. Define the auxiliary aggregate functions

$$\check{p}_k : \widehat{P}_\epsilon^{J+} \times \widehat{\Phi}^{I+} \times K^+ \to \mathbb{R}_+ \quad \check{p}_e^{I+} : \widehat{P}_\epsilon^{J+} \times \widehat{\Phi}^{I+} \times K^+ \to \mathbb{R}_+$$
$$\check{k}^{I+} : K^+ \to \mathbb{R}_+ \quad \check{p}_\kappa : K^+ \to \mathbb{R}_+$$
$$\check{\delta} : \widehat{P}_\epsilon^{J+} \times \widehat{\Phi}^{I+} \times K^+ \to \mathbb{R}_+ \quad \check{\pi}_\epsilon^{J+} : \widehat{P}_\epsilon^{J+} \times \widehat{\Phi}^{I+} \times K^+ \to \mathbb{R}_+$$

for each $(\xi^+, k_-^{I+}) \in \widehat{P}_\epsilon^{J+} \times \widehat{\Phi}^{I+} \times K^+$ as

$$\check{p}_k(\xi^+, k_-^{I+}) = \frac{\bar{\beta}\hat{c}^{J+}(k_-^{I+}) - (1-\bar{\beta})\hat{p}_\epsilon^{J+}(k_-^{I+}) - \bar{\beta}\sigma_\ell^{J+} - \hat{\varphi}^{I+}(k_-^{I+})}{(1-\bar{\beta})(\gamma k_-^{I+} + \bar{k}^{J+}) + \bar{\beta}(\sigma_l^{J+} + \sigma_m^{J+})} \tag{12}$$

$$\check{p}_e^{I+}(\xi^+, k_-^{I+}) = (\sigma_l^{J+} + \sigma_m^{J+})\check{p}_k(\xi^+, k_-^{I+}) + \sigma_\ell^{J+} + e_c^{I+} \tag{13}$$

$$\check{k}^{I+}(k_-^{I+}) = \gamma k_-^{I+} + \bar{k}^{J+} \tag{14}$$

$$\check{p}_\kappa(k_-^{I+}) = \frac{\sigma_\kappa^{J+}}{k_-^{I+}} \tag{15}$$

$$\check{\delta}(\xi^+, k_-^{I+}) = \frac{\check{p}_k(\xi^+, k_-^{I+})}{\gamma\check{p}_k\big(\xi^+, \check{k}^{I+}(k_-^{I+})\big) + \check{p}_\kappa(\check{k}^{I+}(k_-^{I+}))} \tag{16}$$

$$\check{\pi}_\epsilon^{J+}(\xi^+, k_-^{I+}) = \check{p}_k(\xi^+, k_-^{I+})(\bar{k}^{J+} - \sigma_m^{J+} - \sigma_l^{J+}) + \hat{c}^{J+}(k_-^{I+}) - \sigma_\kappa^{J+} - \sigma_\ell^{J+} \tag{17}$$

Remark 4.3. Suppose that $\bar{k}_-^{I+} = \bar{k}^{J+}/(1-\gamma)$. Then $\check{k}^{I+}(k_-^{I+}) \in K^+$ for all $k_-^{I+} \in K^+$. Indeed, $\check{k}^{I+}(\bar{k}_-^{I+}) = \bar{k}_-^{I+}$ and hence

$$|\check{k}^{I+}(k_-^{I+}) - \bar{k}_-^{I+}| = |\check{k}^{I+}(k_-^{I+}) - \check{k}^{I+}(\bar{k}_-^{I+})| = \gamma|k_-^{I+} - \bar{k}_-^{I+}| < \Delta_{\mathrm{K}}$$

Notation 4.2. Write $\widehat{\mathbb{R}}^n$ as the set of all continuous functions from K^+ to $\mathbb{R}^n$ for each $n \in \mathbb{N}$.

Definition 4.6. Define the aggregate operator $\hat{\xi}^+ : \widehat{P}_\epsilon^{J+} \times \widehat{\Phi}^{I+} \to \widehat{\mathbb{R}}^2$ as following. Let $\xi^+ = (\hat{p}_\epsilon^{J+}, \hat{\varphi}^{I+})$ be an arbitrary vector function. Write $\hat{\xi}^+ = (\hat{\xi}_\epsilon^+, \hat{\xi}_\varphi^+)$ where

$$\hat{\xi}_\epsilon^+(\xi^+)(k_-^{I+}) = \check{\delta}(\xi^+, k_-^{I+})\hat{p}_\epsilon^{J+}(\check{k}^{I+}(k_-^{I+})) + \check{\delta}(\xi^+, k_-^{I+})\check{\pi}_\epsilon^{J+}(\xi^+, \check{k}^{I+}(k_-^{I+}))$$
$$\hat{\xi}_\varphi^+(\xi^+)(k_-^{I+}) = \check{\delta}(\xi^+, k_-^{I+})\big(\check{p}_e^{I+}(\xi^+, \check{k}^{I+}(k_-^{I+})) + \hat{\varphi}^{I+}(\check{k}^{I+}(k_-^{I+}))\big) - \bar{\beta}\check{p}_e^{I+}(\xi^+, k_-^{I+})$$

The assumption below establishes conditions on the exogenous parameters and it will be used in the existence theorem.

Assumption 4.1. Suppose that

$$\frac{(1+\bar{\beta}\gamma^2)(\sigma_\ell^{J+}+e_c^{I+})}{\sigma_\kappa^{J+}(1-\bar{\beta}\gamma)}+\frac{\bar{\beta}(1-\gamma)(2+\gamma+\bar{\beta}\gamma^2)}{(1-\bar{\beta}\gamma)^2}<1$$
$$\frac{\bar{\beta}(1-\gamma)(3+\bar{\beta}\gamma^2)}{(1-\bar{\beta}\gamma)(2-\bar{\beta})^{-1}}+\frac{\left(1+\bar{\beta}\gamma^2\right)\left(e_c^{I+}+\sigma_\ell^{J+}\right)}{\sigma_\kappa^{J+}(2-\bar{\beta})^{-1}}<1-\bar{\beta}.$$

The next theorem states that Assumption 4.1 ensures existence of recursive equilibrium. Note that $\sigma_\ell^{J+}+e_c^{I+}<<\sigma_\kappa^{J+}$ and $\bar{\beta}<<\gamma\approx 1$ are crucial for that conditions to be satisfied.

Theorem 4.1. *Suppose that Assumption 4.1 holds. Define the*[21] *aggregate capital* $\bar{k}_-^{I+}=(\sigma_l^{J+}+\sigma_m^{J+})/(1-\gamma)$ *and assume that firms have zero profits at* $\bar{k}_-^{I+}$*. Then for* Δ_{K} *sufficiently small we have*

1. $\hat{\xi}^+(\widehat{P}_\epsilon^{J+}\times\widehat{\Phi}^{I+})\subset\widehat{P}_\epsilon^{J+}\times\widehat{\Phi}^{I+}$
2. $\hat{\xi}^+$ *is a contraction*
3. $\hat{\xi}^+$ *has a single fixed point.*

Proof: See Section 6.3 in the appendix □

The next result states that the recursive equilibrium can be constructed using the fixed point of $\hat{\xi}^+$.

Definition 4.7. Consider $\bar{\xi}^+=(\widehat{p}_\epsilon^{J+},\widehat{\varphi}^{I+})$ the fixed point of $\hat{\xi}^+$. Write for each

[21] Actually, $\bar{k}_-^{I+}$ will be the steady state.

$k_-^{I+} \in K^+$

$$\check{p}_\ell(k_-^{I+}) = \frac{\sigma_\ell^{J+}}{e_\ell^{I+}} \quad \check{p}_l(k_-^{I+}) = \frac{\sigma_l^{J+}\check{p}_k(\bar{\xi}^+, k_-^{I+})}{e_l^{I+}} \quad \check{p}_m(k_-^{I+}) = \frac{\sigma_m^{J+}\check{p}_k(\bar{\xi}^+, k_-^{I+})}{e_m^{I+}}$$

$$\check{p}_e(k_-^{I+}) = [\check{p}_\kappa(k_-^{I+}), \check{p}_\ell(k_-^{I+}), \check{p}_l(k_-^{I+}), \check{p}_m(k_-^{I+}), \bar{p}_c, \check{p}_k(\bar{\xi}^+, k_-^{I+})]$$

$$\check{\pi}_\epsilon^j(k_-^{I+}) = \begin{cases} \hat{c}^j(k_-^{I+}) - \sigma_\kappa^j - \sigma_\ell^j \text{ if } j \in J_c \\ \check{p}_k(\bar{\xi}^+, k_-^{I+})(\bar{k}^j - \sigma_m^j - \sigma_l^j) \text{ if } j \in J_k \end{cases}$$

$$\check{\pi}_\epsilon(k_-^{I+}) = [\check{\pi}_\epsilon^j(k_-^{I+}) : j \in J_k \cup J_c]$$

Remark 4.4. Note that by definition

$$\check{p}_e(k_-^{I+})e^{I+} = \check{p}_e^{I+}(\bar{\xi}, k_-^{I+}) \text{ for all } k_-^{I+} \in K^{I+}. \tag{18}$$

Definition 4.8. Consider $\bar{\xi}^+ \in \widehat{P}_\epsilon^{J+} \times \widehat{\Phi}^{J+}$ a fixed transition. Define the vector operator $\hat{\xi} : \widehat{P}_\epsilon \times \widehat{\Phi} \to \widehat{P}_\epsilon \times \widehat{\Phi}$ as following. Let $\xi = (\hat{p}_\epsilon, \hat{\varphi})$ be an arbitrary vector function. Write $\hat{\xi} = (\hat{\xi}_\epsilon, \hat{\xi}_\varphi)$ where

$$\hat{\xi}_\epsilon(\xi)(k_-^{I+}) = \check{\delta}(\bar{\xi}^+, k_-^{I+})\hat{p}_\epsilon(\check{k}^{I+}(k_-^{I+})) + \check{\delta}(\bar{\xi}^+, k_-^{I+})\check{\pi}_\epsilon(\check{k}^{I+}(k_-^{I+})) \tag{19}$$

$$\hat{\xi}_\varphi(\xi)(k_-^{I+}) = \check{\delta}(\bar{\xi}^+, k_-^{I+})\big(\check{p}_e(\check{k}^{I+}(k_-^{I+}))e + \hat{\varphi}(\check{k}^{I+}(k_-^{I+}))\big) - \bar{\beta}\check{p}_e(k_-^{I+})e \tag{20}$$

Remark 4.5. We claim that $\hat{\xi}$ is well defined. Actually, suppose that $\xi = (\hat{p}_\epsilon, \hat{\varphi}) \in \widehat{P}_\epsilon \times \widehat{\Phi}$. Then by definition, $(\hat{p}_\epsilon^{J+}, \hat{\varphi}^{I+}) \in \widehat{P}_\epsilon^{J+} \times \widehat{\Phi}^{I+}$. Therefore, adding equations (19) and (20) over $j \in J$ and $i \in I$ respectively, we obtain that $(\widehat{p}_\epsilon, \widehat{\varphi}) := \hat{\xi}(\xi) \in \widehat{P}_\epsilon \times \widehat{\Phi}$ by (18). Indeed, the conditions

$$|\widehat{\varphi}^{I+}(k_-^{I+})| \leq \text{Lp}_\varphi |k_-^{I+} - \bar{k}_-^{I+}| \text{ for all } k_-^{I+} \in K^+$$

$$|\widehat{p}_\epsilon^{J+}(k_-^{I+})| \leq \text{Lp}_{p\epsilon} |k_-^{I+} - \bar{k}_-^{I+}| \text{ for all } k_-^{I+} \in K^+$$

comes from the proof of Theorem 4.1 by replacing ξ^+ with $\bar{\xi}^+$. This is the same as state that $(\widehat{p}_\epsilon^{J+}, \widehat{\varphi}^{I+}) \in \widehat{P}_\epsilon^{J+} \times \widehat{\Phi}^{I+}$.

Theorem 4.2. *Suppose that Assumption 4.1 holds. Then the operator $\hat{\xi}$ defined in Definition 4.7 has a single fixed point.*

Proof. We know by Theorem 4.1 that $\sup\{\check{\delta}(\bar{\xi}^+, k_-^{I+}) : k_-^{I+} \in K^+\} < 1$. Therefore, $\hat{\xi}$ is a contraction over a complete metric space since $\bar{\xi}$ is fixed and Equations (19) and (20) are independent. The result follows using the Contraction Map Theorem. □

Theorem 4.3. *Suppose that Assumption 4.1 holds. Let Δ_{K} be the aggregate capital boundary given in Theorem 4.1 for $\bar{k}_-^{I+}$. Define*

$$K^+ = \{k_-^{I+} \in \mathbb{R}_+ : |k_-^{I+} - \bar{k}_-^{I+}| \leq \Delta_{\text{K}}\}.$$

Consider $\xi^+ = (\hat{p}_\epsilon^{J+}, \hat{\varphi}^{I+})$ the fixed point of $\hat{\xi}^+$ and let $\xi = (\hat{p}'_\epsilon, \hat{\varphi}')$ be the fixed point of $\hat{\xi}$. Define for each[22] *$(a_-^i, \bar{a}_-) \in A^i \times A$ and $(i, j) \in I \times J$*

$$\hat{p}_\epsilon(\bar{a}_-) = \hat{p}'_\epsilon(\bar{k}_-^{I+}) \quad \hat{\varphi}(\bar{a}_-) = \hat{\varphi}'(\bar{k}_-^{I+}) \quad \hat{p}_\kappa(\bar{a}_-) = \frac{\sigma_\kappa^{J+}}{\bar{k}_-^{I+}} \tag{21}$$

$$\hat{p}_k(\bar{a}_-) = \check{p}_k(\xi^+, \bar{k}_-^{I+}) \quad \hat{p}_\ell(\bar{a}_-) = \frac{\sigma_\ell^{J+}}{e_\ell^{I+}} \quad \hat{p}_e(\bar{a}_-) = \check{p}_e(k_-^{I+}) \tag{22}$$

$$\hat{p}_l(\bar{a}_-) = \frac{\sigma_l^{J+}\hat{p}_k(\bar{a}_-)}{e_l^{I+}} \quad \hat{p}_m(\bar{a}_-) = \frac{\sigma_m^{J+}\hat{p}_k(\bar{a}_-)}{e_m^{I+}} \quad \hat{p} = [\hat{p}_e, \hat{p}_\epsilon] \tag{23}$$

$$\hat{k}^i(\bar{a}_-) = \frac{\bar{\beta}(\hat{q}(\bar{a}_-) + \hat{\pi}(\hat{p}_e, \bar{a}_-))\bar{a}_-^i - \hat{\varphi}^i(\bar{a}_-) - \hat{p}_\epsilon(\bar{a}_-)\bar{\epsilon}_-^i}{\hat{p}_k(\bar{a}_-)} \tag{24}$$

$$\hat{c}^i(\bar{a}_-) = (1 - \bar{\beta})(\hat{q}(\bar{a}_-) + \hat{\pi}(\hat{p}_e, \bar{a}_-))a_-^i + \hat{p}_e(\bar{a}_-)e^i + \hat{\varphi}^i(\bar{a}_-) \tag{25}$$

$$\hat{a}^i(\bar{a}_-) = (\hat{k}^i(\bar{a}_-), \bar{\epsilon}_-^i) \quad \hat{\epsilon}^i(\bar{a}_-) = \bar{\epsilon}_-^i \tag{26}$$

$$\hat{\delta}(\bar{a}_-) = \check{\delta}(\xi^+, \bar{k}_-^{I+}) \quad \hat{v}^i = \hat{\mu}_v^i(\hat{v}, \hat{p}, \hat{a}) \quad \hat{v}^j = \hat{\mu}_v^j(\hat{p}) \quad \hat{y}^j = \hat{\mu}_y^j(\hat{p}) \tag{27}$$

Then $(\hat{x}, \hat{y}, \hat{p}, \hat{v})$ is an equilibrium.[23]

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[22] Recall that $\hat{q} = (\hat{p}_k, \hat{p}_\epsilon)$.

[23] Note that for each $\bar{a}_- \in A$ we can choose any asset transition $\hat{k}^I(\bar{a}_-)$ with the same capital aggregate transition, say $\gamma\bar{k}_-^{I+} + \bar{k}^{J+}$.

Proof. See Section 6.4 in the appendix. □

The next corollary specifies the shape of the minimal state space in the recursive equilibrium.

Corollary 4.1. *Suppose that Assumption 4.1 holds. Let K^+ be the auxiliary domain given in Theorem 4.3. Define $\bar{K}^i \subset \mathbb{R}_+$ for $i \in I$ as*

$$\bar{K}^i = \mathrm{pr}^i\{k^I \in \mathbb{R}_+^I : k^{I+} \in K^+\}$$

where $\mathrm{pr}^i : \mathbb{R}^I \to \mathbb{R}$ is the i-th coordinate projection defined by $\mathrm{pr}^i(k^I) = k^i$ for all $k^I \in \mathbb{R}^I$. Then the domain A of the recursive equilibrium $(\hat{x}, \hat{y}, \hat{p}, \hat{v})$ could be replaced by $\bar{A} = \prod_{i \in I} \bar{A}^i$ where

$$\bar{A}^i = \{(\bar{k}^i, \bar{\epsilon}^i) : \bar{k}^i \in \bar{K}^i\} \subset A^i \text{ for all } i \in I.$$

Proof. Consider $\hat{k}^I : A \to K^I$ the recursive capital transition equilibrium. Then by Remark 4.3

$$\bar{k}^{I+} \in K^+ \text{ implies } \sum_{i \in I} \hat{k}^i(\bar{k}^i, \bar{\epsilon}^i) = \gamma \bar{k}^{I+} + \bar{k}^{J+} \in K^+$$

therefore, $\hat{k}^i(\bar{k}^i, \bar{\epsilon}^i) \in \bar{K}^i$ for all $i \in I$. □

4.4 Steady state

The close form solution of the steady state is based on the following theorem.

Theorem 4.4. *Suppose that $\bar{a}_- = (\bar{k}_-, \bar{\epsilon}_-)$ is a steady state allocation of the equilibrium $(\hat{x}, \hat{y}, \hat{p}, \hat{v})$, that is, $\hat{a}^i(\bar{a}_-) = \bar{a}^i_-$ for all $i \in I$. Then*

$$\hat{\delta}(\bar{a}_-) = \bar{\beta} \text{ and } \hat{\varphi}^i(\bar{a}_-) = 0$$

$$\hat{p}_k(\bar{a}_-) = \frac{\bar{\beta}\sigma_\kappa^{J+}}{(1 - \bar{\beta}\gamma)\bar{k}_-^{I+}} \text{ and } \hat{p}_\kappa(\bar{a}_-) = \frac{\sigma_\kappa^{J+}}{\bar{k}_-^{I+}} \text{ and } \hat{q}(\bar{a}_-) = \frac{\bar{\beta}\hat{\pi}(\hat{p}_e, \bar{a}_-)}{1 - \bar{\beta}}$$

Proof: See Section 6.5 in the appendix. □

4.5 Example

We assume that the economy has only three firms and two agents sharing the endowment of assets. We consider three assets, that is, $A^i \subset \mathbb{R}^3_+$. Recall that $a^i = (k^i, a^i_1, a^i_2)$ for $i \in I$. More precisely, $\hat{a}^i(\bar{a}_-) = \bar{a}^i = (\bar{k}^i_-, \bar{\epsilon}^i_1, \bar{\epsilon}^i_2)$ for $i \in I$, $\bar{\epsilon}^1 = (1, 0)$ and $\bar{\epsilon}^2 = (0, 1)$. Denote the steady state as[24] $\bar{s} = [\bar{a}^i_-]_{i \in I}$. Agent one works on capital sector and agent two on good sector. Assume also that only agent one has endowments of primary capital, that is, $e^i_m = 0$ for $i = 2$.

Assume that $\bar{\beta} = 0.8$,$\gamma = 0.7$, $e_\ell = [0, 2.718]$, $e_l = [2.718, 0]$, $e_m = [2.718, 0]$ and $e_c = [1/2, 1/2]$. Suppose also that

$$(\sigma^j_l)_{j \in J} = (0.4077, 0);\;\; (\sigma^j_m)_{j \in J} = (0.4077, 0);\;\; (\sigma^j_\kappa)_{j \in J} = (0, 10);\;\; (\sigma^{J+}_\ell)_{j \in J} = (0, 1).$$

Then the equilibrium price is given by

$$p = (0.04719, 0.1, 0.0186276, 0.0223532, 1, 0.223532, 11.0173, 135.558)$$

the optimal agents' allocations

$$\hat{x}^i(\bar{a}_-) = (0, 8, 0, 0, 4.63984, 30, 1, 0) \text{ for } i = 1$$
$$\hat{x}^i(\bar{a}_-) = (0, 0, 6, 0, 21.4222, 54.7637, 0, 1) \text{ for } i = 2$$
$$\hat{y}^j(\bar{a}_-) = (0, 0, -12, -20, 0, 8.47637, 0, 0) \text{ for } j = 1$$
$$\hat{y}^j(\bar{a}_-) = (-84.7637, -10, 0, 0, 20.0621, 0, 0, 0) \text{ for } j = 2.$$

5 Conclusion

This paper presents a closed form recursive equilibrium solution of a multi-sector growth model with heterogeneity. We exhibit a Lipschitz recursive equilibrium

[24] Note that there is no exogenous uncertainty.

with a minimal state space that could be computed through a contraction on a Banach space and which implements the sequential equilibrium through the composition of the transition functions. We conclude that the aggregate capital can be viewed as a minimal state space.

6 Appendix

Remark 6.1. Note that[25] $\hat{\pi}^j(\hat{p}_e, \bar{a}_-) = \hat{p}_\kappa(\bar{a}_-) - (1-\gamma)\hat{p}_k(\bar{a}_-)$ for $j = 1$ and since $a^i_- = (k^i_-, \epsilon^i_-)$ then

$$\begin{aligned}\hat{\pi}(\hat{p}_e, \bar{a}_-)a^{I+}_- &= (\hat{p}_\kappa(\bar{a}_-) - (1-\gamma)\hat{p}_k(\bar{a}_-))k^{I+}_- + \hat{\pi}_\epsilon(\hat{p}, \bar{a}_-)\epsilon^{I+}_- \\ &= (\hat{p}_\kappa(\bar{a}_-) - (1-\gamma)\hat{p}_k(\bar{a}_-))k^{I+}_- + \hat{c}^{J+}(\bar{a}_-) - \sigma^{J+}_\kappa - \sigma^{J+}_\ell \\ &+ \hat{p}_k(\bar{a}_-)(\bar{k}^{J+} - \sigma^{J+}_m - \sigma^{J+}_l)\end{aligned}$$

and hence

$$\begin{aligned}(\hat{q}(\bar{a}_-) + \hat{\pi}(\hat{p}_e, \bar{a}_-))a^{I+}_- &= \big(\hat{p}_k(\bar{a}_-) + \hat{p}_\kappa(\bar{a}_-) - (1-\gamma)\hat{p}_k(\bar{a}_-)\big)k^{I+}_- \\ &+ (\hat{p}_\epsilon(\bar{a}_-) + \hat{\pi}_\epsilon(\hat{p}, \bar{a}_-))\epsilon^{I+}_- \\ &= (\gamma\hat{p}_k(\bar{a}_-) + \hat{p}_\kappa(\bar{a}_-))k^{I+}_- + \hat{c}^{J+}(\bar{a}_-) - \sigma^{J+}_\kappa - \sigma^{J+}_\ell \\ &+ \hat{p}_k(\bar{a}_-)(\bar{k}^{J+} - \sigma^{J+}_m - \sigma^{J+}_l) + \hat{p}_\epsilon(\bar{a}_-)\epsilon^{I+}_-\end{aligned} \tag{28}$$

Moreover, if $\hat{p}$ is a recursive equilibrium

$$\begin{aligned}\hat{p}_l(\bar{a}_-)e^{I+}_l = \sigma^{J+}_l\hat{p}_k(\bar{a}_-) \text{ and } \hat{p}_m(\bar{a}_-)e^{I+}_m = \sigma^{J+}_m\hat{p}_k(\bar{a}_-) \\ \hat{p}_\kappa(\bar{a}_-)\bar{k}^{I+}_- = \sigma^{J+}_\kappa \text{ and } \hat{p}_\ell(\bar{a}_-)e^{I+}_\ell = \sigma^{J+}_\ell\end{aligned} \tag{29}$$

and by the definition of $\hat{p}_e$

$$\begin{aligned}\hat{p}_e(\bar{a}_-)e^{I+} &= \hat{p}_l(\bar{a}_-)e^{I+}_l + \hat{p}_m(\bar{a}_-)e^{I+}_m + \hat{p}_\ell(\bar{a}_-)e^{I+}_\ell + p_c(\bar{a}_-)e^{I+}_c \\ &= \hat{p}_k(\bar{a}_-)(\sigma^{J+}_l + \sigma^{J+}_m) + \sigma^{J+}_\ell + e^{I+}_c\end{aligned}$$

Therefore,

$$\hat{\pi}(\hat{p}_e, \bar{a}_-)\bar{a}^{I+}_- + \hat{p}_e(\bar{a}_-)e^{I+} = \hat{c}^{J+}(\bar{a}_-) + \hat{p}_k(\bar{a}_-)(\bar{k}^{J+} - (1-\gamma)\bar{k}^{I+}_-) + e^{I+}_c$$

and

$$\hat{p}_e(\bar{a}_-)e^{I+} = \hat{p}_k(\bar{a}_-)(\sigma^{J+}_l + \sigma^{J+}_m) + \sigma^{J+}_\ell + e^{I+}_c \text{ for all } \bar{a}_- \in A. \tag{30}$$

Remark 6.2. The following items are useful in the arguments of Theorem 4.1

[25] Later on we will show that $\tilde{\pi}^j(p_e, \bar{k}^{I+}_-) \geq 0$ at the equilibrium.

1. Note that if $g : K^+ \to \mathbb{R}$ is locally Lp_g-Lipschitz at $\bar{k}_-^{I+}$ then

$$-\mathrm{Lp}_g\,|k_-^{I+} - \bar{k}_-^{I+}| \le g(k_-^{I+}) - g(\bar{k}_-^{I+}) \le \mathrm{Lp}_g\,|k_-^{I+} - \bar{k}_-^{I+}| \text{ for all } k_-^{I+} \in K^+.$$

and hence

$$g(\bar{k}_-^{I+}) - \mathrm{Lp}_g\,\Delta_{\mathrm{K}} \le g(k_-^{I+}) \le g(\bar{k}_-^{I+}) + \mathrm{Lp}_g\,\Delta_{\mathrm{K}} \text{ for all } k_-^{I+} \in K^+.$$

2. Consider a locally Lp_h-Lipschitz $h : K^+ \to K^+$ at $\bar{k}_-^{I+}$ and assume that $g : K^+ \to \mathbb{R}$ is locally Lp_g-Lipschitz at $\bar{k}_-^{I+}$. Suppose that $h(\bar{k}_-^{I+}) = \bar{k}_-^{I+}$. Then

$$\begin{aligned}
|g(h(k_-^{I+})) - g(h(\bar{k}_-^{I+}))| &= |g(h(k_-^{I+})) - g(\bar{k}_-^{I+})| \\
&\le \mathrm{Lp}_g\,|h(k_-^{I+}) - \bar{k}_-^{I+}| \\
&= \mathrm{Lp}_g\,|h(k_-^{I+}) - h(\bar{k}_-^{I+})| \\
&\le \mathrm{Lp}_g\,\mathrm{Lp}_h\,|k_-^{I+} - \bar{k}_-^{I+}|.
\end{aligned}$$

6.1 Proof of Proposition 4.1.

(...)

6.2 Proof of Proposition 4.2

(...)

6.3 Proof of Theorem 4.1

(...)

6.3.1 Condition on the locally Lipschitz constant $\mathrm{Lp}_{p\epsilon}$

(...)

6.3.2 Condition on the locally Lipschitz transition $\hat{p}_\kappa$

(...)

6.3.3 Condition on the locally Lipschitz transition $\hat{c}^{J+}$

(...)

6.3.4 Condition on the locally Lipschitz transition $\check{p}_k$

(...)

6.3.5 Condition on the locally Lipschitz transition $\check{\delta}$

(...)

6.3.6 Condition on the locally Lipschitz transition $\hat{\xi}^+_\varphi(\xi^+)$

(...)

6.3.7 Condition on the locally Lipschitz transition $\hat{\xi}^+_{p\epsilon}(\xi^+)$

(...)

Proof of Item 2

(...)

6.3.8 Condition on the contraction $\hat{\xi}^+_\varphi$

(...)

6.3.9 Condition on the contraction $\hat{\xi}^+_\epsilon$

(...)

6.4 Proof of Theorem 4.2

(...)

6.5 Proof of Theorem 4.4

(...)

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