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Common Asset Impact on Default Contagion

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by

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To my wife Camilla, with love. Your constant support is the reason I was able to get here.

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Curriculum Vitæ

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Awards

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Abstract

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In this work we present a simulation study to show that a shock in a common asset can be very impactful to default contagion, and we extend some analytic concepts to this scenario with common assets. We use an inhomogeneous random graph to represent the banking network, and, based on the possible exposures between banks, we define a minimum amount of capital each bank must hold in order to make the system stable to a shock that affects only a few banks. Then, we consider the case when a shock hits all banks at the same time, making them weaker and some of them initially in default. We analyze the final fraction of banks in default and compare it with other cases when the shock hits only a small proportion of banks. We show that a common shock can cause severe damage to the system.

Key words: Default Contagion, Common Assets, Inhomogeneous Random Graph, Banking Network.

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Chapter 1

Introduction

Risk management has been a research topic in financial mathematics for a long time. In 1952, Markowitz developed a theory to maximize the expected returns of a portfolio with a given risk, which was defined as the variance of the portfolio. This pioneered research in risk of a portfolio of assets. Later, Value at Risk became a widely used tool to manage the risk of an asset or a pool of assets. There were many contributions to the risk of a single asset, including some extensions of Value at Risk. A summary of these risk measures is described in [12].

Researchers then became interested in the risk of the failure of many banks at the same time, for instance when the default of a few banks triggers a cascade that can harm the entire financial system. This area received special attention after the financial crisis in 2008. In this particular example, the loss in one asset, mortgage-backed securities, could have started a cascade process that put the entire financial system in risk of failure. However, in this case, the default cascade was contained to just a few companies possibly due to regulator intervention.

There is no unique definition of systemic risk. Almost every single paper cited in this work will have their own understanding of systemic risk and what it means according to their model. However, a generic definition given in [22] gives a good understanding of what is systemic risk and most definitions are in agreement with this one. The authors define systemic risk as a risk of a break of capital flow in the market which leads to reduction in growth of the system's GDP. Our understanding of systemic risk in this work is the risk that a large portion of a banking system will be in default after a contagion process started by a few banks in default. It makes sense that if a large portion of banks are in default, the GDP of this system will have a slower growth rate, because there will be less banks investing in the real economy, for instance.

Tom Hurd in his recent book [26] summarizes systemic risk as follows: a triggering event, the propagation of shock through the financial system and the impact on the macroeconomy. The triggering event can come from outside of the financial system, for example a terrorist attack, or it can come from inside, for instance a spontaneous default of a major financial institution. For details on these definitions see [26].

Tom Hurd, in [26], also defines possible channels of contagion: *Asset correlation, default contagion, liquidity contagion, market illiquidity and fire sales.* The system is susceptible to *asset correlation* when banks have some of the same assets in their portfolios, if these assets suddenly drop price, the system can be harmed. For instance, in the years 2000, most large banks held positions in the US sub-prime mortgage market. *Default contagion* arises from interbank loans. A bankrupted bank which can no longer pay its liabilities to other banks in the system can trigger a default cascade, in which several other banks may go bankrupt as well. *Liquidity contagion* happens when banks are short of cash or other liquid assets and have to shrink their balance sheet. Finally, *market illiquidity and fire sales* happen when banks have to sell shares of a given asset and its prices drop, which can affect other banks. This is a common source of indirect contagion, as discussed in [14].

There are several approaches on how to model systemic risk. In [10] the authors define a system of diffusion processes coupled by their drifts and analyze the probability of a large number of banks reaching a default threshold. They define a game where each bank controls the amount of money they borrow or lend to a central bank and study the mean field game in the limit of a large number of banks. Other approaches include modeling the banking network with graphs, for instance the Eisenberg and Noe model in [20], or random graphs, for instance in the work of Gai and Kapadia in [23].

There are several works and extensions which deal with a graph to model the banking system. In 2000, Allen and Gale in [2] analyzed a network of four banks and show that the spread of the contagion highly depends on the interconnectedness between the four banks. In 2001, Eisenberg and Noe in [20] proposed a model with a graph given by the interbank loans and provided, under the assumption of no-bankruptcy cost, the existence of a unique clearing vector of payments after one or more banks default. In a recent study in [31], the authors greatly generalize the Eisenberg-Noe model to include bankruptcy cost and other channels of contagion: fire sales and cross holdings. Other extensions of the Eisenberg-Noe contagion model are discussed in [4], [5], and [6].

Recent studies have considered not the actual banking network but a random graph. In this context it is possible to make a rigorous analysis and obtain asymptotic results when the network is large enough. In [23], Gai and Kapadia introduce a random graph representing the financial network and assume a zero recovery rate. This work analyzes the dynamics of the contagion process and how it is affected by the network structure. The authors in [29] analyze how the network structure affects systemic risk by varying several parameters of the network. In Cont *et al.*, [3], the authors analyze the fraction of defaulted banks and obtain some first resilience conditions when using a configuration model for the random graph. In [16] and [18], Detering *et al.* introduce a directed inhomogeneous random graph that has similar characteristics as a real banking network, i.e., strong inhomogeneity and absence of second moment of the degree sequence. The authors analyze the default contagion on such a network and obtain analytical results, such as resilience conditions and capital requirements for the banks in the system. Empirical evidence of the properties of the banking network can be found in [13] and [8] in which the authors do an extensive data analysis of the Brazilian and Austrian networks, respectively. Extending the work in [16], in [18] the authors define a random graph with weighted edges representing the exposures between two banks in the system. The authors analyze the final fraction of defaulted banks and give conditions on the each bank's capital to make resilient network, that is, the system is stable when subject to small shocks in its initial condition.

Another channel of contagion is fire sales. In [15], Cont *et al.* propose a model for systemic risk with fire sales as a contagion mechanism and illustrate their model with data for hedge fund losses of August 2007 and the Great Deleveraging following the default of Lehman Brothers in Fall 2008. In [14], Cont *et al.* use data on European banks to analyze indirect contagion through deleveraging effects.

Common assets, as stated in [26], is one of the main sources of systemic risk. It has been studied in several papers, each with its own approach on how to model the impact of common assets on systemic risk.

Systemic risk with common assets and asset fire sales, that is, after an asset has dropped its prices, banks want to sell it quickly which leads to the price dropping even further, has been studied in [24]. The authors use the holdings of each financial institution in the network, a rule applied to these institutions on how to react after a shock and liquidity of assets, to find how fire sales add up to the entire financial system. They find that a shock in one asset has a larger impact if a levered bank holds said asset. The mechanism of systemic risk arising from asset fire sale has also been studied in [11], in which the authors propose a model where every bank is invested in several assets with different prices and exogenous risk factors. The asset prices drop due to exogenous shock and the authors identify two financial responses from the network: direct effects which is the immediate response to the shock and network effects which is the amplification of the shock through changes in the prices of assets. The authors then identify conditions to minimize the risk for all the banking system. In [17], the authors analyze the joint effect of default contagion and asset fire sales in a random graph similar to the one defined in [18].

In [1] the authors propose a model in which banks have projects with random returns and one bank can invest in another banks project, which would create overlapping portfolios. One project could have a low return and the bank would not have the money to pay back the amount it invested in the project. The paper analyzes what happens to the system of banks in such a scenario. The authors conclude that the clustering of the network has huge impact on systemic risk. In [9], the authors propose a network where the nodes represent portfolios chosen from multiple assets and edge w_{ij} captures the loss bank j suffers when bank i liquidates its portfolio. Liquidation drops down the prices of the stocks and other banks will suffer losses as well. The authors also propose some measures of vunerability based on the network they created. In [25] the authors have a similar network model and perform a statistical analysis the topology of the underlying network in a system of hedge funds.

In [28] the authors analyze shared risk between agents. There is no banking network or common asset in this paper, however the authors define a vector V of risk variables which are shared amongst various agents. They assume each entry of V has a Pareto distribution and find bounds for the individual risk of each agent. Even though this work does not deal with common assets, it studies common sources of risk, similar to what a common portfolio would do in a system of banks.

Similarly to [13], in [21] the authors analyze the Austrian banking system in the presence of correlated assets. They integrate financial risk management for a single bank with a network analysis and apply a simulation study of their proposed cascade model with the data of Austrian banks. Their results show that, despite the low probability of default, the contagion process can affect a major portion of the network.

1.1 Outline and Results

This work will use the results in [16] and [18] to analyze the fraction of banks in default in a scenario including assets common to many banks.

In Chapter 2 we will define an inhomogeneous random graph which models the banking network, define the interbank exposures and add to the banks balance sheet an investment in assets that are external to the banking network. We will present the main results from [16] which analyze default contagion in an inhomogeneous random graph and give some results regarding resilience of the network.

In Chapter 3 we will present the results from [18] which expand the results in [16] to include the interbank exposures. We will assume that an asset common to all banks in the system suffers a sudden price drop, which will make some banks default and trigger a default cascade. We show how a shock in the common asset affects each banks capital and we analyze the default contagion after the shock. We use a two-period model: before and after the shock in the common asset, which will highlight the effects of a common shock into default contagion. We also propose a way to define a dynamic network model, which is a possible extension of the two-period model analyzed in this work, where the capital is a stochastic process which depends on the price of a given stock.

In Chapter 4 we present a simulation study where we analyze default contagion after a shock in the common asset for different networks and shocks. We analyze how shocks to the banking system arising from common assets holdings affects default contagion. We also compare this scenario with the example presented in previous studies, for instance

[18], in which some banks default initially but all the others stay as they were at first. Our results show that when default contagion is triggered by a common shock to all banks, the proportion of the network which ends the contagion process in default is much larger than when just a few banks suffer a loss.

The authors in [13] analyze the Brazilian network and conclude that a shock common to all banks increases the proportion of contagion considerably. Their study was based on one sample of an existing network. Our results show the same conclusion with a probabilistic approach and again, ignoring how the market outside the banking network when analyzing default contagion can lead to underestimation of the proportion of the network that will be affected. Similarly to [13], our results show that a common shock introduces several *contagious links*, i.e., banks who will default as soon as one neighbor is in default. In [3] the authors show that the presence of contagious links are a source of systemic risk in networks in which the degree sequence has finite second moment. We observe the same in the examples presented in this work. If the degree sequence of the network does not have finite second moment, we observe that the instability is created by shocking larger banks. In the last example, we assume that the shock to the common asset is comonotone with the connectivity of the banks, that is, larger banks will suffer bigger losses. We show in this case that a big shock is required to start the default contagion, but it has catastrophic consequences to the network. This agrees with the findings in [21]. The shock required to make the network default and trigger the cascade is unlike to happen, however if it does it will affect a very large portion of the network.

Chapter 2

Banking Network Structure

In this chapter we will cover the results needed for the remainder of this work. We will briefly introduce a random graph and some of its characteristics. Then, we will summarize the theory presented in [16] and [18] in which the authors define an inhomogeneous random graph and analyze the default contagion process in this network after a small proportion of vertices are initially in default, or infected, and this infection spreads through the system. We will also summarize some conditions, given in [18], under which the network is stable under small shocks.

2.1 Random Graphs - Basic Definitions

A *directed graph* $G = (V, E)$ consists of a set of vertices V and a set of edges E. The set V is a finite set of size $n \in \mathbb{N}$. We usually number the vertices as $1, 2, \ldots, n$, and therefore we denote $V = [n] := \{1, 2, \ldots, n\}$. An edge connects two vertices and represent some relation between them. A directed edge from vertex i to vertex j is an ordered pair $(i, j) \in E$, $i \neq j$. The set E is the collection of all edges in $G = (V, E)$.

We can also define two important quantities based on a directed graph: the *in-degree* and *out-degree*.

Definition 1 *The in-degree $d^-(u)$ of a vertex u is equal to the number of edges received by vertex u:*

$$d^-(u) = \#\{v \in V : \{v, u\} \in E\}. \tag{2.1.1}$$

The out-degree $d^+(u)$ of a vertex u is equal to the number of edges sent by vertex u:

$$d^+(u) = \#\{v \in V : \{u, v\} \in E\}. \tag{2.1.2}$$

A graph can also be represented by its adjacency matrix $M(G)$, defined by:

$$M_{vu}(G) = \mathbf{1}((v, u) \in G) = \begin{cases} 1, & \text{if } (v, u) \in G, \\ 0, & \text{if } (v, u) \notin G. \end{cases} \tag{2.1.3}$$

From the adjacency matrix (2.1.3) we can easily find the in-degree and out-degree: $d^-(v) = \sum_u M_{uv}(G)$ and $d^+(v) = \sum_u M_{vu}(G)$.

With a graph well defined over n vertices, we can define a random graph.

Definition 2 ***Random Graph**: A random graph with n vertices is a probability space $(E, \mathcal{F}, \mathbb{P})$ on the finite set $G(V, E)$. The sigma-algebra $\mathcal{F}$ is just the power set of E.*

The most common example of a random graph is the Erdős-Rényi random graph. In Example 1 we make a slight modification to the original Erdős-Rényi random graph to make it a directed random graph.

Example 1 *In an Erdős-Rényi random graph of size n, denoted by $G(n, p)$, all edges are present independently of each other with probability p, except self-edges, which are not present. So $\mathbb{P}((u, u) \in G(n, p)) = 0$ for all $u \in [n]$ and $\mathbb{P}((u, v) \in G(n, p)) = p$ for all $(u, v) \in [n]^2$, $u \neq v$.*

We can easily write the adjacency matrix for $G(n,p)$ in terms of independent and identically distributed Bernouli random variables, $X_{u,v}$ with parameter p:

$$M_{u,v}(G) = \begin{cases} X_{u,v} & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases} \tag{2.1.4}$$

From the adjacency matrix (2.1.4) we can see that the distribution for both in-degree and out-degree for the $G(n,p)$ have a Binomial distribution $X \sim Bin(n,p)$. A usual choice for p is $p = \frac{\lambda}{n}$, for some $\lambda > 0$, then the limiting distribution for both in-degree and out-degree is a Poisson distribution with parameter λ.

The Erdős-Rényi random graph has been extensively studied, see, for instance, [7] and [30]. One downside of this random graph when modeling financial networks is that it generates homogeneous networks, i.e., all vertices have the same in-degree and out-degree distribution. Moreover, the in-degree and out-degree distribution have finite moments. In a financial network, it is observed that some banks are much more connected than others and their in-degree and out-degree may not have second moment. Therefore, in order to model a financial network accurately we need to use another random graph.

2.2 Inhomogeneous Random Graph

In [13] the authors do an extensive statistical analysis of the Brazilian network in which they found that the banking network has some nodes which are highly connected to others whereas some vertices have very few connections. This property can be captured with the inhomogeneous random graph.

The inhomogeneous random graph model used here follows the definition in [16] and [18].

(Inhomogeneous Random Graph Model) For each $n \in \mathbb{N}$ we consider the vertex set $[n] = \{1, ..., n\}$ and the set of directed edges $D := \{(i, j)|i, j \in [n], i \neq j\}$. Let $\Omega_1 := \{0, 1\}^{|D|}$ and $\mathcal{F}_1 := 2^{\Omega_1}$. We define a probability measure $\mathbb{P}_1$ on $(\Omega_1, \mathcal{F}_1)$ in the following way. To each vertex $i \in [n]$ there are two deterministic weights $w_i^- = w_i^-(n) \in \mathbb{R}_+$ and $w_i^+ = w_i^+(n) \in \mathbb{R}_+$ and the probability $p_{i,j} = p_{i,j}(n)$ for $i \neq j$ that there exists a directed edge from vertex i to vertex j by is given by

$$p_{i,j} = \min \left\{1, w_i^+ w_j^- /n\right\}. \tag{2.2.1}$$

Furthermore, we assume that the event that an edge is present happens independent of the presence of all other edges. Call the resulting random graph $G_n(w^+, w^-)$.

The quantities w_i^+ and w_i^- determines the tendency of vertex $i \in [n]$ to have incoming or outgoing edges, respectively. The inhomogeneous random graph can capture well the inhomogeneity observed in many networks, since vertices with higher weights are more likely to have many neighbors than vertices with small weights.

Remark 1 *Note that we are only using w_i^+ and w_i^- which determines a tendency. So far nothing was said about the actual in-degree or out-degree of a vertex. These quantities are related and we will describe this relationship in Section 2.3.*

Figures 2.1 and 2.2 show examples of inhomogeneous random graphs with $n = 50$ nodes. Our examples show two inhomogeneous random graphs with different choices on how we sample w_i^+ and w_i^-. In the first example, in Figure 2.1, the weights w_i^+ and w_i^- are sampled from a Pareto (see Definition 3) distribution with $\beta = 3.1$. The second example, in Figure 2.2, the weights w_i^+ and w_i^- are sampled from a Pareto distribution with $\beta = 2.1$ which has no second moment.

Definition 3 ***Pareto Distribution****: A random variable X has Pareto distribution with* *parameter β when the pdf is given by:*

$$f_X(x) = \begin{cases} (\beta-1)(x_m)^{\beta-1}x^{-\beta}, & \textit{if } x \geq x_m, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.2.2}$$

for $\beta > 0$ and $x_m > 0$.

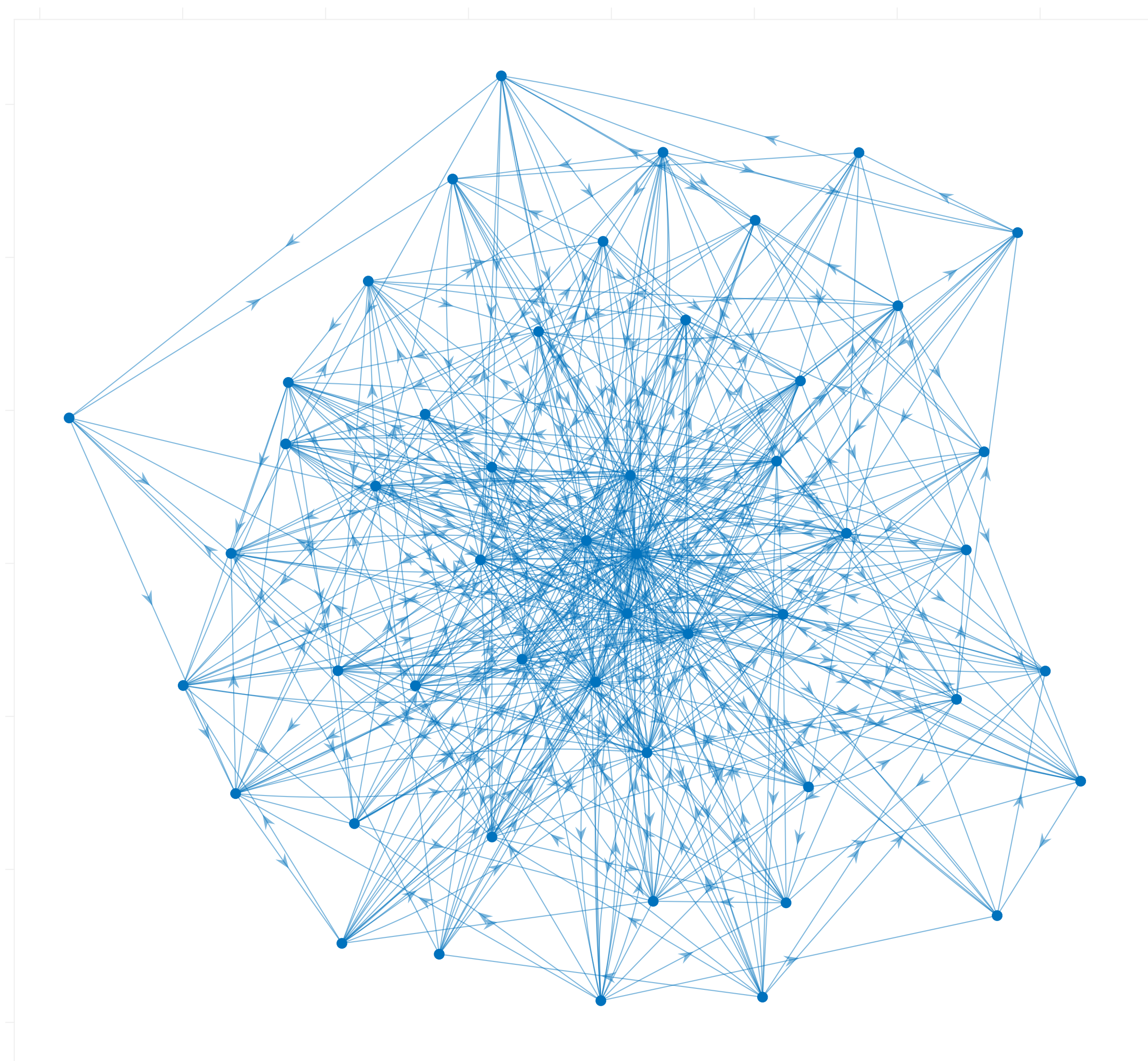

Figure 2.1: Inhomogeneous random graph with w^+ and w^- sampled from a Pareto distribution with parameter $\beta = 3.1$.

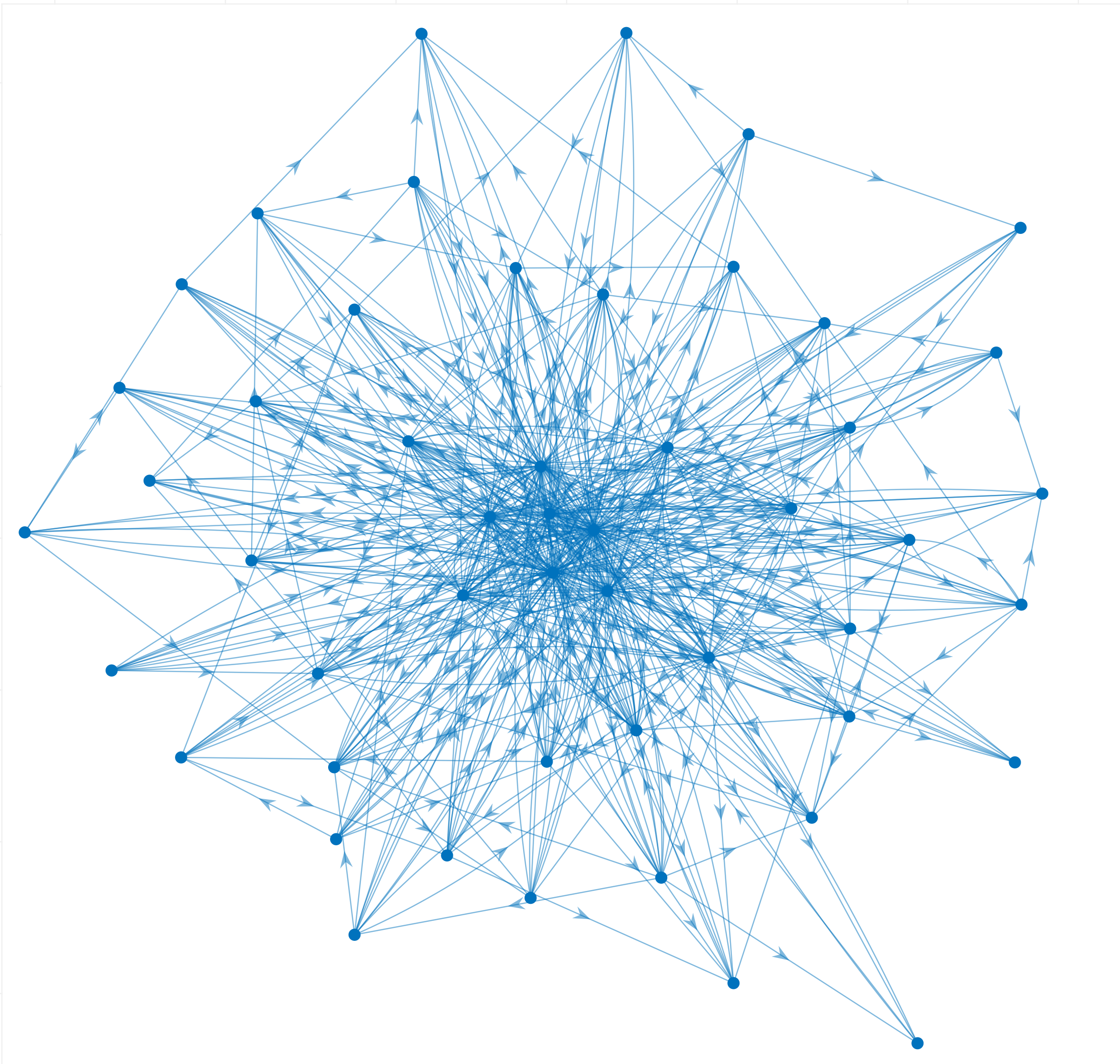

Figure 2.2: Inhomogeneous random graph with w^+ and w^- sampled from a Pareto distribution with parameter $\beta = 2.1$.

We chose $x_m = 1$ for the graph in Figure 2.2. The choice for x_m in the first graph was made in such a way that the number of edges in both graphs are close. In fact, the graph in Figure 2.1 has 560 total edges and the graph in Figure 2.2 has 553 total edges.

However we notice the presence of highly connected vertices in the second example. This is one of the advantages of the inhomogeneous random graph. It can generate random graphs with vertices that are highly connected and some vertices with only a few edges. From our examples it is clear that a random graph with weights sampled from a Pareto distribution without second moment we can generate such graphs.

We need to ensure that the proportion of vertices of degree k approaches a limit when we let the size of the network n tend to infinity. We will assume that the vertex weights satisfy the following regularity conditions state below. It turns out these conditions imply convergence of the degree distribution in the generalized random graph.

Consider the empirical distribution of the deterministic weights $(w^-(n), w^+(n))_{n\geq 1}$:

$$F_n(x,y) = n^{-1}\sum_{i\in[n]} \mathbf{1}\left\{w_i^-(n) \leq x, w_i^+(n) \leq y\right\} \quad \text{for all} (x,y) \in \mathbb{R}^2. \tag{2.2.3}$$

In the following (W_n^-, W_n^+) is a random vector with distribution function $F_n(x,y)$.

Definition 4 *The sequence $(w^-(n), w^+(n))$ is regular if it satisfies the following conditions:*

1. *There exists a distribution function F such that for all (x,y) where F is continuous, $\lim_{n\to\infty} F_n(x,y) = F(x,y)$.*

2. *Let (W^-, W^+) be a random variable with distribution F. Then $\lim\limits_{n\to\infty} \mathbb{E}[(W_n^-, W_n^+)] = \mathbb{E}[(W^-, W^+)]$.*

3. *$w_i^+(n)$ and $w_i^-(n)$ are lower bounded by a positive constant.*

Here is an illustration of a regular weight sequence, adapted from [30]:

Example 2 *Let F be a distribution function for which $F(0) = 0$. Define*

$$w_i^- = w_i^+ = [1-F]^{-1}(i/n), \tag{2.2.4}$$

where $[1-F]^{-1}$ *is the generalized inverse function of* $1-F$ *defined, for* $u \in (0,1)$*, by*

$$[1-F]^{-1}(u) = \inf\{x : [1-F](x) \leq u\}. \tag{2.2.5}$$

By convention, we set $[1-F]^{-1}(1) = 0$*. The definition of* $[1-F]^{-1}$ *is chosen such that*

$$[1-F]^{-1}(1-u) = F^{-1}(u) = \inf\{x : F(x) \geq u\}. \tag{2.2.6}$$

For the weights chosen as in equation (2.2.4)*, we can compute the empirical distribution function* F_n*:*

$$\begin{aligned} F_n(x) &= \frac{1}{n}\sum_{i\in[n]} \mathbb{1}_{\{w_i \leq x\}} = \frac{1}{n}\sum_{i\in[n]} \mathbb{1}_{\{[1-F]^{-1}(i/n)\leq x\}} = \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{\{[1-F]^{-1}(1-\frac{j}{n})\leq x\}} \\ &= \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{\{F^{-1}(\frac{j}{n})\leq x\}} = \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{\{\frac{j}{n}\leq F(x)\}} = \frac{1}{n}(\lfloor n(F(x)\rfloor + 1) \wedge 1 \end{aligned}$$

We can see that $\lim_{n\to\infty} F_n(x) = F(x)$*. Note that* $\lim_{n\to\infty} \frac{\lfloor n(F(x)\rfloor+1}{n} = F(x)$*. A particular example when* F *has a power law distribution is given below.*

$$F(x) = \begin{cases} 0, & \textit{for } x \leq a \\ 1-(a/x)^{\tau-1}, & \textit{for } x > a, \end{cases}$$

with $a \geq 0$ *and* $\tau > 1$*. It follows that*

$$[1-F]^{-1}(u) = au^{-1/(\tau-1)}$$

and

$$w_i^- = w_i^+ = a(\frac{i}{n})^{-1/(\tau-1)}.$$

The example above is in one dimension. We will need to simulate two-dimensional regular weight sequences. We start with an example where the two random variables are independent.

Example 3 *Let F_{W^-} and F_{W^+} be distribution functions for which $F_{W^-}(0) = 0$ and $F_{W^+}(0) = 0$. Define*

$$w^-_{j\lfloor\sqrt{n}\rfloor+k} = F^{-1}_{W^-}(\frac{j}{\lfloor\sqrt{n}\rfloor}), \quad j = 0, 1, \ldots, \lfloor\sqrt{n}\rfloor - 1; \quad k = 1, 2, \ldots, \lfloor\sqrt{n}, \rfloor \tag{2.2.7}$$

and

$$w^+_{j\lfloor\sqrt{n}\rfloor+k} = F^{-1}_{W^+}(\frac{k-1}{\lfloor\sqrt{n}\rfloor}), \quad j = 0, 1, \ldots, \lfloor\sqrt{n}\rfloor - 1; \quad k = 1, 2, \ldots, \lfloor\sqrt{n}.\rfloor \tag{2.2.8}$$

Suppose W^- and W^+ are independent. For the weights chosen as defined above, we have

$$\begin{aligned}
F_n(x,y) &= \frac{1}{n}\sum_{i\in[n]} \mathbb{1}_{\{w_i^- \le x, w_i^+ \le y\}} = \frac{1}{n}\sum_{i\in[n]} \mathbb{1}_{\{w_i^- \le x\}} \mathbb{1}_{\{w_i^+ \le y\}} \\
&= \frac{1}{n}\sum_{j=0}^{\lfloor\sqrt{n}\rfloor-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \mathbb{1}_{\{w^-_{j\lfloor\sqrt{n}\rfloor+k} \le x\}} \mathbb{1}_{\{w^+_{j\lfloor\sqrt{n}\rfloor+k} \le y\}} \\
&= \frac{1}{n}\sum_{j=0}^{\lfloor\sqrt{n}\rfloor-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \mathbb{1}_{\{F^{-1}_{W^-}(\frac{j}{\lfloor\sqrt{n}\rfloor}) \le x\}} \mathbb{1}_{\{F^{-1}_{W^+}(\frac{k-1}{\lfloor\sqrt{n}\rfloor}) \le y\}} \\
&= \frac{1}{n}\sum_{j=0}^{\lfloor\sqrt{n}\rfloor-1}\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \mathbb{1}_{\{\frac{j}{\lfloor\sqrt{n}\rfloor} \le F_{W^-}(x)\}} \mathbb{1}_{\{\frac{k-1}{\lfloor\sqrt{n}\rfloor} \le F_{W^+}(y)\}} \\
&= \frac{1}{n}\sum_{j=0}^{\lfloor\sqrt{n}\rfloor-1} \left(\mathbb{1}_{\{\frac{j}{\lfloor\sqrt{n}\rfloor} \le F_{W^-}(x)\}}\right) \left(\sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \mathbb{1}_{\{\frac{k-1}{\lfloor\sqrt{n}\rfloor} \le F_{W^+}(y)\}}\right) \\
&= \frac{1}{n}\big(\lfloor\sqrt{n}\rfloor F_{W^-}(x) \lfloor\sqrt{n}\rfloor F_{W^+}(y) + 1\big) \wedge 1
\end{aligned} \tag{2.2.9}$$

It follows that $\lim_{n\to\infty} F_n(x,y) = F_{W^-}(x)F_{W^+}(y)$.

Remark 2 *We can find $\lim_{n\to\infty} \frac{(\lfloor\sqrt{n}\rfloor)^2}{n}$ using $\lfloor\sqrt{n}\rfloor = \sqrt{n} - \delta$, for $\delta \in [0,1)$.*

$$\lim_{n\to\infty} \frac{(\lfloor\sqrt{n}\rfloor)^2}{n} = \lim_{n\to\infty} \frac{(\sqrt{n}-\delta)^2}{n} = \lim_{n\to\infty} \frac{n-2\sqrt{n}\delta+\delta^2}{n} = 1.$$

Often in the simulations done in this work, we will need W^- and W^+ to be comonotone. Let us introduce some properties of comonotone random variables.

Proposition 1 *Suppose we have a pair of random variables (X, Y) with continuous marginal distributions F_X and F_Y. The pair is comonotone if and only if one of the following equivalent properties hold:*

1. $F_{X,Y}(x,y) = \min\{F_X(x), F_Y(y)\}$, *for all* $x, y \in \mathbb{R}$;
2. *For* $U \sim Unif(0,1)$, $(X,Y) \stackrel{\mathcal{D}}{=} (F_X^{-1}(U), F_Y^{-1}(U))$;

The proof of the properties in Proposition 1 can be found in [19] in the multivariate case. A summary of the properties of comonotone and counter-monotone random variables in the bivariate case can be found in [27].

In Example 4 we build a regular weight sequence for comonotone random variables.

Example 4 *Let F_{W^-} be a distribution function for which $F_{W^-}(0) = 0$ and F_{W^+} be a distribution function for which $F_{W^+}(0) = 0$. Define the weights w_i^- and w_i^+ as*

$$w_i^- = F_{W^-}^{-1}\left(\frac{i-1}{n}\right),$$

and

$$w_i^+ = F_{W^+}^{-1}\left(\frac{i-1}{n}\right).$$

The empirical distribution function F_n is

$$\begin{aligned} F_n(x,y) &= \frac{1}{n}\sum_{i\in[n]} \mathbb{1}_{\{w_i^-\le x, w_i^+\le y\}} = \frac{1}{n}\sum_{i\in[n]} \mathbb{1}_{\{\frac{i-1}{n}\le F_{W^-}(x), \frac{i-1}{n}\le F_{W^+}(y)\}} \\ &= \frac{1}{n}\sum_{i\in[n]} \mathbb{1}_{\{\frac{i-1}{n}\le \min\{F_{W^-}(x), F_{W^+}(y)\}\}} \end{aligned}$$

$$= \frac{1}{n} \left(\lfloor n \min \{F_{W^-}(x), F_{W^+}(y)\} \rfloor + 1 \right) \wedge 1 \tag{2.2.10}$$

We can see that $\lim_{n\to\infty} F_n(x, y) = \min \{F_{W^-}(x), F_{W^+}(y)\} = F_{X,Y}(x, y)$.

2.3 Banking Network Statistics

In Chapter 4 we will simulate a default contagion process in a inhomogeneous random graph, that is, we infect one or a few vertices in the random graph, and observe how that infection spreads through the graph.

We will simulate a banking network with an inhomogeneous random graph where the vertices represent banks and the edges represent the presence of interbank loans. To each possible directed edge (i, j) with $i, j \in [n]$ and $i \neq j$, we assign another random variable $E_{i,j}$ which represents the amount of money bank i owes to bank j in case edge (i, j) is present in the network.

In [13] the authors did an extensive statistical analysis the Brazilian banking network. Their findings show that the distributions of the in-degree, out-degree and interbank exposures have a heavy-tail behavior.

2.3.1 Degree Distribution

In our model, the degree distribution is obtained through the limiting distribution of vertex weights (W^-, W^+), which according to [13] have a heavy-tailed distribution. More precisely, the authors perform a goodness of fit test for the degree sequence to a Pareto distribution and fail to reject that the data fits said distribution. Additionally they show that the distribution is stable over time. To find the degree distribution we will need the following definition:

Definition 5 *A random variable X has a mixed Poisson distribution with mixing distribution F when, for every $k \in \mathbb{N}_0$,*

$$\mathbb{P}(X = k) = \mathbb{E}\left[e^{-W}\frac{W^k}{k!}\right], \tag{2.3.1}$$

where W is a random variable with distribution function F.

In Chapter 6 of [30], the author shows that for an undirected inhomogeneous random graph, the degree distribution is a mixed Poisson with mixing distribution W (see Definition 5). In Section 3 of [16] the authors extend this result for a directed random graph.

The following Proposition shows that a mixed Poisson distribution with a Pareto mixing distribution has a heavy-tail.

Proposition 2 *Let X be a mixed Poisson random variable as in Definition 5. If the mixing distribution of W is a power law, then the tail of the distribution of X is a power law.*

Proof: Assume the density function $f_W(w)$ of the mixing distribution of W is the following:

$$f_W(w) = \begin{cases} cw^{-\alpha-1}, & \text{if } w > w_{\min}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3.2}$$

for $\alpha, w_{\min} > 0$. Then

$$\begin{aligned}\mathbb{P}(X = k) &= \mathbb{E}\left[e^{-W}\frac{W^k}{k!}\right] \\ &= \int_{w_{\min}}^{\infty} e^{-w}\frac{w^k}{k!}cw^{-\alpha-1}dw \\ &= \frac{c}{k!}\Gamma(k-\alpha, w_{\min}).\end{aligned}$$

$$(2.3.3)$$

Here $\Gamma(s,x) = \int_x^\infty t^{s-1}\exp^{-t} dt$ is the upper incomplete gamma function. From equation (2.3.3), if we let $k \to \infty$, we obtain

$$\lim_{k\to\infty} \frac{c}{k!}\Gamma(k-\alpha, w_{\min}) = \lim_{k\to\infty} \frac{c}{k!}(\Gamma(k-\alpha) - \gamma(k-\alpha, w_{\min})) = ck^{-(\alpha+1)}. \tag{2.3.4}$$

Here $\gamma(s,x) = \int_0^x t^{s-1}\exp^{-t} dt$ is the lower incomplete gamma function and $\Gamma(s,x) = \int_0^\infty t^{s-1}\exp^{-t} dt$ is the gamma function. We used the property that $\Gamma(s) = \Gamma(s,x) + \gamma(s,x)$. To find the limit we use the following asymptotic properties of the gamma functions:

$$\lim_{k\to\infty} \frac{\Gamma(k-\alpha)}{\Gamma(k)k^{-\alpha}} = \lim_{k\to\infty} \frac{k\Gamma(k-\alpha)}{k!k^{-\alpha}} = 1, \text{ for } \alpha \in \mathbb{R},$$

and

$$\lim_{k\to\infty} \frac{\gamma(k,w)}{k!} = \lim_{k\to\infty} \frac{1}{k!} \sum_{j=0}^{\infty} \frac{w^{k+j}\exp^{-w}}{j(j+1)\dots(j+k)} = 0.$$

■

Therefore, in our simulations both W^- and W^+ will have Pareto distributions.

2.3.2 Exposures Distribution

In [18] the authors study default contagion in a banking network where they consider the interbank loans. In this work we are interested in seeing how a loss in an asset common in many bank's balance sheet can trigger and/or increase the cascade effect. We will use the exposure matrix $E_{i,j}$, where each element represents the amount of money bank i owes to bank j in case the edge (i,j) is present in the network. We will also consider that each bank has amount A_i invested in a single asset and a debt D_i. These quantities are summarized in Table 2.1.

Assets	Liabilities
Interbank assets $\sum_j E_{ji}$	Interbank liabilities $\sum_j E_{ij}$
Other assets A_i: derivatives, bonds, real state	Other debt D_i: deposits, short sales

Table 2.1: Example of bank's balance sheet

We will assume that $E_{i,j}$ is pairwise independent and Exponentially distributed with parameter λ. As mentioned in [13] the authors show that each element in $E_{i,j}$ follows a Pareto distribution. We assume Exponential distribution for analytical tractability. In Chapter 3 we will use the interbank exposures to find the distribution of the threshold level, an important quantity used to study the default contagion process.

2.4 Default Contagion for Threshold Model

The main Theorem in [16] gives us a tool to analyze the final fraction of banks in default after a contagion process. First we shall define a threshold level.

Definition 6 *To each vertex $i \in [n]$ we associate a number $c_i \in \mathbb{N}_0^\infty$ which we call threshold level.*

A vertex i becomes infected after a c_i number of vertices that have directed edge to vertex i are infected. Vertices that can never become infected have a threshold level equal infinity. Vertices with threshold level 0 are infected.

We need to assume some further conditions in the sequence $(w^-(n), w^+(n), c(n))$, which includes the threshold level.

Definition 7 *Let $(w^-(n), w^+(n))$ be a regular weight sequence and $c(n)$ a sequence of threshold values. We say that the sequence $(w^-(n), w^+(n), c(n))$ is a regular vertex sequence if there exists a distribution function $F : \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0^\infty \to [0,1]$ such that for all points $(x, y, l) \in \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0^\infty$ for which $F(x, y, l)$ is continuous in (x, y) we have* $\lim_{n\to\infty} F_n(x, y, l) = F(x, y, l)$ *where $F_n(x, y, l)$ is the empirical distribution function*

$$F_n(x, y, l) = n^{-1} \sum_{i \in [n]} \mathbf{1}\left\{w_i^-(n) \leq x, w_i^+(n) \leq y, c_i(n) \leq l\right\} \quad \forall (x, y, l) \in \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0^\infty. \tag{2.4.1}$$

Note that in Definition 7, we do not impose integrability conditions as in Definition 4.

Assume we have a random graph G, as defined in the previous section, where the sequence $(w^-(n), w^+(n), c(n))$ converges in distribution to (W^-, W^+, C). Suppose there is an initial set of vertices that are infected, which we denote as $\mathcal{A}_0 := \{i \in [n] | c_i = 0\}$. After $k \in \mathbb{N}$ rounds of infection, the set of infected vertices is given by:

$$\mathcal{A}_n := \{i \in [n] | N_G^-(i) \bigcap \mathcal{A}_{k-1} \geq c_i\} \tag{2.4.2}$$

where $N_G^-(i)$ is the set of the in-neighbors of vertex i.

Equation (2.4.2) can be interpreted in the following way: if in round $k-1$ a vertex has more than c_i neighbors in default, then vertex i itself becomes defaulted. We call $\mathcal{A}_n$ the final set of infected vertices, as the contagion process stops after at most $n-1$ rounds.

Theorem 1, from [16], gives us an analytical way to determine the final fraction of infected vertices.

First let us introduce the following notation. For $r \in \mathbb{N}_0 \cup \{\infty\}$, let $\psi_r(x)$ be defined as

$$\psi_r(x) = \begin{cases} \mathbb{P}(Poi(x) \geq r) = \sum_{j \geq r} \exp^{-x} x^j / j!, & \text{for } r \geq 0 \\ \\ 0, & \text{for } r = \infty \end{cases} \tag{2.4.3}$$

Theorem 1 *Let $(w^-(n), w^+(n), c(n))_{n\geq 1}$ be a regular vertex sequence with limiting distribution $F : \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0^\infty \to [0,1]$. Let (W^-, W^+, C) be a random vector with distribution F. Assume $\mathbb{P}(C = 0) > 0$. Denote by $\hat{z}$ the smallest positive solution of*

$$f(z; (W^-, W^+, C)) := \mathbb{E}[W^+ \psi_C(W^- z)] - z = 0 \tag{2.4.4}$$

Let $\mathcal{A}_n$ denote the final set of infected vertices in $G_n(w^-, w^+, c)$. Then

1. *For all $\epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(n^{-1}|\mathcal{A}_n| \geq \mathbb{E}[\psi_C(W^- \hat{z})] - \epsilon) = 1$.*

2. *If there exists $\delta > 0$ and a $\kappa < 1$ such that $\mathbb{E}[W^+ W^- \mathbb{P}(Poi(zW^-) = C - 1)\mathbf{1}_{C\geq 1}] < \kappa$ for $z \in (\hat{z} - \delta, \hat{z} + \delta)$, then*

$$n^{-1}|\mathcal{A}_n| \xrightarrow{p} g(\hat{z}; (W^-, C)) := \mathbb{E}[\psi_C(W^- \hat{z})], \quad \textit{as } n \to \infty. \tag{2.4.5}$$

In Theorem 1, the condition $\mathbb{P}(C = 0) > 0$ means that the probability of default is larger than 0. For the graph $G_n(w^-, w^+, c)$ that means that a linear number of vertices is infected. To see this, note that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i\in[n]} \mathbb{1}_{\{c_i = 0\}} \to \mathbb{P}(C = 0).$$

We can see from Theorem 1 that, in a particular case, the proportion of vertices infected in the end of the cascade process is determined by functional g, in equation (2.4.5). This functional depends on $\hat{z}$, the root of equation (2.4.4), and is increasing with $\hat{z}$. In Chapter 4 we will calculate numerically equation (2.4.4) and evaluate its root for different random vectors (W^-, W^+, C).

Theorem 1 gives a lower bound for the proportion of banks infected by the contagion process when the functional f in equation (2.4.4) touches 0 but then becomes positive again. In [18] the authors have an extension which allows us to also find an upper bound.

Theorem 2 *Under the same assumptions of Theorem 1, denote by $\hat{z}$ the smallest positive solution of $f(z)$ in equation (2.4.4), and let z^* be the smallest value of $z > 0$ at which $f(z)$ crosses zero, i.e.,*

$$z^* := \inf\{z > 0 : f(z) < 0\}. \tag{2.4.6}$$

Then the following holds:

1. *For all $\epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(n^{-1}|\mathcal{A}_n| \geq \mathbb{E}[\psi_C(W^-\hat{z})] - \epsilon) = 1$*

2. *If further $\mathbb{E}[W^+W^-\mathbb{P}(Poi(zW^-) = C - 1)\mathbf{1}_{C\geq 1}] - 1$ is continuous on some neighborhood of z^*, then for all $\epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(n^{-1}|\mathcal{A}_n| \leq \mathbb{E}[\psi_C(W^-z^*)] - \epsilon) = 1$. In particular, if $\hat{z} = z^*$, then*

$$n^{-1}|\mathcal{A}_n| \xrightarrow{p} g(\hat{z}; (W^-, C)) := \mathbb{E}[\psi_C(W^-\hat{z})], \quad \text{as } n \to \infty \tag{2.4.7}$$

The figures below show examples for each case in Theorem 1.

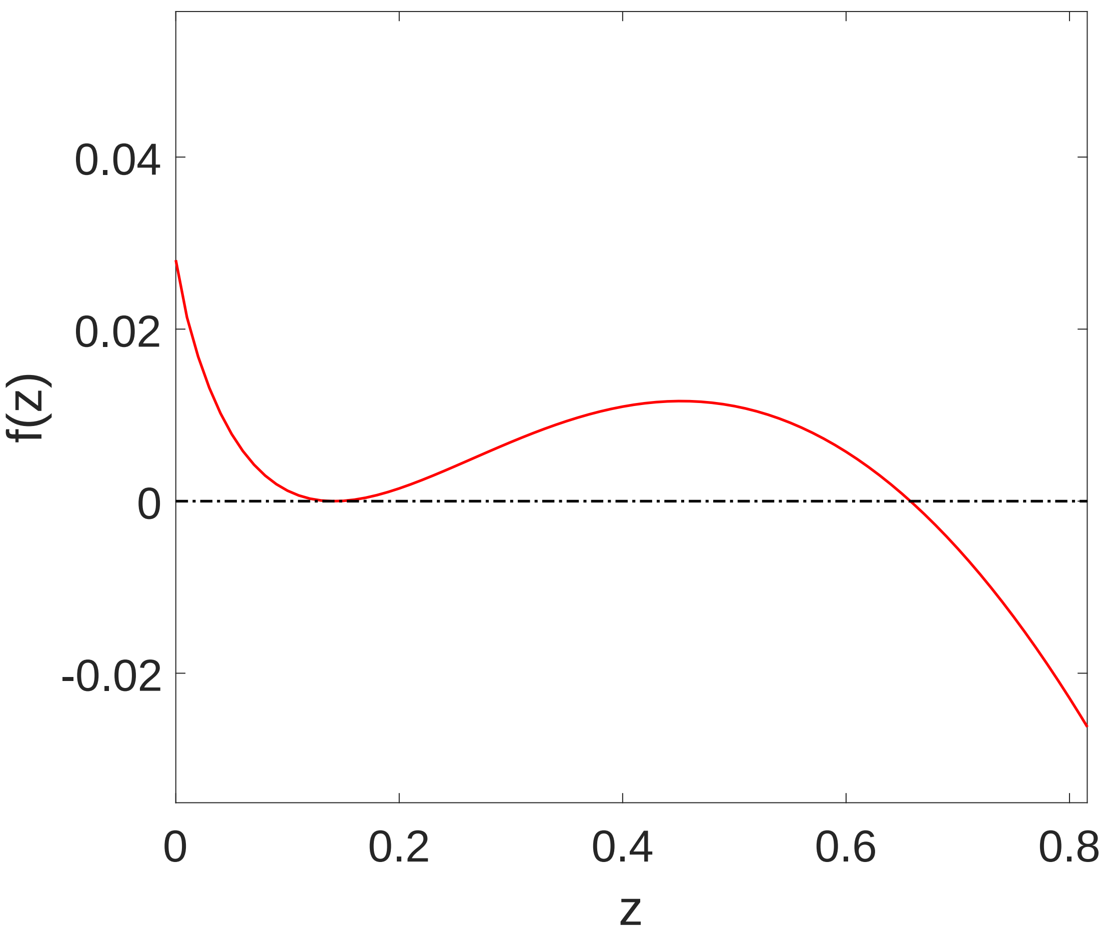

Figure 2.3: Example to explain case 1 in Theorem 1.

Figure 2.3 is an example of the first case in Theorem 1, when the functional z touches 0 and then becomes positive again. In this case Theorem 1 does not specify if the contagion process stops or not at $\hat{z}$, the first root. We can say that the final proportion of infected vertices is bounded below by $g(\hat{z})$, where $\hat{z}$ is the first positive root of f. Additionally, by Theorem 2 we can say that the final proportion of infected vertices is bounded above by $g(z^*)$, where z^* is the smallest value for which $f(z)$ is negative.

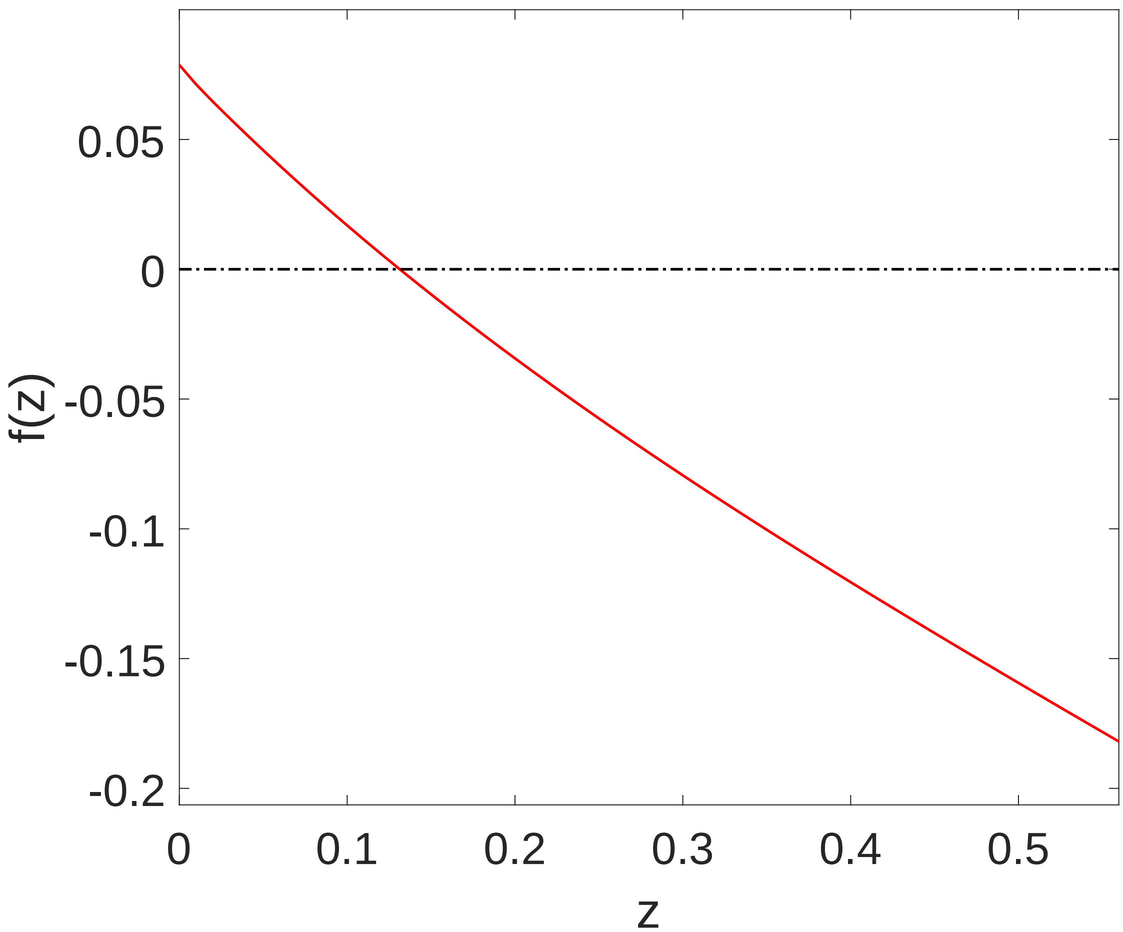

Figure 2.4: Example to explain case 2 in Theorem 1.

Figure 2.4 shows a more typical example which illustrates case 2 in Theorem 1. In this example the functional becomes negative after its first positive root $\hat{z}$. In this case, the theorem states that the contagion process stops. Moreover, the final proportion of infected vertices is given by $g(\hat{z})$.

2.5 Resilience Condition

In a *resilient* network, only a small proportion of the network should be affected after an initial shock has been applied to it. In a *non-resilient* network, even small shocks can possibly harm the entire system.

Our study considers default contagion in a network. In Chapter 4, we will apply a small shock to the network and see how it affects the entire network, that is, we will find what is the final proportion of infected vertices. In order to classify a network between *resilient* and *non-resilient*, we will use an adapted definition from [18]. The proofs for Theorems 3 and 4 can be found in [18] as well.

In Definitions 8 and 9, as well as Theorems 3 and 4, small shocks are understood as follows: we have $\mathbb{P}(C = 0) = 0$. We assume that a shock affects the network and the new threshold level becomes $\tilde{C} = C(1 - M)$, where M is a Bernoulli random variable with parameter p, that is, each bank is in default with probability p. We will assume that p is small or going to 0.

Definition 8 *A financial system is non-resilient if there exists a constant* $\delta > 0$ *such that*

$$n^{-1}|\mathcal{A}_n| > \delta \text{ with high probability,} \tag{2.5.1}$$

for any random variable M *with parameter* $p > 0$.

The following theorem gives a criterion to classify a system as *non-resilient*.

Theorem 3 *Assume that* $(w^-(n), w^+(n), c(n))$ *is a regular vertex sequence with limiting distribution* (W^-, W^+, C)*,* $\mathbb{P}(C = 0) = 0$*, and there exists* $z_0 > 0$ *such that*

$$f(z) > 0, \text{ for all } 0 < z < z_0. \tag{2.5.2}$$

Then for all M with parameter $p > 0$

$$n^{-1}|\mathcal{A}_n| > g(z_0) \text{ with high probability} \tag{2.5.3}$$

independent of p. Therefore, the system is non-resilient.

Definition 9 *A financial system is resilient if for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $p < \delta$ (i.e., $\mathbb{P}(\tilde{C} = 0) < \delta$),*

$$n^{-1}|\mathcal{A}_n| \leq \epsilon \text{ with high probability.} \tag{2.5.4}$$

The following Theorem gives a criterion to classify a system as *resilient*.

Theorem 4 *Assume that $(w^-(n), w^+(n), c(n))$ is a regular vertex sequence with limiting distribution (W^-, W^+, C), $\mathbb{P}(C = 0) = 0$, and there exists $z_0 > 0$ such that*

$$\mathbb{E}[W^- W^+ \mathbb{P}(Poi(W_- z) = C - 1)\mathbb{1}_{C \geq 1}] - 1 < 0, \text{ for all } 0 < z < z_0. \tag{2.5.5}$$

For every ϵ, there exists a p_0 such that if $\mathbb{P}(M = 0) < p_0$ then

$$n^{-1}|\mathcal{A}_n| < \epsilon \text{ with high probability.} \tag{2.5.6}$$

The system is resilient.

Figures 2.5 and 2.6 illustrates Theorems 3 and 4.

The resilience criteria given in this Section analyzes the behavior of the functional f in Theorem 1. In Chapter 3, we will show resilience conditions for the capital of each bank, i.e., a minimum amount of capital each bank must hold such that the network is resilient based on the definitions above.

We can slightly generalize Theorem 3 to the cases we will study in Chapters 3 and 4.

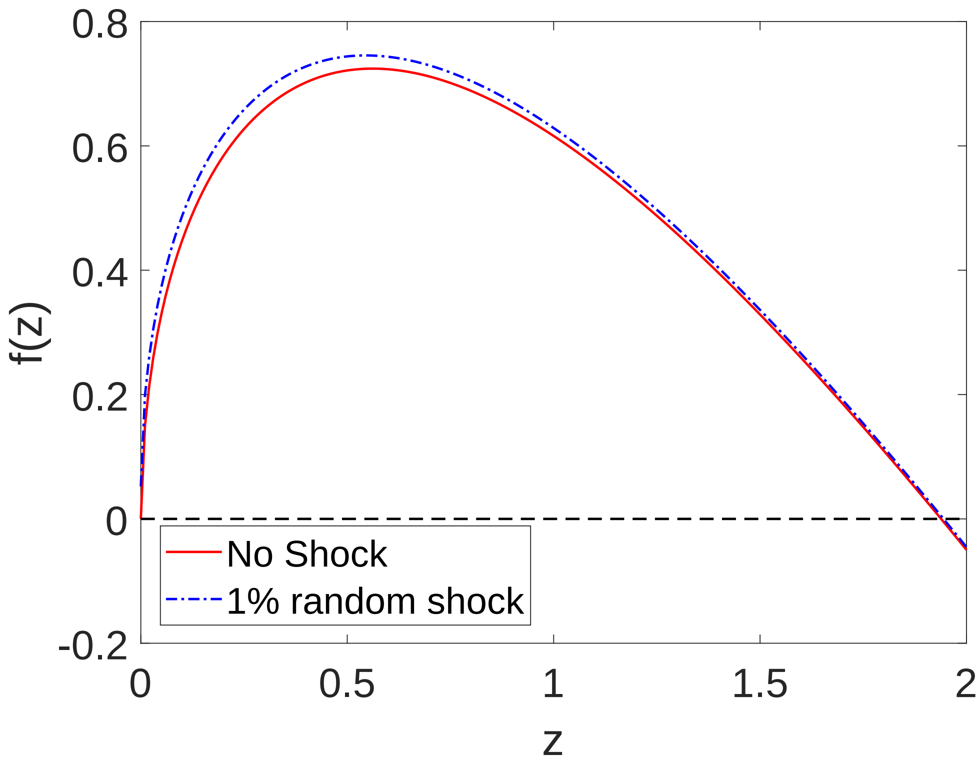

Figure 2.5: The contagion process for the unshocked system (red) starts at 0 and has positive derivative at 0. For the shocked system (blue), the functional is always slightly larger, and therefore has a larger root.

Theorem 5 *Assume that $(w^-(n), w^+(n), c(n))$ is a regular vertex sequence with limiting distribution (W^-, W^+, C), $\mathbb{P}(C = 0) = 0$, and there exists $z_0 > 0$ such that*

$$f(z; W^+, W^-, C) > 0, \text{ for all } 0 < z < z_0. \tag{2.5.7}$$

Now assume that we have a new $\tilde{C}$ such that $\mathbb{P}(\tilde{C} \leq C) = 1$. Then

$$f(z; W^+, W^-, \tilde{C}) \geq f(z; W^+, W^-, C) \tag{2.5.8}$$

therefore $\tilde{z}_0 > z_0$, where $\tilde{z}_0$ is the first positive root of $f(z; W^+, W^-, \tilde{C})$. Finally,

$$g(\tilde{z}_0) \geq g(z_0) \tag{2.5.9}$$

and the new system is still non-resilient.

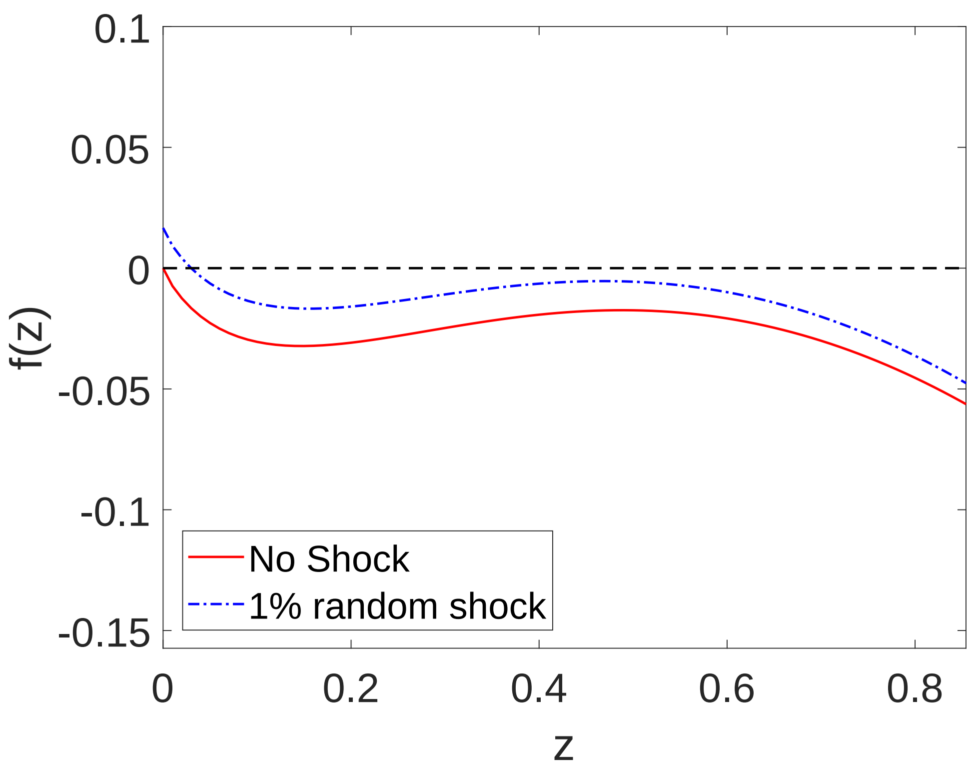

Figure 2.6: The contagion process for the unshocked system (red) starts at 0 and has negative derivative at 0. For the shocked system (blue), the contagion process starts positive but the derivative is still negative at 0.

Proof: If $\mathbb{P}(\tilde{C} < C) = 1$ then $\mathbb{P}(Poi(W^- z) \geq C) \leq \mathbb{P}(Poi(W^- z) \geq \tilde{C})$. It follows that

$$
\begin{aligned}
f(z; W^+, W^-, \tilde{C}) &= \mathbb{E}[W^+ \mathbb{P}(Poi(W^- z) \geq \tilde{C})] - z \\
&\geq \mathbb{E}[W^+ (\mathbb{P}(Poi(W^- z) \geq C))] - z \\
&= f(z; W^+, W^-, C).
\end{aligned}
$$

Since $f(z; W^+, W^-, \tilde{C}) \geq f(z; W^+, W^-, C)$, we have that $\tilde{z}_0 > z_0$, and by the monotonicity of $g(z)$, we have that $g(\tilde{z}_0) \geq g(z_0)$.

If $\mathbb{P}(\tilde{C} = 0) > 0$, then $f(z; W^+, W^-, \tilde{C}) > 0$, and the contagion process ends when

$z = \tilde{z}_0$ and the final proportion of infected vertices is $g(\tilde{z}_0) \geq g(z_0)$.

If $\mathbb{P}(\tilde{C} = 0) = 0$, there is no contagion process since there is no initial infection. Using Theorem 3 with $\tilde{C}$, we infect a proportion p of vertices and the final proportion of vertices in default is $n^{-1}|\mathcal{A}| > g(\tilde{z}_0)$.

■

In [16] and [18], the authors analyzed a network in which only a small proportion of vertices were infected, whereas the other vertices remain unchanged. In this work we are interested in analyzing the network after a shock in the common asset. In this case, not only will a few vertices be infected, but the other vertices can be weakened, so the threshold value after a shock is smaller (or equal) the initial threshold value. In Proposition 3 we show a condition under which the network with this new threshold level $\tilde{C}$, after a shock in the common asset, is stable, that is, the first positive root of the functional f is close to 0, and, therefore, the contagion process stops soon after it starts.

Proposition 3 *Assume that $(w^-(n), w^+(n), c(n))$ is a regular vertex sequence with limiting distribution (W^-, W^+, C), $\mathbb{P}(C = 0) = 0$, and there exists $z_0 > 0$ such that*

$$f(z; W^+, W^-, C) < 0, \textit{ for all } 0 < z < z_0. \tag{2.5.10}$$

Let $\min f(z) = \epsilon$, for all $0 < z < z_0$. Assume further that we have a new $\tilde{C}$ such that $\mathbb{P}(\tilde{C} \leq C) = 1$ and that $\mathbb{E}[W^+W^-\mathbb{P}(Poi(zW^-) = \tilde{C} - 1)\boldsymbol{1}_{\tilde{C} \geq 1}] - 1$, the weak derivative of $f(z; W^+, W^-, C)$, is continuous . If

$$\mathbb{P}(\tilde{C} \leq Poi(zW^-) \leq C - 1) < \delta \tag{2.5.11}$$

for some $\delta > 0$, then $f(z; W^+, W^-, \tilde{C}) < 0$ for some $z \in (0, z_0)$, and, therefore, the contagion process for $(W^+, W^-, \tilde{C})$ will stop before z_0.

Proof: We estimate $f(z; W^+, W^-, \tilde{C}) - f(z; W^+, W^-, C)$:

$$
\begin{aligned}
& f(z; W^+, W^-, \tilde{C}) - f(z; W^+, W^-, C) \\
&= \mathbb{E}[W^+ \mathbb{P}(Poi(W^- z) \geq \tilde{C})] - z - \left(\mathbb{E}[W^+ \mathbb{P}(Poi(W^- z) \geq C)] - z\right) \\
&= \mathbb{E}\left[W^+ \left(\mathbb{P}(Poi(W^- z) \geq \tilde{C}) - \mathbb{P}(Poi(W^- z) \geq C)\right)\right] \\
&= \mathbb{E}\left[W^+ \mathbb{P}\left(\tilde{C} \leq Poi(zW^-) \leq C - 1\right)\right] \\
&< \mathbb{E}[W^+ \delta] < \epsilon
\end{aligned}
\tag{2.5.12}
$$

In this case, $f(z; W^+, W^-, \tilde{C}) - f(z; W^+, W^-, C) < \epsilon$, therefore for some $z \in (0, z_0)$ we have that $f(z; W^+, W^-, \tilde{C}) < 0$, and, from Theorem 2, we know that the contagion process stops, and the final proportion of infected vertices is bounded by $g(z^*)$, where $z^* < z_0$ is the smallest value of $z > 0$ at which $f(z)$ crosses zero.

■

Remark 3 *Proposition 3 assumes that $f(z; W^+, W^-, C)$ is negative for $0 < z < z_0$. It states that under some conditions, $f(z; W^+, W^-, \tilde{C})$ is also negative for some $z \in [0, z]$. Therefore, the contagion process stops and the new system is also resilient.*

2.6 Systemic Threshold Requirements

A central bank or regulatory agency would be interested in determining a minimum threshold level to ensure that the network is resilient to external shocks. In Section 2.5 we showed some conditions under which the system is resilient, and in [18] the authors deduce a minimal threshold level to ensure that the system will meet these conditions. In fact, the authors present conditions which depend on the *in-weight* w^- and *out-weight* w^+.

Theorem 6 *Assume that* $(w^-(n), w^+(n), c(n))$ *is a regular vertex sequence with limiting distribution* (W^-, W^+, C) *and that* W^- *and* W^+ *are Pareto distributed (see Definition 3) with parameters* (β^-, w^-_{min}) *and* (β^+, w^+_{min})*, respectively, with* $\beta^-, \beta^+ > 2$ *and* $w^-_{min}, w^+_{min} > 0$*. For each bank* $i \in [n]$ *let the threshold value* τ_i *depend on the in-weight* w^- *by some functional form* $\tau_i = \tau_i(w^-)$*, where* $\tau : \mathbb{R}_+ \to \mathbb{N}\backslash\{0,1\}$*. Set*

$$\gamma_c := 2 + \tfrac{\beta^-+1}{\beta^++1} - (\beta^-) \text{ and } \alpha_c = \tfrac{\beta^+-1}{\beta^+-2} w^+_{min} (w^+_{min})^{(1-\gamma_c)}$$

Then the system is resilient if one of the following holds:

1. $\gamma_c < 0$.

2. $\gamma_c = 0$ *and* $\liminf_{w\to\infty} \tau(w) > \alpha_c + 1$.

3. $\gamma_c > 0$ *and* $\liminf_{w\to\infty} w^{-\gamma_c}\tau(w) > \alpha_c$.

Remark 4 *Part 1 of Theorem 6 states that the system is resilient for constant* $\tau(w) = 2$ *in case* $\gamma_c < 0$*, that is, when* $\beta^-, \beta^+ > 3$*. This is also stated in [3], where the authors say a network in which both* W^- *and* W^+ *have finite second moment is resilient if there are no contagious links, i.e., banks who will default as soon as one neighbor is in default. A contagious link is, in the notation used in this work, a vertex with threshold level* $\tau = 1$*.*

Remark 5 *In the case where* W^- *and* W^+ *do not have finite second moment, that is,* $\beta^-, \beta^+ \in (2,3)$*, we will use the following threshold level, proposed in [18]:*

$$\tau_i(w^-) = \max\left\{2, \alpha_c(1+\delta_1)(w^-_i)^{\gamma}_c(1+\delta_2)\right\},$$

where $\delta_1, \delta_2 > 0$*. This choice, according to part 3 of Theorem 6, makes the network resilient.*

Chapter 3

Common Asset Impact

The main goal of this chapter is to find out how a common shock to all banks, i.e., a shock in a common asset, will affect each bank and, consequently, the network, in case of a default cascade. As mentioned in Chapter 2, we will impose a random shock to each bank's capital and find the new threshold level for each bank. We will show, based on [18], how the threshold model presented in Chapter 2 can be used when we include bank exposures. The resulting cascade will be analyzed and the proportion of banks in default will be quantified by Theorem 1.

We are interested in analyzing how the capital of each bank relates to the threshold level. We will do this by introducing the *hypothetical threshold level* (see equation (3.1.2)), as in [18]. We will assume that a regulatory agency, for instance a central bank, imposes a minimum amount of capital bank i must hold initially, which we will call $\nu_{i,0}$. Using each bank's capital we will use equation (3.1.2) to find a *hypothetical threshold level* $c_{i,t}$ for any time $t \geq 0$. The time t here can be considered as a two-period model $t \in \{0, 1\}$, where $t = 0$ is the initial state, which is determined by the regulatory agency, and $t = 1$ is the state after a shock is applied to the system.

Looking back at Table 2.1, we see that all banks could be invested in other assets

A_i, which are not part of the banking system. However, if an asset common to many banks suffer a sudden loss in its value, than the bank's capital would decrease which may trigger a default cascade in the system. This is the main point of interest of this study. We assume that, at time 1, there will be a shock in a common asset and, therefore, the new value $\nu_{i,1}$ will be smaller than the initial capital $\nu_{i,0}$. We suppose that the shock is of the form $\nu_{i,1} = \nu_{i,0} - \alpha_i$, where $\alpha_i > 0$ is either a constant or another random variable, independent of the edges or exposures. Note that it is possible that the new capital is negative, making the threshold function equal 0 with the bank now in default, since this bank can no longer pay its liabilities. This is the scenario in which we are interested in as the banks in default will trigger a cascade.

3.1 Default Contagion for Exposure Model

In the Chapter 2, all banks had a threshold level c_i which means bank i would default after a number c_i of its in-neighbors were in default, regardless of the amount of money banks owed each other. Now we would like to expand the model presented before to include the exposures between banks, as in [18].

We will define the exposures, or edge-weights, on a probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, which is different from $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ where the inhomogeneous random graph was defined. The random graph now has edges, which is the skeleton of the network, and edge-weigths which represent the exposure between banks and is defined on the product space of $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. The edges and exposures are independent of each other in the product space (for details, see [18]). We will represent the possible exposure between each pair of vertices $(i, j) \in [n]^2$, $i \neq j$ with a random variable $E_{i,j} > 0$. Since there are no exposures between one bank with itself, $E_{i,i} = 0$ for all $i \in [n]$. This gives us a matrix E of possible exposures.

We will further assume that $E_{i,j}$ are pairwise independent for any $(i,j) \in [n]^2$, $i \neq j$.

Remark 6 *In [18] the authors only assume that the sequence $E_{1,j}, ..., E_{j-1,j}, E_{j+1,j}, ..., E_{n,j}$ is exchangeable. Our assumption here is stronger, since independence implies exchangeability. The proof in [18] will still hold in this case.*

To understand default contagion now, we need to know the bank's capital. Let $\boldsymbol{\nu_t} = (\nu_{1,t}, \nu_{2,t}, \ldots, \nu_{1,t})$ be a vector that represents each bank's capital at time $t \geq 0$. Bank i is in default if $\nu_{i,t} = 0$. The set of banks initially in default can be defined as $\mathcal{D}_{0,t} = \{i \in [n] | \nu_{i,t} \leq 0\}$. These banks initially in default trigger a cascade. After k rounds, the set of banks in default is given by:

$$\mathcal{D}_{k,t} = \{i \in [n] | \nu_{i,t} \leq \sum_{j \in \mathcal{D}_{k-1,t}} E_{j,i}\}. \tag{3.1.1}$$

As before, equation (3.1.1) can be interpreted as bank i is in default if its capital $\nu_{i,t}$ is smaller than the amount owed to bank i by the banks that are in default and no longer paying its liabilities.

In the threshold model, defined in Section 2.4, each bank had a threshold level c_i and bank i would default after c_i in-neighbors were in default. The case here is different. From equation (3.1.1) we can see that to determine if a bank is in default we need the interbank exposures. However, we assumed that all exposures $E_{i,j}$ are independent and identically distributed. Since we are dealing with possible exposures, it should not make a big difference which neighbors are in default, since the distribution of their exposures is the same. This will be clear once we define the following quantity:

Definition 10 (Hypothetical Threshold Function) *Let $i \in [n]$, $\boldsymbol{\nu_t}$ be a vector of bank's capital and E be the matrix of exposures, all previously defined. We can define the following random threshold level for $t \in \{0,1\}$, which we will name hypothetical threshold level.*

$$c_{i,t}(n) = \begin{cases} 0, & \textit{if } \nu_{i,t} < 0 \\ \\ \inf\{s \in 0 \cup [n] | \sum_{l=1}^{s} E_{il} \geq \nu_{i,t}\}, & \textit{it } \nu_{i,t} \geq 0 \end{cases} \tag{3.1.2}$$

with the convention that $\inf \emptyset := \infty$.

We can see from equation (3.1.2) that the order in which we choose the exposures $E_{i,j}$ will result in different threshold level $c_{i,t}$. However, since all the exposures are independent and identically distributed random variables, and therefore exchangeable, we can use the same argument as in [18] to ensure that the distribution of the threshold function in equation (3.1.2) is the same for any given order of selection of the exposures $E_{i,j}$. First consider a permutation $\pi_j : [n-1] \rightarrow [n] \setminus \{j\}$ of all the possible exposures of vertex j, $E_{1,j}, E_{2,j}, ..., E_{j-1,j}, E_{j+1,j}, ..., E_{n,j}$. The multivariate distribution of this vector is the same as the multivariate distribution of any deterministic permutation π of this same vector. Due to this fact, we can see that no matter the order that we choose the exposures $E_{i,j}$ the distribution of the random variable $c_{i,t}$ is the same.

In [18], the authors show that the hypothetical threshold level defined (3.1.2) captures the actual dynamics of the contagion process if the exposures are exchangeable. In fact, the characteristics of the contagion process of the exposure model is the same as the threshold model with threshold level $c_{i,t}$.

So, as in [18], the network is described by the vector $(w^-(n), w^+(n), \nu_{i,t})$. Using the *hypothetical threshold level* in equation (3.1.2) we find another vector $(w^-(n), w^+(n), c_t(n))$. Since we assumed that the exposures are independent, and therefore exchangeable, we can use the threshold model in Theorem 1 with the limiting distribution (W^-, W^+, C_t) to analyze the contagion process of our initial network.

In order to use Theorem 1, we need the random vector (W^-, W^+, C_t). The first two entries of this random vector, W^- and W^+ are chosen to fit the data, as stated in Chapter 2.

We need to find the distribution of the hypothetical threshold $c_{i,t}(n)$ before and after a shock has affected the capital of the banks in the network. If $c_{i,t}(n) < 0$ for any $i \in [n]$ then these banks are in default and the contagion can spread through the network. The size of the contagion will be quantified by Theorem 1. We also need convergence of the hypothetical threshold level to the limiting distribution C_t. In our examples this will be ensured by independence of the interbank exposures $E_{i,j}$.

Using equation (3.1.2), we can find the pdf of $c_{i,t}$ as a function of the exposures E_{ij} and the bank's capital $\nu_{i,t}$ at time $t \in \{0, 1\}$. Therefore,

$$\mathbb{P}(c_{i,t}(n) = k) = \begin{cases} \mathbb{P}(\nu_{i,t} < 0), & \text{if } k = 0, \\ \mathbb{P}(0 < \nu_{i,t} < E_{i,l}), & \text{if } k = 1, \\ \mathbb{P}(\sum_{l=1}^{k-1} E_{i,l} < \nu_{i,t} < \sum_{l=1}^{k} E_{i,l}), & \text{if } 2 \leq k < n. \end{cases} \tag{3.1.3}$$

The formula in equation (3.1.3) depends on the joint pdf of the sum of exponential random variable.

In the following sections we will describe how to find the distribution in equation (3.1.3) for a few different choices of initial capital.

3.2 Capital Requirement for Banks

This section discusses how to define the initial capital $\nu_{i,0}$ for each bank. This choice is based on Theorem 6 in order to ensure the resilience of the network. We assume that $E_1, E_2, ..., E_n$ is a sequence of independent and identically distributed random variables with $\mathbb{E}[E_i] = \mu_i = \mu$.

In [18], the authors suggests a way to define the minimum capital requirements based on $\mathbb{E}[E_i] = \mu_i = \mu$. We also propose another way to define the capital which makes a tractable example.

Remark 7 *In [18] the authors actually assume that for bank i, $\mathbb{E}[E_{j,i}] = \mu_i$. In our case, we assume that all $E_{i,j}$'s are independent and identically distributed, therefore $\mu_i = \mu$ for all $i \in [n]$.*

Assume that the regulatory agency imposes the minimum capital requirements through a threshold function τ_i, different from $c_{i,t}$ above. The value τ_i, which could be a constant or defined as a function of other parameters of the network, i.e. $\tau_i = \tau(w_i^-)$, is used to calculate the capital for each bank i in the following ways:

$$\nu_{i,0} = \max\{\tau(w_i^-)\mu_i, \max_{j\in[n]\setminus\{i\}} E_{j,i}\}, \tag{3.2.1}$$

and

$$\nu_{i,0} = \sum_{l=1}^{\tau_i} E_{i,l}. \tag{3.2.2}$$

Remark 8 *In equation (3.2.1) the condition $\nu_{i,0} > \max_{j\in[n]\setminus\{i\}} E_{j,i}$ ensures that there are no contagious links, i.e., no bank will default when only one of its in-neighbors default.*

The capital requirement in equation (3.2.2), henceforth called the *hypothetical capital requirement* is proposed in this work. One can see that it takes the exposures in a natural order. From a regulatory view this is pointless, because two banks may have very different capital requirements even though their exposures have the same distribution. However, for very large networks one can expect to see intuitive results with this particular choice. In fact we will show with simulations in Chapter 4 that the behavior of the network in this case is similar to the capital requirements in equation (3.2.1).

Remark 9 *The hypothetical capital requirement in* (3.2.2) *was chosen such that the hypothetical threshold function $c_{i,0}$ is always equal to $\tau(w_i^-)$, by construction.*

The capital requirements in equation (3.2.1), henceforth called the *average capital requirement*, was proposed in [18]. *Average capital requirement* is a smart choice since it takes into consideration just the average of the exposures for bank i. If τ_i is large, we expect *hypothetical capital requirement* and *average capital requirement* to be close, because of the law of large numbers.

The following Theorem from [18] shows that the capital requirements in equation (3.2.1) ensures the resilience of the network for a certain choice of $\tau(w_i^-)$. The quantity $c(n)$ in Theorem 7 is a sequence of threshold levels as in Definition 6.

Theorem 7 *Assume that $(w^-(n), w^+(n), c(n))$ is a regular vertex sequence with limiting distribution (W^-, W^+, C). Assume further that the empirical distribution $F_n(x, y, l)$ of $(w^-(n), w^+(n), c(n))$ converges almost surely to $F(x, y, l)$ for all points $(x, y, l) \in \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0^\infty$ for which $F(x, y, l)$ is continuous in (x, y). Let W^- and W^+ be Pareto distributed (see Definition 3) with parameters (β^-, w^+_{min}) and (β^+, w^+_{min}) respectively, with $\beta^-, \beta^- > 2$ and $w^-_{min}, w^+_{min} > 0$. The quantities γ_c and α_c are defined as in Theorem 6. Assume further that $\nu_i > \max_{j \in [n] \setminus \{i\}} E_{j,i}$ almost surely for all $i \in [n]$ and that $\mathbb{E}[E_{i,j}] = \mu$ for all $i, j \in [n]^2$. Then the system is resilient if one the following holds:*

1. *If $\gamma_c < 0$*

2. *$\gamma_c = 0$ and there exist a function $\tau : \mathbb{R}_+ \to \mathbb{N}\backslash\{0,1\}$ and some $\gamma > 0$ such that $\liminf_{w\to\infty} w^{-\gamma_c}\tau(w) > 0$ and for all $i \in [n]$, $\nu_i \geq \tau(w_i^-)\mu$ almost surely.*

3. *$\gamma_c > 0$ and there exist $\tau : \mathbb{R}_+ \to \mathbb{N}\backslash\{0,1\}$ such that $\liminf_{w\to\infty} w^{-\gamma_c}\tau(w) > \alpha_c$ and for all $i \in [n]$, $\nu_i \geq \tau(w_i^-)\mu$ almost surely.*

Remark 10 *Since the capital in equation (3.2.1) does not allow for contagious links, it ensures the resilience of the network in case W^- and W^+ have finite second moment.*

3.3 Distribution of the Threshold function for *Hypothetical Capital requirement*

In Section 3.1 we mentioned that we need to obtain the random vector (W^-, W^+, C). In this section we will discuss how to find the distribution of the hypothetical threshold level C for a fixed given capital as in equation (3.2.2). The random variable C will later be used to calculate the functional f given in Theorem 1.

Previously we stated that the interbank exposures $E_{i,j}$ are independent and identically distributed. For the remainder of this work we will also assume that they are exponentially distributed with parameter λ.

The hypothetical capital requirement was defined in equation (3.2.2). In Remark 9 we stated that the initial hypothetical threshold level in this case is exactly $\tau(w_i^-)$.

The capital is defined as $\nu_{i,0} = \sum_{l=1}^{\tau_i} E_{i,l}$. If we plug this into equation (3.1.2), we get the following:

$$c_{i,t}(n) = \begin{cases} 0, & \text{if } \sum_{l=1}^{\tau_i} E_{i,l} < 0, \\ \inf\{s \in 0 \cup [n] | \sum_{l=1}^{s} E_{il} \geq \sum_{l=1}^{\tau_i} E_{i,l}\}, & \text{if } \sum_{l=1}^{\tau_i} E_{i,l} \geq 0. \end{cases} \tag{3.3.1}$$

Therefore we can see that $c_{i,0} = \tau_i$, for all $i \in [n]$, by construction of the initial capital $\nu_{i,0}$.

Now we need to find the distribution for the hypothetical threshold level after a shock is applied to each bank.

3.3.1 Distribution of Threshold Function After Constant Shock

We want find the distribution of the hypothetical threshold level at time $t = 1$ after a shock is applied to each bank. We assume that the new capital is given by $\nu_{i,1} = \nu_{i,0} - \alpha$, where α is a constant and $\nu_{i,0}$ is the initial capital for bank i.

From equation (3.2.2), we know that $\nu_{i,0} = \sum_{l=1}^{\tau_i} E_{il}$. We start by calculating the probability that $c_{i,1}(n) = 0$.

$$\begin{aligned}
&\mathbb{P}(c_{i,1}(n) = 0) = \mathbb{P}(\nu_{i,1} < 0)\\
&= \mathbb{P}(\sum_{l=1}^{\tau_i} E_{il} - \alpha < 0)\\
&= \mathbb{P}(\sum_{l=1}^{\tau_i} E_{il} < \alpha) = F_X(\alpha)\\
&= e^{-\lambda\alpha} \sum_{j=\tau_i}^{\infty} \frac{(\lambda\alpha)^j}{j!},
\end{aligned} \tag{3.3.2}$$

where $X \sim Gamma(\tau_i, \lambda)$ and $F_X(x)$ is the cdf of random variable X at point x. Now for $k = 1$, we need to find the following probability:

$$\begin{aligned}
&\mathbb{P}(c_{i,1}(n) = 1) = \mathbb{P}(0 < \nu_{i,1} < E_{i1})\\
&= \mathbb{P}(0 < \sum_{l=1}^{\tau_i} E_{il} - \alpha < E_{i1})\\
&= \mathbb{P}(\sum_{l=1}^{\tau_i} E_{il} > \alpha, \sum_{l=2}^{\tau_i} E_{il} < \alpha).
\end{aligned} \tag{3.3.3}$$

Let $X_1 = E_1 \sim Exp(\lambda)$ and $X_2 = \sum_{l=2}^{\tau_i} E_{il} \sim Gamma(\tau_i - 1, \lambda)$. We need the joint pdf of (Y_1, Y_2), $Y_1 = X_1 + X_2$ and $Y_2 = X_2$. The joint pdf of (Y_1, Y_2) is given by:

$$f_{Y_1,Y_2}(y_1, y_2) = |J| f_{X_1}(y_1 - y_2) f_{X_2}(y_2), \quad \text{if } 0 < y_2 < y_1 < \infty \tag{3.3.4}$$

where $|J|$ is the Jacobian of the transformation from (X_1, X_2) to (Y_1, Y_2), which is equal to 1. Now substituting the known pdf's of a Gamma and Exponential random variables, we get the following joint pdf:

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{\lambda^{\tau_i}}{\Gamma(\tau_i - 1)} y_2^{c_{i,0}-2} e^{-\lambda y_1}, \quad \text{if } 0 < y_2 < y_1 < \infty. \tag{3.3.5}$$

Back to the probability of $c_{i,1} = 1$:

$$\begin{aligned}
&\mathbb{P}(c_{i,1}(n) = 1) = \mathbb{P}(Y_2 < \alpha, Y_1 > \alpha) \\
&= \int_0^{\alpha} \int_{\alpha}^{\infty} \frac{\lambda^{\tau_i}}{\Gamma(\tau_i - 1)} y_2^{\tau_i - 2} e^{-\lambda y_1} dy_1 dy_2 \\
&= \frac{(\lambda\alpha)^{\tau_i - 1}}{(\tau_i - 1)!} e^{-\lambda\alpha}.
\end{aligned} \tag{3.3.6}$$

Now for $k = 2, ..., \tau_i$:

$$\begin{aligned}
&\mathbb{P}(c_{i,1}(n) = k) = \mathbb{P}(\sum_{l=1}^{k-1} E_{il} < \nu_{i,1} < \sum_{l=1}^{k} E_{il}) \\
&= \mathbb{P}(\sum_{l=1}^{k-1} E_{il} < \sum_{l=1}^{\tau_i} E_{il} - \alpha < \sum_{l=1}^{k} E_{il}) \\
&= \mathbb{P}(\sum_{l=k}^{\tau_i} E_{il} > \alpha, \sum_{l=k+1}^{\tau_i} E_{il} < \alpha)
\end{aligned} \tag{3.3.7}$$

We need to find the joint pdf of $Y_1 = \sum_{l=k}^{\tau_i} E_{il}$ and $Y_2 = \sum_{l=k+1}^{\tau_i} E_{il}$. Let $X_1 = E_{ik}$ and $X_2 = \sum_{l=k+1}^{\tau_i}$. The joint pdf of (Y_1, Y_2) can be obtained by a simple transformation: $Y_1 = X_1 + X_2$ and $Y_2 = X_2$. We know that $X_1 \sim Exp(\lambda)$ and $X_2 \sim Gamma(\tau_i - k, \lambda)$. As we did before, the joint pdf of Y_1, Y_2 is given by

$$f_{Y_1,Y_2}(y_1, y_2) = |J| f_{X_1}(y_1 - y_2) f_{X_2}(y_2), \quad \text{if } 0 < y_2 < y_1 < \infty, \tag{3.3.8}$$

where $|J|$ is the Jacobian of the transformation from (X_1, X_2) to (Y_1, Y_2), which is equal to 1 as before. Now substituting the known pdf's of a Gamma and Exponential random variables, we get the following joint pdf:

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{\lambda^{\tau_i+1-k}}{\Gamma(\tau_i - k)} y_2^{\tau_i-k-1} e^{-\lambda y_1}, \quad \text{if } 0 < y_2 < y_1 < \infty. \tag{3.3.9}$$

Now we can use this joint pdf to calculate the pmf of $c_{i,1}$:

$$\begin{aligned}
&\mathbb{P}(c_{i,1}(n) = k) = \mathbb{P}(Y_2 < \alpha, Y_1 > \alpha) \\
&= \int_0^\alpha \int_\alpha^\infty \frac{\lambda^{\tau_i+1-k}}{\Gamma(\tau_i - k)} y_2^{\tau_i-k-1} e^{-\lambda y_1} dy_1 dy_2 \\
&= \frac{(\lambda\alpha)^{\tau_i-k}}{(\tau_i - k)!} e^{-\lambda\alpha},
\end{aligned} \tag{3.3.10}$$

which is the probability that a Poisson random variable with parameter $\lambda\alpha$ is equal to $\tau_i - k$, for $k = 1, ..., \tau_i$

In summary we have obtained the distribution of the hypothetical threshold level at time $t = 1$ after a constant shock is applied to the bank's capital.

3.3.2 Distribution of the Threshold Function with Random Shock

In Section 3.3.1, we assumed that the shock size α was a constant. It is very unrealistic to expect that all banks in a given network experience the same drop in their balance sheet. We will now assume that α is a random variable. We will obtain the distribution of the hypothetical threshold level for α following two distributions, namely Exponential and Gamma distributions.

Remark 11 *We choose α to have Exponential and Gamma distribution because the exposures also have Exponential distribution. It would make sense that the shock in the bank's balance sheet have a similar distribution.*

Remark 12 *In this section we could use any distribution for α and obtain the distribution of the hypothetical threshold value analytically. We can easily see this from equations* (3.3.3), (3.3.7) *and* (3.3.11)*, where we have the probability distribution for the hypothetical threshold value after a constant shock. To find this distribution with a random shock α, we just need to integrate over the distribution of α, as will be discussed in this section.*

3.3.2.1 Shock with Exponential Distribution

Now we assume that the shocks are exponentially distributed with parameter γ. We will further assume that the shock α is independent of the exposures $E_{i,j}$.

The probabilities calculated in the previous section can be interpreted as a conditional probability given the shock size α. For instance, the equation (3.3.3) is $\mathbb{P}(c_{i,1}(n) = 0|\alpha)$.

We would like to calculate the unconditional default probability, i.e., $\mathbb{P}(c_{i,1}(n) = 0)$. To find that, we will take the expectation of this probability with respect to the distribution of α.

$$
\begin{aligned}
&\mathbb{E}[\mathbb{E}[\mathbb{1}_{c_{i,1}(n)=0}|\alpha]] = \mathbb{E}_\alpha\left[\mathbb{P}(c_{i,1}(n) = 0|\alpha)\right] \\
&= \mathbb{E}_\alpha\left[\exp^{-\alpha\lambda}\sum_{j=\tau_i}^{\infty}\frac{(\alpha\lambda)^j}{j!}\right] = \int_0^\infty \exp^{-\alpha\lambda}\gamma\exp^{-\gamma\alpha}\sum_{j=\tau_i}^{\infty}\frac{(\alpha\lambda)^j}{j!}d\alpha \\
&= \sum_{j=\tau_i}^{\infty}\frac{\gamma\lambda^j}{j!}\int_0^\infty \exp^{-\alpha(\gamma+\lambda)}\alpha^j d\alpha
\end{aligned}
$$

$$= \sum_{j=\tau_i}^{\infty} \frac{\gamma\lambda^j}{j!}(\gamma+\lambda)^{-j-1}\Gamma(j+1), \tag{3.3.11}$$

where Γ is the gamma function and we will use the well-known property that $\Gamma(n) = (n-1)!$ for integer n. Thus

$$\sum_{j=\tau_i}^{\infty} \frac{\gamma}{\gamma+\lambda}(\frac{\lambda}{\gamma+\lambda})^j = \frac{\gamma}{\gamma+\lambda}\sum_{j=\tau_i}^{\infty}(\frac{\lambda}{\gamma+\lambda})^j. \tag{3.3.12}$$

Substituting $k = j - \tau_i$, we get:

$$\begin{aligned}
&\frac{\gamma}{\gamma+\lambda}\sum_{j=\tau_i}^{\infty}(\frac{\lambda}{\gamma+\lambda})^j \\
&= \frac{\gamma}{\gamma+\lambda}\sum_{k=0}^{\infty}(\frac{\lambda}{\gamma+\lambda})^{k+\tau_i} \\
&= \frac{\gamma}{\gamma+\lambda}(\frac{\lambda}{\gamma+\lambda})^{\tau_i}\sum_{k=0}^{\infty}(\frac{\lambda}{\gamma+\lambda})^k \\
&= \frac{\gamma}{\gamma+\lambda}(\frac{\lambda}{\gamma+\lambda})^{\tau_i}\frac{1}{1-\frac{\lambda}{\gamma+\lambda}} \\
&= \frac{\gamma}{\gamma+\lambda}(\frac{\lambda}{\gamma+\lambda})^{\tau_i}\frac{\gamma+\lambda}{\gamma} \\
&= (\frac{\lambda}{\gamma+\lambda})^{\tau_i}.
\end{aligned} \tag{3.3.13}$$

We found the unconditional probability $\mathbb{P}(c_{i,1}(n) = 0)$, which depends on the parameter γ of the distribution of the shock. We can now choose this parameter to fit the desired default probability in our simulation.

Now we would like to calculate the unconditional default probability, i.e., $\mathbb{P}(c_{i,1}(n) = k)$ for $k > 0$.

We will do the same calculations as above.

$$
\begin{aligned}
\mathbb{E}_\alpha \left[\mathbb{P}(c_{i,1}(n) = k|\alpha)\right] &= \mathbb{E}_\alpha \left[\exp^{-\alpha\lambda} \frac{(\lambda\alpha)^{\tau_i - k}}{(\tau_i - k)!}\right] \\
&= \int_0^\infty \exp^{-\alpha\lambda} \gamma \exp^{-\gamma\alpha} \frac{(\lambda\alpha)^{\tau_i - k}}{(\tau_i - k)!} d\alpha \\
&= \frac{\gamma}{(\tau_i - k)!} \left\{-\alpha(\lambda\alpha)^{\tau_i - k}((\gamma+\lambda)\alpha)^{-1-\tau_i+k}\Gamma(1+\tau_i - k, (\gamma+\lambda)\alpha)\right\} \\
&= \left(\frac{\lambda}{\gamma+\lambda}\right)^{(\tau_i - k)} \frac{\gamma}{\gamma+\lambda},
\end{aligned}
\tag{3.3.14}
$$

for $k > 0$.

3.3.2.2 Shock with Gamma Distribution

Assume now that $\alpha \sim Gamma(k, \theta)$. We want to find $\mathbb{E}_\alpha \left[\mathbb{P}(c_{i,1}(n) = 0|\alpha)\right]$. We will do similar calculations as in Section 3.3.2.

Following the calculations on equation (3.3.12) above, but using the Gamma pdf, we obtain:

$$
\begin{aligned}
&\mathbb{E}_\alpha \left[\mathbb{P}(c_{i,1}(n) = 0|\alpha)\right] = \sum_{j=\tau_i}^{\infty} \frac{1}{\Gamma(k)\theta^k} \frac{\lambda^j}{j!} \int_0^\infty \exp^{-\alpha(\lambda + \frac{1}{\theta})} \alpha^{(k+j-1)} d\alpha \\
&= \sum_{j=\tau_i}^{\infty} \frac{1}{\Gamma(k)\theta^k} \frac{\lambda^j}{j!} \left(\left(\lambda + \frac{1}{\theta}\right)^{-j-k} \Gamma(j+k)\right) \\
&= \sum_{j=\tau_i}^{\infty} \binom{j+k-1}{j} \lambda^j \frac{1}{\theta^k} \left(\frac{\theta}{\lambda\theta + 1}\right)^{j+k} \\
&= \left(\frac{1}{\lambda\theta+1}\right)^k \sum_{j=\tau_i}^{\infty} \binom{j+k-1}{j} \left(\frac{\lambda\theta}{\lambda\theta+1}\right)^j \\
&=_2 F_1(1, \tau_i + k; \tau_i + 1; \frac{\lambda\theta}{\lambda\theta+1}),
\end{aligned}
\tag{3.3.15}
$$

where the ${}_2F_1$ is the hypergeometric function.

To find $\mathbb{E}_\alpha\left[\mathbb{P}(c_{i,1}(n)=k|\alpha)\right]$ for $k>0$, we do calculations analogous to equation (3.3.14):

$$\begin{aligned}
\mathbb{E}_\alpha\left[\mathbb{P}(c_{i,1}(n)=k|\alpha)\right] &= \int_0^\infty \frac{\exp^{-\alpha\lambda}(\lambda\alpha)^{\tau_i-k}}{(\tau_i-k)!}\frac{1}{\Gamma(\gamma)\theta^\gamma}\alpha^{\gamma-1}\exp^{\frac{-\alpha}{\theta}}d\alpha \\
&= \frac{\lambda^{\tau_i-k}}{(\tau_i-k)!\Gamma(\gamma)\theta^\gamma}\int_0^\infty \exp^{-\alpha(\lambda+\frac{1}{\theta})}\alpha^{\tau_i+\gamma-k-1}d\alpha \\
&= \frac{\lambda^{\tau_i-k}}{(\tau_i-k)!\Gamma(\gamma)\theta^\gamma}\left[-\alpha^{\tau_i+\gamma-k}\left(\alpha\left(\lambda+\frac{1}{\theta}\right)\right)^{-\tau_i-\gamma+k}\Gamma\left(\tau_i+\gamma-k,\left(\lambda+\frac{1}{\theta}\right)\alpha\right)\right]_0^\infty \\
&= \frac{\lambda^{\tau_i-k}}{(\tau_i-k)!\Gamma(\gamma)\theta^\gamma}\left(\lambda+\frac{1}{\theta}\right)^{-\tau_i-\gamma+k}\Gamma(\tau_i+\gamma-k) \\
&= \frac{\lambda^{\tau_i-k}}{(\tau_i-k)!\Gamma(\gamma)\theta^\gamma}\left(\frac{\theta\lambda+1}{\theta}\right)^{k-\tau_i-\gamma}(\tau_i+\gamma-k-1)! \\
&= \binom{\tau_i+\gamma-k-1}{\gamma-1}\frac{\lambda^{\tau_i-k}}{\theta^\gamma}\left(\frac{\theta}{\lambda\theta+1}\right)^{\tau_i-k}\left(\frac{\theta}{\lambda\theta+1}\right)^\gamma \\
&= \binom{\tau_i+\gamma-k-1}{\gamma-1}\left(\frac{\lambda\theta}{\lambda\theta+1}\right)^{\tau_i-k}\left(\frac{1}{\lambda\theta+1}\right)^\gamma. \qquad (3.3.16)
\end{aligned}$$

3.4 Distribution of C for *Average Capital Requirement*

In this section we are interested in finding the distribution of the hypothetical threshold level C for the capital defined in equation (3.2.1). In this case, since τ_i is a given constant, the capital $\nu_{i,0}$ is also constant, in contrast to the previous section where the capital was a random variable.

To find $\mathbb{P}(c_{i,0}(n)=k)=\mathbb{P}(\sum_{l=1}^{k-1}E_{il}<\nu_{i,0}<\sum_{l=1}^{k}E_{il})$, we need the joint pdf of $\sum_{l=1}^{k-1}E_{il}$ and $\sum_{l=1}^{k}E_{il}$.

Let $X_1=\sum_{l=1}^{k-1}E_{il}$ and $X_2=E_{kl}$. The joint pdf of $\sum_{l=1}^{k-1}E_{il}, \sum_{l=1}^{k}E_{il}$ can be obtained by a simple transformation: $Y_1=X_1$ and $Y_2=X_1+X_2$. The joint pdf is given by

$$f_{Y_1,Y_2}(y_1,y_2)=|J|f_{X_1}(y_1)f_{X_2}(y_2-y_1), \text{if } 0<y_1<y_2<\infty \qquad (3.4.1)$$

where $|J|$ is the Jacobian of the transformation from (X_1, X_2) to (Y_1, Y_2), which is 1. Now substituting the PDF of a $Gamma(k-1, \lambda)$, $f(x) = \frac{\lambda^{k-1}}{\Gamma(k-1)} x^{k-2} e^{-\lambda x}$ and the pdf of an $Exp(\lambda)$: $f(x) = \lambda e^{\lambda x}$, we get the following joint pdf:

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{\lambda^k}{\Gamma(k-1)} y_1^{k-2} e^{-\lambda y_2}, \text{if } 0 < y_1 < y_2 < \infty. \tag{3.4.2}$$

Now we can use this pdf to calculate the desired probability:

$$\begin{aligned} &\mathbb{P}(c_{i,0}(n) = k) = \mathbb{P}(Y_1 < \nu_{i,0} < Y_2) \\ &= \int_0^{\nu_{i,0}} \int_{\nu_{i,0}}^{\infty} \frac{\lambda^k}{\Gamma(k-1)} y_1^{k-2} e^{-\lambda y_2} dy_2 dy_1 \\ &= \frac{(\lambda \nu_{i,0})^{k-1}}{(k-1)!} e^{-\lambda \nu_{i,0}}, \end{aligned} \tag{3.4.3}$$

where $\nu_{i,0} = \max\{\tau(w_i^-)\mu_i, \max_{j \in [n] \setminus \{i\}} E_{j,i}\}$.

Remark 13 *The distribution in equation (3.4.3) holds for any fixed capital choice.*

Remark 14 *In fact, the capital defined in equation (3.2.1) is a random variable because of the presence of* $\max_{j \in [n] \setminus \{i\}} E_{j,i}$*. However, this is just a technical condition and most times the maximum will be the constant* $\tau(w_i^-)\mu_i$*.*

To find the hypothetical threshold level after a shock has been applied to the system we will use simulations. We describe briefly how to obtain the threshold level in this case.

First, for each bank i, we simulate all of its possible exposures $E_{i,j}$. Then we calculate the initial capital $\nu_{i,0}$ according to equation (3.2.1) and simulate the random shock α_i for different distributions. We calculate the new capital $\nu_{i,1} = \nu_{i,0} - \alpha_i$. Finally we use equation (3.1.2) to find $c_{i,1}$, the hypothetical threshold level after the shock.

Finding the distribution in this case is theoretically possible but the calculations are extensive. For this particular choice of capital, we do not have the simplifications which made the first example tractable. Therefore, we will rely on simulations to find the distribution of the threshold level in this case.

We will use Exponential and Gamma distributions for α. The results for the hypothetical threshold function both before and after the shock will be presented in Chapter 4 in the form of histograms. We will be able to see how a shock affects the distribution of the hypothetical threshold function, and, as a consequence, it will affect the default contagion in the network.

3.5 Common Asset Impact to Bank's Capital

Previously we showed how to obtain the distribution of the *hypothetical threshold level* after the initial capital $\nu_{i,0}$ is decreased by a shock α_i, where α_i are independent random variables and are also independent of the exposures. In this section we will show how we can use conditionally independent random variables to model a shock in a common asset, assuming we know the value of the common asset after a shock.

3.5.1 Capital After Loss in a Common Asset

In Section 3.2 we proposed two different ways to define the capital for each bank. We have a vector $\{\nu_{i,0}\}_{i\in[n]}$ where $\nu_{i,0}$ is the capital of bank i at time $t = 0$. We assume that $\nu_{i,0}$ is independent of $\nu_{j,0}$ for $i \neq j$. We assume further that $\nu_{i,0}$ depends on $A_{i,0}$, which is the monetary investment of bank i in the common asset. Also, $A_{i,0}$ is independent of $A_{j,0}$ for $i \neq j$. Suppose that $\nu_{i,0}$ is defined as:

$$\nu^*_{i,0} = \nu_{i,0} - A_{i,0}, \tag{3.5.1}$$

where $\nu^*_{i,0}$ is just the part of the capital which is not invested in the common asset.

Assume that the initial price of the common asset is S_0 and at time $t = 1$ the value is $S_1 = PS_0$, where P is a random variable between 0 and ∞. Since we are interested in a crisis scenario, i.e. $S_1 < S_0$, the values of P should be between 0 and 1. The capital of each bank is now given by:

$$\nu_{i,1} = \nu^*_{i,0} + PA_{i,0} = \nu_{i,0} + (P-1)A_{i,0}, \tag{3.5.2}$$

which of course depends on S_1 or P. If the value S_1 is known, or the value of P, we can find each bank's capital conditioned on this value:

$$\nu_{i,1}|P = \nu_{i,0} - (1-p)A_{i,0}. \tag{3.5.3}$$

Since we assumed that $A_{i,0}$'s, $i \in [n]$, are independent, then we can see that the capital after the shock is conditionally independent and the term $pA_{i,0}$ can be substituted by α_i. Therefore, we can use the formulas given in Sections 3.3 and 3.4.

Finally, since the capitals $\nu_{i,1}|P = p$, $i \in [n]$, are independent and identically distributed, the sequence $(w^-(n), w^+(n), c_1(n))$ is a regular vertex sequence and converges almost surely to (W^-, W^+, C_1) for a fixed value of p.

3.5.2 Distribution of the Shock in the Common Asset

In equation (3.5.3) we showed that the shock in the common asset is given by $\alpha = (1-p)A_{i,0}$, where $A_{i,0}$ is the initial monetary amount invested in assets outside the banking network and p is a constant in $(0, 1)$. Assume now that each bank i invests the entire amount $A_{i,0}$ in one asset with price S_0, so each bank holds $\frac{A_{i,0}}{S_0}$ shares of this asset. After the shock, at time $t = 1$, the investment bank i has in the asset is $A_{i,1} = \frac{A_{i,0}}{S_0}S_1$, where S_1 is the price of the asset. The capital for bank i is given by:

$$\nu_{i,1} = \nu_{i,0} - \frac{A_{i,0}}{S_0}\left(S_0 - S_1\right). \tag{3.5.4}$$

Given the dynamics of the process S_t, we can fit the shock distribution with appropriate parameters for a loss distribution of the asset S_t at time $t = 1$. In [13] the authors use real data and impose a shock in bank's assets based on quantiles of the price distribution. Using data, the same could be done in this model.

In our simulations, which will be presented in Chapter 4, we will assume that the distribution of the shock α is Exponential or Gamma. Both these random variables are closed under scaling by a positive constant, that is, if X is a random variable with Exponential or Gamma distribution, then cX has the same distribution with a new parameter.

The shock is $\alpha = pA_{i,0}$, where p is a positive constant, therefore we can see that $A_{i,0}$ has the same conditional distribution as α. This is part of our model. We assumed that the exposures $E_{i,j}$ are exponentially distributed. It is a reasonable assumption that other quantities follow similar distributions. The reason for choosing Gamma is simply the relationship with the exponential distribution. If X and Y are independent exponentially distributed random variables with parameter λ, then $X + Y \sim Gamma(2, \lambda)$.

Remark 15 *Note that the distribution of the shock does not need to be Exponential or Gamma. From equation (3.5.4), the shock is defined as* $\alpha_i = \frac{A_{i,0}}{S_0}(S_0 - S_1)$. *As long as the distribution of the quantity on the right-hand side is known and is conditionally independent and identically distributed, we can obtain the new value for the bank's capital and, as in Section 3.5.1, we can ensure almost sure convergence for* C_1.

3.6 Dynamic Network Model

Previously, we were dealing with a two-period model where $t = 0$ denoted the initial state of the system and $t = 1$ the state of the system after a shock was applied to the common asset. Even though this is enough for the purposes of this work, this setting can be generalized to a continuous time model with $t \geq 0$.

In Section 3.5.1 we considered that the initial capital was given by $\nu_{i,0} = \nu^*_{i,0} + A_{i,0}$, which is just the common asset $A_{i,0}$ plus an amount the remainder of the capital which is not invested in the common asset. We assume that the amount $A_{i,0}$ is invested in a stock, or pool of stocks, with price at time $t = 0$ equal S_0. Let the price of the stock $\{S_t\}_{t>0}$ be a stochastic process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t>0}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t>0}$ is the filtration generated by the stochastic process. Note that $\{S_t\}_{t>0}$ could be a random vector of many assets. We consider the case with only one asset for simplicity.

As in the previous Section, assume that we have only one asset and bank i owns $\frac{A_{i,0}}{S_0}$ shares of this asset. At time $t > 0$, the value of the investment bank i has in this asset is $A_{i,t} = \frac{A_{i,0}}{S_0} S_t$, where S_t is the price of the asset. The capital for bank i at time $t > 0$ is given by

$$\nu_{i,t} = \nu_{i,0} + \frac{A_{i,0}}{S_0} \left(S_t - S_0\right). \tag{3.6.1}$$

So, given the value of the underlying asset $\{S_t\}_{t>0}$, we can obtain the capital of each bank at any time $t > 0$.

Now we have a capital for each bank i for every time $t > 0$. We can, similarly to what we did before in Sections 3.3 and 3.4, obtain the distribution of the hypothetical threshold for every time $t > 0$ for a given value of S_t. Note that as Section 3.5.1, the capitals $\nu_{i,t}$, $i \in [n]$ are independent conditionally on the value S_t.

Since we assume that $A_{i,0}$'s are independent for each bank, that means that ν_i is independent of ν_j for $i \neq j$. Therefore, by the strong law of large numbers, we have that, given S_t, $c_t(n) \xrightarrow{a.s.} C_t$ as $n \to \infty$. Then we have a random vector $\{(W^-, W^+, C_t)\}_{t>0}$, where $\{C_t\}_{t>0}$ is a probability measure valued stochastic process, that is, at each time $t > 0$ we have a distribution C_t.

Remark 16 *We could also make W^- and W^+ time-dependent in (W^-, W^+, C_t), which means the global parameters of the network skeleton are changing over time. However this is not in the context of study of this work. In addition, the authors in [13] show empirical evidence and perform a statistical test that suggest that W^- and W^+ are stable over time.*

Remark 17 *Note that we only have almost sure convergence of $c_t(n)$ to C_t conditionally on the value S_t. Although this is not a general result, in our work we deal with stress-testing scenarios, in which we assume that the value of the common asset has dropped a certain amount which is known. If we want to generalize this in terms of quantiles, see for instance [13], one would need the unconditional convergence of $c_t(n)$.*

Chapter 4

Simulation Study

In this Chapter we will focus on simulation of default contagion for different networks and shocks on the common asset. We are interested in finding the final fraction of banks in default. This is done by calculating functional f given in Theorem 1 and finding its first zero. In Chapter 2 we described the contagion process in a random network with a threshold model and then with an exposure based model. We showed in Chapter 3, based on [18], that the contagion process for the exposure model behaves like the threshold model for a given limiting distribution C for the threshold level.

We showed how to obtain the distribution of C for one particular choice of the initial capital, in which each bank's capital is specifically chosen such that the hypothetical threshold level is exactly the threshold level imposed by the regulator. For other choices of capital we have to rely on simulation to estimate this distribution. We will use a bootstrap method to estimate the distribution of the threshold level in these cases.

Our simulations are split into two different cases: weight sequence with and without second moment. In [3], the authors study the contagion process in a random network with second moment. These are generally easier to deal analytically. In [16] and [18], the authors generalize the contagion process for a random network without second moment.

We will analyze the simulations based on the results presented in [16] but we shall also compare them with findings in [3].

We will assume that all exposures are independent and identically distributed with an Exponential distribution. As stated in Chapter 2, the authors in [13] have empirical evidence to suggest that the exposures are distributed according to a power-law. However, dealing with Exponential distributions makes the problem analytically tractable and we hope that, given the similar shape of these distributions, the results presented here will give some insight for a real network.

In order to fit our simulations according to the data presented in [13], we will choose the parameter of the Exponential distribution such that its expected value is the same as the average exposure presented in that paper.

4.1 Simulation Study with Second Moment

In this section we will simulate default contagion on financial networks whose degree sequence, as defined in Chapter 2, have a finite second moment. The weights W^+ and W^- are sampled from a Pareto distribution (see Definition 3). Also we make W^+ and W^- comonotone for all simulations.

To simulate the default contagion we need to calculate, for each choice of initial capital, the hypothetical threshold before and after the shock, respectively, C_0 and C_1. We always choose C_0 such that the network is resilient according to definition 9, that is, for the initial threshold level C_0, small shocks to the network will not affect a large portion of banks.

With the triples (W^+, W^-, C_0) and (W^+, W^-, C_1) we will calculate f and compare how a shock common to all banks affects the default contagion. Our simulations will show that,for the same initial proportion of banks in default, a common shock to all networks will have a much deeper impact into the contagion process compared to the case when just a few banks are in default but the others remains unharmed. We will show this using Theorems 3 and 4and compare the shocked systems with Figures 2.5 and 2.6.

4.1.1 Hypothetical Capital Choice

These simulations were done by choosing the capital according to equation (3.2.2), which we name the *hypothetical capital*. This is because the hypothetical threshold level $c_{i,0}$ for each bank before the shock in this case will be exactly τ_i, which is a deterministic number given by a regulatory agency.

In [3] the authors showed that in a random network where the degree sequence has finite second moment, the system will be resilient if there are no *contagious links*. In our setting which deals with threshold levels, this means that if there are no banks with a threshold level equals to 1, the system will be resilient, i.e., a small shock will not create a large cascade.

To simulate a network under these conditions we assume that $\tau_i = 2$ for all $i \in [n]$, that is, all banks can withstand the default of their first two counterparts. Since we are dealing with the *hypothetical capital*, it is straightforward that $C_0 = 2$.

Remark 18 *We do not have an explicit resilience condition for a network with capital requirement as in (3.2.2). We will check the resilience of the initial network using Theorem 4 and the system will, in fact, be resilient, as we expected. Finding a general resilience condition for the capital defined in (3.2.2) is beyond the scope of this work.*

We obtain C_1 for several different distributions for the shock α. First we apply a constant shock to all banks. Next we apply random shocks to all banks, with different distributions, namely the Exponential and Gamma distributions.

We also obtain the distribution of C_1 after shocking just a small percentage of banks and leaving all other banks unshocked, as in [18]. In this scenario, we make a proportion p of banks default, i.e., $\mathbb{P}(C_1 = 0) = p$ and the other banks remain unchanged, i.e., $\mathbb{P}(C_1 = 2) = 1 - p$. We would like to compare this case, which has no shock common to all banks, with the case where the default is caused by a shock in a common asset.

In our simulations we will set $p = 1\%$. We set $\mathbb{P}(C_1 = 0) = p$ for all different shocks, that is, no matter what the distribution of α is, the initial proportion of banks in default will always be p.

The histograms for all C_1 obtained are presented in Figures 4.1 to 4.4. We plotted the histogram for C_1 after a common shock with C_1 after a shock to just a small percentage of banks, to highlight the effects of the common shock. We can see that the major difference is that a common shock introduces several *contagious links*.

The parameters used for each shock distribution are given in Table 4.1 below.

Type of shock	Parameter	Mean	Variance
Exponential	$\lambda = 5$	0.2	0.04
Gamma $k = 2$	$\beta = 8.6207$	0.232	0.0269
Gamma $k = 3$	$\beta = 12.5$	0.24	0.0192
Constant	$\alpha = 0.2655$	0.2655	0

Table 4.1: Parameters for shock distributions.

From Table 4.1 we can see that the shocks with higher expected value have smaller variance in order to have the same initial proportion of banks in default. This pattern will be recurring in the next simulations as well.

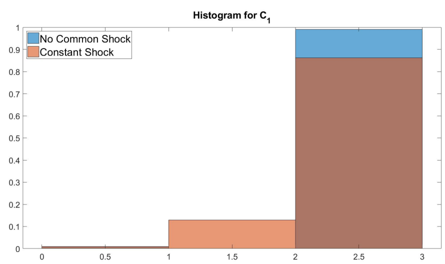

Figure 4.1: Histogram without common shock (blue) and with constant shock (red).

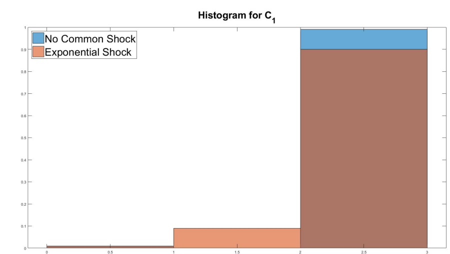

Figure 4.2: Histogram without common shock (blue) and with Exponential shock (red).

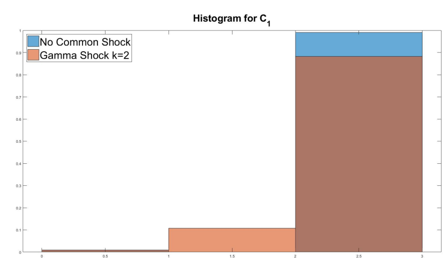

Figure 4.3: Histogram without common shock (blue) and with Gamma shock (red).

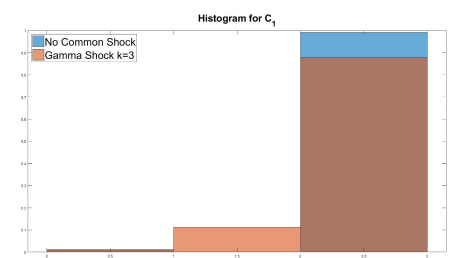

Figure 4.4: Histogram without common shock (blue) and with Gamma shock (red).

For the C_1 obtained above we calculated functional f in equation (2.4.4) from Theorem 1.

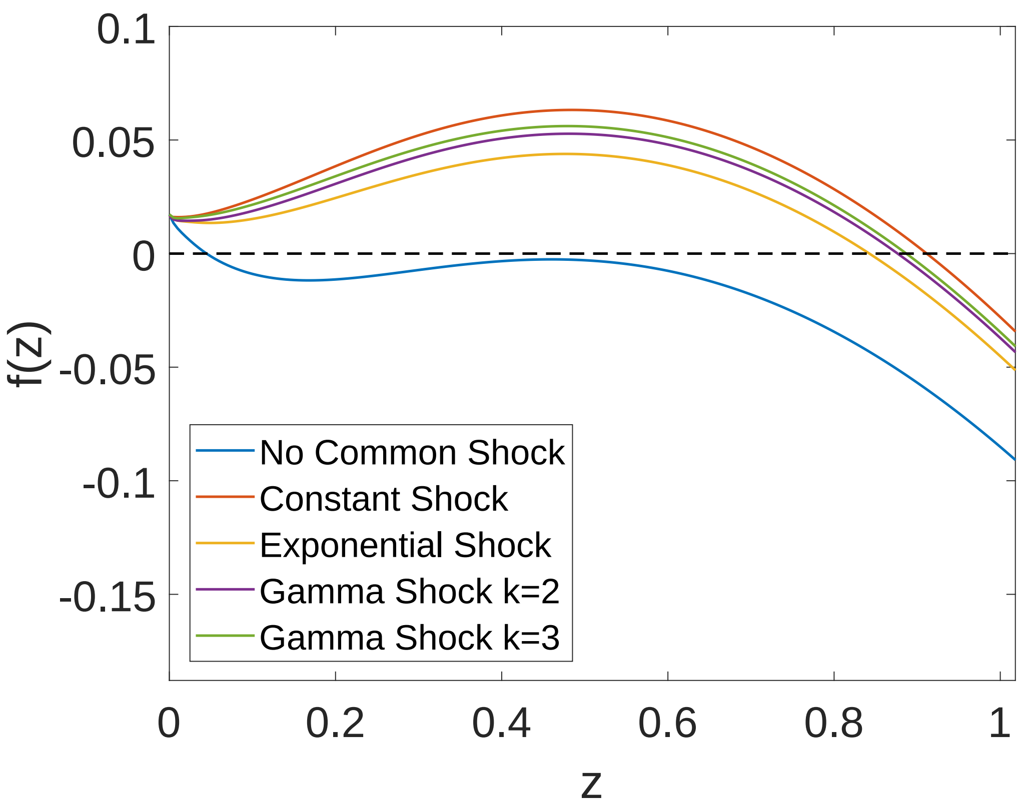

Figure 4.5: Functional f after different shocks are applied to initial capital.

As discussed in Chapter 2, functional f describes the default contagion process, which ends at the first positive root $\hat{z}$ of f if it becomes negative after $\hat{z}$ (see discussion after Theorem 1). Also, the final fraction of banks in default is given by g and is increasing with $\hat{z}$.

From Figure 4.5 we can see that all cases when a common shock was applied to the network have a much larger root than in the case when the shock is applied only to 1% of the banks, even though the initial proportion of banks in default is the same. We observe that introducing a few *contagious links* makes the contagion process reach a much larger portion of the network.

The final proportion of banks in default $g(\hat{z})$ is given in Table 4.2 below for all different shocks.

Type of shock	$g(\hat{z})$
No Common Shock	0.0137
Exponential	0.4036
Gamma $k = 2$	0.4261
Gamma $k = 3$	0.4325
Constant	0.4488

Table 4.2: Proportion of banks in default after cascade process ends.

From Table 4.2 it is clear that the default contagion becomes much more threatening to the system after a common shock is applied.

We can also see both from Figure 4.5 and Table 4.2 that the distribution of the shock applied to the network is not very relevant as all different common shocks get similar results in the end. The critical part is the introduction of *contagious links*, which are very impactful to the contagion process.

4.1.2 Average Capital Choice

Now we will show simulations for the *average capital*: $\nu_{i,0} = \tau(w_i^-)\mu_i$. We start again with $\tau_i = 2$ for all $i \in [n]$. Note that from the definition of the *Average capital* any choice of τ_i would make the network resilient, since it does not allow for contagious links. The choice for $\tau_i = 2$ is simply to make a comparison with the previous example.

We follow the same procedure as in the previous section. Find C, which in this case will not be equal τ for all vertices. Then we obtain C_1 for different types of shocks applied to the initial capital.

In this setting, it is possible that $c_i = 1$ for some bank i, that is, the probability of having *contagious links* is not zero, and therefore, we need to check if the network is resilient for the initial choice of C_0. In [18], the authors show that in a network with degree sequeuce with finite second moment is be resilient if $\mathbb{E}[W^+W^-\mathbf{1}_{C=1}] < 1$. Note that this is weaker then saying the network has no contagious links, since it allows for them to exist but it bounds the total number. We will check this condition numerically and, in fact, for all our simulations it is true that the initial network is resilient.

We fix again an initial proportion of 1% infected vertices, regardless of the type of shock applied.

As in the previous section, the histograms are plotted for C_1 after a common shock and C_1 when just a proportion p of banks are in default.

The parameters for each of the distributions of the shocks are given in Table 4.3.

Type of shock	Parameter	Mean	Variance
Exponential	$\lambda = 1.2820$	0.78	0.6084
Gamma $k = 2$	$\beta = 1.8518$	1.08	0.5832
Gamma $k = 3$	$\beta = 2.3255$	1.29	0.5547

Table 4.3: Parameters for shock distributions for *average capital*.

Note that in this example we did not present constant shock. Since $\tau(w_i^-)$ is equal 2 for all $i \in [n]$, the capital is now a constant and equal for all banks, so a constant shock is not reasonable since it would set either all or no banks to default.

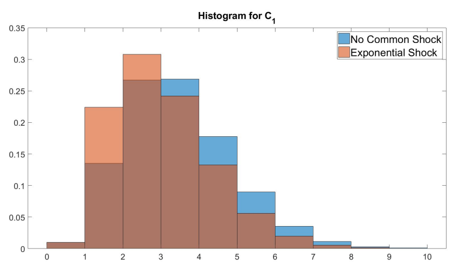

Figure 4.6: Histogram without common shock (blue) and with Exponential shock (red).

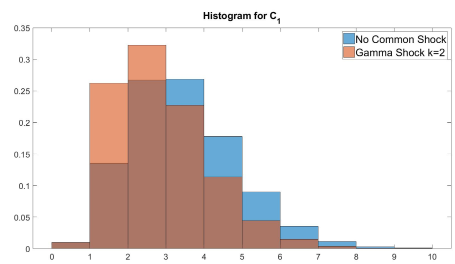

Figure 4.7: Histogram without common shock (blue) and with Gamma shock (red).

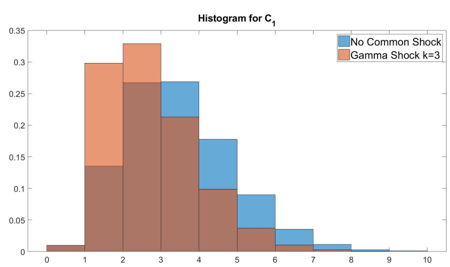

Figure 4.8: Histogram without common shock (blue) and with Gamma shock (red).

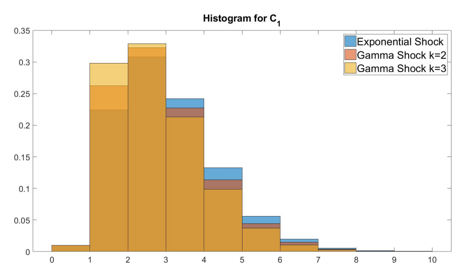

Figure 4.9: Histogram Exponential shock (blue), Gamma shock $k = 2$ (red) and Gamma shock $k = 3$ (yellow).

Functional f was calculated for all C_1 obtained. The result is shown in Figure 4.10.

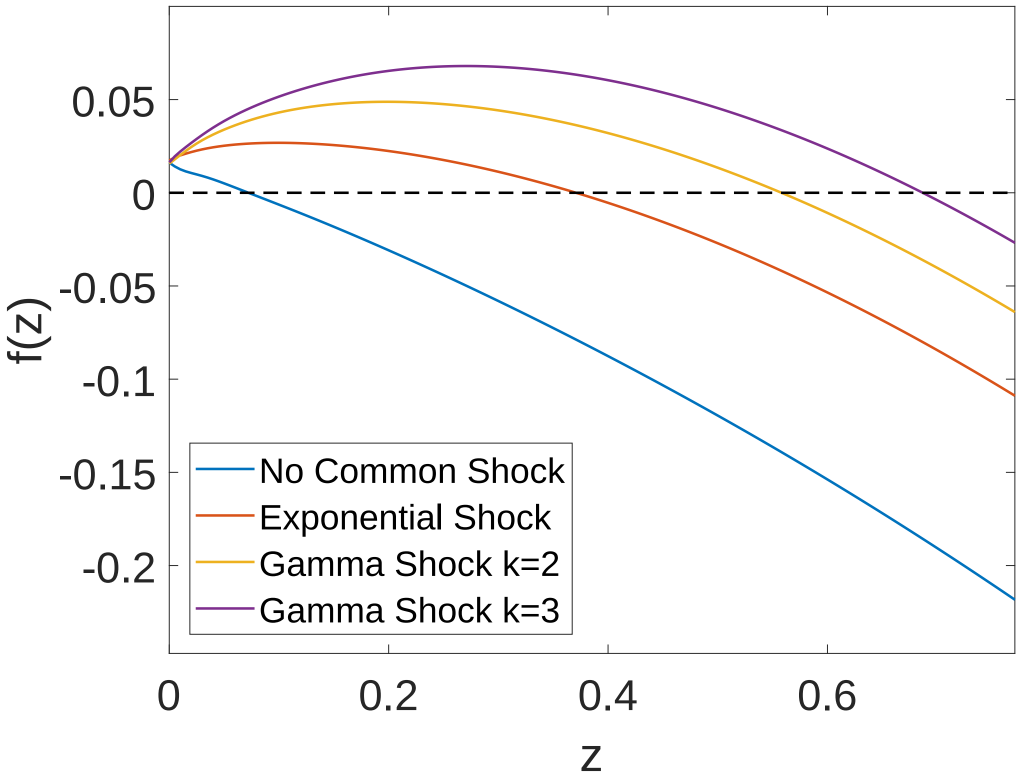

Figure 4.10: Functional f after different shocks are applied to initial capital.

The final proportion of banks in default is given in Table 4.4.

Type of shock	$g(\hat{z})$
No Common Shock	0.0270
Exponential	0.1558
Gamma $k = 2$	0.2527
Gamma $k = 3$	0.3238

Table 4.4: Proportion of banks in default after cascade process ends.

From Figure 4.10 and Table 4.4 we can see that a common shock in fact makes the initial default spread to a larger portion of the network.

4.1.3 Average Capital for Fixed Expected Loss

In Section 4.1.2 the simulation considered different distributions for the shock α to each bank's capital such that the initial proportion of banks in default was equal p. This means that each distribution for the shock α had different expectation and variance. In this section we will fix the expected value of the shock α to each bank's capital. This way the expected loss throughout the network will be the same for different distributions due to the Law of Large Numbers.

The same procedure is followed. We start with C_0 which makes the network resilient. We can verify this by checking if $\mathbb{E}[W^+W^-\mathbf{1}_{C=1}] < 1$, according to [18]. Then we calculate C_1 after different shocks by bootstrap estimation.

In the first example, we choose $\alpha \sim Gamma(k = 3, \beta)$ and choose β such that the initial proportion of banks in default is 1%. The parameters of the distribution for the other shocks are chosen such that they all have the same expected value. The parameters used are presented in Table 4.5. Functional f was calculated for all C_1 obtained. The result is shown in Figure 4.11.

Type of shock	Parameter	Mean	Variance
Exponential	$\lambda = 0.7751$	1.29	1.665
Gamma $k = 2$	$\beta = 1.5504$	1.29	0.832
Gamma $k = 3$	$\beta = 2.3255$	1.29	0.5547

Table 4.5: Parameters for shock distributions for *average capital*. All distributions have the same expected value.

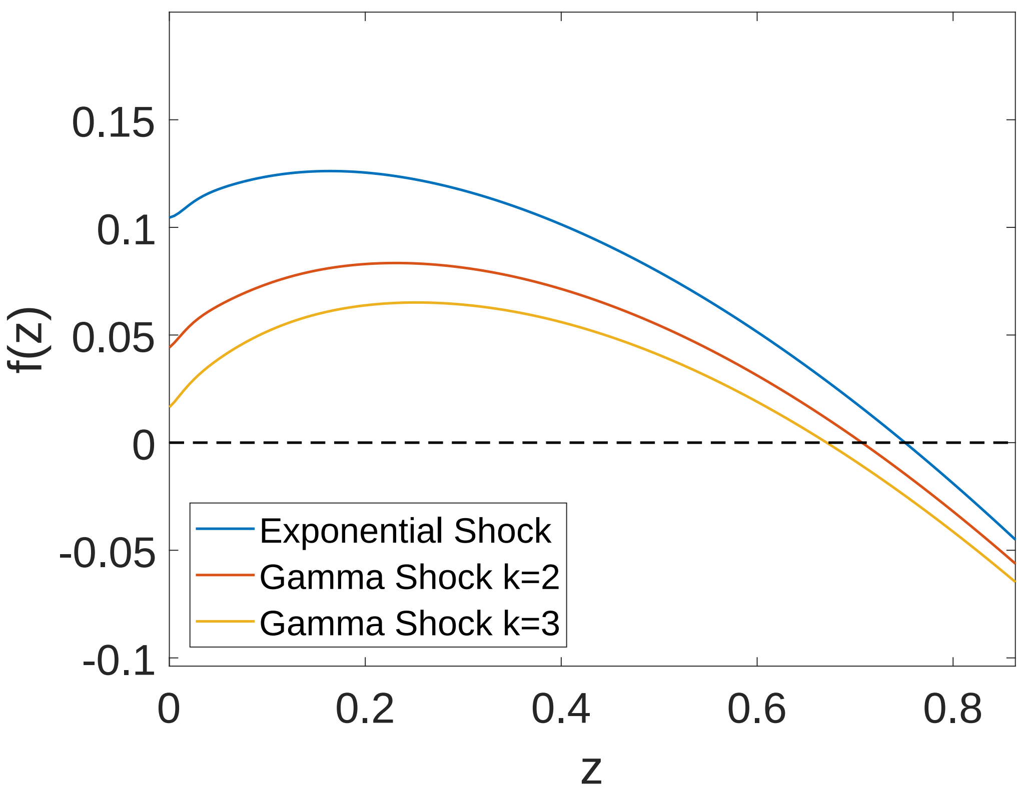

Figure 4.11: Functional f after different shocks are applied to initial capital. Expected value of different shocks have the same mean.

The initial and final proportion of banks in default, $g(0)$ and $g(\hat{z})$ respectively, are given in Table 4.6.

Type of shock	$g(0)$	$g(\hat{z})$
Exponential	0.0624	0.3669
Gamma $k = 2$	0.0259	0.3373
Gamma $k = 3$	0.0100	0.3155

Table 4.6: Proportion of banks in default after cascade process ends.

From Figure 4.11 and Table 4.6 we see that the initial proportion of infected vertices is much larger for the Exponential shock but the final proportion of infected vertices is similar for all different shocks.

We did the same procedure but now we choose $\alpha \sim Exponential(\lambda)$ and choose β such that the initial proportion of banks in default is 1%. The parameters for each distribution are given in Table 4.7.

Type of shock	Parameter	Mean	Variance
Exponential	$\lambda = 1.2820$	0.78	0.608
Gamma $k = 2$	$\beta = 2.5642$	0.78	0.3042
Gamma $k = 3$	$\beta = 3.8462$	0.78	0.2028

Table 4.7: Parameters for shock distributions for *average capital*. All distributions have the same expected value.

Functional f was calculated for all C_1 obtained. The result is shown in Figure 4.12.

The initial and final proportion of banks in default $g(0)$ and $g(\hat{z})$, respectively, are given in Table 4.8.

Type of shock	$g(0)$	$g(\hat{z})$
Exponential	0.0100	0.1618
Gamma $k = 2$	0.0012	0.1083
Gamma $k = 3$	0.00016	0.1076

Table 4.8: Proportion of banks in default after cascade process ends.

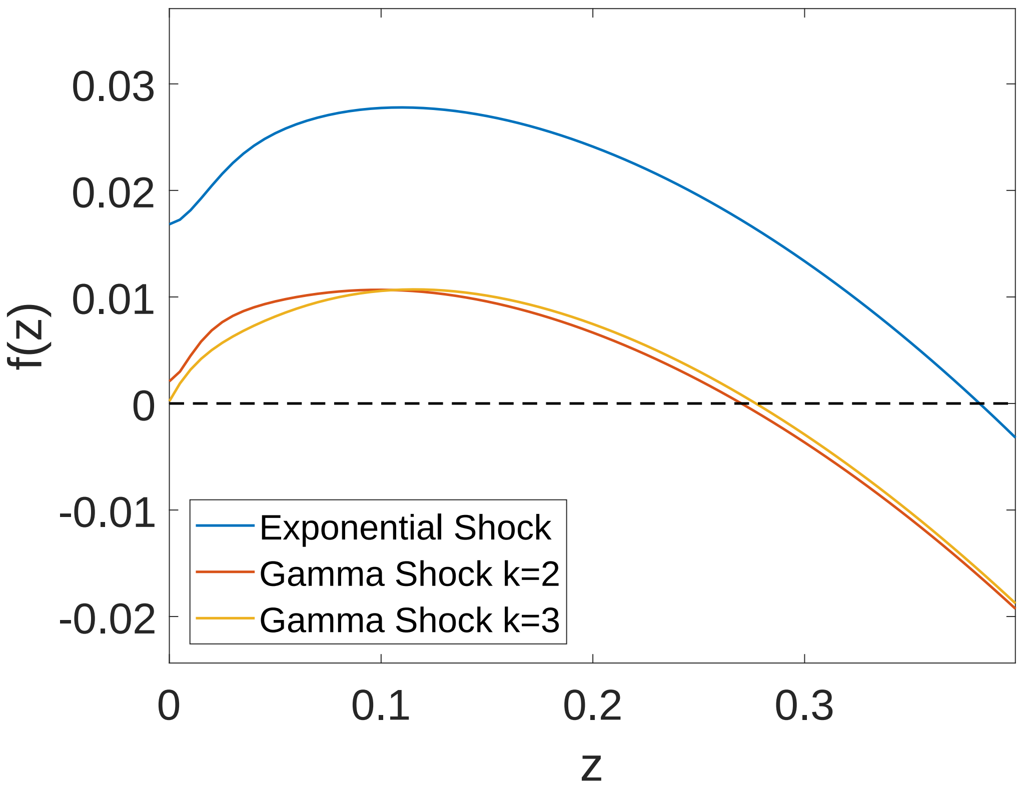

Figure 4.12: Functional f after different shocks are applied to initial capital. Expected value of different shocks have the same mean.

From Figure 4.12 and Table 4.8 we can see, once again, that the common asset has a large impact in the default cascade, no matter how small the initial infection is.

Also, by comparing Figures 4.11 and 4.12 as well as Tables 4.6 and 4.8, we can see that even though the distributions for the shock α give sometimes very different initial infections, the final proportion of banks in default is somewhat similar. Once again, the common shock creating *contagions links* is very impactful to the contagion process.

4.1.4 Average Capital with Comonotone Shock

In the previous sections we obtained the distribution C_1 after a shock α which was always independent of the weights W^+ and W^- and the exposures $E_{i,j}$. In this section we will make the distribution α comonotone with W^- (it will also be comonotone with

W^+ since W^+ and W^- are comonotone). This simulates a more realistic scenario where big banks suffer bigger shocks.

The parameters chosen for the shock α in this simulation are exactly the same as in Table 4.3. The distributions for C_1 look the same, since the parameters for α are the same. However instead of being independent of everything else, they are now comonotone with W^-.

The functional f was calculated for all C_1. The result is shown in Figure 4.13.

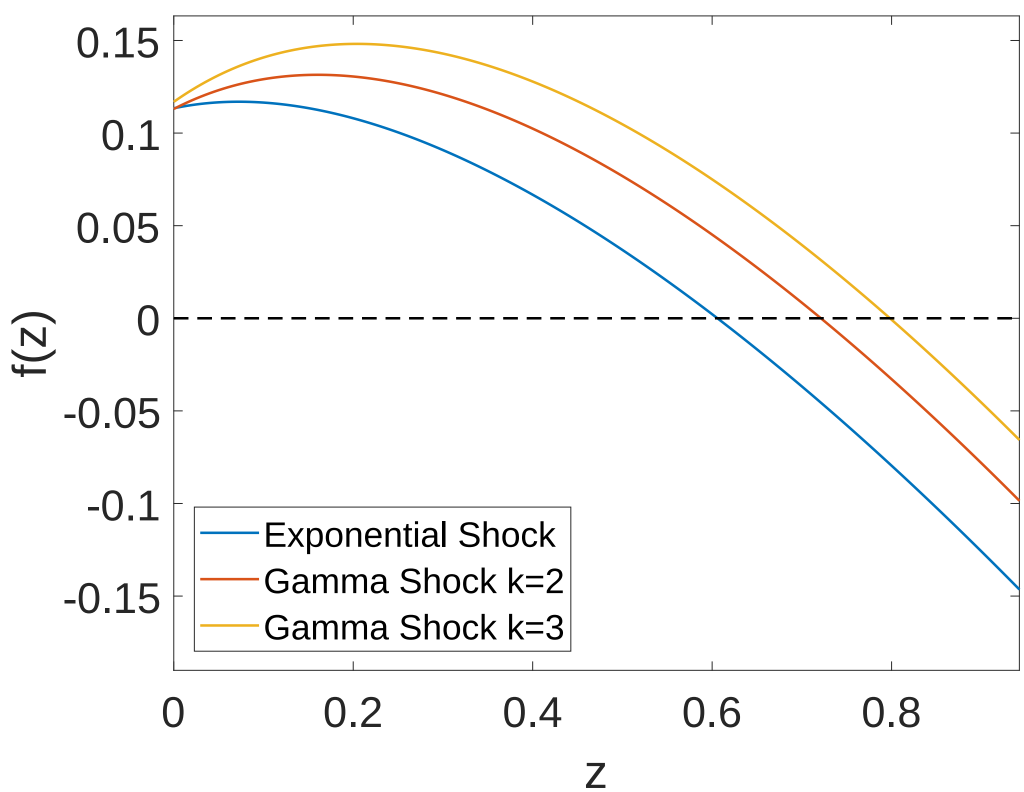

Figure 4.13: Functional f after different comonotone shocks are applied to initial capital.

It is important to point out that even though $f(0)$ is larger in this case then in Figure 4.10, the initial proportion of banks in default is the same. Value $f(0)$ can be calculated from equation (2.4.4):

$$f(0) = \mathbb{E}[W^+\mathbb{P}(Poi(0) \geq C)] = \mathbb{E}[W^+\mathbf{1}_{C=0}] \tag{4.1.1}$$

If W^+ and α are independent, then from equation (4.1.1), we can see that $f(0) = p\mathbb{E}[W^+]$, where p is the initial proportion of banks in default. In the comonotone case, however, this probability that $c_i = 0$ is higher for banks with large weight w_i^+,and therefore $f(0)$ should be larger because $W^+\mathbf{1}_{C=0}$ will be non-zero for larger values of W^+.

The final proportion of banks in default is given in Table 4.9.

Type of shock	$g(\hat{z})$
Exponential	0.2464
Gamma $k = 2$	0.3130
Gamma $k = 3$	0.3612

Table 4.9: Proportion of banks in default after cascade process ends.

If we compare the proportion of banks in default given in Tables 4.9 and 4.4, we can see that the cascade affects a larger portion of the network when the shock is comonotone with W^-. This result agrees with the findings in [18] which mention that larger banks have a more significant impact to the contagion process in the network. In our example, making larger banks having a higher probability of default increases the final proportion of banks in default.

4.2 Simulation Study Without Second Moment

As in section 4.1, we simulate the default contagion process on a financial network, but now we assume that the degree sequence does not have finite second moment. Again, the weights W^+ and W^- are comonotone and sampled from a Pareto distribution (see Definition 3).

We will start our simulation by imposing a minimum capital requirement for each bank using a threshold function $\tau_i(w^-)$, as proposed in [18], which makes the network resilient (see Theorem 7) for the capital choice in equation (3.2.1). We do not have a resilience criterion if the capital is defined as in equation (3.2.2). In this case we will check if the system is resilient using Theorem 4. The function $\tau_i(w^-)$ is given by:

$$\tau_i(w^-) = \max\left\{2, \alpha_c(1+\delta_1)(w_i^-)^{\gamma_c(1+\delta_2)}\right\}, \tag{4.2.1}$$

where

$$\gamma_c = 2 - \frac{\beta^- - 1}{\beta^+ - 1} - \beta^-,$$

and

$$\alpha_c = \frac{\beta^+ - 1}{\beta^+ - 2} w^+_{min} (w^+_{min})^{(1-\gamma_c)}.$$

The parameters δ_1 and δ_2 are positive constants. According to Theorem 7, any positive δ_1 and δ_2 will make the network resilient. Unless otherwise specified, we always use $\delta_1 = \delta_2 = 0.08$.

Once we have each bank's capital, as in equations (3.2.1) and (3.2.2), we start the same procedure described in Section 4.1: obtain the distribution of C_0, the initial hypothetical threshold level, and C_1, the hypothetical threshold level after a shock is applied to the common asset, which will be used to calculate functional f which describes the contagion process.

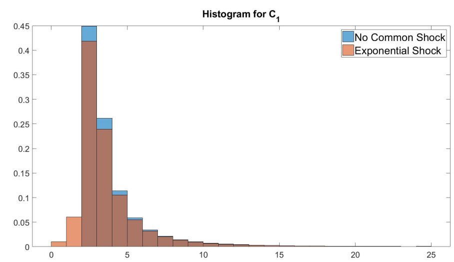

Figure 4.14: Histogram without common shock (blue) and with Exponential shock (red).

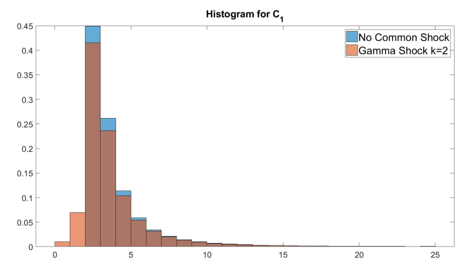

Figure 4.15: Histogram without common shock (blue) and with Gamma shock (red).

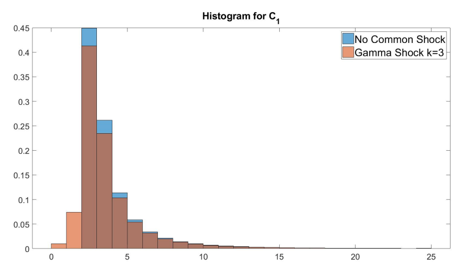

Figure 4.16: Histogram without common shock (blue) and with Gamma shock (red).

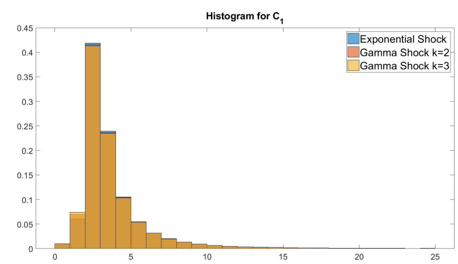

Figure 4.17: Histogram Exponential shock(blue), Gamma shock k=2 (red), Gamma shock k=3 (yellow).

4.2.1 Hypothetical Capital Choice

In this section we analyze the default contagion process when each bank's initial capital is given in equation (3.2.2), with $\tau_i(w^-)$ as in equation (4.2.1). As mentioned in Section 4.1.1, we do not have a resilience criterion for this particular definition of the capital. We will check resilience using Theorem 4 by plotting functional f without any initial banks in default and observe the derivative around 0. It will in fact be resilient in this case.

We start by presenting the histograms of C_0 and C_1 in Figures 4.14-4.17 for different distributions of the shock.

Once again, note that the common shock introduces not only default but also contagious links. Also note that, for the same initial infection, the Exponential shock introduces fewer contagious links and Gamma with $k = 3$ introduces the most contagious links in our examples. The parameters for each distribution used are presented in Table 4.10.

Type of shock	Parameter	Mean	Variance
Exponential	$\lambda = 3.333333$	0.3	0.09
Gamma $k = 2$	$\beta = 5.952381$	0.336	0.0564
Gamma $k = 3$	$\beta = 8.474576$	0.354	0.0418

Table 4.10: Parameters for shock distributions for *hypothetical capital*.

From Table 4.10 we see that, for the same initial infection, distributions with higher expected value have smaller variances. This was also observed in the same example when the network has finite second moment (see Table 4.1).

For the hypothetical threshold levels showed in Figures 4.14-4.17 we simulate the default contagion process by calculating functional f. The result is presented in Figure 4.18

In Figure 4.18 the blue line represents an unshocked network. Note that the derivative is negative for small z, so the system is resilient according to Theorem 4. We can see that the contagion process has nearly the same behavior for different distributions of the shock. Also we note that the derivative of the functional around $z = 0$ is negative, which indicates that the networks is resilient. This suggests that the choice of capital is sufficiently strong and the network will be stable if it subject to small shocks. However, note that despite the initial negative derivative, it starts growing rapidly and, if the shock is large enough, the system will be widely affected.

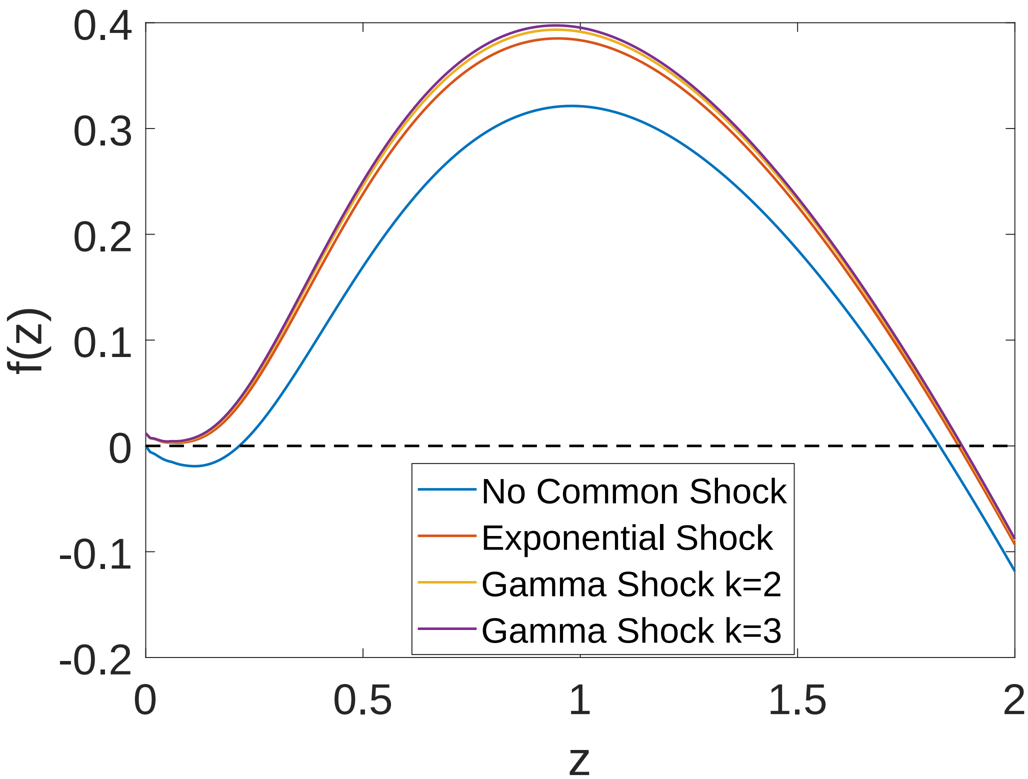

Figure 4.18: Functional f after different shocks are applied to initial capital.

The final proportion of banks in default $g(\hat{z})$ is given in Table 4.11 for all different distributions of the shock.

Type of shock	$g(\hat{z})$
Exponential	0.8051
Gamma $k = 2$	0.8101
Gamma $k = 3$	0.8116

Table 4.11: Proportion of banks in default after cascade process ends.

From Table 4.11 it is clear that the contagion process affects a huge portion of the

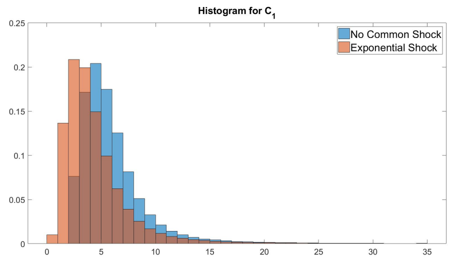

Figure 4.19: Histogram without common shock (blue) and with Exponential shock (red).

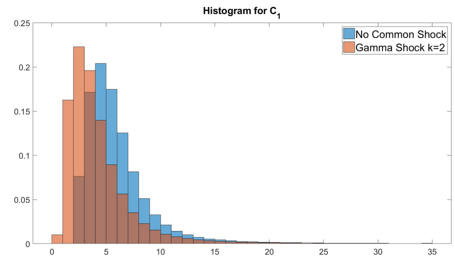

Figure 4.20: Histogram without common shock (blue) and with Gamma shock (red).

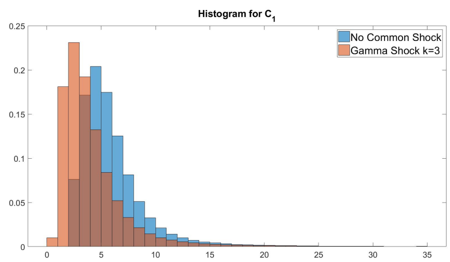

Figure 4.21: Histogram without common shock (blue) and with Gamma shock (red).

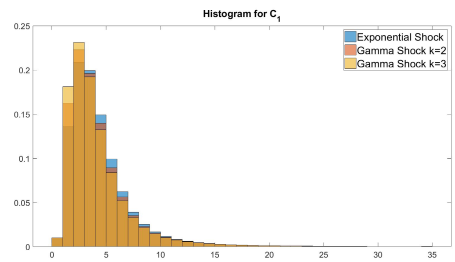

Figure 4.22: Histogram Exponential shock(blue), Gamma shock k=2 (red), Gamma shock k=3 (yellow).

network. Once again we see that the distribution of the shock does not matter too much since the final proportion is similar for all different shocks.

4.2.2 Average Capital Choice

We will now simulate default contagion for initial capital defined as in equation (3.2.1), and $\tau_i(w^-)$ defined in equation (4.2.1). In this case, this particular choice for the capital and $\tau_i(w^-)$ makes the network resilient. This is shown in Theorem 7 and more details can be found in [18].

We follow the same steps: obtain C_0 and C_1 for different distributions of the shock and simulate the default contagion by calculating functional f. The histograms for C_0 and C_1 are shown in Figures 4.19-4.22.

In this case we can see that all shocks create a large number of contagious links. We will see that this will impact heavily on the default contagion process. Again we see that the Gamma shock with $k = 3$ seems to create more contagious links and the Exponential shock creates fewer. The parameters for each shock distribution are given in Table 4.12.

Type of shock	Parameter	Mean	Variance
Exponential	$\lambda = 1.086957$	0.92	0.846
Gamma $k = 2$	$\beta = 1.612903$	1.24	0.7688
Gamma $k = 3$	$\beta = 2.074689$	1.446	0.697

Table 4.12: Parameters for shock distributions for *average capital*.

For the hypothetical threshold levels showed in Figures 4.19-4.22 we simulate the default contagion process by calculating functional f. The result is presented in Figure 4.23

The blue line in Figure 4.23 represents the unshocked system. We can see that the derivative is negative around $z = 0$ and the system is resilient. However, the shocked systems do not seem to have negative derivatives around 0, and, as we can see, the contagion process stops very late. Once again the behavior of the system for different shocks is not very different. Note that this system seems less resilient than the previous example in Figure 4.18. However, comparing Tables 4.10 and 4.12 we can see that it takes a higher average shock to make this system have the same initial proportion of banks in default.

The final proportion of banks in default $g(\hat{z})$ is given in Table 4.13 for all different distributions of the shock.

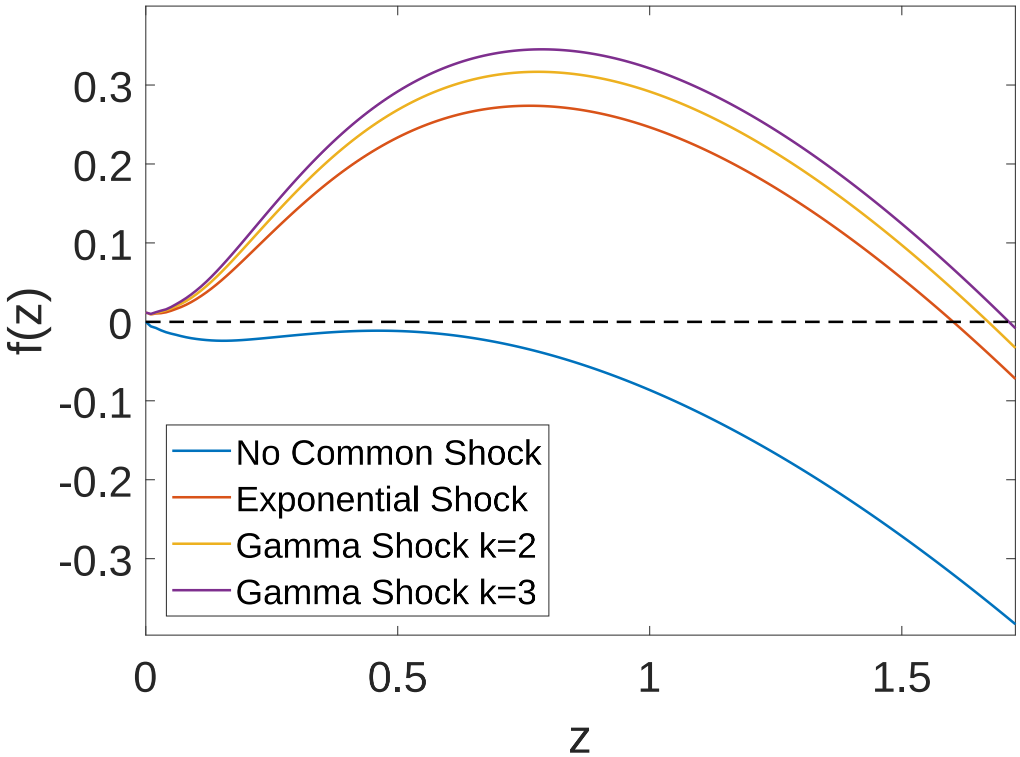

Figure 4.23: Functional f after different shocks are applied to initial capital.

Type of shock	$g(\hat{z})$
Exponential	0.6276
Gamma $k = 2$	0.6740
Gamma $k = 3$	0.7016

Table 4.13: Proportion of banks in default after cascade process ends.

Comparing Tables 4.10 and 4.12 we can see that the average shock is higher in the second case. However, comparing , Tables 4.11 and 4.11 we see that the final fraction of banks in default is smaller in the second case. The example given in Section 4.10 has several banks with hypothetical threshold level 2 and even small shocks can lead to banks in default or introduce contagious links. In the later example, in Section 4.11 we can see that banks have a higher initial hypothetical threshold level, and a larger shock is required to lead banks to default. Also, these banks with very high initial threshold level do not suffer a big loss in this case where the shocks are independent and identically distributed, which suggests that big banks have a heavier impact on the stability of the system.

4.2.3 Average Capital for Fixed Expected Loss

In Section 4.2.2 we had different expected values for the different distributions of the shock. In this Section, as in 4.1.3, we will fix the expected value of one distribution and fit the parameters of the others such that all systems suffer the same expected loss after the shock.

We start by fitting the parameter of the shock with Exponential distribution such that the initial proportion of banks in default is 1%. The parameters for the other distributions are chosen such that they have the same expected value.

The parameters are given in Table 4.14.

Type of shock	Parameter	Mean	Variance
Exponential	$\lambda = 1.086957$	0.92	0.846
Gamma $k = 2$	$\beta = 2.173913$	0.92	0.4232
Gamma $k = 3$	$\beta = 3.260869$	0.92	0.2821

Table 4.14: Parameters for shock distributions for *average capital.*

From Table 4.14 we can see that the Exponential distribution has higher variance. We expect that the initial infection is higher for this distribution for the shock. Functional f for the shocks in Table 4.14 is presented in Figure 4.24.

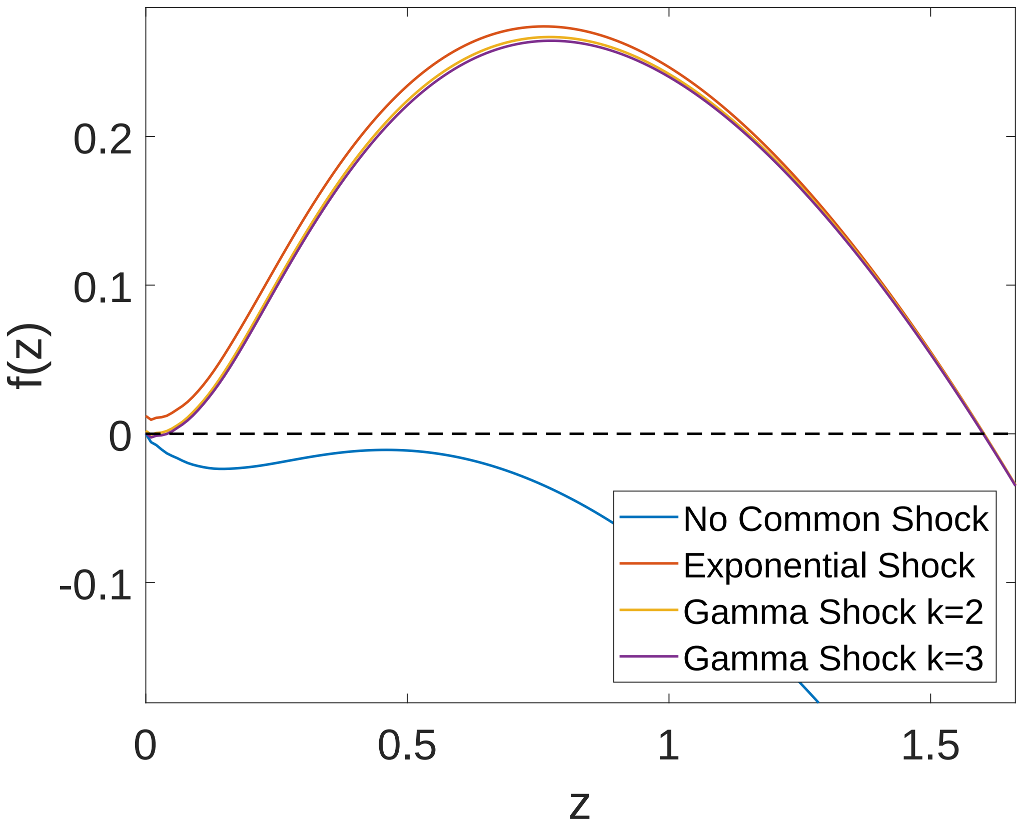

Figure 4.24: Functional f after different shocks are applied to initial capital. Expected value of different shocks have the same mean.

The initial and proportion of banks in default is given in Table 4.15.

Type of shock	$g(0)$	$g(\hat{z})$
Exponential	0.0100	0.6277
Gamma $k = 2$	0.0017	0.6270
Gamma $k = 3$	0.0003	0.6263

Table 4.15: Proportion of banks in default after cascade process ends.

We can see that the initial proportion of banks in default after a shock with Gamma distribution affects the network is negligible. However, despite some numerical instabilities around $z = 0$, we can see that the functional starts with positive derivative, which means that the network is not resilient in this case. This result is similar to Figure 4.12.

Remark 19 *The evaluation of f around $z = 0$ for very small initial proportion of infection is not precise and usually result in $f(z) = -z$ because the expected value will be 0. In Figure 4.24 we disregard the first couple of values for the purple and yellow lines because of this.*

Analogously, we did the same example but now we fit the shock with Gamma distribution with $k =$ such that the initial proportion of banks in default is 1%. The parameters for the other distributions are chosen such that they have the same expected value.

The parameters are given in Table 4.15.

Type of shock	Parameter	Mean	Variance
Exponential	$\lambda = 0.691563$	1.446	2.091
Gamma $k = 2$	$\beta = 1.383125$	1.446	1.0455
Gamma $k = 3$	$\beta = 2.074689$	1.446	0.697

Table 4.16: Parameters for shock distributions for *average capital*.

From Table 4.16 we can see that again the Exponential shock has higher variance and, therefore, should have a higher initial proportion of banks in default. We calculate functional f for the shocks given in Table 4.16.

The initial and final proportion of banks in default is given in Table 4.17.

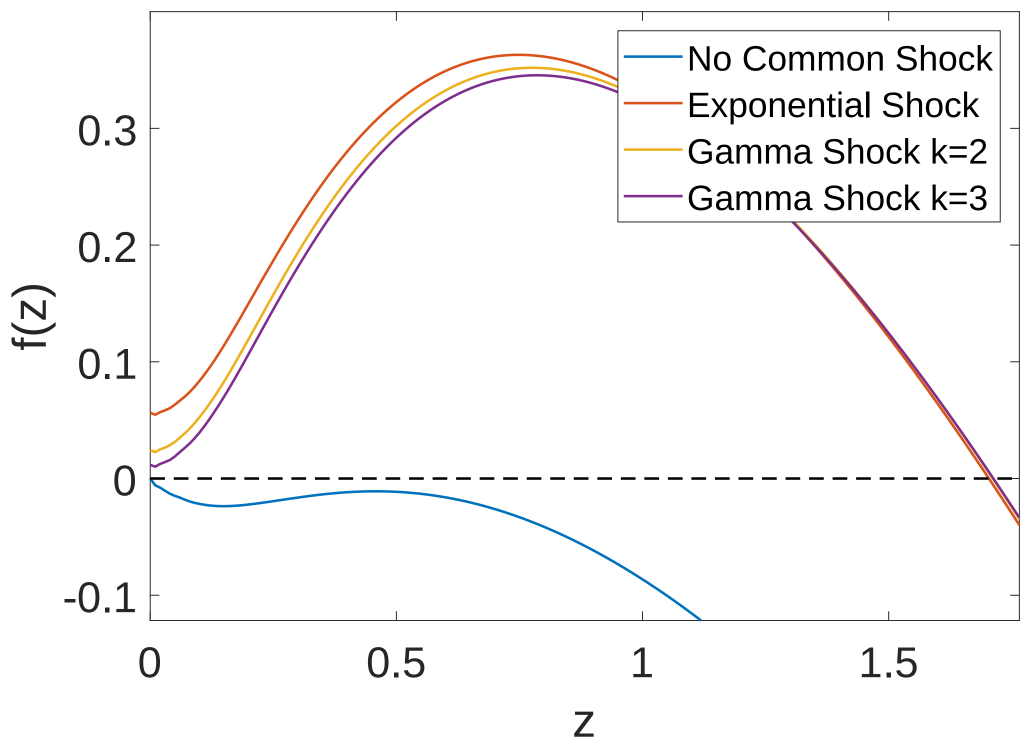

Figure 4.25: Functional f after different shocks are applied to initial capital. Expected value of different shocks have the same mean.

Type of shock	$g(0)$	$g(\hat{z})$
Exponential	0.0453	0.6963
Gamma $k = 2$	0.0204	0.7016
Gamma $k = 3$	0.0100	0.7017

Table 4.17: Proportion of banks in default after cascade process ends.

We can see from Figure 4.25 that the initial proportion of banks in default is indeed higher for the exponential shock. However, once again, regardless of the type of shock the final proportion of infected vertices is quite similar. This result is similar to Figure 4.11.

4.2.4 Average Capital with Comonotone Shock

In Sections 4.2.1-4.2.3 we assumed that the shock α was independent of everything else. While the examples gave some insight into which banks are more important, it is unrealistic to expect that big banks, which have a large balance sheet, will suffer the same loss as a bank with a much smaller balance sheet. In this Section, as in Section 4.1.4, we will make the distribution of α comonotone with W^-. Of course, since W^+ and W^- are comonotone themselves, α will also be comonotone with W^+.

This is the most realistic scenario considered in this work. W^+ and W^- both have infinite second moment in this case, which is compatible with the findings in [13], and big banks will suffer bigger losses.

We proceed as we did several times. First we find C_0 and C_1 and then we analyze the default contagion process by calculating functional f. Histograms for these distributions are presented in Figures 4.26-4.29.

From Figures 4.26-4.29 we can see that these shocks are quite severe. Furthermore, we see that after the shock, there are very few banks with high hypothetical threshold level.

In this example, since the big banks have a much larger capital requirement, the shocks also have a much larger expected value. The parameters for the shock distributions are given in Table 4.18.

It is clear when we compare Table 4.18 with Tables 4.12, 4.14 and 4.16 that the shock in this case will be, on average, much higher. However, since it is comonotone with W^-, bigger banks, i.e., banks with larger hypothetical threshold level, will suffer bigger losses.

Functional f for these examples is presented in Figure 4.30. From Figure 4.30 we can see that the initial system, represented by the blue line, is in fact resilient. However, the

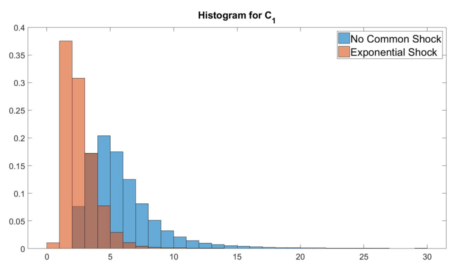

Figure 4.26: Histogram without common shock (blue) and with Exponential shock (red).

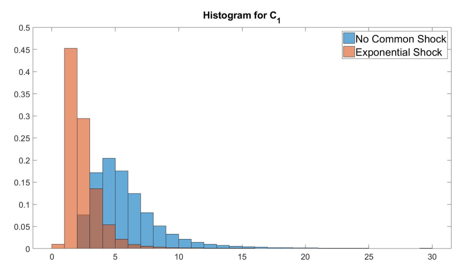

Figure 4.27: Histogram without common shock (blue) and with Gamma shock (red).

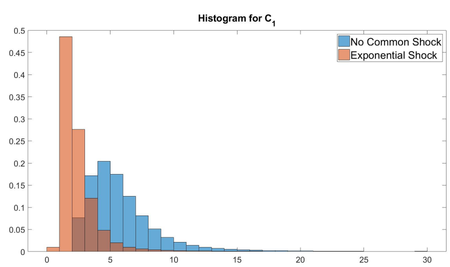

Figure 4.28: Histogram without common shock (blue) and with Gamma shock (red).

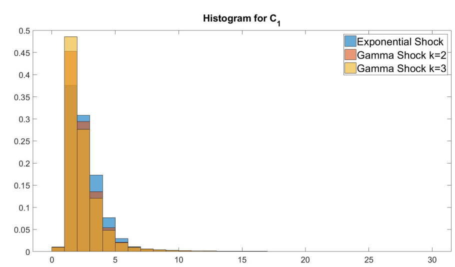

Figure 4.29: Histogram Exponential shock(blue), Gamma shock k=2 (red), Gamma shock k=3 (yellow).

Type of shock	Parameter	Mean	Variance
Exponential	$\lambda = 0.237529$	4.2100	17.724
Gamma $k = 2$	$\beta = 0.455373$	4.392	9.6448
Gamma $k = 3$	$\beta = 0.683060$	4.392	6.4299

Table 4.18: Parameters for shock distributions for *average capital*.

shocked systems are non-resilient. Furthermore we see that the derivative around $z = 0$ is positive and the initial growth of f is quite significant for small z, contrary to other exa

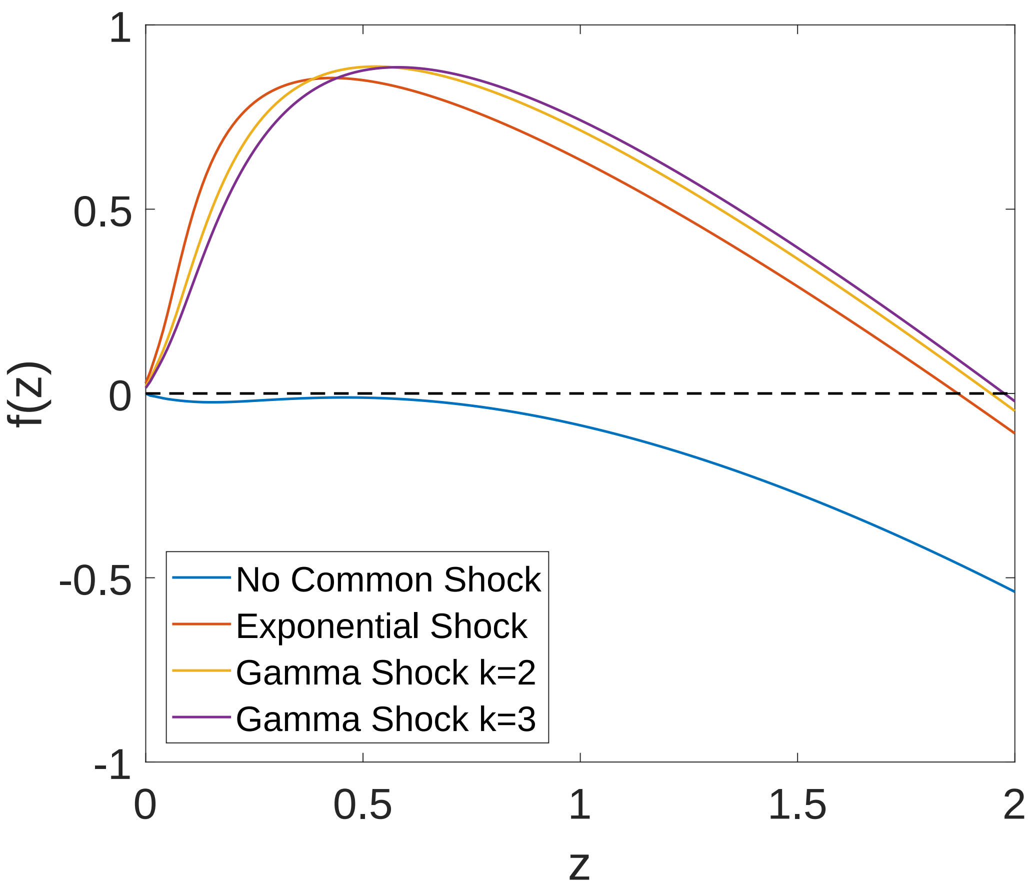

Figure 4.30: Functional f after different comonotone shocks are applied to initial capital with $\delta_1 = \delta_2 = 0.08$.

The final fraction of banks in default is presented in Table 4.19

Type of shock	$g(\hat{z})$
Exponential	0.7828
Gamma $k = 2$	0.8433
Gamma $k = 3$	0.8704

Table 4.19: Proportion of banks in default after cascade process ends.

This is the example in which the largest proportion of the network is affected. It is clear by observing the behavior of the functional f that this contagion process is indeed very harmful to the network. Once again we show that large banks are very impactful to the contagion process, and in order to insure stability of the system these large banks should remain solvent. This is consistent with the findings in [21], which show that even though the contagion process is rare, when it happens it can affect a very large portion of the system.

Using the same quantities from this example, we increased parameters δ_1 and δ_2 in equation (4.2.1) to check how a larger imposed threshold level affects the default contagion. Now we choose $\delta_1 = \delta_2 = 0.092$. To keep the example consistent, the same parameters for the shock α were used. The resulting functional f is presented in Figure 4.31. The difference between Figures 4.30 and 4.31 is that in the later there are no banks in default initially. We can see from the behavior of the functional that the system is non-resilient, that is, even very small shocks to this system will lead to a large portion of the system being affected. Again we show that a common shock is very impactful to the contagion process.

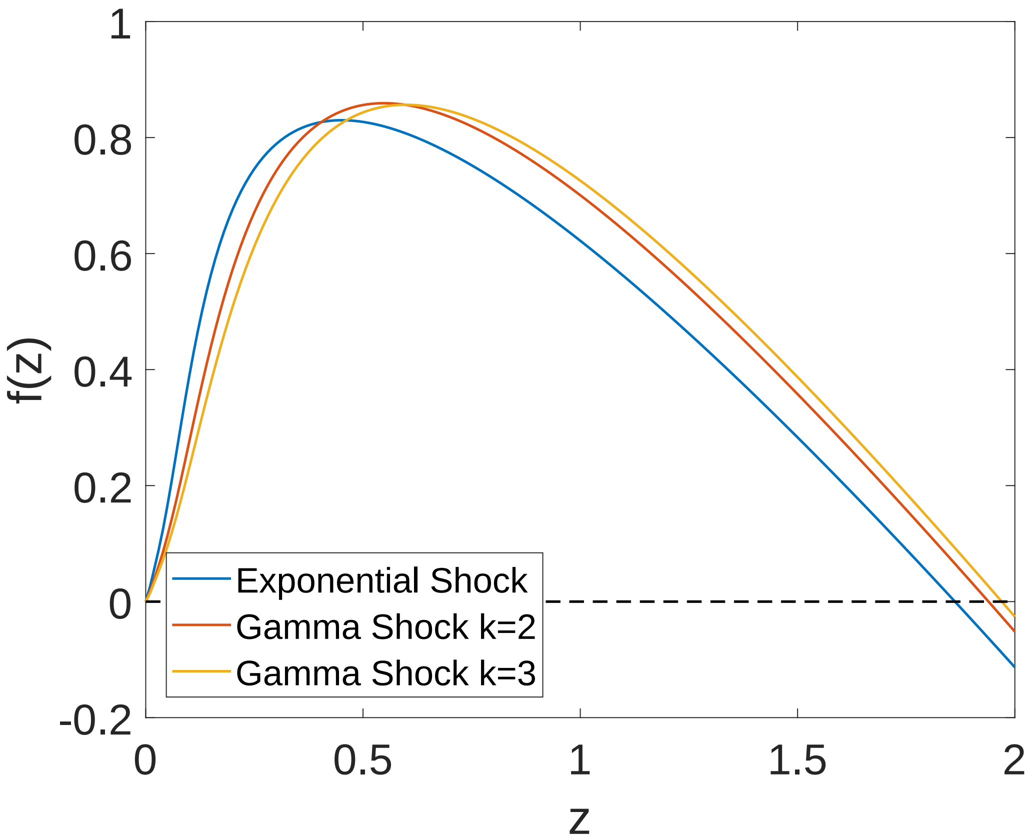

Figure 4.31: Functional f after different comonotone shocks are applied to initial capital with $\delta_1 = \delta_2 = 0.092$.

If we increase the parameters δ_1 and δ_2 even further, it is possible to make the functional f have negative derivative at $z = 0$. In Figure 4.31 we show and example with $\delta_1 = 0.2$ and $\delta_2 = 0.3$. We chose $\delta_2 > \delta_1$ to make larger banks have even higher threshold level. Again, the same parameters for the shock α were used. From Figure 4.32 we can see that a higher capital requirement will make the network resilient. However, in this scenario where very large shocks are applied to all banks, the capital requirement needed in order to make the system resilient is also very large. From equation (4.2.1) we can see that $\tau \propto (1 + \delta_1)(w_i^-)^{1+\delta_2}$, so increasing δ_2 will have a large impact, particularly for the larger banks.

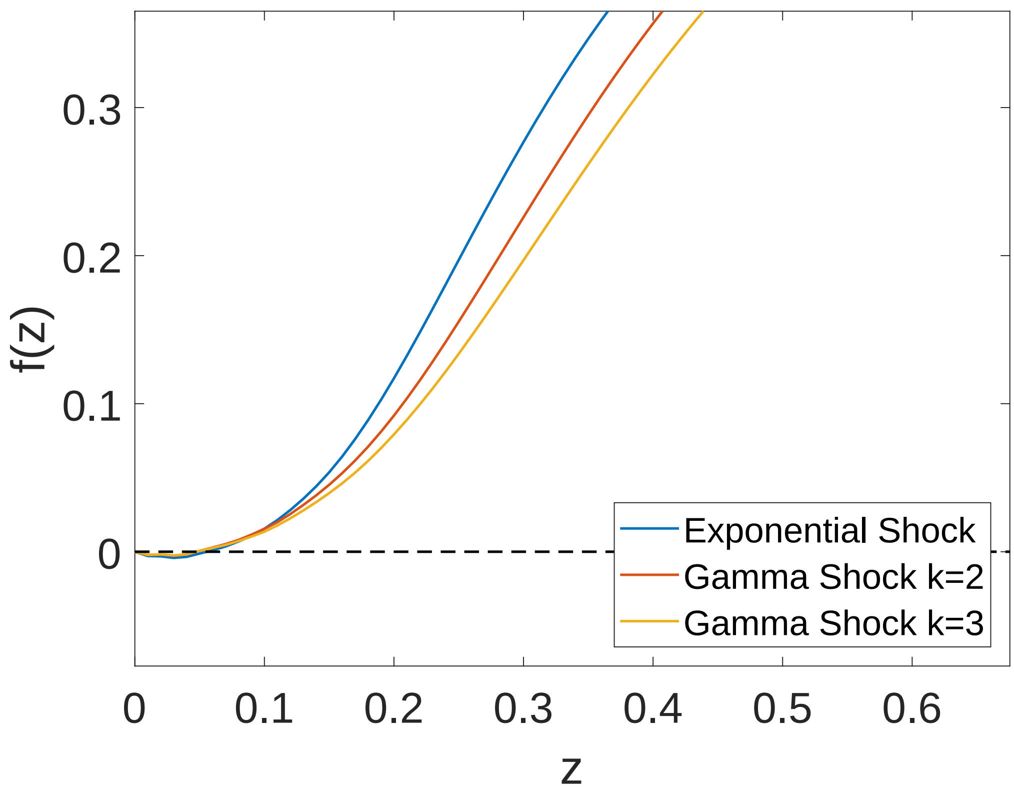

Figure 4.32: Functional f after different comonotone shocks are applied to initial capital with $\delta_1 = 0.2$ and $\delta_2 = 0.3$.

4.3 Methodology

In this Section we will show briefly the codes used to make the simulations in Sections 4.1 and 4.2. Note that not the entire code will be show here, just the main commands and functions.

We analyzed the final fraction of banks in default in a given network by calculating functional f, defined in equation (2.4.4). We need to simulate the triple (W^-, W^+, C_0), which describes the network prior to a shock, and (W^-, W^+, C_1), which is after the shock.

In the first part of the code we define all the parameters of the problem: λ, which is the parameter for the exposure distribution; the parameter for the distribution of W^-

and W^+, as defined in (3); $\tau(w_i^-)$, as defined in (4.2.1). Next we simulate the comonotone vector (w_i^-, w_i^+) and the shock α_i:

```
% power law exponent according to the definition in this work
exp_plus  = 2.8861;
exp_minus = 2.132;

%fix parameters for the gprnd function in Matlab
alpha_plus=exp_plus-1;
alpha_minus=exp_minus-1;

% simulate w_plus and w minus
%set seed = 0 for both
xmin=1;
rand('seed',0);
w_plus  = gprnd(1/alpha_plus ,xmin/alpha_plus ,xmin,1,size);
rand('seed',0);
w_minus = gprnd(1/alpha_minus,xmin/alpha_minus,xmin,1,size);

%paramater lambda for exposures, chosen to fit the data
alpha_exposures=2.27;
lambda=( (alpha_exposures)/(alpha_exposures-1) )^-1;

%define parameters for tau
delta1=0.08;
```

```
delta2=0.08;
alp_c1 = alp_c*(1+delta1);
gam_c1 = gam_c*(1+delta2);

%define tau
tau=floor( alp_c1*(w_minus).^(gam_c1) );
tau=max(2,tau);

%iid shock alpha
%k=1,2,3
%values beta change
alpha=gamrnd(k,1/beta,1,size);

%alpha comonotone with w^-
alpha=gaminv(gpcdf(w_minus,1/alpha_minus,xmin/alpha_minus,
    xmin),k,1/beta);
```

The final input to calculate functional f are C_0 and C_1. We used the following function which simulates both vectors:

```
function [c0,c1]=cmean(size,tau,alpha,lambda)

c0=zeros(1,size);
c1=zeros(1,size);

%initial capital
capital=(1/lambda)*tau.*ones(1,size);
```

```

%capital after the shock
newcapital = nu_mean - alpha;

        for i=1:size
    E=exprnd(1/lambda,10*tau(i),1);
    cummexpo=cumsum(E);

%this step is usually skiped, we just need to ensure that the sum of the exposures is smaller than banks capital the ensure that we are not capping the hypothetical threshold level. There are fancier ways of doing this, but this seems eficient since creating vectors in Matlab is fast and the definition above is almost always enough.
        if capital(i)>sum(E)
            E=[E;exprnd(1/lambda,20*tau(i),1)];
        end

        %find new threshold function for each vertex
        [~,C0]=histc(capital(i),cummexpo(:,1));
    c0(1,i)=C0+1;



    [~,C1]=histc(newcapital(i),cummexpo(:,1));
```

```
        c1(1,i)=C1+1;

            if newcapital(i)<0
                c1(1,i)=0;
            end

        end
end
```

Finally, we need another function to calculate functional f:

```
function [f,g,zhat]=ff(C,w_minus,w_plus)

%grid for z
z=0:0.01:2;
maxit=length(z);

%initialize functional f
f=zeros(1,maxit);

%routine to calculate functional f at each point from the
    grid above
for i=1:maxit
    psi= 1-poisscdf(C-1,w_minus*z(i)) ; %for each vertex,
        multiply w- by z, and find the probability that a
        Poisson with parameter z*w- is larger than C
    f(i)=mean(w_plus.*psi)-z(i);
```

```
end

%finding the root of f
[M,I]=min(abs(f));
zhat=z(I);

%find g
g=mean(1-poisscdf( C-1,w_minus*zhat ));
```

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