COMPROMISE RULES TO SELECT GROUPS OF FIXED SIZE

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Preliminary version: March 28, 2022

Abstract

We propose three mechanisms for two parties to jointly select a group of fixed size. We show that if the parties' preferences over sets are leximin extensions of the parties' preferences over candidates then these mechanisms implement the Unanimity Compromise Set. This work extends the concepts and the results in Barberà and Coelho (2022), in which the parties had to choose a single candidate, to cover a wide class of natural applications.

Keywords: The Unanimity Compromise Set, Compromise Rule of k Names, Shortlisting Contest, Alternate Shortlists, Shortlisting, Voting by Alternating Offers and Vetoes and Fallback Bargaining. **JEL classification:** D02, D71, D72.

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1 Introduction

In a recent paper, Barberà and Coelho (2022) propose three methods to achieve compromise between two concerned parties. These methods apply to cases where someone has to be appointed to develop a task, and two parties with conflicting, but not always diametrically opposed, are given a say about who should be chosen. Examples of such situations include the appointment of an arbitrator, or the selection of a judge to fill a vacancy in a court.

Even if the case where only one candidate must be selected is important, for these and other examples, there are also many instances where the concerned parties must end up choosing a set of individuals.

In the case of arbitration, when high amounts are at stake, the claims of two opponents are heard by a panel composed of three arbitrators instead of a sole one. In such cases, many arbitral institutions stipulate that if a panel of three is to be selected, each party shall choose one member, and then the two party-appointed arbitrators shall agree on a third one, who will act as president. However, other arbitration agreements require that all three arbitrators should be jointly selected by the two parties. For instance, one of the main providers of arbitration services in the US, JAMS (JAMS Arbitration, Mediation and ADR services), proceeds as follows. It sends to both parties in dispute a list of at least ten candidates in the case of a tripartite panel. Each party may then strike three names and shall rank the remaining ones in order of preference. Finally, the three candidates with the highest composite ranking are appointed. Here, we show that the JAMS' method may induce a Pareto inefficient outcome and we propose alternatives to this procedures.

For another example, take the case of judiciary appointments. The Brazilian constitution establishes that, when there is one vacancy in the Superior Court of Justice, the same court will present three candidates to the President of the Republic, who will nominate a candidate from that list. Yet, not infrequently, more than one vacant position in that court needs to be filled simultaneously, due to deaths or retirements close in time. In such situations, it would be natural to expect the court to apply the procedure established by the constitution successively until all positions were filled, but the Brazilian court has not followed this path. Instead, it has presented a list of size equal to the number of vacant positions plus two, for the president of the republic to appoint simultaneously, out of this list, the needed number of candidates to the vacant positions. Notice that when there is one position vacant the size of the list is equal to three names as established by the constitution. We see two possible explanations for the court to act this way. This first one is expediency: it is faster than applying the constitutional proviso sequentially. The second one is that it is a way to increase the court's power, because it reduces the number of candidates that can be vetoed by the president, as the number of magistrates to be elected increases.

A recent event in Spanish parliamentary life also raises a question that is very much related to this one. The renewal of different bodies of the judiciary was frozen because there was no way to get the required two thirds of votes necessary to appoint new members unless two dominant parties agreed, and the party who appointed the existing juries was interested in keeping the status quo. The fact is that these bodies are of a good size (say ten people), and that a number of possibilities are open. One is to let each decisive party to choose five. This procedure has been used in the past and it is a simple one, but it induces very polarized outcomes. Again, in this case, and also in that of Brazil using mechanisms of the sort that we propose could lead to more satisfactory compromise solutions than the ones resulting from present practices.

In this paper we generalize the three methods proposed by Barberà and Coelho (2022) to select $\theta \in \{1, ..., \#\mathbf{C}\}$ candidates from a set of candidates $\mathbf{C}$. These methods are based on the Rule of k names. Before presenting our methods, we need first present the Rule of k Names and the Unanimity Compromise Set adapted to this context of selecting more than one candidate**.**

The $\theta-$**Rule of k Names** works as follows: one of the parties (the proposer) selects k candidates out of those in an original list $\mathbf{C}$, and then the other party (the chooser) chooses θ winners out of those selected by the opponent.

We characterize the unique SPNE outcome of the game induced by $\theta-$**Rule of k Names** and prove that if the preferences over sets are leximin extensions of the preferences over alternatives that this outcome is always Pareto Efficient.

For example, let $k = 3$, $\theta = 2$ and this preference profile over alternatives ($c1 \succ_1 c2 \succ_1 c3 \succ_1 c4 \succ_1 c5$ and $c5 \succ_2 c2 \succ_2 c3 \succ_2 c4 \succ_2 c1$). Each party knows its opponent's preferences. Suppose that the preferences over sets are leximin extensions of the preferences over alternatives as follows: $\{c1, c2\} \succ_1 \{c1, c3\} \succ_1 \{c2, c3\} \succ_1 \{c1, c4\} \succ_1 \{c2, c4\} \succ_1 \{c3, c4\} \succ_1 \{c1, c5\} \succ_1 \{c2, c5\} \succ_1 \{c3, c5\} \succ_1 \{c4, c5\}$ and

$\{c2, c5\} \succ_2 \{c3, c5\} \succ_2 \{c2, c3\} \succ_2 \{c4, c5\} \succ_2 \{c2, c4\} \succ_2 \{c3, c4\} \succ_2 \{c1, c5\} \succ_2 \{c1, c2\} \succ_2 \{c1, c3\} \succ_2 \{c1, c4\}$

The following strategy profile is a subgame perfect Nash equilibrium (SPNE) of the game induced by the θ-Rule of k Names: the proposer (Party 1) proposes $\{c1, c2, c3\}$ and the chooser (Party 2) picks its θ best alternatives from it which is $\{c2, c3\}$. In case that Party 2 is the proposer, she proposes $\{c2, c3, c5\}$ and Party 1 picks $\{c2, c3\}$ from it. The $\{c2, c3\}$ is the unique equilibrium outcome and is a Pareto efficient set. Surprisingly, if the preferences over sets were leximax extensions, the set $\{c2, c3\}$ would still be the unique equilibrium outcome but would be Pareto dominated by $\{c1, c5\}$.[1]

We can now introduce our proposed mechanisms.

The $\theta-$**Compromise Rule of k Names** ($\theta-$**CRK**) works as follows: Party 1 chooses $k \in \{\theta, ...\mathbf{c}\}$. Once this choice is made public, Party 2 decides whether to play as the proposer or the chooser. Then the two parties play according to the rule of k names adapted to select θ candidates. For symmetry, we can think of a previous uniform lottery as having determined the first-mover.

The $\theta-$**Alternate Shortlists** ($\theta-$**ASL**) works as follows: Party 1 first proposes a non-empty subset of $\mathbf{C}$ with cardinality greater than or equal to θ. Then party 2 decides whether to immediately select the θ winning candidates from the subset proposed by party 1, or else counteract with a subset of cardinality one plus that of the set it rejects choosing from. In that case, 1 selects the θ winning candidates out of those presented by 2. For symmetry, we can think of a previous uniform lottery as having determined the first-mover.

The $\theta-$**Shortlisting Contest** ($\theta-$**SLC**) works as follows: both parties simultaneously propose a non-empty subset of $\mathbf{C}$ with cardinality greater than or equal to than or equal to θ. The subset with the highest cardinality prevails and whoever proposed the discarded subset shall select the θ winning candidates from the prevailing subset. If the cardinalities are the same and odd, the parties know that Party 1's proposed subset

[1] In this case the preferences would be: $\{c1, c2\} \succ_1 \{c1, c3\} \succ_1 \{c1, c4\} \succ_1 \{c1, c5\} \succ_1 \{c2, c3\} \succ_1 \{c2, c4\} \succ_1 \{c2, c5\} \succ_1 \{c3, c4\} \succ_1 \{c3, c5\} \succ_1 \{c4, c5\}$ and
$\{c2, c5\} \succ_2 \{c3, c5\} \succ_2 \{c4, c5\} \succ_2 \{c1, c5\} \succ_2 \{c2, c3\} \succ_2 \{c2, c4\} \succ_2 \{c2, c5\} \succ_2 \{c3, c4\} \succ_2 \{c1, c3\} \succ_2 \{c1, c4\}$

prevails, otherwise Party 2's proposed subset prevails.

Now let us give an intuitive explanation of how our mechanisms induce the parties to a compromise decision under complete information assumption. Following Barberà and Coelho (2022), we consider that this assumption is appropriate in the case of choosing arbitrators.

Consider again this preference profile ($c1 \succ_1 c2 \succ_1 c3 \succ_1 c4 \succ_1 c5$ and $c5 \succ_2 c2 \succ_2 c3 \succ_2 c4 \succ_2 c1$), its leximin extension and the $2-$Compromise Rule of k Names mechanism. The last two stages of this mechanism consist of different subgames that are characterized by a value of k and by who submits the k candidates (the proposer). If Party 2 played as the proposer, the equilibrium outcome would be $\{c2, c5\}$ if $k = 2$,$\{c2, c3\}$ if $k = 3$, $\{c2, c3\}$ if $k = 4$ or $\{c1, c2\}$ if $k = 5$. And if the proposer was Party 1, the equilibrium outcome would be $\{c1, c2\}$ if $k = 2$,$\{c2, c3\}$ if $k = 3$, $\{c2, c3\}$ if $k = 4$ or $\{c2, c5\}$ if $k = 5$. Knowing it, Party 2 would opt to be the proposer if $k = 4$. Consequently, Party 1's best strategy is to choose $k = 4$ in order to ensure the election of $\{c2, c3\}$. Under the $2-$Alternate Shortlists method, in equilibrium, Party 1 proposes $\{c1, c2, c3, c4\}$ and Party 2 decides to pick $\{c2, c3\}$ since she knows that she cannot induce a better outcome by proposing a subset with five alternatives. Under the $2-$Shortlisting Contest mechanism, in equilibrium Party 1 proposes $\{c1, c2, c3, c4\}$ and Party 2 proposes $\{c2, c3, c4, c5\}$, in the last stage Party 1 picks $\{c2, c3\}$ out of $\{c2, c3, c4, c5\}$.

Notice that $\{c2, c3\}$ is the unique equilibrium outcome of our three mechanisms. It is a natural compromise solution since it is Pareto efficient set and maximizes the welfare of the worst-off party, when each party's payoff from an subset X is the cardinality of that party's lower contour set at X. This is not a coincidence, as we show in the paper when we characterize the equilibria of the games induced by our proposed methods, and prove the uniqueness of their outcomes. The main desirable property share by these methods is the following one: under leximin preferences over sets, these three mechanisms weakly implement the Unanimity Compromise Set.

The Unanimity Compromise Set over sets of size θ can be obtained by applying a procedure, called Fallback Bargaining, proposed and studied by Hurwicz and Sertel (1997) and Brams and Kilgour (2001). The definitions that follow are expressed for any number of parties but their use in our case will only be applied to two parties. Consider as the alternatives the elements of all possible subsets of size θ contained in $\mathbf{C}$. Start by considering the set of alternatives that are best for some party. If all prefer the same alternative to all others, there is a depth 1 agreement, the procedure stops and that alternative is the Unanimity Compromise Set. If not all the parties agree on a most-preferred alternative, then their next-most preferred alternatives are also considered. If there exist some alternatives that are within the top two of every party, these would provide a depth 2 agreement, and the intersection of such alternatives become the Unanimity Compromise Set. Otherwise, the procedure continues, and as long as there is no common agreement of lower depth, the parties descend to lower and lower levels in their rankings, one at a time, until the intersection of their top-ranked alternatives becomes non-empty for the first time, at depth d^*. That set of common agreements, which always exists for some d^*, is the Unanimity Compromise Set.

The Unanimity Compromise Set has attracted a lot of attention on its normative grounds. Any element of this set is always Pareto efficient, is never ranked below the median alternatives of any of the two parties whenever c is odd, and maximizes the welfare of the worst-off party, when each party's payoff from an alternative x is the cardinality of that party's lower contour set at x. It has been proven to contain at most two elements. For example, consider this preference profile over alternatives ($c1 \succ_1 c2 \succ_1 c3 \succ_1 c4 \succ_1 c5$ and $c5 \succ_2 c4 \succ_2 c2 \succ_2 c3 \succ_2 c1$) and its leximin extension. In this case, the Unanimity Compromise set is $\{\{c2, c4\}, \{c2, c3\}\}$ and $d^* = 5$.[2]

2 The Model and Results

Consider any finite set of candidates, $\mathbf{C} = \{1, ..., \mathbf{c}\}$. There are two parties, 1 and 2. Let $\mathbf{P}$ be the set of all strict orders on $\mathbf{C}$.[3] Preferences profiles are elements of $\mathbf{P} \times \mathbf{P}$, denoted as $(\succ_1, \succ_2)$. These two components are interpreted to be the preferences of parties 1 and 2,

[2] The preferences over sets of size two are the following: $\{c1, c2\} \succ_1 \{c1, c3\} \succ_1 \{c2, c3\} \succ_1 \{c1, c4\} \succ_1 \{c2, c4\} \succ_1 \{c3, c4\} \succ_1 \{c1, c5\} \succ_1 \{c2, c5\} \succ_1 \{c3, c5\} \succ_1 \{c4, c5\}$ and
$\{c4, c5\} \succ_2 \{c2, c5\} \succ_2 \{c2, c4\} \succ_2 \{c3, c5\} \succ_2 \{c3, c4\} \succ_2 \{c2, c3\} \succ_2 \{c1, c5\} \succ_2 \{c1, c4\} \succ_2 \{c1, c2\} \succ_2 \{c1, c3\}$

[3] Transitive: For all $x, y, z \in \mathbf{C}$: ($x \succ y$ and $y \succ z$) implies that $x \succ z$. Asymmetric: For all $x, y \in \mathbf{C}$: $x \succ y$ implies that $\neg(y \succ x)$. Irreflexive: For all $x \in \mathbf{C}, \neg(x \succ x)$. Complete: For all $x, y \in \mathbf{C}$: $x \neq y$ implies that ($y \succ x$ or $x \succ y$).

respectively. Suppose that the two parties need to jointly select $\theta \in \{1, ..., \mathbf{c}\}$ candidates from the set $\mathbf{C}$. We assume that parties preferences over sets are strict orders and satisfy Axiom 1.

Definition 1 *Axiom 1. For any $X, Y \subseteq C$ and any $a, b \in C$, we have $X \succ Y$ if $a \succ b$, $b \in Y$ and $X = \{a\} \cup Y \backslash \{b\}$.*[4]*.*

The following concepts will be useful for our later analysis.

Definition 2 *Given any preference profile $(\succ_1, \succ_2) \in \mathbf{P} \times \mathbf{P}$ and any $\theta \in \{1, ..., \mathbf{c}\}$, θ-depth of a preference profile over candidates is denoted by $d^*(\succ_1, \succ_2, \theta)$ and it is the smallest value of q in $\{1, ..., \mathbf{c}\}$ in which the intersection between the agents q-top candidates has at least θ candidates.*

Definition 3 *Given any preference profile $(\succ_1, \succ_2) \in \mathbf{P} \times \mathbf{P}$ and any $\theta \in \{1, ..., \mathbf{c}\}$, the θ-mirrored depth of the preference profile is denoted by $k^*(\succ_1, \succ_2, \theta)$ and it is defined as follows:*

$k^*(\succ_1, \succ_2, \theta) \equiv \mathbf{c} - d^*(\succ_1, \succ_2, \theta) + \theta$

Remark 1 *The depth $d^*(\succ_1, \succ_2, \theta) \leq \frac{\mathbf{c}+\theta}{2}$ if $\mathbf{c} + \theta$ is even, $d^*(\succ_1, \succ_2, \theta) \leq \frac{\mathbf{c}+\theta+1}{2}$, otherwise. This implies that $k^*(\succ_1, \succ_2, \theta) \geq \frac{\mathbf{c}+\theta}{2}$ if $\mathbf{c} + \theta$ is even and $k^*(\succ_1, \succ_2, \theta) \geq \frac{\mathbf{c}+\theta-1}{2}$, otherwise.*

2.1 Characterizing the equilibria of the θ-Rules of k Names

Proposition 1 *Consider any preference profile $(\succ_1, \succ_2) \in \mathbf{P} \times \mathbf{P}$ and any $\theta \in \{1, ..., \mathbf{c}\}$. Let agent $i \in \{1, 2\}$ be the proposer and $j \in \{1, 2\} \backslash \{i\}$ be the chooser. The unique subgame perfect equilibrium outcome of the game of complete information induced by the θ-Rule of k Names is the set of agent i's preferred θ candidates among agent j's $(\mathbf{c} - k + \theta)$-top candidates. There may be several subgame perfect strategy profiles leading to the unique common outcome. A strategy profile is a subgame perfect equilibrium of this game if and only if its strategies satisfy the following two conditions:*
C1. Party j (the chooser) always selects his θ preferred candidates in any subset submitted

[4]Axiom 1 is a modified version of the monotonicity axiom of Kannai and Peleg (1984), used among others by Kaymak and Sanver (2003) and Barberà and Coelho (2008).

by party i (the proposer).

C2. Party i always submits a subset that contains her θ preferred candidates among Party j´s $(\mathbf{c} - k + \theta)$-top candidates and any other $k - \theta$ lower ranked candidates than those θ candidates according to the Party j's preferences.

Here is the intuition for this result. Under the game induced by θ**-**Rule of k Names, only one player moves at each stage. Hence, subgame perfect equilibria and backward induction equilibria coincide, and any backward induction equilibrium outcome is unique as long as the parties' preferences over sets of size θ are strict. Denote by X this unique subgame perfect equilibrium outcome. Because X is a subset of the proposer's screened set and Axiom 1 implies that X contains all θ best candidates of this screened set according to the Party j's preferences. Any candidate in X must be among the Party j´s $(\mathbf{c} - k + \theta)$-top candidates. Because even in the worse of cases, where both agents have reverse preferences, the proposer cannot lead the chooser to pick any candidate worse than her $(\mathbf{c} - k + \theta)$-th ranked alternatives.

Corollary 1 *If the preferences over sets are leximin extensions of the preferences over candidates then the unique SPNE outcome of the θ-Rule of $k(\succ_1, \succ_2, \theta)$ Names is Pareto Efficient.*

Proof. Let us prove by contradiction. Denote by X the equilibrium outcome. Suppose that there is a $Y \neq X$ of size θ that Pareto dominated X, so $Y \succ_i X$ and $Y \succ_j X$. Denote by y_i and y_j (x_i and x_j) parties i and j worst elements of Y (X), respectively. The leximin extension assumption implies that $y_i \succsim_i x_i$ and $y_j \succsim_j x_j$. But then the elements of Y is also among the Party j´s $(\mathbf{c} - k + \theta)$-top candidates. Thus, we reach a contradiction because, by proposition 1, X is the set of Party i's θ preferred candidates among Party j´s $(\mathbf{c} - k + \theta)$-top candidates and, so by leximin extension assumption, we have $X \succ_i Y$. ■

Corollary 2 *If the preferences over sets are leximin extensions of the preferences over candidates then the unique SPNE outcome of the θ-Rule of $k^*(\succ_1, \succ_2, \theta)$ Names is the proposer's θ best element in the Unanimity Compromise Set.*

Proof. Denote by X the equilibrium outcome, let Party i be the proposer and Party j be the chooser. By Proposition 1 and Definition 1, X is the set of Party i's θ preferred candidates among the Party j´s $d^*(\succ_1, \succ_2, \theta)$-top candidates since $d^*(\succ_1, \succ_2, \theta) = \mathbf{c} - k^*(\succ_1, \succ_2, \theta) + \theta$. It implies and by the definition of $d^*(\succ_1, \succ_2, \theta)$ that X is the set of Party i's θ best candidates in the intersection between the parties i and j's $d^*(\succ_1, \succ_2, \theta)$-top candidates. Take any element of the Unanimity Compromise Set and denote it by Y. It implies that Y is Pareto Efficient. By leximin extension assumption, Y must be a subset of the intersection between the parties i and j's $d^*(\succ_1, \succ_2, \theta)$-top candidates. Otherwise, it would be Pareto dominated by X and it is a contradiction. Given that Y is a subset of that intersection than $Y = X$ or Y is Party j's θ best candidates in the intersection. Otherwise it would be Pareto dominated by X. So, we prove that any element of Unanimity compromise set is the set of Party i or Party j's θ best candidates in the intersection between the parties i and j's $d^*(\succ_1, \succ_2, \theta)$-top candidates. ■

Proposition 2 *Consider any $\theta \in \{1, ..., \mathbf{c}\}$, any preference profile $(\succ_1, \succ_2) \in \mathbf{P} \times \mathbf{P}$ and any $k' \in \{\theta, ..., \mathbf{c}\}$. If the preferences over sets are leximin extensions of the preferences over candidates then, under any subgame perfect equilibrium strategy profile, if k' is not greater than the mirrored depth, $k^*(\succ_1, \succ_2, \theta)$, then both parties are weakly better off when playing as the proposer under the θ-Rule of $k^*(\succ_1, \succ_2, \theta)$ Names than playing as the chooser under the θ-Rule of k' Names. Otherwise, both parties are better off playing as the chooser under the Rule of $k^*(\succ_1, \succ_2, \theta)$ Names than playing as the proposer under the θ-Rule of k' names.*

It is worth noting that Proposition 2 implies that, given any k and preference profile over candidates, both agents share the same preferences between the roles of proposer and chooser.

Our proofs of the characterizations of the subgame perfect equilibria of our mechanisms will be heavily based on propositions 1 and 2. The proofs of Proposition 2 and of theorems 1, 2 and 3 below are in the appendix.

2.2 Characterizing the equilibria of our proposed methods

Theorem 1 *Consider any preference profile $(\succ_1, \succ_2) \in \mathbf{P} \times \mathbf{P}$ and any $\theta \in \{1, ..., \mathbf{c}\}$. If the preferences over sets are leximin extensions of the preferences over candidates then the game induced by the θ-Compromise Rule of k Names method adapted to select θ candidates has a subgame perfect equilibrium such that*

(i) *in the first stage, Party 1 chooses* $k^*(\succ_1, \succ_2, \theta)$, *and*

(ii) *in the second stage, for any value of* $k \in \{\theta, ..., \mathbf{c}\}$ *chosen by Party 1, Party 2 opts to be the proposer unless* $k > k^*(\succ_1, \succ_2, \theta)$, *and*

(iii) *in the third stage, for any value of* k *chosen by Party 1, whoever is the proposer proposes a subset that contains its* θ *preferred candidates among the chooser's* $(c - k + \theta)$ *top candidates, plus the chooser's* $k - \theta$ *worst candidates, and*

(iv) *in the fourth stage, whoever is the chooser picks its* θ *preferred candidates out of opposing party's proposed subset.*

As a consequence, a set is a subgame perfect equilibrium outcome if and only if it is the Party 2's best element of the Unanimity Compromise Set.

The following results refer to the θ-Alternate Shortlists method.

Theorem 2 *Consider any preference profile* $(\succ_1, \succ_2) \in \mathbf{P} \times \mathbf{P}$ *and any* $\theta \in \{1, ..., \mathbf{c}\}$. *If the preferences over sets are leximin extensions of the preferences over candidates then the game induced by the* θ*-Alternate Shortlists method adapted to select* θ *has a subgame perfect equilibrium such that*

(i) *in the first stage, Party 1 submits a subset with cardinality equal to* $k^*(\succ_1, \succ_2, \theta)$ *that contains its* θ *preferred candidates among Party 2's* $(\mathbf{c} - k^*(\succ_1, \succ_2, \theta) + \theta)$ *top candidates, plus Party 2's* $k^*(\succ_1, \succ_2, \theta) - \theta$ *worst candidates, and*

(ii) *in the second stage, for any Party 1's proposed subset* $\mathbf{S}$, *Party 2 picks the subset of its* θ *preferred candidates in* $\mathbf{S}$ *only if this subset is weakly preferred to the subset of its* θ *preferred candidates among Party 1's* $(\mathbf{c} - \#|\mathbf{S}|)$ *top candidates. Otherwise, it counter-offers a subset with cardinality* $\#|\mathbf{S}| + 1$ *that contains its preferred* θ *candidates among Party 1's* $(\mathbf{c} - \#|\mathbf{S}|)$ *top candidates plus Party 1's* $\#|S| + 1 - \theta$ *worst candidates, and*[5]

(iii) *in the third stage, whenever Party 1 assumes the role of the chooser it picks the subset of its* θ *preferred candidates out of the opposing party's proposed subset.*

[5] The reader can get an intuition for this counter offer strategy by comparing it with the proposer's equilibrium strategy of the game induced by the rule of k names characterized by Proposition 1.

As a consequence, a set is a subgame perfect equilibrium outcome if and only if it is the Party 1's best element of the Unanimity Compromise Set.

The following results refer to the θ-Shortlisting Contest method.

Theorem 3 *Consider any preference profile* $(\succ_1, \succ_2) \in \mathbf{P} \times \mathbf{P}$ *and any* $\theta \in \{1, ..., \mathbf{c}\}$. *If the preferences over sets are leximin extensions of the preferences over candidates then the game induced by the* θ*-Shortlisting Contest method adapted to select* θ *has a subgame perfect equilibrium such that*

(i) *in the first stage, each party proposes a subset with cardinality equal to* $k^*(\succ_1, \succ_2, \theta)$ *that contains its preferred* θ *candidates among the opposing party's* $(\mathbf{c} - k^*(\succ_1, \succ_2, \theta) + \theta)$ *top candidates plus the* $k^*(\succ_1, \succ_2, \theta) - \theta$ *worst candidates according to the opposing party's preference, and*

(ii) *in the second stage, whoever party is the chooser picks the subset of its* θ *preferred candidates out of the opposing party's proposed subset.*

As a consequence, a set of candidate is a subgame perfect equilibrium outcome when $k^*(\succ_1, \succ_2, \theta)$ *is odd (respectively,* $k^*(\succ_1, \succ_2, \theta)$ *is even) if and only if it is the Party 1's (respectively Party2's) best element of the Unanimity Compromise Set.*

Example 1 *Consider the* 3*-Compromise Rule of k Names,* $\mathbf{C} = \{c1, c2, c3, c4, c5\}$, *the following preference profile over candidates:*$c1 \succ_1 c2 \succ_1 c3 \succ_1 c4 \succ_1 c5$ *and* $c4 \succ_2 c3 \succ_2 c2 \succ_2 c5 \succ_2 c1$. *Let us assume that the preferences over sets are leximin extensions of this preference profile. The first step to find its SPNE is to identify* $\mathbf{UC}(\succ_1, \succ_2, 3)$, $d^*(\succ_1, \succ_2, 3)$ *and* $k^*(3, \succ_1, \succ_2, 3)$. *Inspecting the preference profile above, we have that:*$\mathbf{UC}(\succ_1, \succ_2, 3) = \{\{c2, c3, c4\}\}$, $d^*(\succ_1, \succ_2, 3) = 4$ *and* $k^*(\succ_1, \succ_2, 3) = \mathbf{c} - d^*(\succ_1, \succ_2, 3) + 3 = 5 - 4 + 3 = 4$.

Suppose that Party 1 was selected to choose the value of k. The following pair of strategies is a subgame perfect equilibrium of the 3*-Compromise Rule of k Names method: Party 1 proposes* $k = k^*(\succ_1, \succ_2, 3)$. *As for Party 2, at stage 2, its strategy is to opt to be the proposer unless* $k > k^*(\succ_1, \succ_2, 3)$. *Once the parameters of the* θ*-Rule of k Names are fully determined at stages 1 and 2, the parties play the stages 3 and 4 according to the subgame equilibrium of the game induced by this rule. Thus, at stage 3, Party 2 proposes* $\{c2, c3, c4, c5\}$ *and stage 4, Party 1 chooses* $\{c2, c3, c4\}$.

We now turn to the 3*-Alternate Shortlists method and suppose that Party 1 is the first to move. The following strategy profile is a subgame perfect equilibrium: Party 1 proposes* $\{c1, c2, c3, c4\}$ *and, at stage 3, it chooses its three preferred candidates out of Party 2's proposed set, if called to do so. As for Party 2, at stage 2, its strategy is to choose its three preferred candidate out of Party 1's proposed set only if its cardinality is larger than four or it contains* $\{c2, c3, c4\}$*. Otherwise, it counteracts by proposing* $\{c2, c3, c4, c5\}$*. The subgame perfect equilibrium outcome is Party 1's three preferred candidates in Party 2's proposed set,* $\{c2, c3, c4, c5\}$*, which is* $\{c2, c3, c4\}$
Finally, let us consider the 3*-Shortlisting Contest method. The following pair of strategies is a subgame perfect equilibrium: party 1 proposes* $\{c1, c2, c3, c4\}$ *at stage 1, and at the second stage will choose its three preferred candidates out of Party 2's proposed set, if called to do so. Party 2 proposes* $\{c2, c3, c4, c5\}$ *at the first stage, and at the second stage will choose its three preferred alternatives out of Party 1's proposed set, if called to do so. The equilibrium outcome is Party 1's three preferred candidates in Party 2's proposed set,* $\{c2, c3, c4, c5\}$*, which is* $\{c2, c3, c4\}$*.*

In case of leximax extension, the next example shows that these three mechanisms can induce a Pareto inefficient SPNE outcome.

Example 2 *For example, let* $\theta = 3$ *and this preference profile over alternatives (*$c1 \succ_1 c2 \succ_1 c3 \succ_1 c4 \succ_1 c5$ *and* $c5 \succ_2 c2 \succ_2 c4 \succ_2 c3 \succ_2 c1$*). Suppose that the preferences over sets are leximax extensions of the preferences over alternatives as follows:*
$\{c1, c2, c3\} \succ_1 \{c1, c2, c4\} \succ_1 \{c1, c3, c4\} \succ_1 \{c1, c2, c5\} \succ_1 \{c1, c3, c5\} \succ_1 \{c1, c4, c5\} \succ_1 \{c2, c3, c4\} \succ_1 \{c2, c3, c5\} \succ_1 \{c2, c4, c5\} \succ_1 \{c3, c4, c5\}$ *and*
$\{c2, c4, c5\} \succ_2 \{c2, c3, c5\} \succ_2 \{c3, c4, c5\} \succ_2 \{c1, c2, c5\} \succ_2 \{c1, c4, c5\} \succ_2 \{c1, c3, c5\}_1 \succ_2 \{c2, c3, c4\} \succ_2 \{c1, c2, c4\}\{c1, c2, c3\} \succ_2 \{c1, c3, c4\}$
Inspecting the preference profile above, we have that:$\mathbf{UC}(\succ_1, \succ_2, 2) = \{\{c1, c2, c5\}\}$, $d^*(\succ_1, \succ_2, 2) = 4$ *and* $k^*(\succ_1, \succ_2, 2) = \mathbf{c} - d^*(\succ_1, \succ_2, 3) + 3 = 5 - 4 + 3 = 4$.

The unique SPNE outcome of all three mechanisms is $\{c2, c3, c4\}$ *which is dominated by* $\{c1, c2, c5\}$*. Let us consider first the 3-Compromise Rule of k Names. If Party 2 played as the proposer, the equilibrium outcome would be* $\{c2, c4, c5\}$ *if* $k = 3$, $\{c2, c3, c4\}$ *if* $k = 4$ *and* $\{c1, c2, c3\}$ *if* $k = 5$*. And if the proposer was Party 1, the equilibrium outcome would be* $\{c1, c2, c3\}$ *if* $k = 3$, $\{c2, c3, c4\}$ *if* $k = 4$, $\{c2, c4, c5\}$ *if* $k = 5$*. Knowing it,* *Party 2 would opt to be the proposer if* $k = 4$*. Consequently, Party 1's best strategy is to choose* $k = 4$ *in order to ensure the election of* $\{c2, c3, c4\}$*. Under the* $2-$*Alternate Shortlists method, in equilibrium, Party 1 proposes* $\{c1, c2, c3, c4\}$ *and Party 2 decides to pick* $\{c2, c3, c4\}$ *since she knows that she cannot induce a better outcome by proposing a subset with five alternatives. Under the* $3-$*Shortlisting Contest mechanism, in equilibrium Party 1 proposes* $\{c1, c2, c3, c4\}$ *and Party 2 proposes* $\{c2, c3, c4, c5\}$*, in the last stage Party 1 picks* $\{c2, c3, c4\}$ *out of* $\{c2, c3, c4, c5\}$*.*

The next example shows that the method recommended by JAMS (JAMS Arbitration, Mediation and ADR services) may induce a Pareto inefficient outcome. It is not a surprise since Hurwicz and Schmeidler (1978) and Maskin (1999) prove that there exists no deterministic mechanism with two players, except for dictatorship, guaranteeing that every Nash equilibrium is Pareto efficient.

Example 3 *Consider the set of candidates* $\{c1, c2, c3, c4, c5, c6, c7, c8, c9, c10\}$*. JAMS's method works as follows: each party may then strike three names and shall rank the remaining ones in order of preference. Finally, the three candidates with the highest Borda score are appointed. Consider this preference profile over alternatives (*$c1 \succ_1 c2 \succ_1 c3 \succ_1 c4 \succ_1 c5 \succ_1 c6 \succ_1 c7 \succ_1 c8 \succ_1 c9 \succ_1 c10$ *and* $c1 \succ_2 c2 \succ_2 c7 \succ_2 c4 \succ_2 c5 \succ_2 c6 \succ_2 c3 \succ_2 c8 \succ_2 c9 \succ_2 c10$*).*
Notice that under the assumption of leximin extension, the Unanimity Compromise set is $\{\{c1, c2, c4\}\}$*. However, the following Nash strategy equilibrium induces the election of* $\{c1, c2, c5\}$ *which is Pareto dominated by* $\{c1, c2, c4\}$*: Party 2 vetoes* $c3, c4$ *and* $c10$ *and ranks* $c1 > c2 > c5 > c6 > c7 > c8 > c9$ *and Party 1 vetoes* $c4, c7$ *and* $c10$ *and ranks* $c1 > c2 > c3 > c5 > c6 > c8 > c9$*. So,* $c1, c2$ *and* $c5$ *are alternatives with highest borda scores.*

3 Final remarks

We proposed three mechanisms for two parties to jointly select a group of fixed size. We showed that if the parties' preferences over sets are leximin extensions of the parties' preferences over candidates then these mechanisms implement the Unanimity Compromise Set. This work extends the concepts and the results in Barberà and Coelho (2022), in which the parties had to choose a single candidate, to cover a wide class of natural applications.

As for future research on mechanism design for selecting a group of fixed size that implements the Unanimity Compromise Set, it would be relevant to consider cases where the parties' preferences over sets are different from leximin extensions or when there are more than two parties involved in the decision. Another promising direction would be to verify if a generalized version of Voting Alternating Offers and Vetoes proposed by Anbarci (1993 and 2006) would also implement the Unanimity Compromise Set, as happens with the selection of a single alternative.

4 References

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APPENDIX

Proof of Proposition 2. Suppose that $k^*(\succ_1, \succ_2, \theta) \geq k'$. Denote by X the subgame equilibrium outcome when Party i is the proposer under Rule of $k^*(\succ_1, \succ_2, \theta)$ Names. Denote by Y the subgame equilibrium outcome when Party i is the chooser under Rule of k' Names. Suppose by contradiction that Party i prefers Y to X. By definition of $k^*(\succ_1, \succ_2, \theta)$, the elements of X is among Party i's $(\mathbf{c} - k^*(\succ_1, \succ_2, \theta) + \theta)$ top candidates. Since $k^*(\succ_1, \succ_2, \theta) \geq k'$, it implies that the elements of X is also among Party i's $(\mathbf{c} - k' + \theta)$ top candidates. Thus, by Axiom 1, we have that Party j prefers Y to X, it follows because by Proposition 1, the elements of Y is Party j's preferred candidates among Party i's $(\mathbf{c} - k' + \theta)$ top candidates. Therefore, we have that both parties prefer Y to X. Notice that it implies that X is Pareto Inefficient. By Corollary 1, it cannot be a subgame perfect equilibrium outcome under rule of Rule of $k^*(\succ_1, \succ_2, \theta)$ Names. This is a contradiction.

Now, suppose that $k' > k^*(\succ_1, \succ_2, \theta)$. Denote by X the subgame equilibrium outcome when Party i is the chooser under Rule of $k^*(\succ_1, \succ_2, \theta)$ Names. Denote by Y the subgame equilibrium outcome when Party i is the proposer under Rule of k' Names. Suppose by contradiction that Party i weakly prefers Y to X. By Proposition 1, the elements of Y is among Party j's $(\mathbf{c} - k' + \theta)$ top candidates. Since $k' > k^*(\succ_1, \succ_2, \theta)$, it implies that the elements of Y is also among Party j's $(\mathbf{c} - k^*(\succ_1, \succ_2, \theta) + \theta)$ top candidates. Notice also by Proposition 1, the elements of X is among Party i's $(\mathbf{c} - k^*(\succ_1, \succ_2, \theta) + \theta)$ top candidates. By Axiom 1, since Party i weakly prefers Y to X, it implies that Y is also among Party i's $(\mathbf{c} - k^*(\succ_1, \succ_2, \theta) + \theta)$ top candidates. Notice that these facts imply that X and Y belong to the Unanimity Compromise Set and, by Corollary 1, we have that Party j weakly prefers X to Y. Given that Party i weakly prefers Y to X and Party j weakly prefers X to Y, by Remark 1, it implies that y, the Party j's worst element in Y, is ranked at $d^*(\succ_1, \succ_2, \theta)$ th position according to her preference. Notice that $k' > k^*(\succ_1, \succ_2, \theta)$ imply that $d^*(\succ_1, \succ_2, \theta) \equiv \mathbf{c} - k^*(\succ_1, \succ_2, \theta) + \theta > \mathbf{c} - k' + \theta$. Thus, it implies that y does not belong to Party j's $(\mathbf{c} - k' + \theta)$ top candidates because . This is a contradiction. ■

Proof of Theorem 1. Denote by X the subgame perfect equilibrium outcome of the strategy profile described in Theorem 1. First notice that given the first-mover's choice $k^*(\succ_1, \succ_2)$ and Corollary 2, X is the proposer's best element of the Unanimity Compromise Set and, at the fourth stage, the chooser picks it out of the subset proposed by the proposer.

First let us prove that the strategy profile stated in Theorem 1 is a subgame perfect equilibrium. Notice that the strategies adopted in the second, third and fourth stages are direct consequences of propositions 1 and 2.

Now, let us prove that Party 1 does not have a profitable deviation. Given Party 2's strategy, it is enough to consider only $k' > k^*(\succ_1, \succ_2, \theta)$. If $k' > k^*(\succ_1, \succ_2, \theta)$, Party 1 will become the proposer. It follows by Proposition 2 that it would be not a profitable deviation. Therefore, our initial strategy profile is a subgame perfect equilibrium.

Now let us argue that the equilibrium outcome X is unique. Under this mechanism, only one player moves at each stage. Hence, subgame perfect equilibria and backward induction equilibria coincide, and any backward induction equilibrium outcome is unique as long as the parties' preferences over sets of size θ are strict. ■

Proof of Theorem 2. Given the characterization of the strategy profile described in Theorem 2, let us show that Party 2 opts to not counter-offer and the winning subset is Party 1's best element of the Unanimity Compromise Set. Denote by X this subset. First note that, according to Party 2's strategy, it picks its preferred subset out of the proposed subset if this subset is weakly preferred to the subset of its θ preferred candidates among Party 1's $(\mathbf{c} - \#|\mathbf{S}|)$ top candidates. Suppose that Party 2 does not counter-offer. It follows by Corollary 1 that the winning subset will be X. We need to prove that Party 2 weakly prefers X to the set of its preferred θ candidates among Party 1's $(\mathbf{c} - k^*(\succ_1, \succ_2, \theta) + \theta - 1)$ top candidates. This condition is satisfied because Proposition 2 implies that Party 2 is better off playing as the chooser under the Rule of $k^*(\succ_1, \succ_2, \theta)$ Names than playing as the proposer under the θ-Rule of $k^*(\succ_1, \succ_2, \theta) + 1$ names. Notice in the first game the equilibrium outcome would be X and, in the second, it would be the set of its preferred θ candidates among Party 1's $(\mathbf{c} - k^*(\succ_1, \succ_2, \theta) + \theta - 1)$ top candidates.

Now let us prove that the strategy profile stated in Theorem 2 is a subgame perfect equilibrium. We use propositions 1 and 2. Suppose by contradiction that there is a profitable deviation. This means that if Party 2 counteroffers with a subset of cardinality equal to $k^*(\succ_1, \succ_2, \theta) + 1$, the winning subset is preferred to X according to that Party 2's preferences. Proposition 2 states that when $k = k^*(\succ_1, \succ_2, \theta) + 1$, it is better to be the chooser. So, this winning subset is also preferred to X according to the first-mover preferences. Hence, X is a Pareto inefficient candidate and, by corollaries 1 and 2, it is a contradiction.

Now, let us prove that there is no profitable deviation to the first-mover. Denote by $\mathbf{Z}$ the subset proposed by the first-mover. We need to prove there is no subset $\mathbf{S} \subset \mathbf{C}$, such that $\#|\mathbf{S}| \geq \theta$ and $\mathbf{S} \neq \mathbf{Z}$, that would make Party 1 better off by choosing $\mathbf{S}$ instead of $\mathbf{Z}$, when Party 2's strategy remains unchanged. Proposition 1 implies that it is enough to consider only deviations with subsets such that $\mathbf{S} \subset \mathbf{C}$ with $\#|\mathbf{S}| \geq \theta$, containing the set of Party 1's θ preferred candidates among Party 2's $(\mathbf{c} - \#|\mathbf{S}| + \theta)$ top candidates plus the second-mover*'s* $\#|\mathbf{S}| - \theta$ worst candidates*.*

Given the rules of our method and Party 2's strategy, if Party 1 deviates by choosing a subset with cardinality $k' < k^*(\succ_1, \succ_2, \theta)$, it will become the chooser, because Party 2 will propose a subset with cardinality $k' + 1$. Finally, if $k' > k^*(\succ_1, \succ_2, \theta)$, Party 2 will choose the winning subset from this subset. It follows by Proposition 2 that none of these two possible types of deviations would be profitable. Therefore, our initial strategy profile is a subgame perfect equilibrium.

Now let us argue that the equilibrium outcome X is unique. Under this mechanism, only one player moves at each stage. Hence, subgame perfect equilibria and backward induction equilibria coincide, and any backward induction equilibrium outcome is unique as long as the parties' preferences over sets of size θ are strict. ■

Proof of Theorem 3. Without loss of generality suppose that $k^*(\succ_1, \succ_2, \theta)$ is odd. First let us prove that the strategy profile stated in Theorem 3 is a subgame perfect equilibrium. Denote by $\mathbf{X}$ its outcome and by $\mathbf{Z}^i$ the subset proposed by Party $i \in \{1, 2\}$ under this strategy profile. First notice that given $k^*(\succ_1, \succ_2, \theta) = \mathbf{c} - d^*(\succ_1, \succ_2, \theta) + \theta$ and Corollary 2, X is Party 1's best element in Unanimity Compromise Set.

We need only to prove for each $i' \in \{1, 2\}$ there exists no subset $\mathbf{S} \subset \mathbf{C} \backslash \mathbf{Z}^{i'}$, such that $\#|\mathbf{S}| \geq \theta$ and $S \neq \mathbf{Z}^{i'}$, that would make Party i' better off by choosing S instead of $\mathbf{Z}^{i'}$, while the other player's strategy remains unchanged. Proposition 1 implies that it is enough to consider only deviations with subsets $\mathbf{S}$ such that: $\mathbf{S} \subset \mathbf{C}$,with $\#|\mathbf{S}| \geq \theta$, that contains Party $i' \in \{1, 2\}$ preferred subset of size θ among the opposing party's $(\mathbf{c} - \#|\mathbf{S}| + \theta)$ top candidates plus the opposing party's $\#|\mathbf{S}| - 1$ worst candidates.

Given the rules of the mechanism and the other player strategy, if Party i deviates by choosing a subset with cardinality smaller than $k^*(\succ_1, \succ_2, \theta)$, it will pick the winning subset out of the subset proposed by its opponent. And if it deviates by choosing a subset with cardinality higher than $k^*(\succ_1, \succ_2, \theta)$, its opponent will pick the winning subset out of its subset. It follows from Proposition 2 that neither of these two possible types of deviations would be profitable.

Having proved that our proposed strategy profile is a subgame perfect equilibrium, let us show that X is the unique subgame perfect equilibrium outcome of the game. Note that any subgame perfect equilibrium outcome of this method needs to be Pareto efficient. Otherwise, by Corollary 1, the party who proposed the subset from which the outcome is picked would have a profitable deviation, by changing its composition and avoid the election of the Pareto inefficient subset. Given that fact, we suppose by contradiction that besides the equilibrium outcome described in Theorem 3 there was another one. Now, we will prove that no strategy profile could sustain it.

Let us denote by **SX** the strategy profile described in Theorem 3 that sustains X as an equilibrium outcome. Suppose by contradiction that X is not unique. Let $Y \neq X$ be another subgame perfect equilibrium outcome. Denote by **SY** the strategy profile that sustains Y as a subgame perfect equilibrium outcome and by k' the cardinality of the subset from which one of the parties picks Y on its equilibrium path. Suppose that $k' < k^*(\succ_1, \succ_2, \theta)$. It implies that the cardinality of the other subset proposed by the opponent is equal or smaller than k'. Given that X and Y are Pareto efficient, denote by i the party that prefers X to Y and by j the party that prefers Y to X. Notice that Party i would have a profitable deviation by proposing a subset with cardinality $k^*(\succ_1, \succ_2, \theta)$, because X would be elected from this subset. So, we reach a contradiction. If $k' > k^*(\succ_1, \succ_2, \theta)$, there is also a contradiction because **SX** would not be a subgame perfect equilibrium given that Party j would have a profitable deviation by proposing k' instead of $k^*(\succ_1, \succ_2, \theta)$. Finally, given **SX**, if $k' = k^*(\succ_1, \succ_2, \theta)$, it implies that Y was chosen from the subset proposed by Party 2, the one whose proposed subset does not prevail in case of ties. In addition, it implies that Party 2 prefers Y to X and Party 1 prefers X to Y. This is a contradiction because Party 1 would have incentive to deviate by proposing a subset with $k^*(\succ_1, \succ_2, \theta)$ to induce the election of X. ■