# Resource Allocation on Networks of Externalities*

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**Abstract**

We study the problem of allocating a scarce resource to individuals using a network to describe the allocative externalities among them. The policymaker seeks an allocation mechanism that maximizes welfare, which equals the sum of private values and externalities. We show that this problem is NP-hard as it generalizes a version of the Max-Cut Problem with size restrictions. In cases where the policymaker has complete information on agents' values and externalities, we design an allocation algorithm that guarantees at least $75\%$ of the optimal for any instance of the problem. This algorithm is based on a method of rounding linear relaxations and the connection to the Max-Cut problem. Additionally, we derive conditions under which allocating in a greedy manner is close to optimal. For scenarios in which the policymaker has no information, we provide a truthful $(1 - 1/e)$–approximation randomized mechanism that is based on the convex rounding scheme presented at Dughmi (2011). Moreover, we analyze a simple (non-truthful) item bidding mechanism with VCG payments and show that there always exists an optimal pure strategy Nash equilibrium. We also provide efficiency guarantees for both pure strategy and Bayes-Nash equilibria.

**Keywords:** Allocative Externalities, Approximation Algorithms, Truthful mechanisms, Item Bidding.

**JEL Codes:** D44, D62, D82.

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# 1 Introduction

Policymakers sometimes need to decide how to optimally allocate scarce resources among individuals when the allocation entails externalities. For instance, when determining the recipients of a certain educational resource such as textbooks, tablets, or computers, the evaluation should encompass both the direct advantages these individuals gain from receiving the resource, as well as the broader impacts these individuals could have on others, arising from sharing the resource (e.g., Frölich and Michaelowa (2011)). Similarly, when policymakers decide how to allocate healthcare treatments among individuals, they must not only consider who stands to benefit most from treatment but also what externalities individual treatments have on others. Miguel and Kremer (2004) show, for example, that medical treatments provided to schoolchildren not only substantially improved the health and school participation outcomes for the treated individuals but also had positive effects on those who were untreated.[1]

In this paper, we study the problem of allocating a good or service, that is in limited supply, to agents with varying private values who are linked by a network of non-negative externalities. Our framework revolves around centralized choice and focuses on the case of allocative externalities, wherein the externalities emerge from the impact one agent exerts on others when she receives a unit of the good. The precise interpretation of these externalities depends on the particular setting. For instance, in the context of healthcare treatments, these externalities might arise from the fact that as more agents receive the treatment, the overall risk of infection decreases. Whereas, in the case of educational resources, externalities might result from the possibility of agents sharing the allocated resource.

Our setup is as follows: There are $n \in \mathbb{N}$ agents and $k < n$ units of an indivisible, identical good. Agents have unit demand for the good and the externalities among them are described by a weighted network, where the weight is interpreted as the externality that one agent imposes on another when receiving the good. The policymaker chooses a subset of $k$ agents who will receive the good with the objective of maximizing welfare, which is the sum of the private values of those who were allocated and the externalities they impose on others. We do not impose any restriction on the structure of the network, and allow for heterogeneous private values and externalities.

[1] Another instance worth noting is that of vaccines. For vaccine distribution, many countries implement some form of a priority mechanism that considers factors like age groups, health vulnerabilities, and occupations. Allocating vaccines initially to individuals aged 65 and above or those with compromised immune systems, for instance, aims to grant priority to those with a higher private value for the vaccine. Conversely, directing vaccines to healthcare professionals or educators prioritizes individuals who impose greater externalities on others.

Finding the optimal allocation, even with full knowledge of individuals' private values and the externalities they impose on others, is a hard problem. We show that a variant of the Max-Cut problem with size restrictions can be reduced to the policymaker's problem. The Max-Cut problem is one of Karp's 21 original NP-complete problems (see Karp (1972)), which immediately implies that the policymaker's problem is NP-hard. Therefore, exact optimization is not tractable and there is no choice but to devise allocation mechanisms that approximate the optimal solution.

We begin our analysis by assuming that the policymaker has complete information on individuals' private values and externalities. This allows us to focus on the algorithmic problem at hand, abstracting from incentive considerations. We employ a method of rounding linear relaxations known as *pipage rounding* (Ageev and Sviridenko (1999)) to design a $3/4$–approximation algorithm for the policymaker's problem.[2] Moreover, we show that a simple greedy allocation strategy provides a performance guarantee of $1 - 1/e \approx 0.63$, and that in scenarios where agents' private values significantly outweigh externalities, using the greedy strategy is close to optimal.[3]

In the second part of our analysis, we relax the complete information assumption and assume that both private values and externalities are not known to the policymaker. This introduces the additional challenge of devising a mechanism that also incentivizes individuals to reveal their true valuations. The Vickery-Clarke-Groves (VCG) mechanism (Vickrey (1961), Clarke (1971), Groves (1973)) offers a general solution to such welfare maximization problems. However, it requires an exact solution to the policymaker problem, which is not feasible in our case.[4] Moreover, the approximation algorithms provided for the complete information setting cannot be converted into incentive-compatible mechanisms. Thus, we need to develop new algorithms that take agents' strategic incentives into account.

We show that the policymaker's problem can be mapped to a combinatorial public project (CPP) problem (Papadimitriou et al. (2008)). The CPP problem is a canonically hard mechanism design problem for which no truthful, deterministic, constant ratio approximation mechanisms are known (see Papadimitriou et al. (2008), Buchfuhrer et al. (2010), Dobzinski (2011)). Thus, we turn to randomized mechanisms and show that we can apply the convex randomized rounding scheme presented by Dughmi (2011) and obtain a truthful $(1 - 1/e)$–approximation randomized mechanism for the policymaker's problem.

The truthful mechanism presented might be considered too complicated for

[2]An algorithm is said to be a $\rho$–approximation algorithm if, for any instance of the policymaker's problem, the algorithm achieves at least $\rho$ of the maximum welfare.

[3]The greedy algorithm allocates, in each step, an additional unit of the good to the agent that leads to the largest increase in welfare at that step. See Example 1 for an illustration.

[4]In fact, VCG requires solving many instances of the problem for the calculation of payments.

practical implementation as it might require exponential communication for combinatorial valuations. Therefore, we also analyze a "simple" and more practical alternative - *item bidding with VCG payments* (Lucier et al. (2013), Markakis and Telelis (2017)).[5] Under the item bidding mechanism, an individual submits a bid for each agent (including herself) to be allocated a unit of the good. The mechanism then allocates goods to those agents who received the $k$ highest sum of bids.

We show that there always exists an optimal Pure Strategy Nash Equilibrium, where agents bid their valuations for the optimal allocation and incur a payment of zero. However, the ratio between the optimal solution and the worst-case equilibrium (i.e. the Price of Anarchy) is linear in the number of agents. Additionally, we bound the loss of efficiency for Bayes-Nash equilibria and show that it is also linear in the number of agents. Our choice of VCG-based payments as opposed to first or second prices is due to the fact that the latter payment rules can lead to scenarios where there is no optimal equilibrium, and the worst-case performance does not improve under these payment rules. We note that using an item bidding mechanism comes at the cost of losing truthfulness unless valuations are additive.

**Related literature**

There is a large body of literature on the design of efficient mechanisms. However, the vast majority of this literature does not consider situations in which the allocation entails externalities. Our work contributes to the literature on mechanism design with allocative externalities. Ostrizek and Sartori (2023) and Akbarpour et al. (2024) propose frameworks to consider externalities in screening models. More related to our work, Jehiel et al. (1996), Jehiel et al. (1999), Jehiel and Moldovanu (2001), and Jehiel et al. (2003) analyze auctions with allocative externalities. Specifically, Jehiel et al. (1996) and Jehiel et al. (1999) consider the problem of auctioning one unit of a single good in order to maximize revenue, while Jehiel et al. (2003) looks at multi-object auctions with multi-unit demand agents, analyzing the tension between efficiency and revenue. Jehiel and Moldovanu (2001) studies a setting with heterogeneous objects and interdependent values. They show that in the presence of allocative externalities, there does not exist an incentive compatible mechanism that allocates the goods efficiently. We consider a different setting with several units of a single (homogeneous) good. Additionally, we consider the case where an agent reports the externalities imposed on her (i.e., the agent reports her own valuation), while the incomplete information analysis in Jehiel et al. (1996) involves agents reporting the externalities they impose on others.

A related problem to the one we address is the problem of optimal seeding (see

[5]By "simple" here we mean that it reduces communication complexity relative to the truthful randomized mechanism.

Domingos and Richardson (2001), Richardson and Domingos (2002), and Kempe et al. (2003) for seminal papers on the algorithmic problem, as well as Banerjee et al. (2013), Cai et al. (2015), and Beaman et al. (2021) for applications). In this problem, given a fixed network and a number of seeds, one needs to select agents (seeds) to maximize diffusion on the network. Our problem is similar in the sense that there exists a resource constraint for the number of seeds (which can be viewed as our constraint of the number of goods to be distributed), but differs in that the diffusion in our setting is constrained, occurring exclusively between those who were allocated and the agents to whom they are connected to. Additionally, we also allow for positive spillovers within the set of allocated agents. This has important implications on algorithmic performance: for example, while Akbarpour et al. (2018) show that random seeding may perform close to optimal or even better by randomly allocating a few additional seeds beyond the resource constraint, randomly allocating goods to agents in our setting can lead to arbitrarily bad performance.

The literature on networked markets focuses mainly on the private provision of public goods (Bramoullé and Kranton (2007), Elliott and Golub (2019), Galeotti et al. (2020)). In most of these papers, individuals decide how much they wish to contribute to the production of the public good, taking into account that the good is non-excludable among linked agents. We ask a different question: How should a central planner optimally allocate a resource given a fixed network of externalities?

Finally, in terms of methods, our paper is closely related to the literature on monotone submodular maximization with cardinality constraints (Nemhauser et al. (1978), Conforti and Cornuéjols (1984), Vondrák (2010), Sviridenko et al. (2017)). In their seminal paper, Nemhauser et al. (1978) show that a simple greedy algorithm provides a $(1-1/e)$–approximation and prove that this is the best performance guarantee achievable in polynomial time for general monotone submodular functions. We show that, even for our more restricted class of valuations, the greedy algorithm does not yield a better performance guarantee. However, by leveraging the connection to a variant of the Max-Cut problem (Karp (1972), Goemans and Williamson (1995)), we are able to provide a polynomial algorithm that improves upon the $1-1/e$ performance and provides a $3/4$–approximation for the policymaker's problem. Notably, the best known approximation ratio for the related variant of the Max-Cut problem, The Max Directed Cut problem with size restriction, is $1/2$ (see Ageev et al. (2001)). The monotonicity of our problem allows us to improve upon that.

**Organization of the paper.** In Section 2, we present the definitions that constitute our general framework and formally define the policymaker's optimization problem. In section 3, we establish the connection between the policymaker's optimization problem and a version of the Max-Cut Problem with size restrictions. Building on this connection we show that the policymaker's problem is NP-hard. In Section 4, we present approximation algorithms for the policymaker's problem, assuming that she has complete information. We develop a 3/4-approximation algorithm and also show that a greedy algorithm provides an approximation ratio of $1-1/e$ for the general case, and specify conditions under which allocating in a greedy manner is close to optimal. In section 5, we relax the assumption of complete information and take a mechanism design approach. We show that there exists a truthful (randomized) $(1-1/e)$-approximation mechanism to the policymaker's problem. Additionally, we analyze a non-truthful item bidding mechanism with VCG payments and provide performance guarantees for several equilibrium concepts. Section 6 provides concluding remarks.

## 2 Framework

A policymaker needs to allocate $k \in \mathbb{N}$ units of an indivisible, identical good to a set of $N = \{1, \ldots, n\}$ agents, each with unit demand.[6]

***Allocations:*** Let $X = \{0,1\}^n$ be the set of $n$-dimensional binary vectors. An allocation $x \in X$ is a binary vector, where $x_i = 1$ if agent $i$ was allocated a unit and $x_i = 0$ otherwise. We say that an allocation is feasible if $\sum_{i=1}^n x_i \leq k$.

***Agents' private values and externalities:*** Agents differ in their private value for the good and the externalities they impose on other agents. The vector $v^0 = (v_1^0, \ldots, v_n^0) \in \mathbb{R}_+^n$ denotes the private values assigned by different agents to receiving a unit of the good. The matrix $E \in \mathbb{R}_+^{n \times n}$ describes the adjacency matrix of (non-negative) externalities. Its entry $E_{ij} \in \mathbb{R}_+$ is the externality that agent $i$ imposes on agent $j$ when the former receives a unit of the good ($x_i = 1$) and the latter does not ($x_j = 0$). If $j$ is also allocated the good ($x_j = 1$), we assume that $E_{ij}$ is scaled by a type-dependent parameter $\alpha_{ij} \in [0,1]$.[7] If $\alpha_{ij} = 0$, for example, then allocating a good to $i$ imposes a positive externality on $j$ only if $j$ was not allocated the good. The other extreme case of $\alpha_{ij} = 1$ implies that $j$ gets the same externality from $i$ receiving the good, regardless of whether $j$ also received it or not.

Agents do not impose externalities on themselves, and so for every $i \in N$ we set $E_{ii} = 0$. We view the agents as being linked by a directed, weighted network, where the weights of the edges describe the externalities agents impose on each other.[8]

***Valuations:*** Each agent $i$ has a valuation function $v_i : X \to \mathbb{R}_+$. Let $x \in X$ be an

[6] We assume that $k < n$, otherwise the problem is trivial.

[7] The magnitude of these type-dependent scaling parameters is context-dependent. For example, comparing allocation instances of educational resources versus healthcare treatment, one might assume that the scaling parameters are larger in the latter compared to the former.

[8] See Example 1 for an illustration.

allocation. Agent's $i$ valuation for allocation $x$, denoted by $v_i(x)$, is given by

$$v_i(x) = \begin{cases} v_i^0 + \sum_{j=1}^n \alpha_{ji} E_{ji} x_j, & x_i = 1 \\ \sum_{j=1}^n E_{ji} x_j, & x_i = 0 \end{cases} \tag{1}$$

where $0 \leq \alpha_{ji} \leq 1$.[9]

We assume that for every agent $i$, $v_i^0 \geq \sum_{j=1}^n (1 - \alpha_{ji}) E_{ji}$. This assumption prevents a situation in which, given an allocation, an agent prefers not to receive the good over being allocated. Therefore, this implies that for each agent $i$ the valuation function is non-decreasing, i.e., for any two allocations $x$ and $\hat{x}$ such that $x_j \geq \hat{x}_j$ for all $j \in N$ we have that $v_i(x) \geq v_i(\hat{x})$. We also consider more general monotone valuation functions and show that several of our results extend to larger classes of valuations.

There are a few observations worth noting regarding the functional form specified above. First, note that for all $i \in N$ we have that $v_i(0, \ldots, 0) = 0$. That is, valuations for not allocating any unit of the good are zero. Second, agents' outside options are endogenously determined by the allocation mechanism, i.e. $v_i(0, x_{-i})$ depends on how the good was allocated among agents other than $i$. Note that when there are no externalities, i.e. $E_{ij} = 0$ for all $i, j \in N$, we are back to the standard setting of private values and a zero outside option. Finally, our specification also allows for externalities within the allocated set of agents. These externalities are scaled by the type-dependent parameter $\alpha_{ji}$.

***The objective function:*** The *welfare* of an allocation $x$ is the sum of agents' valuations, which equals the sum of private values of those who were allocated and the externalities that they impose on others:

$$\begin{aligned} W(x) = \sum_{i=1}^n v_i(x) &= \sum_{i=1}^n v_i^0 x_i + \sum_{i,j \in N} \alpha_{ij} E_{ij} x_i x_j + \sum_{i,j \in N} E_{ij} x_i (1 - x_j) \\ &= \sum_{i=1}^n v_i^0 x_i + \sum_{i,j \in N} E_{ij} \left( x_i - (1 - \alpha_{ij}) x_i x_j \right) \end{aligned} \tag{2}$$

The policymaker's objective is to provide a feasible allocation $x$ that maximizes

[9] Note that, for an agent $i$ with $\alpha_{ji} = 1$ for all $j \in N$, the valuation function is additive in $x$. The other extreme case where $\alpha_{ji} = 0$ for all $j \in N$ implies that once allocated, agent $i$'s valuation does not depend on who else was allocated, i.e. $x_i = x'_i = 1$ implies $v_i(x) = v_i(x') = v_i^0$.

welfare.

$$\max_{x \in X} W(x) = \sum_{i=1}^{n} v_i^0 x_i + \sum_{i,j \in N} E_{ij} \left( x_i - (1-\alpha_{ij}) x_i x_j \right) \quad \text{(PM-k)}$$
$$s.t. \sum_{i=1}^{n} x_i \le k,$$
$$x_i \in \{0,1\}, \quad \forall i \in N.$$

Note that the welfare function is non-decreasing in $x$. This implies that the feasibility constraint, $\sum_{i=1}^{n} x_i \le k$ holds with equality for the optimal allocation.

To illustrate the presented concepts, we conclude this section with the following example.

**Example 1.** A policymaker needs to allocate $k = 2$ units of an indivisible, identical good to $n = 3$ agents. The agents have unit demand for the good and are connected in the network of externalities described in Figure 1. We assume that $\alpha_{ij} = 0$ for all $i, j \in N$. This implies that agent's $i$ valuation is $v_i(x) = v_i^0$ if $x_i = 1$ and $v_i(x) = \sum_{j=1}^{n} E_{ji} x_j$ otherwise.

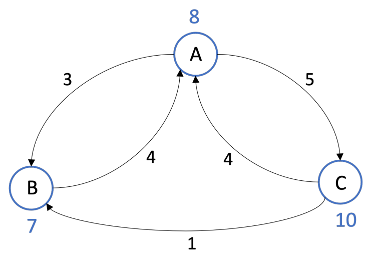


Figure 1: Consider the following directed, weighted network with $n = 3$ agents. The numbers colored in blue represent agents' private values for a unit of the good and the edge weights correspond to the externalities agents impose on one another.

The vector of private valuations and the adjacency matrix of externalities for the above network are given by:

$$v^0 = \begin{pmatrix} 8 \\ 7 \\ 10 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 3 & 5 \\ 4 & 0 & 0 \\ 4 & 1 & 0 \end{pmatrix}$$

The welfare of allocation $x$ is given by: $W(x) = \sum_{i=1}^{n} v_i^0 x_i + \sum_{i,j \in N} E_{ij} x_i (1 - x_j)$. Therefore, if we choose to allocate the units using a greedy strategy which assigns the goods in a sequential order such that, in each step, the individual that leads to the highest increase in welfare is assigned, we will obtain a welfare of 22 with agent $A$ receiving the first unit of the good and agent $C$ receiving the second. However, this allocation does not maximize welfare. The optimal allocation consists of allocating the units to agents $B$ and $C$, which results in a welfare of $25$. ◇

# 3 Optimal Allocations and the Max-Cut Problem

In this section, we present a variant of the Max-Cut Problem and establish the connection between that problem and the policymaker's one. We build on this connection to provide the hardness result of the policymaker's problem.

The Max-Cut Problem is a graph theoretic optimization problem and one of Karp's original $21$ NP-complete problems (Karp (1972)).[10] We present a concise description of a variant of the problem known as the Max-Directed Cut Problem with given sizes of parts, henceforth max-dicut with gsp (see Ageev and Sviridenko (1999) and Ageev et al. (2001) for approximation algorithms to this problem).

Let $G = (V, E)$ be a directed graph. A directed cut in $G$ is defined to be the set of edges leaving some vertex subset $V_1 \subseteq V$, i.e. $\{ij \in E | i \in V_1 \text{ and } j \notin V_1\}$. Given a directed graph $G$ and a weight edge function $w : E(G) \to \mathbb{R}_+$, where $E(G)$ denotes the set of edges in $G$, the maximum directed cut problem is that of finding a directed cut with maximum total weight. That is, finding a partition $(V_1, V_2)$ of $V$, such that the total edge weights from nodes in $V_1$ to nodes in $V_2$ is maximized. The max-dicut with gsp adds the restriction that $|V_1| = k$ for some integer number $k \leq |V|$.

The max-dicut with gsp can be written as the following quadratic binary program:

$$\max_x \sum_{ij \in E(G)} w_{ij} x_i (1 - x_j) \tag{MC-k}$$

$$s.t. \sum_{i \in V} x_i = k,$$

$$x_i \in \{0, 1\}, \quad \forall i \in V.$$

The NP-hardness of the max-dicut problem with gsp follows from the fact that the Max-Cut problem is a special case of it, in which the graph is undirected, unweighted, and there is no size constraint.

**Theorem 1.** *The policymaker's problem (PM-k) is at least as hard as the max-directed cut with given sizes of parts (MC-k).*

*Proof.* Reduction from max-dicut with gsp: Given a graph $G = (V, E)$ and a weight

[10] In Karp (1972) the decision problem related to the Max-Cut optimization problem is stated as an NP-complete problem. In the decision variant one needs to verify the following: Given a graph $G = (V, E)$, weight function $w : E(G) \to \mathbb{R}_+$, and a positive integer $W$, is there a partition $(V_1, V_2) \subseteq V$ such that the total edge weight from nodes in $V_1$ to nodes in $V_2$ is at least $W$.

edge function $w : E(G) \to \mathbb{R}_+$ we construct the following instance of the policymakers' problem. We assume that each agent $i$ has the same private value for the good, which is given by $v_i^0 = v \equiv \max_{j \in V} \sum_{m \in V} w_{mj}$, for all $i \in N$. We set agents' externalities to be equal to the corresponding edge weights, i.e. $E_{ij} = w_{ij}$ if $ij \in E(G)$, and $E_{ij} = 0$ otherwise. Finally, set $\alpha_{ij} = 0$ for all $i, j \in N$. Then, the policymaker's problem (PM-k) can be written as:

$$\max_x \; W(x) = v \sum_{i=1}^{n} x_i + \sum_{ij \in E(G)} w_{ij} x_i (1 - x_j)$$
$$s.t. \; \sum_{i=1}^{n} x_i = k,$$
$$x_i \in \{0, 1\}, \quad \forall i \in N.$$

It follows that the solution to the policymaker's problem with equal private values coincides with the solution to the max-dicut problem with gsp (MC-k), given by the graph $G = (V, E)$ and the weight edge function $w : E(G) \to \mathbb{R}_+$. Thus, the max-directed cut problem with given sizes of parts can be viewed as a special case of the policymaker's problem. This immediately implies that the policymaker's problem is at least as hard. □

The following Corollary is an immediate implication of Theorem 1.

**Corollary 1.** *The policymaker's problem (PM-k) is NP-hard.*

We note that the policymaker's problem remains NP-hard, even when agents' private values are identical, externalities among all linked agents are equal, and the graph is undirected and regular.[11] The NP-hardness for this case follows from the fact that the Max-Cut problem with given sizes of parts remains NP-hard even for regular graphs and equal weights.

This observation underscores that the computational complexity of the policymaker's problem arises primarily from the network structure of the externalities rather than from heterogeneity in private values or externalities.

# 4 The Algorithmic Problem

The NP-hardness of the policymaker's problem implies that we cannot solve for an exact optimum. Therefore, in this section, we develop two approximation algorithms for the optimal allocation and provide provable performance guarantees for both of

[11] That is setting $v_i^0 = v \in \mathbb{R}_+$ for all $i \in N$, $\alpha_{ij} = \alpha \in [0, 1]$ for all $i, j \in N$, $E_{ij} = E_{ji} \in \{0, E\}$ for all $i, j \in N$, and a given $E \in \mathbb{R}_+$. Moreover, each agent has the same number of direct connections.

them. We assume throughout this section that the policymaker has full knowledge of individuals' private values and the externalities they impose on each other. We relax this assumption in Section 5.

**Definition 1 (Performance Guarantee).** An algorithm has a performance guarantee of $\rho \in [0, 1]$ if it achieves at least $\rho$ of the optimal for any instance of the problem, i.e.,

$$ALG(v) \geq \rho \cdot OPT(v)$$

for any instance, $v,$ of the problem.

## 4.1 The LP-Rounding Algorithm

We start with an approximation algorithm (the *LP-Rounding Algorithm*) that employs a method of rounding a linear relaxation (Pipage rounding) and provides a performance guarantee of at least $3/4$.

The key ideas behind this method are associating with the welfare function $W(x) = W(x_1, \ldots, x_n)$ another function $L(x_1, \ldots, x_n)$ that coincides on binary vectors and can be polynomially computable on the $n$-dimensional cube $[0, 1]^n$. Additionally, we show that the following two properties hold:

(i) There exists $\rho > 0$ such that $W(x_1, \ldots, x_n) \geq \rho L(x_1, \ldots, x_n)$ for each $x \in [0, 1]^n$.

(ii) $\epsilon$-Convexity: The function $\phi(\epsilon, x, i, j) = W(x_1, \ldots, x_i + \epsilon, \ldots, x_j - \epsilon, \ldots, x_n)$ is convex with respect to $\epsilon \in [-\min\{x_i, 1 - x_j\}, \min\{1 - x_i, x_j\}]$ for any pair of indices $i$ and $j$ and each $x \in [0, 1]^n$.

These two properties imply that we can construct a $\rho$-approximation algorithm. The first property provides a non-trivial performance guarantee for the relaxed problem and the second ensures that we can round the relaxed solution to a binary one, preserving the guarantee.

**Theorem 2.** *Let $\alpha_{min} = \min_{i,j \in N} \alpha_{ij}$, and let $x^{LP}$ be the allocation induced by the LP-Rounding algorithm and $x^{OPT}$ the allocation that maximizes $W(\cdot)$. Then, the LP-Rounding algorithm provides a performance guarantee of*

$$W(x^{LP}) \geq \begin{cases} 3/4 W(x^{OPT}), & \alpha_{min} \leq 0.5 \\ (1 - \alpha_{min} + \alpha_{min}^2) W(x^{OPT}), & \alpha_{min} > 0.5 \end{cases}$$

*Proof.* See appendix.

The *LP-Rounding algorithm* provides a performance guarantee that is weakly increasing in $\alpha_{min}$ and is bounded below by $3/4$. An increase in $\alpha_{min}$ implies that agents' valuations approach the additive case and as a result, the policymaker's problem becomes more tractable. In the limit case where $\alpha_{min} = 1$, the *LP-Rounding algorithm* provides the optimal allocation. We provide a proof overview for Theorem 2 and describe the proposed algorithm. The complete proof can be found in Appendix A.

***Proof Overview***

We start by re-writing the welfare function in the following way:

$$
\begin{aligned}
W(x) &= \sum_{i=1}^{n} v_i^0 x_i + \sum_{i,j\in N} E_{ij}\left(x_i - (1-\alpha_{ij})x_i x_j\right) \\
&= \sum_{i,j\in N} E_{ij}(x_i + (1-\alpha_{ij})x_j - (1-\alpha_{ij})x_i x_j) + \sum_{i=1}^{n}\left(v_i^0 - \sum_{j=1}^{n}(1-\alpha_{ji})E_{ji}\right) x_i
\end{aligned}
$$

where the second equality follows from plugging $v_i^0 = v_i^0 - \sum_{j=1}^{n}(1-\alpha_{ji})E_{ji} + \sum_{j=1}^{n}(1-\alpha_{ji})E_{ji}$, and re-arranging terms. Thus, the policymaker's problem can be written as:

$$
\begin{aligned}
\max_x \; W(x) = \sum_{i,j\in N} & E_{ij}(x_i + (1-\alpha_{ij})x_j - (1-\alpha_{ij})x_i x_j) + \sum_{i=1}^{n}\left(v_i^0 - \sum_{j=1}^{n}(1-\alpha_{ji})E_{ji}\right) x_i \\
s.t. \; & \sum_{i=1}^{n} x_i = k, \qquad \text{(PM-k')} \\
& x_i \in \{0,1\}, \quad \forall i \in N.
\end{aligned}
$$

Next, we define the following integer linear program, which coincides with (PM-k') on binary vectors:

$$
\begin{aligned}
\max_x \; L(x) = \sum_{i,j\in N} & E_{ij}\min\{(x_i + (1-\alpha_{ij})x_j), 1\} + \sum_{i=1}^{n}\left(v_i^0 - \sum_{j=1}^{n}(1-\alpha_{ji})E_{ji}\right) x_i \\
s.t. \; & \sum_{i=1}^{n} x_i = k, \qquad \text{(ILP)} \\
& x_i \in \{0,1\}, \quad \forall i \in N.
\end{aligned}
$$

Since these two problems coincide on binary vectors it is immediate that an optimal solution $x^{OPT}$ to (PM-k') must also be an optimal solution to (ILP).

Relaxing (ILP) to a continuous linear program by replacing the binary constraints to continuous ones, i.e., replacing $x_i \in \{0,1\}$ to $0 \le x_i \le 1$ for all $i \in N$, we show that any optimal fractional solution, $x^*$, to the relaxed linear program satisfies:

$$W(x^*) \geq \begin{cases} 3/4L(x^*), & \alpha_{min} \leq 0.5 \\ \left(1 - \alpha_{min} + \alpha_{min}^2\right) L(x^*), & \alpha_{min} > 0.5 \end{cases}$$

This follows from the observation that for all pairs $0 \leq x_i, x_j \leq 1$ the following inequality holds:

$$\begin{aligned} & x_i + (1-\alpha_{ij})x_j - (1-\alpha_{ij})x_i x_j \\ \geq & \begin{cases} 3/4\left(\min\{(x_i + (1-\alpha_{ij})x_j), 1\}\right), & \alpha_{ij} \leq 0.5 \\ (1 - \alpha_{ij} + \alpha_{ij}^2)\left(\min\{(x_i + (1-\alpha_{ij})x_j), 1\}\right), & \alpha_{ij} > 0.5 \end{cases} \end{aligned} \tag{3}$$

Finally, we show that the welfare function satisfies the $\epsilon$-convexity property which implies that we can round the fractional solution, $x^*$, to a binary one, $x^{LP}$, that provides weakly greater welfare, i.e.,

$$W\left(x^{LP}\right) \geq W(x^*)$$

Putting everything together we have that:

$$W(x^{LP}) \geq \begin{cases} 3/4W(x^{OPT}), & \alpha_{min} \leq 0.5 \\ \left(1 - \alpha_{min} + \alpha_{min}^2\right) W(x^{OPT}), & \alpha_{min} > 0.5 \end{cases}$$

which completes the proof. □

## 4.2 The Greedy Algorithm

The next approximation algorithm we develop (the Greedy Algorithm) uses a natural greedy strategy and allocates, in each step, an additional unit to the agent that leads to the largest increase in welfare.

**Definition 2 (Greedy Algorithm).** The greedy algorithm for the policymaker's problem:

**Initialization:** Start with the null allocation, $x = (0, 0, \ldots, 0)$.

**Step 1:** Allocate a unit of the good to agent $i^*$ that satisfies:

$$i^* = \arg\max_{\{i \in N\,;\, x_i = 0\}} \left[v_i^0 + \sum_{j=1}^n E_{ij}(1 - (1-\alpha_{ij})x_j) - \sum_{j=1}^n (1-\alpha_{ji})E_{ji}x_j\right]$$

**Step 2:** Set $x_{i^*} = 1$. If $\sum_{i=1}^n x_i = k$ stop. Otherwise, repeat Step 1.

In order to prove that the greedy algorithm provides a constant performance guarantee, we first show that the welfare function satisfies the following "diminishing returns" property.

**Definition 3 (Submodular Set Function).** A set function $f : 2^N \to \mathbb{R}$ is submodular if it satisfies

$$f(S \cup \{i\}) - f(S) \geq f(T \cup \{i\}) - f(T)$$

for all pairs $S \subseteq T$ and all elements $i \notin T$.

**Lemma 1.** *The welfare function $W(\cdot)$ is submodular.*

*Proof.* See appendix.

This result implies that the marginal welfare gain of allocating a unit of the good to an agent weakly decreases as the set of allocated agents increases in the set inclusion order. The following proposition shows that the Greedy Algorithm provides a performance guarantee of $(1 - 1/e) \approx 0.63$.

**Proposition 1.** *Let $x^G$ be the allocation induced by the greedy algorithm and $x^{OPT}$ the welfare-maximizing allocation. Then, the greedy algorithm provides a $(1 - 1/e)-$ approximation, i.e.,*

$$W\left(x^G\right) \geq (1 - 1/e)\, W\left(x^{OPT}\right)$$

*Proof.* The proof follows from Lemma 1 and from Theorem 4.3 in Nemhauser et al. (1978) which shows that for any non-negative, monotone, and submodular function $W(\cdot)$ with a cardinality constraint on the size of the set, the greedy algorithm provides a $(1 - 1/e)-$approximation. □

The Greedy algorithm guarantees a minimum of approximately 63% of the welfare achieved by the optimal allocation. Note that this result, as well as the other results presented in this section, only requires that the valuation function is submodular. Thus, the analysis presented here extends to a much larger class of valuation functions.[12]

Since Proposition 1 holds for any valuation function that is submodular, a natural question is whether the $1-1/e$ bound is tight for our specific case. In the following result, we show that this is indeed the case.

**Proposition 2.** *The $1 - 1/e$ performance guarantee is tight for the policymaker's problem. That is, for every $\rho' > 1 - 1/e$, there exists $\{v_i(x)\}_{i=1}^n$ and $k \leq n$ such that:*

$$W\left(x^G\right) < \rho' W\left(x^{OPT}\right)$$

[12]For example, the results continue to hold if agents only value the largest externality imposed on them, instead of summing across all externalities. This is captured by the following valuation function: $v_i(x) = v_i^0 + \alpha_i \max_{\{j\,;\,x_j=1\}} E_{ji}$ if $x_i = 1$ and $v_i(x) = \max_{\{j\,;\,x_j=1\}} E_{ji}$ otherwise.

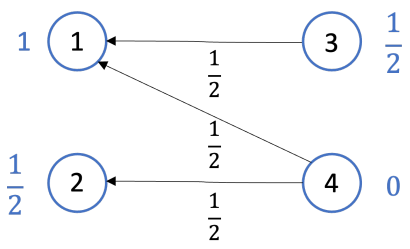


Figure 2: Figure 2 displays the case for $k = 2$. The numbers colored in blue represent agents' private values for a unit of the good and the edge weights correspond to the externalities agents impose on one another.

*Proof.* We will prove that the bound is tight by providing an example in which the Greedy algorithm provides exactly $1 - \left(1 - \frac{1}{k}\right)^k$ of the welfare attained in the optimal allocation, which converges from above to $1 - 1/e$ as $k \to \infty$.

Let $N = N_1 \cup N_2$ where $N_1 = \{1, \ldots, k\}$ and $N_2 = \{k+1, \ldots, 2k\}$. The initial vector of private values $v^0 \in \mathbb{R}^n_+$ is given by:

$$v_i^0 = \begin{cases} (\frac{k-1}{k})^{i-1}, & i \in N_1 \\ (\frac{k-1}{k})^{k-1}, & i \in N_2 \setminus \{2k\} \\ 0, & i = 2k. \end{cases}$$

The adjacency matrix of externalities $E \in \mathbb{R}^{2k \times 2k}_+$ is defined as follows:

$$E_{ij} = \begin{cases} \frac{v_j}{k}, & i \in N_2 \text{ and } j \in N_1 \setminus \{k\} \\ v_j, & i = 2k \text{ and } j = k \\ 0, & otherwise. \end{cases}$$

Finally, we set $\alpha_{ij} = 0$ for all $i, j \in N$. Figure 2 illustrates the example for $k = 2$.

The optimal allocation $x^{OPT}$ is $x_i^{OPT} = 1$ if and only if $i \in N_2$, and the welfare is $W(x^{OPT}) = \sum_{i=1}^{2k} v_i^0 = k$. Next, note that for any $i \in N_2$ and $m \in N_1$, we have $v_i^0 + \sum_{j=m}^{2k} E_{ij} = v_m^0$. Thus, a possible allocation of the greedy algorithm is to allocate the $m-$th unit of the good to agent $m \in N_1$. We then have that $x^G = 1$ if and only if $i \in N_1$, and the welfare achieved is:

$$W(x^G) = \sum_{i \in N_1} v_i^0 = k \left[ 1 - \left( \frac{k-1}{k} \right)^k \right]$$

Therefore, $\frac{W(x^G)}{W(x^{OPT})} = \left[1 - \left(1 - \frac{1}{k}\right)^k\right] \to 1 - \frac{1}{e}$ as $k \to \infty$.[13]

$\square$

While the bound in Proposition 1 is tight, the performance guarantee of the greedy algorithm increases substantially when agents' private values are large compared to the externalities they impose on each other. In these scenarios allocating in a greedy manner becomes close to optimal. The following definition formalizes the comparison between private values and externalities.

**Definition 4.** Let $v^0 \in \mathbb{R}^n_+$ and $E \in \mathbb{R}^{n\times n}_+$ be the initial valuations and externalities. We define the following parameters:

(i) $\gamma_{in} = \max_{j\in N} \frac{\sum_{i\in N} E_{ij}}{v^0_j}$

(ii) $\gamma_{out} = \max_{j\in N} \frac{\sum_{i\in N} E_{ji}}{v^0_j}$

The first parameter, denoted as $\gamma_{in}$, represents the maximal ratio of the aggregate externalities imposed on an agent to that agent's private value for a unit of the good. Let $\alpha_{max} = \max_{i,j\in N} \alpha_{ij}$. Note that, $\gamma_{in} \in [0, 1/(1-\alpha_{max})]$ as it is assumed that the sum of scaled externalities imposed on any agent does not surpass their private value for the good, $\sum_{i=1}^n (1-\alpha_{ij})E_{ij} \leq v^0_j$ for any agent $j \in N$.

The second parameter, denoted as $\gamma_{out} \in \mathbb{R}_+$, represents the maximal ratio of the aggregate externalities an agent imposes on all other agents to that agent's private value for a unit of the good.

Thus, these two parameters imply that both the externalities imposed on each agent and the total externalities that each agent imposes on all other agents are bounded by her private value for the good, scaled respectively by $\gamma_{in}$ or $\gamma_{out}$.

**Proposition 3.** *Let $v^0 \in \mathbb{R}^n_+$ and $E \in \mathbb{R}^{n\times n}_+$ be agents' private values and externalities. Let $c = \min\{\frac{(1-\alpha_{min})(\gamma_{in}+\gamma_{out})}{1+\gamma_{out}}, 1\}$, and let $x^G$ and $x^{OPT}$ be the allocations induced by the greedy algorithm and the welfare maximizing allocation respectively. Then,*

$$W\left(x^G\right) \geq \frac{1}{c}\left(1 - \frac{1}{e^c}\right) W\left(x^{OPT}\right)$$

*Proof.* See appendix.

Setting $\alpha_{min} = 0$ and $\gamma_{in} \geq 1$ (which yields $c = 1$) we obtain the approximation ratio presented in Proposition 1. For the case where $\gamma_{in} = \gamma_{out} = 0$, which is a scenario

[13] At each step, the Greedy algorithm will be indifferent between allocating to an agent from $N_2$ or to the agent with the largest private value from $N_1$. We can break these indifferences by adding an arbitrarily small $\epsilon > 0$ to the private value of each $i \in N_1$, which will result in a unique allocation for the Greedy algorithm. For this unique allocation, as we take $\epsilon \to 0$ and $k \to \infty$ we get the $1 - 1/e$ bound.

of no externalities, or for $\alpha_{min} = 1$ which is the case of additive valuations, the Greedy algorithm attains an approximation ratio of $1$ and produces an optimal allocation. Proposition 3 leads to the following comparative statics result for the performance guarantee of the Greedy algorithm.

**Corollary 2.** *The performance guarantee of the Greedy algorithm is:*

*(i) Monotone increasing in* $\alpha_{min}$.

*(ii) Monotone decreasing in* $\gamma_{in}$ *and* $\gamma_{out}$.

The intuition for Corollary 2 is straightforward. An increase in $\alpha_{min}$ implies that agents' valuations are "closer" to additive ones, which results in an increase in the performance guarantee of the Greedy algorithm. An increase in $\gamma_{in}$ or $\gamma_{out}$, on the other hand, implies an increase in externalities compared to private values, which leads to a decrease in the performance guarantee.

Finally, whether the Greedy Algorithm provides a better approximation ratio than the LP-Rounding algorithm depends on the values of $\alpha_{min}, \gamma_{in}$ and $\gamma_{out}$. If $\gamma_{in} = 1$, for example, then $c = 1 - \alpha_{min}$. In this case, the Greedy Algorithm yields a higher performance guarantee whenever $\alpha_{min} \geq \bar{\alpha}$ where $\bar{\alpha} \approx 0.4$.[14]

# 5 The Mechanism Design Problem

The analysis so far was carried out under the assumption that the policymaker has complete information and fully observes agents' private values, externalities, and the scaling parameters $\alpha_{ij}$. In this section, we relax this assumption and assume that the policymaker possesses no knowledge of agents' valuation functions.[15] We provide a truthful, randomized $(1 - 1/e)$–approximation mechanism for the policymaker's problem. Furthermore, we analyze a (non-truthful) item bidding mechanism and provide performance guarantees for Pure Strategy and Bayes-Nash equilibria.

## 5.1 A Truthful Randomized $(1 - 1/e)$-Approximation Mechanism

We begin our analysis by showing that the policymaker's problem can be mapped to a combinatorial public project (CPP) problem (Papadimitriou et al. (2008)), where implementing a set of projects or allocating a set of resources can be thought of as

[14] The cutoff value $\bar{\alpha}$ satisfies the following condition:

$$\frac{1}{1-\bar{\alpha}}\left(1 - \frac{1}{e^{1-\bar{\alpha}}}\right) = \frac{3}{4}$$

.

[15] That is, the policymaker does not know agents' private values, the externalities they impose on each other, and the scaling parameters.

allocating units of the good to a set of agents. Building on this connection we employ the convex rounding scheme presented in Dughmi (2011) to obtain a truthful, randomized $(1-1/e)$–approximation mechanism for the policymaker's problem.

In a Combinatorial Public Project problem, there exists a set of agents $N = \{1,\ldots,n\}$, a set of projects $M = \{1,\ldots,m\}$, and an integer $k \leq m$. Each agent $i$ has a valuation function $v_i : 2^M \to \mathbb{R}_+$ over all possible subsets of projects. The valuation functions satisfy two standard assumptions. First, $v_i$ is normalized such that $v_i(\emptyset) = 0$. Second, $v_i$ is monotone, i.e. for every $S \subseteq T \subseteq M$, $v_i(S) \leq v_i(T)$. The objective is to choose a subset of projects $S^* \subseteq M$ of size $k$ that maximizes total welfare, $\sum_{i\in N} v_i(S^*)$. We consider a flexible version of CPP, where the chosen set can be of size at most $k$, i.e., a feasible solution is any subset $S$ such that $|S| \leq k$. The CPP problem can be written as the following optimization problem:

$$\begin{aligned} \max_{S\subseteq M} &\sum_{i\in N} v_i(S) \\ s.t.\ & |S| \leq k. \end{aligned} \tag{4}$$

It is often assumed that agents' valuations are submodular functions (as in definition 3). A prominent subset of submodular functions, which will be of key interest to our setting, is coverage functions.

**Definition 5.** Let $(\mathcal{Y},\mu)$ be a measure space. A valuation function $v_i : 2^M \to \mathbb{R}_+$ is a coverage function if there exists $m$ measurable subsets $A_1, A_2, \ldots, A_m \subseteq \mathcal{Y}$ such that $v_i(S) = \mu(\cup_{l\in s} A_l)$.

It is easy to check that the policymaker's problem (PM-k) corresponds to a CPP problem in which $M = N$ and, for each agent $i$, $v_i(S) = v_i^0 + \sum_{j\in S} \alpha_{ji} E_{ji}$ if $i \in S$ and $v_i(S) = \sum_{j\in S} E_{ji}$ otherwise.[16] Moreover, we show that agents' valuations in the policymaker's problem belong to the class of coverage functions.

**Lemma 2.** *The policymaker problem (PM-k) corresponds to a CPP problem such that for each $i$, $v_i$ is a coverage valuation function.*

*Proof.* We prove Lemma 2 for the case where $\alpha_{ij} = 0$ for all $i,j \in N$. The complete proof can be found in Appendix A and uses similar arguments for the general case.

For each agent $i$, consider $v_i$ such that $v_i(S) = v_i^0$ if $i \in S$ and $v_i(S) = \sum_{l\in S} E_{li}$ otherwise. Let $\mathcal{Y} = [0, v_i^0]$ and $\mu$ be the Lebesgue measure.[17] Define $A_i = \mathcal{Y} = [0, v_i^0]$ and $A_j = \left(\sum_{t=1}^{j-1} E_{ti}, \sum_{t=1}^{j} E_{ti}\right]$ for all $j \neq i$.[18] Note that for $j \neq i$, $\mu(A_j) = E_{ji}$ and

[16] In our setting, the chosen projects can be thought of as the agents being allocated a unit of the good.

[17] The Lebesgue measure is such that for all $(a,b], b \geq a, \mu((a,b]) = b-a$.

[18] We use the standard convention that the sum over the empty set equals 0. Therefore, $A_1 = [0, E_{1i}]$ for all $i \neq 1$.

$A_j \cap A_l = \emptyset$ for all $l \neq j, i$. Take an arbitrary set $S \subseteq N$. If $i \in S$, then $\cup_{l \in s} A_l = A_i$. Thus, $v_i(S) = \mu(A_i) = v_i^0$ if $i \in S$. If $i \notin S$, then $\cup_{l \in s} A_l$ is the union of pairwise disjoint sets. Hence, $v_i(S) = \sum_{l \in S} \mu(A_l) = \sum_{l \in S} E_{li}$. □

We consider direct-revelation mechanisms for the policymaker's problem. For given parameters $n$ and $k$, denote by $\mathcal{S}$ the set of feasible allocations, i.e., $\mathcal{S} = \{S \subseteq N\,;\, |S| \leq k\}$. A direct revelation mechanism consists of an allocation rule $\mathcal{A}$ that maps a vector of reported valuation functions $(\hat{v}_1, \ldots, \hat{v}_n)$ to a feasible allocation $S \in \mathcal{S}$ and a payment rule $p$ which maps a vector of reported valuation functions to payments for each player. We allow for randomized allocation and payment rules.

We say that a randomized mechanism is truthful if each agent maximizes her expected payoff by reporting her valuation function truthfully, regardless of other agents' reports.

**Definition 6.** A randomized mechanism is said to be truthful if $\mathcal{A}$ and $p$ satisfy:

$$E[v_i(\mathcal{A}(v_i, \hat{v}_{-i})) - p_i(v_i, \hat{v}_{-i})] \geq E[v_i(\mathcal{A}(v_i', \hat{v}_{-i})) - p_i(v_i', \hat{v}_{-i})]$$

for every agent $i$, reported valuation $v_i'$ and other agents reported valuations $\hat{v}_{-i}$.

Lemma 2 together with Dughmi (2011) implies that there exists a truthful, randomized mechanism that provides a $(1 - 1/e)$-approximation and runs in expected polynomial time.

**Proposition 4.** *There exists a truthful, randomized* $(1 - 1/e)$*-approximation mechanism for the policymaker's problem.*

*Proof.* The proof follows from lemma 2 and Theorem 3.1 in Dughmi (2011), which shows that such a mechanism exists for CPP with coverage valuations. □

For completeness, we provide a succinct description of the mechanism proposed in Dughmi (2011). Consider the following integer programming formulation of the CPP problem faced by the policymaker:

$$\max_x \; W(x) = \sum_{i=1}^{n} v_i(x); \qquad \text{(PM-k)}$$
$$s.t. \; \sum_{i=1}^{n} x_i \leq k,$$
$$x_i \in \{0, 1\}, \quad \forall i \in N.$$

where $v_i(x) = v_i^0 x_i + \sum_{j \in N} \alpha_{ji} E_{ji} x_j x_i + \sum_{j \in N} E_{ji} x_j (1 - x_i)$.

We consider a natural relaxation to the polytope $\mathcal{P} = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leq k\ , 0 \leq x \leq 1\}$. Solving the relaxed problem efficiently is not feasible as the above optimization problem is not convex. The novelty in Dughmi (2011) and Dughmi et al. (2011) approach is defining a randomized rounding scheme $r(x)$ that renders the objective concave and maximizes directly on the output of the rounding scheme rather than on its input. The rounding scheme maps points from $\mathcal{P}$ to feasible solutions $\mathcal{S}$. Since $r$ is randomized it induces a distribution over $\mathcal{S}$ for each $x \in \mathcal{P}$.[19] We can then write (PM-k) as:[20]

$$\begin{aligned} &\max_x \ E_{S\sim r(x)}\left[\sum_{i=1}^n v_i(S)\right] \qquad (5)\\ &s.t. \sum_{i=1}^n x_i \leq k,\\ &x_i \in [0,1], \quad \forall i \in N. \end{aligned}$$

The rounding scheme proposed in Dughmi (2011), referred to as the $k$-bounded-lottery rounding scheme, works as follows: given $x \in \mathcal{P}$, take $k$ independent random draws on $N \cup \{0\}$ from a distribution where, in each draw, $i$ is chosen with probability $x_i/k$ (and $0$ is chosen with probability $1 - \sum_{i\in N} x_i/k$). The resulting allocation $S$ consists of agents $i$ that were chosen in at least one draw (where choosing $0$ in one draw is interpreted as not choosing any agent). By construction, this rounding scheme yields a set $S$ of size of at most $k$. Moreover, note that agent $i \in N$ is included in the realized set $S$ with probabilibility $1-(1-\frac{x_i}{k})^k$.[21]

Dughmi (2011) shows that for coverage valuations the objective in (5) is concave. Moreover, this rounding scheme yields a $1-1/e$ approximation for submodular valuations. Taken together with VCG payments this yields a truthful, randomized $(1-1/e)$-approximation mechanism for the policymaker's problem.

## 5.2 Item Bidding with VCG Payments

In this section, we study a more "practical" item bidding mechanism with VCG-based payments. The practicality of this mechanism stems from the reduced communication complexity imposed on agents when reporting valuations and the

[19] The standard approach is to maximize a simple extension of $W(\cdot)$ to $\mathcal{P}$. This yields an optimal fractional solution $x^*$ which is then rounded to an integer solution by $r(x^*)$. However, this approach is often incompatible with truthful mechanisms. The truthfulness in expectation follows immediately from the allocation rule being *maximal-in-distributional-range* (MIDR).

[20] Note that we reformulate $v_i$ to be defined over feasible sets $S$.

[21] The probability that $i$ is not chosen in one particular draw is $1-\frac{x_i}{k}$. Thus, the probability that $i$ is not chosen in any of the $k$ independent draws is $(1-\frac{x_i}{k})^k$.

straightforward allocation rule.[22] However, being non-truthful, this mechanism requires agents to take strategic actions and reason through the mechanism.[23]

The items in our setting correspond to agents receiving the resource. Each agent $i \in N$ submit a bid vector $b_i = (b_i^1, \cdots, b_i^n)$, where $b_i^j$ is the bid $i$ places on agent $j$ being allocated a unit of the good. Let $B_j = \sum_{i=1}^n b_i^j$ be the sum of bids placed on agent $j$ receiving a unit of the good. The item bidding mechanism chooses the $k$ agents with the highest sum of bids. For a profile of bids $b \in \mathbb{R}_+^{n \times n}$ let $S(b)$ be the chosen set of agents. The payment of an agent $i$ is given by the Clarke pivot rule:

$$p_i(b) = \sum_{k \neq i} \sum_{j \in S(0, b_{-i})} b_k^j - \sum_{k \neq i} \sum_{j \in S(b_i, b_{-i})} b_k^j \tag{6}$$

The utility of agent $i$ from a profile of bids $b$ is defined as $u_i(b) = v_i(S(b)) - p_i(b)$, where $v_i(S(b)) = v_i^0 + \sum_{j \in S(b)} \alpha_{ji} E_{ji}$ if $i \in S(b)$ and $v_i(S(b)) = \sum_{j \in S(b)} E_{ji}$ otherwise. We impose a standard no overbidding assumption, meaning that for any subset of items, the sum of bids submitted by an agent cannot exceed his valuation for it.[24]

We analyze two main solution concepts: Pure Strategy Nash Equilibrium (PSNE) and Bayes-Nash Equilibrium (BNE). The definitions and analysis for Bayes-Nash Equilibrium can be found in Appendix B.

A bid profile $b$ is a Pure Strategy Nash Equilibrium if for all agents $i$ and any feasible deviation $b_i'$ the following holds:

$$u(b_i, b_{-i}) \geq u(b_i', b_{-i})$$

Pure Strategy Nash Equilibrium assumes a full information model, where agents' valuations are commonly known.

We denote the welfare function by $W(S) = \sum_{i=1}^n v_i(S)$. Let $S^{OPT}$ be the optimal subset of agents to be allocated given the cardinality constraint $|S| \leq k$. The *Price of Anarchy (PoA)* for PSNE is the ratio of the worst-case welfare in equilibrium and that of the optimal allocation:

$$\min_{b:\text{a PSNE}} \frac{W(S(b))}{W(S^{OPT})}$$

When we refer to the Price of Anarchy of the item bidding mechanism, we mean the minimum PoA across all possible valuation functions $\{v_i(S)\}_{i=1}^n$ and number of items $k$.

[22] In the item bidding mechanism, each agent needs to provide $n$ values, where $n$ is the number of agents. For agents' valuations in the policymaker's problem provided in equation 1, each agent needs to provide $2n$ values, and for general submodular valuations $2^n - 1$ values are needed.

[23] Furthermore, in the item bidding mechanism, the computational burden is transferred from the policymaker to the agents.

[24] Formally, we require that $\sum_{j \in S} b_i^j \leq v_i(S)$ for every agent $i \in N$ and $S \subseteq N$.

We begin the analysis of the item bidding mechanism by showing that there exists an optimal PSNE for the policymaker's problem. This result does not require the no overbidding assumption

**Proposition 5.** *There exists a pure strategy Nash equilibrium $b$ such that $S(b) = S^{OPT}$ for the policymaker's problem under the item bidding mechanism with VCG payments. Moreover, at this equilibrium agents bid their valuation for the optimal allocation, i.e. $\sum_{i=1}^{n} b_i^j = v_i(S^{OPT})$ for every $i \in N$.*

*Proof.* Let $S^{OPT}$ be the optimal subset of agents to be allocated. We will show that the following bidding profile

$$b_i^j = \begin{cases} v_i^0, & i, j \in S^{OPT}, i = j \\ \alpha_{ji} E_{ji}, & i, j \in S^{OPT}, i \neq j \\ E_{ji}, & i \notin S^{OPT}, j \in S^{OPT} \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

is a PSNE. It is immediate that $S(b) = S^{OPT}$ and that $p_i(b) = 0$ for all $i \in N$ since $B_j = 0$ for any $j \notin S^{OPT}$. Assume by contradiction that $b$ is not a PSNE. Then there exists an agent $i$ with a profitable deviation $b_i'$ such that $u_i(b_i', b_{-i}) = v_i(S(b_i', b_{-i})) - p_i(b_i', b_{-i}) > v_i(S^{OPT}) = u_i(b)$. From the definition of the payment rule we have that $p_i(b_i', b_{-i})) = \sum_{k \neq i} \sum_{j \in S(0, b_{-i})} b_k^j - \sum_{k \neq i} \sum_{j \in S(b_i', b_{-i})} b_k^j$. Plugging agent $i$'s payment after the deviation and re-arranging terms we have

$$\begin{aligned} v_i(S(b_i', b_{-i})) + \sum_{k \neq i} \sum_{j \in S(b_i', b_{-i})} b_k^j &> v_i\left(S^{OPT}\right) + \sum_{k \neq i} \sum_{j \in S(0, b_{-i})} b_k^j \\ \implies v_i(S(b_i', b_{-i})) + \sum_{k \neq i} \sum_{j \in S(b_i', b_{-i})} b_k^j &> \sum_{i=1}^{n} v_i\left(S^{OPT}\right) \end{aligned}$$

Where the second inequality follows from the observation that $S(0, b_{-i}) = S^{OPT}$ and that under the bidding profile $b$ all agents are bidding their true valuation for the optimal allocation. Next, we show that

$$\sum_{k \neq i} \sum_{j \in S(b_i', b_{-i})} b_k^j \leq \sum_{k \neq i} v_k(S(b_i', b_{-i}))$$

The inequality above follows from the observation that for any agent $k \neq i$:

$$\sum_{j \in S(b_i', b_{-i})} b_k^j = \begin{cases} v_k^0 + \sum_{j \in S^{OPT} \cap S(b_i', b_{-i})} \alpha_{jk} E_{jk}, & k \in S^{OPT} \cap S(b_i', b_{-i}), \\ \sum_{j \in S^{OPT} \cap S(b_i', b_{-i})} \alpha_{jk} E_{jk}, & k \in S^{OPT} \text{ and } k \notin S(b_i', b_{-i}) \\ \sum_{j \in S^{OPT} \cap S(b_i', b_{-i})} E_{jk}, & k \notin S^{OPT} \end{cases}$$

Which implies that for any agent $k \neq i$, $\sum_{j \in S(b_i', b_{-i})} b_k^j \leq v_k(S(b_i', b_{-i}))$. Thus, we have that

$$\sum_{i=1}^{n} v_i(S(b_i', b_{-i})) > \sum_{i=1}^{n} v_i \left(S^{OPT}\right)$$

Contradicting the optimally of $S^{OPT}$. □

This result shows that the Price of Stability, the ratio of the best-case equilibrium and that of the optimal allocation, is 1. Moreover, it proves existence of PSNE for the policymaker's problem under the proposed mechanism. However, this does not rule out the existence of "bad" equilibria in terms of efficiency loss.

**Proposition 6.** *The PSNE price of anarchy of the item bidding mechanism with VCG payments is $\frac{1}{n}$.*

*Proof.* The proof is carried out in two parts. We first give a $\frac{1}{n}$ lower bound on the PSNE PoA and then provide an example to show that the bound is tight.

Let $b$ be a PSNE and denote by $S(b) = S$ the resulting allocation. Since $b$ is a PSNE, no agent has a profitable deviation. This implies that for any agent $i$ and any deviation $b_i'$ that results in the optimal allocation, i.e., $S(b_i', b_{-i}) = S^{OPT}$, the following holds

$$v_i(S) - p_i(b_i, b_{-i}) \geq v_i(S^{OPT}) - p_i(b_i', b_{-i})$$
$$\implies v_i(S) + \sum_{k \neq i} \sum_{j \in S(b_i, b_{-i})} b_k^j \geq v_i(S^{OPT}) + \sum_{k \neq i} \sum_{j \in S(b_i', b_{-i})} b_k^j$$

Summing over all agents we have that

$$\sum_{i=1}^{n} \left( v_i(S) + \sum_{k \neq i} \sum_{j \in S(b_i, b_{-i})} b_k^j \right) \geq \sum_{i=1}^{n} \left( v_i(S^{OPT}) + \sum_{k \neq i} \sum_{j \in S(b_i', b_{-i})} b_k^j \right)$$
$$\implies W(S) + \sum_{i=1}^{n} \sum_{k \neq i} \sum_{j \in S(b_i, b_{-i})} b_k^j \geq W(S^{OPT})$$

Using the no overbidding assumption we note that

$$\sum_{i=1}^{n} \sum_{k \neq i} \sum_{j \in S(b_i, b_{-i})} b_k^j \leq (n-1) \sum_{i=1}^{n} v_i(S) = (n-1) W(S)$$

Thus,

$$nW(S) \geq W(S^{OPT})$$

Which proves a $\frac{1}{n}$ lower bound on the PSNE price of anarchy.

Next, we provide an example that shows that the bound is tight. Let $N = N_1 \cup N_2$ where $N_1 = \{1\}$ and $N_2 = \{2, \ldots, n\}$ and let $k = 1$. Agents have identical private values, i.e., $v_i^0 = v$ for all $i \in N$. The adjacency matrix of externalities is defined as follows:

$$E_{ij} = \begin{cases} v, & i = 1, j \in N_2 \\ 0, & \text{otherwise} \end{cases}$$

Finally, we set $\alpha_{ij} = 0$ for all $i, j \in N$. Let $b$ be defined as follows:

$$b_i^j = \begin{cases} v, & i = j = 2 \\ 0, & \text{otherwise} \end{cases}$$

Then $b$ is a PSNE and $S(b) = \{2\}$ and the ratio between the welfare in this equilibrium and that of the optimal allocation, $S^{OPT} = \{1\}$, is $\frac{1}{n}$. □

The analysis of the item bidding mechanism with VCG payments shows that while the best case equilibrium is efficient, there is a wide range of potential pure strategy Nash equilibria in terms of efficiency guarantee. In Appendix B we assume that agents' valuations are not common knowledge and extend the analysis presented here to Bayes-Nash equilibrium. We show that the worst-case BNE provides a performance guarantee that lies between $\frac{1}{n+1}$ and $\frac{1}{n}$.

# 6 Conclusion

This paper studies the problem of allocating a good or a service that is in limited supply when the allocation entails externalities. The literature on mechanism design has developed an extensive framework for designing efficient allocation mechanisms taking into account individuals' private values. However, far less attention has been given to the role of externalities. In this paper, we focus on the role of allocative externalities in the design of efficient allocation mechanisms. Including externalities results in a computationally hard problem for which exact optimization is not tractable. We thus turn to practical mechanisms that approximate the optimal solution and provide performance guarantees.

The connections established in this paper between designing efficient mechanisms in the face of allocative externalities and the well-known Combinatorial Public Project problem, as well as simple mechanisms such as item bidding, shed light on why mechanism design with externalities is a hard problem and might offer a formal framework for analyzing the role of externalities in mechanism design. We view the work presented in this paper as another step towards a broader understanding of mechanism design in the face of externalities.

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# Appendix A: Proofs

**Proof of Theorem 2.**

The proof proceeds by a series of steps.

**Step 1: Re-writing the Welfare Objective Function.**

We begin by re-writing the welfare objective function in the following way:

$$
\begin{aligned}
W(x) &= \sum_{i=1}^{n} v_i^0 x_i + \sum_{i,j\in N} E_{ij}\left(x_i - (1-\alpha_{ij})x_i x_j\right) \\
&= \sum_{i=1}^{n} \left( v_i^0 + \sum_{j=1}^{n}(1-\alpha_{ji})E_{ji} - \sum_{j=1}^{n}(1-\alpha_{ji})E_{ji} \right) x_i + \sum_{i,j\in N} E_{ij}\left(x_i - (1-\alpha_{ij})x_i x_j\right) \\
&= \sum_{i,j\in N} E_{ij}(x_i + (1-\alpha_{ij})x_j - (1-\alpha_{ij})x_i x_j) + \sum_{i=1}^{n} \left( v_i^0 - \sum_{j=1}^{n}(1-\alpha_{ji})E_{ji} \right) x_i
\end{aligned}
$$

where the second equality follows immediately from re-arranging terms.

Therefore, the policymaker's problem can be written as:

$$
\begin{aligned}
\max_x \; W(x) &= \sum_{i,j\in N} E_{ij}(x_i + (1-\alpha_{ij})x_j - (1-\alpha_{ij})x_i x_j) + \sum_{i=1}^{n} \left( v_i^0 - \sum_{j=1}^{n}(1-\alpha_{ji})E_{ji} \right) x_i \\
s.t. \; & \sum_{i=1}^{n} x_i = k, \\
& x_i \in \{0,1\}, \quad \forall i \in N.
\end{aligned}
$$

**Step 2: A Linear Relaxation and the Performance Guarantee.**

The policymaker's problem can be formulated as the following integer program:

$$
\begin{aligned}
\max_x \; L(x) &= \sum_{i,j\in N} E_{ij} \min\{x_i + (1-\alpha_{ij})x_j, 1\} + \sum_{i=1}^{n} \left( v_i^0 - \sum_{j=1}^{n}(1-\alpha_{ji})E_{ji} \right) x_i \qquad \text{(LP)} \\
s.t. \; & \sum_{i=1}^{n} x_i = k, \\
& x_i \in \{0,1\}, \quad \forall i \in N.
\end{aligned}
$$

It is easy to check that if $x^{OPT}$ is an optimal solution to the policymaker's problem then it must also be an optimal solution to (LP). We relax the integer program to a continuous one by replacing the binary constraints with $0 \leq x_i \leq 1$. Let $x^*$ be an optimal fractional solution to the relaxed linear program. We claim that:

$$W(x^*) \geq \begin{cases} 3/4L(x^*), & \alpha_{min} \leq 0.5 \\ \left(1-\alpha_{min}+\alpha_{min}^2\right)L(x^*), & \alpha_{min} > 0.5 \end{cases}$$

It suffices to check that for all pairs $i, j \in N$ the following holds

$$\begin{aligned} & x_i + (1-\alpha_{ij})x_j - (1-\alpha_{ij})x_i x_j \\ & \geq \begin{cases} 3/4\left(\min\{(x_i+(1-\alpha_{ij})x_j), 1\}\right), & \alpha_{ij} \leq 0.5 \\ \left(1-\alpha_{ij}+\alpha_{ij}^2\right)\left(\min\{(x_i+(1-\alpha_{ij})x_j), 1\}\right), & \alpha_{ij} > 0.5 \end{cases} \end{aligned} \tag{8}$$

Assume first that $x_i + (1-\alpha_{ij})x_j < 1$. For $\alpha_{ij} \leq 0.5$, we get that (8) is equivalent to $x_i + (1-\alpha_{ij})x_j \geq 4(1-\alpha_{ij})x_i x_j$, which follows from:

$$x_i + (1-\alpha_{ij})x_j \geq (x_i + (1-\alpha_{ij})x_j)^2 \geq 4(1-\alpha_{ij})x_i x_j$$

For the case where $\alpha_{ij} > 0.5$ we get that (8) is equivalent to $x_i + (1-\alpha_{ij})x_j \geq \frac{x_i x_j}{\alpha_{ij}}$. The inequality trivially holds for $x_i = 0$ or $x_j = 0$. For $x_i \neq 0$ and $x_j \neq 0$, we can rewrite the inequality as:

$$\frac{x_i + (1-\alpha_{ij})x_j}{x_i x_j} \geq \frac{1}{\alpha_{ij}}$$

The expression on the left side attains a global minimum at $x_i = \alpha_{ij}$ and $x_j = 1$ and, thus, the inequality holds for this case.

Next assume that $x_i + (1-\alpha_{ij})x_j \geq 1$. For the case where $\alpha_{ij} \leq 0.5$, we have that (8) is equivalent to $x_i + (1-\alpha_{ij})x_j - (1-\alpha_{ij})x_i x_j \geq 3/4$. Since $x_i + (1-\alpha_{ij})x_j - (1-\alpha_{ij})x_i x_j$ attains a global minimum at $x_i = \frac{1}{2}$, $x_j = \frac{1}{2(1-\alpha_{ij})}$, the inequality holds for this case.

For $\alpha_{ij} > 0.5$, we have that (8) is equivalent to $x_i + (1-\alpha_{ij})x_j - (1-\alpha_{ij})x_i x_j \geq 1 - \alpha_{ij} + \alpha_{ij}^2$. The expression on the left side attains a global minimum at $x_i = \alpha_{ij}$ and $x_j = 1$ and, thus, the inequality also hold for this case.

**Step 3: Pipage Rounding.**

In this step we round the fractional solution, $x^*$, to an integral one, using a deterministic rounding method known as "pipage" rounding.

We start by defining the following function:

$$\phi(\epsilon, x, i, j) = W(x_1, \ldots, x_i + \epsilon, \ldots, x_j - \epsilon, \ldots, x_n)$$

We say that this function satisfies the **$\epsilon$-convexity condition** if it is convex with respect to $\epsilon \in [-\min\{x_i, 1-x_j\}, \min\{1-x_i, x_j\}]$ for any pair of indices $i$ and $j$ and each $x \in [0,1]^n$.

Notice that for each pair of indices $i$ and $j$ the function $\phi(\epsilon, x, i, j)$ is the sum of

$E_{ij}(-(1-\alpha_{ij})(x_i+\epsilon)(x_j-\epsilon)) + E_{ji}(-(1-\alpha_{ji})(x_j-\epsilon)(x_i+\epsilon))$ and a term linear in $\epsilon$. It follows that $\phi(\epsilon, x, i, j)$ is a quadratic polynomial in $\epsilon$ with a non-negative leading coefficient for each pair of indices $i$ and $j$ and each $x \in [0,1]^n$. Thus, $\phi(\epsilon, x, i, j)$ satisfies the $\epsilon$-convexity condition.

If the solution $x^*$ is not binary, then due to the feasibility constraint $\sum_{i=1}^n x_i = k$, it has at least two different components $x_i$ and $x_j$ with values lying strictly between 0 and 1. By the $\epsilon$-convexity condition, $\phi(\epsilon, x, i, j) \geq W(x)$ either for $\epsilon = -\min\{x_i, 1-x_j\}$ or for $\epsilon = \min\{1-x_i, x_j\}$. Thus, we obtain a new feasible solution $\hat{x} = (x_1, \ldots, x_i + \epsilon, \ldots, x_j - \epsilon, \ldots, x_n)$ with a smaller number of noninteger components and such that $W(\hat{x}) \geq W(x^*)$. Repeating this step at most $n-1$ times yields a binary solution $x'$ with

$$W(x') \geq W(x^*)$$

which completes the proof. □

**Proof of Lemma 1.**

Let $x$ and $\hat{x}$ be two allocations such that $x \leq \hat{x}$. This implies $\hat{x}_i = 1$ if $x_i = 1$, for all $i \in N$. Let $j \in N$ be such that $\hat{x}_j = 0$ and hence $x_j = 0$. Finally, let $x \cup \{j\}$ be the allocation that allocates a unit of the good to agent $j$ as well as to those who were allocated in allocation $x$. Then, we have that,

$$\begin{aligned} W(x \cup \{j\}) - W(x) &= v_j^0 + \sum_{k \in N} E_{jk}(1-x_k) - \sum_{k \in N}(1-\alpha_{kj})E_{kj}x_k \\ &\geq v_j^0 + \sum_{k \in N} E_{jk}(1-\hat{x}_k) - \sum_{k \in N}(1-\alpha_{kj})E_{kj}\hat{x}_k \\ &= W(\hat{x} \cup \{j\}) - W(\hat{x}) \end{aligned}$$

where the inequality follows from $\hat{x}_i = 1$ if $x_i = 1$, for all $i \in N$. □

**Proof of Proposition 3.**

We start with the following definition of the total curvature of a monotone submodular set function $W : 2^x \to \mathbb{R}_+$.

**Definition 7 (Total Curvature).** The total curvature of a set function $W(\cdot)$ is a parameter $c \in [0,1]$ such that

$$c = 1 - \min_{S, j \notin S} \frac{W(S \cup \{j\}) - W(S)}{W(\{j\})}$$

The total curvature reflects how much the marginal value of $W(S \cup \{j\})$ can decrease as a function of $S \subseteq X$. We will use the following result which will be key in our proof.

**Theorem 3 (Conforti and Cornuéjols (1984)).** *If $X$ is a uniform matroid and $U$ has a total curvature of $c$, then*

$$W(a^G) \geq \frac{1}{c}\left(1 - \frac{1}{e^c}\right) W(a^*) \tag{9}$$

In the policymaker's problem (PM-k) the constraint induces a uniform matroid of exactly $k$ elements. The total curvature of the welfare function is given by

$$\begin{aligned} c &= 1 - \min_{j \in N} \frac{W(\{N \setminus \{j\}\} \cup \{j\}) - W(\{N \setminus \{j\})}{W(\{j\})} \\ &= 1 - \min_{j \in N} \frac{v_j^0 - \sum_{i \in N}(1-\alpha_{ij})E_{ij} + \sum_{i \in N} \alpha_{ji} E_{ji}}{v_j^0 + \sum_{i \in N} E_{ji}} \end{aligned}$$

Notice that

$$\begin{aligned} \min_{j \in N} \frac{v_j^0 - \sum_{i \in N}(1-\alpha_{ji})E_{ij} + \sum_{i \in N} \alpha_{ij} E_{ji}}{v_j^0 + \sum_{i \in N} E_{ji}} &\geq \min_{j \in N} \frac{v_j^0 - (1-\alpha_{min}) \sum_{i \in N} E_{ij} + \alpha_{min} \sum_{i \in N} E_{ji}}{v_j^0 + \sum_{i \in N} E_{ji}} \\ &= \min_{j \in N} \frac{(1-\alpha_{min})(v_j^0 - \sum_{i \in N} E_{ij})}{v_j^0 + \sum_{i \in N} E_{ji}} + \alpha_{min} \\ &\geq \min_{j \in N} \frac{(1-\alpha_{min})(v_j^0 - \gamma_{in} v_j^0)}{v_j^0 + \gamma_{out} v_j^0} + \alpha_{min} \\ &= \frac{(1-\alpha_{min})(1-\gamma_{in})}{1+\gamma_{out}} + \alpha_{min} \end{aligned}$$

Therefore we have that,

$$c \leq \frac{(1-\alpha_{min})(\gamma_{in} + \gamma_{out})}{1+\gamma_{out}}$$

Finally, note that $\frac{1}{c}\left(1 - \frac{1}{e^c}\right)$ is a decreasing function of $c$, for $c \in [0,1]$, and thus the inequality in (9) holds for $c = \min\{\frac{(1-\alpha_{min})(\gamma_{in}+\gamma_{out})}{1+\gamma_{out}}, 1\}$. □

# Appendix B: Item Bidding Mechanism: Bayes-Nash Equilibrium

In this section, we examine the price of anarchy of Bayes-Nash equilibrium (BNE). BNE is the standard solution concept when it is assumed that agents' valuations are not common knowledge. Let $V_i$ denote agent $i$'s possible valuations. The valuation profile $v = (v_1, \ldots, v_n)$ is drawn from a product distribution $F = F_1 \times F_2 \cdots \times F_n$ over the set $V = V_1 \times V_2 \cdots \times V_n$, where $F$ and $V$ are common knowledge. A Bayes-Nash equilibrium is a profile of strategies $b(v) = (b_i(v_i))_{i \in N}$ such that for each agent $i \in N$, $v_i \in V_i$ and any deviation $b'_i(v_i)$:

$$\mathbb{E}_{v_{-i} \sim F_{-i}}[u_i(b_i(v_i), b_{-i}(v_{-i})] \geq \mathbb{E}_{v_{-i} \sim F_{-i}}[u_i(b'_i(v_i), b_{-i}(v_{-i})]$$

Let $W(S, v) = \sum_{i=1}^{n} v_i(S)$ be the welfare function given the valuation profile $v$. Denote by $S^{OPT}(v)$ the optimal subset of agents to be allocated given the cardinality constraint $|S| \leq k$ and valuation profile $v$. The *Price of Anarchy (PoA)* for BNE is the ratio of the worst-case expected welfare in equilibrium and that of the optimal allocation.

$$\min_{b\text{: a BNE}} \frac{E_{v \sim F}[W(S(b(v)), v)]}{E_{v \sim F}[W(S^{OPT}(v), v)]}$$

When we refer to the Bayes-Nash Price of Anarchy of the item bidding mechanism, we mean the minimum PoA across all possible valuation sets $V$, distributions $F$ and number of items k.

From Proposition 6 we know that the BNE price of anarchy must be at most $\frac{1}{n}$.[25] We provide an upper bound of $\frac{1}{n+1}$ by utilizing the *smoothness* framework developed by Roughgarden (2012) for incomplete information games. We adjust Roughgarden (2012) definition of smooth games to our setting as presented in Markakis and Telelis (2017).

**Definition 8** (Roughgarden (2012))**.** A mechanism is $(\lambda, \mu)$–smooth with respect to a bidding function $b^* : V \to \mathbb{R}^{n \times n}$ if

$$\sum_{i=1}^{n} u_i(v_i;\ S(b_i^*(v), b'_{-i})) \geq \lambda \sum_{i=1}^{n} v_i(S(b^*(v))) - \mu \sum_{i=1}^{n} v'_i(S(b'))$$

for every valuation profiles $v, v' \in V$, and every bidding profile $b'$ that satisfies no over bidding with respect to $v'$.

[25]This observation is immediate since the analysis of PSNE is a special case of BNE in which the product distribution is degenerate.

**Theorem 4** (Roughgarden (2012))**.** *If a mechanism is $(\lambda, \mu)$–smooth with respect to a bidding function $b(\cdot)$ that maximizes $W(S(b))$, then the Bayes-Nash equilibrium price of anarchy is at least $\frac{\lambda}{\mu+1}$.*

The bidding profile provided in the proof of Proposition 5 can be easily converted into a bidding function that maximizes welfare with respect to any valuation profile. We show that the item bidding mechanism with VCG payments is $(1, n)$–smooth with respect to that function.

**Proposition 7.** *The item bidding mechanism with VCG payments is $(1, n)$–smooth with respect to the bidding function (7).*

*Proof.* Let $v, v' \in V$ be two profiles of valuations. Denote by $b^*(v) = b^*$ be the bidding profile from (7). For any bidding profile $b'$ that satisfies no over bidding with respect to $v'$ we have that

$$\begin{aligned} u_i(v_i\,;\,S(b_i^*, b'_{-i})) &= v_i(S(b_i^*, b'_{-i})) - p_i(b_i^*, b'_{-i}) \\ &= v_i(S(b_i^*, b'_{-i})) - \sum_{k \neq i} \sum_{j \in S(0, b'_{-i})} b_k^j + \sum_{k \neq i} \sum_{j \in S(b_i^*, b'_{-i}))} b_k^j \\ &\geq \sum_{k=1}^{n} \sum_{j \in S(b_i^*, b'_{-i}))} b_k^j - \sum_{k \neq i} \sum_{j \in S(0, b'_{-i})} b_k^j \end{aligned}$$

where the last inequality follows from no overbidding. Next, note that

$$\sum_{k \neq i} \sum_{j \in S(0, b'_{-i})} b_k^j \leq \sum_{k=1}^{n} \sum_{j \in S(b'_i, b'_{-i})} b_k^j$$

This inequality holds due to the allocation rule of the item bidding mechanism. Moreover, from the definition of the bidding profile (7) and using the definition of the allocation rule again, the following also holds:

$$\sum_{k=1}^{n} \sum_{j \in S(b_i^*, b'_{-i}))} b_k^j \geq v_i(S(b^*))$$

Using the two inequalities above:

$$u_i(v_i\,;\;S(b_i^*,b'_{-i}) \geq \sum_{k=1}^{n}\sum_{j\in S(b_i^*,b'_{-i}))} b_k^j - \sum_{k\neq i}\sum_{j\in S(0,b_{-i})} b_k^j$$

$$\geq v_i(S(b^*)) - \sum_{k=1}^{n}\sum_{j\in S(b_i^*,b_{-i})} b_k^j$$

$$\geq v_i(S(b^*)) - \sum_{k=1}^{n} v_k(S(b'))$$

where the last inequality follows from no overbidding. Summing over all agents:

$$\sum_{i=1}^{n} u_i(v_i;\;S(b_i^*(v),b'_{-i})) \geq \sum_{i=1}^{n} v_i(S(b^*(v)) - n\sum_{i=1}^{n} v'_i(S(b'))$$

□

Proposition 7 immediately implies a lower bound on the price of anarchy for BNE.

**Corollary 3.** *The Bayes-Nash price of anarchy of the item bidding mechanism with VCG payments is at least* $\frac{1}{n+1}$.

Corollary 3 together with Proposition 6 imply that the *BNE Price of Anarchy* lies between $\frac{1}{n+1}$ and $\frac{1}{n}$.