Cognition in Preferences and Choice

Maria Betto

7 · 31 · 2024

[Click here for the latest version]

Abstract

This paper models and analyzes the role of cognition in the refinement of preferences and choices. As cognition increases, choices become more selective, resulting in narrower sets of preferred options and finer rankings. To characterize this behavior, the classical rational-choice framework is extended through the introduction of a modified version of the Weak Axiom, called the Weak Axiom of Revealed Preference Difficulty (WARPD). The paper shows that WARPD is equivalent to an interval-valued utility representation, where the size of the interval decreases monotonically with cognition. It also demonstrates that WARPD is equivalent to a fuzzy rationalizability concept, implying that cognition-dependent choices satisfying WARPD can be represented by a complete and transitive fuzzy binary relation. Finally, the paper describes two applications which highlight how consumers' choice coarseness influences firms' strategic pricing decisions in different competitive settings.

1 Introduction

A decision-maker's preferences over items in a finite set X are usually a complete ordering of the elements of X, possibly with ties, ranked from most to least preferred. However, it is improbable that the decision-maker can immediately, effortlessly and precisely articulate this complete ordering upon request. The opposite – that the decision-maker would possess neither intuition nor a vague idea about this ordering – is also unlikely. In practice, one would anticipate that she would be able to articulate a general if "coarse" understanding of how each item in X compares to the others. For instance, she may recognize that an all-expenses-paid trip to Paris is superior to working overtime without any additional compensation, even if she remained uncertain about whether a vacation to Paris is preferable to one to Rome.

A detailed ranking that accurately reflects the decision-maker's tastes, likes, and dislikes often requires a significant amount of thoughtful consideration – a mental resource referred broadly to as "cognition" throughout the paper.

To illustrate this point and introduce the approach in this paper, consider the following example.

Example 1.1. Xavier and Yvonne are friends who share the same taste in movies. Yvonne asks Xavier to recommend a movie from a set of five options labeled a, b, c, d and e – all of which Xavier has recently watched and remembers well. Since Xavier does not know which movies Yvonne has access to or how many she wants to see, he must do his best to provide a ranking of the movies.

In a first scenario, suppose Xavier is busy and only has a few seconds to rank the movies. He remembers disliking e but enjoyed the others, so he splits them into a "bad" category containing only e and a "good" category with the rest.

In a second scenario, Xavier is taking a break from work and thus has a few more minutes of undistracted thought to consider this ranking. Upon further consideration, he realizes he enjoyed a more than both c and d, so he creates a new "okay" category containing the similarly-enjoyed movies b, c, and d. The "good" category now contains both a and b, whereas e remains "bad". Note that b is both "good" and "okay". This shows that the ranking as it stands is not quite complete (nor is it transitive), but it still provides a better characterization of what Xavier's preferences are than the first scenario.

In a third and final scenario, suppose Xavier has a couple of days to come up with a definitive ranking. After much thought, he concludes that a is the best movie, followed by b, c, d, and, lastly, e.

In this example, Xavier starts out with a coarse idea of what the ranking looks like. This notion can then be refined as he applies more thought, or *cognition*, to the problem. Depending on the context, Xavier's cognition levels vary, which in turn makes his recommendation either more or less precise. $\lozenge$

The focus of this paper is on preference and choice refinement through cognition. The underlying premise is that increased cognition is linked with discerning behavior, leading to a narrower selection of options and finer rankings. Conversely, lower levels of cognition suggest that a wider range of options are deemed similarly desirable, resulting in larger and less selective choice sets and therefore coarser rankings.

An implication of this approach is that a cognitively constrained consumer may for example overlook minor qualitative differences, presenting both difficulties and opportunities for advertisers, service providers, and product manufacturers. Furthermore, strategically presenting certain choices in situations or contexts where consumers are more prone to reduced cognition can have advantages, such as enabling products that would otherwise be eliminated from consideration to be selected, or disadvantages, such as when a strong preference for a particular product goes unnoticed.

With that in mind, this paper formalizes the process by which preferences and choice are refined through cognition, while retaining some of the coherence associated with rational choice. The key theoretical result is the equivalence between two axioms and an interval-valued utility functional representation.

The first axiom, denoted the Weak Ordering of Cognition Indices (WOCI), states criteria for choice behavior under which an increase in cognition implies greater selectiveness. Specifically, if, within a given context, the decision-maker's choice results in a smaller and more restrictive set of chosen options compared to another context, it is asserted that her level of cognition in the former case must be higher than in the latter. For instance, suppose the decision-maker selects a particular brand of gluten-free, vitamin enriched bread at 9 A.M. However, by 10 P.M., she becomes less discerning and is willing to pick a broader range of different bread brands. In this scenario, we can assert that her decision-making at 9 A.M. involves higher cognition compared to her 10 P.M. decision-making.

While obtaining direct evidence of WOCI is challenging due to the inherent difficulty in eliciting choice correspondences (Bouacida, 2021, Balakrishnan et al., 2022), illustrative examples from experimental data can be found in Section 2.2. In an experiment conducted in Bouacida (2021), participants were tasked with selecting subsets of tasks under various information treatments, corresponding to different levels of cognitive engagement. The observed choice correspondences at the lower information treatment (indicative of low cognition) were consistently larger, on average, than those at higher information treatments (indicative of high cognition). According to Ross et al. (2020), decision-makers undergoing a budget contraction, even if temporary, engage in more trade-off thinking – i.e. apply more cognition, in the language of the current paper. The experimental results from Ross et al. (2020) support this notion, revealing that bundles chosen after a budget contraction and subsequent restoration contain, on average, fewer items than bundles chosen before any budget contraction.

The second axiom is a generalization of the Weak Axiom of Revealed Preference, denoted Weak Axiom of Revealed Preference *Difficulty*, or WARPD. Unlike the classical Weak Axiom (WARP), WARPD allows for the emergence of intransitive indifference in specific circumnstances. This occurrence is a result of WARPD characterizing the behavior of decision makers who optimize only up to an approximate degree. To understand this, consider that if x is approximately optimal given the set $\{x, y\}$, and if y is similarly approximately optimal in $\{y, z\}$, then it may not be true that x is approximately optimal in $\{x, z\}$. This is because the approximation "errors" can compound to the point where x is no longer considered good enough when compared to z – even if x was to y, and y was to z.

On the other hand, WARPD maintains the independence of irrelevant alternatives (IIA) assumption inherited from classical WARP: if an alternative is approximately optimal in a large set, it should continue to be approximately optimal in a smaller set, provided it remains available. This encapsulates the notion that decision-makers' perception of an alternative as at least approximately as good as another should remain consistent, without being influenced by the introduction of a third option, for any given level of cognition.

Theorem 2.1 demonstrates that the combination of WOCI and WARPD is equivalent to a cognition-dependent interval utility representation. Recall that, given two choices, a rational decision-maker prefers one of them if and only if it gives her strictly more utility. A cognition-dependent decision-maker will not have a clear preference between them unless the utility difference between the two choices is larger than a certain threshold. Moreover, the higher the cognition, the smaller the threshold.

Specifically, given a finite set of alternatives S a rational decision-maker will choose from S a subset $C(S)$ where

$$C(S) = \{x \in S : \text{ for all } y \in S,\ u(y) \leq u(x)\}, \tag{1}$$

according to some real-valued function u. The **cognition-dependent representation** modifies the above as follows: for each level of cognition λ, the decision-maker's coarse, or approximate choices from S will instead be given by:

$$C_\lambda(S) = \{x \in S : \text{ for all } y \in S,\ u(y) \leq u(x) + \varepsilon(\lambda, x)\}, \tag{2}$$

where $\varepsilon(\lambda, x)$ is a nonnegative threshold function satisfying certain properties (see Definition 2.1). In particular, ε decreases in the cognition index λ, reaching 0 for all x at some $\overline{\lambda}$ where this cognition is maximal and classical rationality (i.e. 1) is restored.

Section 3 explores applications of the proposed representation, highlighting instances where consumers' limited cognition results in distinct forms of price insensitivity. The first application considers two firms engaged in Bertrand competition. It is then shown that consumers' inability to perceive minor proportional price differences between identical goods can result in an equilibrium outcome resembling those sustained through collusion. This outcome stands in stark contrast to the traditional equilibrium ourcome whereby firms earn zero profits and price at marginal cost. Indeed, evidence that dates all the way back to Asch and Seneca (1975, 1976) seem to show that low-profitability, homogeneous-goods industries seem to be more prone to collusive-like outcomes.

In the second application, price-insensitive consumers must decide whether or not to purchase a monopolist's product, depending on their degree of "brand loyalty." If the majority of consumers lack brand loyalty, then the lower the level of cognition, the more substantial the discounts that the monopolist must offer to attract customers. Conversely, when most consumers exhibit brand loyalty, customers' limited cognition benefits the monopolist. This is because, in such scenarios, the monopolist can raise prices without experiencing a significant reduction in demand.

Finally, this paper also contributes to the literature on fuzzy preferences and choice. It shows that both WOCI and WARPD can be reformulated by drawing on mathematical tools from Fuzzy Set Theory and Fuzzy Relational Theory.

A fuzzy relation is a mathematical object that assigns values ranging from 0 to 1 to ordered pairs of elements. Unlike traditional, crisp binary relations that provide only binary (0 or 1) indications of whether elements are related or not, a fuzzy relation allows for partial "degrees of relationship". Specifically in this paper, these values are taken to signify the level of difficulty the decision maker experiences in establishing a clear preference for the second element in an ordered pair over the first, on scale from 0 (no challenge) to 1 (infinitely challenging or impossible).

An alternative characterization of WOCI and WARPD, and hence of cognition-dependent representations, is provided in terms of fuzzy concepts: the individual's choices satisfy WARPD if and only if they can be represented by a complete and transitive *fuzzy* relation between alternatives. The core idea underlying this interpretation revolves around the notion that the decision maker's inherent preferences are crisp and rational, but her ability to perceive these preferences is limited. Consequently, the introduction of the fuzzy relation R is used to quantify her ability to discern relative rankings among alternatives.

To summarize, this paper provides two alternative, but equivalent representations of choice correspondences that are affected by cognition, as formalized by WARPD; first, per Theorem 2.1, via the cognition-dependent representation in Equation 2; second, in terms of a complete and transitive fuzzy relation, per Theorem 4.1.

The paper is organized as follows. Section 2 discusses the characterization of choice data augmented by cognitive covariates, focusing on the WOCI/WARPD axioms and their equivalence to the cognition-dependent representation. Section 3 explores the applications. Section 4 introduces key concepts of fuzziness, including fuzzy relations, fuzzy choice, and fuzzy rationalizability. It concludes with the equivalence result between WOCI/WARPD and fuzzy rationalizability. Section 5 contains extensions, including a stronger form of WARPD that imposes WARP that precludes intransitive indifferences. Section 6 reviews and discusses the relevant literature. Finally, Section 7 concludes.

2 Setup and characterization

2.1 Choices and cognition

Fix a finite set X of alternatives, and let $\mathcal{S}$ denote the set of nonempty subsets of X.

Choice data consists of a family of choice correspondences $C_\lambda : \mathcal{S} \longrightarrow \mathcal{S}$, indexed by $\lambda \in \Lambda$. Each λ represents a *cognition* index, interpreted as the amount of mental energy exerted by the decision maker in the process of choosing. Clearly, the analyst cannot observe cognition directly. Nevertheless, we deem it possible to at least identify certain covariates indicative of the decision-maker's mental state and ability to exclude suboptimal options from her chosen set. These covariates or descriptors could be obtained, for example, through a survey (inquiring directly or indirectly about the decision-maker's general mental state), performance measures (e.g. reaction times) or through environmental characteristics (e.g. level of noise, abundance of concurrent visual stimuli, pollution, time of day, etc).

Axiom 1 (stated below) imposes the necessary constraints that these indices must adhere to. Intuitively, we want to be able to rank cognition indices so that higher cognition means smaller, more selective choice correspondences and lower cognition implies larger, less selective ones.

To make this notion precise, given $S \in \mathcal{S}$ and $\lambda \in \Lambda$, let $C_\lambda(S)$ denote the decision-maker's choice correspondence from S at λ, i.e. the set all alternatives regarded as undominated out of S, when judged at λ.

Axiom 1 (Weak Ordering of Cognition Indices (WOCI))**.** For all $S, T \in \mathcal{S}$ and all $\lambda, \lambda' \in \Lambda$,

(i) **Nestedness.** $C_\lambda(S) \subseteq C_{\lambda'}(S)$ or $C_{\lambda'}(S) \subseteq C_\lambda(S)$.

(ii) **Consistency.** $C_\lambda(S) \subsetneq C_{\lambda'}(S)$ only if $C_\lambda(T) \subseteq C_{\lambda'}(T)$.

Define the relation $\geq$ on Λ as follows: $\lambda' \geq \lambda$ if, and only if, $C_{\lambda'}(S) \subseteq C_\lambda(S)$ for all $S \in \mathcal{S}$. Note that Nestedness and Consistency guarantee that $\geq$ is a weak order (See Appendix A.1). Moreover, these two properties imply that "higher" indices generate smaller choice sets, i.e. the decision maker is increasingly discriminating or selective. It is in this sense that a higher cognition index means that more mental effort was put into the choice.

To illustrate this concept, consider the decision maker's task of shopping for a pack of dry spaghetti pasta at the supermarket. Under a level of cognition λ, perhaps due to her being "in a hurry on a Monday morning", the decision maker perceives most options on the shelves as acceptable, with perhaps a few exceptions (such as gluten-free and plant-based options). Due to her busy schedule and external preoccupations, she is likely to quickly pick one arbitrarily out of a large set of seemingly undominated options, and move on. On the other hand, at a level of cognition λ', where she is "well-rested with no other commitments", the decision maker has time and examines the shelves more carefully, considering factors such as recipe choices, organic options, and artisanal brands. Her set of desirable options is now narrower than in λ. Since $C_{\lambda'}(S) \subsetneq C_{\lambda}(S)$ (S for spaghetti), we can conclude that $\lambda' > \lambda$, as the state of being well-rested and having no commitments is associated with higher cognition than being in a hurry on a Monday morning.

Note that, since each $S \subseteq X$ is finite, Λ can be partitioned into a finite number of strictly ordered equivalence classes. In particular, this implies that there exists a $\geq$-maximal state and a $\geq$-minimal state, denoted by $\overline{\lambda}$ and $\underline{\lambda}$, respectively. Importantly, for all $S \in \mathcal{S}$, $C_{\overline{\lambda}}(S) \subseteq C_{\lambda}(S) \subseteq C_{\underline{\lambda}}(S)$. That is, $\overline{\lambda}$ (and any equally ranked state) corresponds to the maximum amount of selectivity the decision maker can express through her choices, whereas $\underline{\lambda}$ represents the minimum.

Consequently, in what follows, the set Λ will be condensed into an ordered, finite set of indices – each representing an equivalence class. Denote the lowest index as $\underline{\lambda}$ and the highest as $\overline{\lambda}$. Additionally, to simplify notation, assign real numbers in the interval $[0,1]$ as labels to the indices in Λ, preserving the usual order of real numbers to reflect the order of indices implied by WOCI. In other words, let $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ with each $\lambda_i \in [0,1]$ and $\underline{\lambda} = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \overline{\lambda}$. Finally, also let $\underline{\lambda} = 0$ and $\overline{\lambda} = 1$ without loss of generality.

2.2 WARP and WARPD

The theory of rational choice assumes decision-makers behave according to well-defined, consistent preferences, expressed in the form of pairwise comparisons between options. Mathematically, these comparisons are encoded by a binary relation $\succsim$ defined on the set of alternatives X, with statements of the sort $x \succsim y$ and read as "x weakly preferred to y", or "x is not worse than y". The concept of rationality then is taken to mean that the relation $\succsim$ is complete (either $x \succsim y$ or $y \succsim x$) and transitive (if $x \succsim y$ and $y \succsim z$, then $x \succsim z$).

One of the fundamental results in economic theory establishes that a unique rational $\succsim$ can be revealed from observed choices if, and only if these choices satisfy the Weak Axiom of Revealed Preference (WARP), restated below for a (cognition-independent) choice correspondence $C : \mathcal{S} \longrightarrow \mathcal{S}$.

Axiom 2 (Weak Axiom of Revealed Preference (WARP))**.** For all $x, y \in X$ and all $S, T \in \mathcal{S}$ with $\{x, y\} \subseteq S \cap T$,

$$\text{If } x \in C(S) \text{ and } y \in C(T), \text{ then } x \in C(T). \tag{3}$$

In this standard rationality framework, agents are always maximally selective, in that there is nothing to be refined or discovered regarding the decision-makers' own preferences and choices. Thus, cognition does not matter, and $C \equiv C_\lambda$ for all λ. This is clearly no longer true if WOCI has any bite, that is, if $C_\lambda(S) \neq C_{\lambda'}(S)$ for some $\lambda, \lambda' \in \Lambda$ and $S \in \mathcal{S}$.

To capture the partial resolution of preferences discussed in the Introduction, WARP is retained only at $\overline{\lambda}$; each C_λ will then increasingly depart from $C_{\overline{\lambda}}$ as λ decreases. Thus, at $\overline{\lambda}$, the decision maker acts according to standard rational preferences – interpreted as her "true", unobscured tastes. At $\lambda < \overline{\lambda}$, however, she is only approximately rational, in the sense described by the following alternate axiom, imposed on the family of choice correspondences $\mathcal{C} := \{C_\lambda : \lambda \in \Lambda\}$.

Axiom 3 (Weak Axiom of Revealed Preference Difficulty (WARPD))**.** For all $\lambda \in \Lambda$, $x, y \in X$ and $S, T \subseteq \mathcal{S}$ with $\{x, y\} \subseteq S \cap T$,

$$x \in C_{\overline{\lambda}}(S), y \in C_\lambda(T) \implies x \in C_\lambda(T) \text{ and } y \in C_\lambda(S). \tag{4}$$

The discussion on revealed preference shall be postponed to section 4. Instead, below the analysis will focus on the interpretation and direct consequences of WARPD, and particularly how it differs from WARP.

Although WARP and WARPD share similarities, they only coincide when the cognition index is maximal (i.e. at $\overline{\lambda}$). WARP states that, if x is chosen from a set S that also contains y, and y is chosen from a set T that also contains x, then x must be chosen from T and y must be chosen from S. On the other hand, under WARPD, the conclusion only holds if x is chosen from S also at $\overline{\lambda}$. In other words, if x is the rational benchmark, or true preference-optimizing alternative in S, and $y \in S$ is perceived as undominated in a set T at a cognition index of λ, with $x \in T$, then two things must be true. First, y cannot be that much worse than x (otherwise it would not be undominated in T, since T contains x) and thus y is not that far off from the true optimum of S. That means y should also be undominated in S at λ. Second, because x is the true optimum of a set that contains y, then x is not worse than y in the rational benchmark. And, if at λ, y is "good enough" in T, then x must be at least that – i.e., at that cognition index of λ, x must also be undominated in T.

The interpretation of WARP is usually facilitated by its decomposition into two properties, commonly known in the literature as Sen's α and β. The following proposition is the classical equivalence between WARP and Sen's α and β properties, reproduced below for clarity of the exposition.

Proposition 2.1. *WARP holds if and only if $C : \mathcal{S} \longrightarrow \mathcal{S}$ satisfies*

(i) ***α (independence of irrelevant alternatives):*** *For all $S, T \in \mathcal{S}$ with*

$S \subseteq T$, and $x \in S$,

$$x \in C(T) \implies x \in C(S).$$

(ii) ***β:*** *For all $S, T \in \mathcal{S}$ with $S \subseteq T$, $x, y \in C(S)$,*

$$y \in C(T) \implies x \in C(T)$$

Sen paraphrases α as: if the world champion of a given competition is Pakistani, then she must also be the national champion of Pakistan.

The paraphrasing of β goes as follows: if the world champion happens to be Pakistani, then all national champions of Pakistan are also world champions.

A similar decomposition exercise can be performed with WARPD, as given in the following proposition.

Proposition 2.2. *WARPD holds if and only if $\mathcal{C}$ satisfies*

(i) ***α (independence of irrelevant alternatives):*** *For all $\lambda \in \Lambda$ $S, T \in \mathcal{S}$ with $S \subseteq T$, and $x \in S$,*

$$x \in C_\lambda(T) \implies x \in C_\lambda(S).$$

(ii) ***λβ:*** *For all $\lambda \in \Lambda$, $S, T \in \mathcal{S}$ with $S \subseteq T$, $x, y \in C_\lambda(S)$,*

$$y \in C_\lambda(T) \text{ and } [x \in C_{\overline{\lambda}}(S) \text{ or } y \in C_{\overline{\lambda}}(T)] \implies x \in C_\lambda(T).$$

Note that α holds unchanged at every index λ, implying cognition-dependent choices preserve IIA and comprise of a particular generalization of β.

To use Sen's analogy, our α property can be expressed as follows: if a Pakistani achieves podium placement in the international edition of a competition, then she must also secure podium placement in the Pakistani edition.

Regarding $\lambda\beta$, this property can be understood as follows: if a Pakistani attains the title of world champion, then everyone who secured podium positions in Pakistan must also be among the podium finishers in the world games. Conversely, if a Pakistani achieves podium placement in the world games (without necessarily being the world champion), then the champion of Pakistan must be among those podium finishers.

To further fix ideas, the next two examples showcase situations where (i) WARPD (but not WARP) holds for $\lambda < \overline{\lambda}$, (ii) neither WARP nor WARPD holds.

Example 2.1 (WARPD holds, but not WARP)**.** Consider again the situation described in Example 1.1. Note the violation of Sen's β at cognition index λ_1:

$b, c \in C_{\lambda_1}(\{a,b,c\})$ but $c \notin C_{\lambda_1}(\{a,b,c\})$: c is dominated by a in $\{a,b,c\}$, while b is not.

WARPD still holds, however. Because $b \notin C_{\overline{\lambda}}(\{a,b,c\})$ and $c \notin C_{\overline{\lambda}}(\{b,c\})$, the $\lambda\beta$ property (unlike β) does not impose that c must be in $C_{\lambda_1}(\{a,b,c\})$. ◊

Example 2.2 (WARPD does not hold)**.** Suppose that you are deciding between three vacation packages: Rome (r), Rome (but $100 cheaper) ($c$, for cheap) and Paris ($p$).

At the maximal level of cognition $\overline{\lambda}$, you would choose:

$$\begin{aligned} C_{\overline{\lambda}}(\{p,r\}) &= \{p\} \\ C_{\overline{\lambda}}(\{r,c\}) &= \{c\} \\ C_{\overline{\lambda}}(\{p,c\}) &= \{p\} \\ C_{\overline{\lambda}}(\{p,r,c\}) &= \{p\}, \end{aligned}$$

i.e. you prefer the Paris vacation, followed by the cheap Roman vacation, and the expensive Roman vacation comes in last.

Now, suppose that you spend less cognition on this decision. It might be clear to you that c is better than r, but comparing either Roman vacation with the Parisian vacation is more complicated. Thus, your choices might follow:

$$\begin{aligned} C_{\overline{\lambda}}(\{p,r\}) &= \{p,r\}, \\ C_{\overline{\lambda}}(\{r,c\}) &= \{c\}, \\ C_{\overline{\lambda}}(\{p,c\}) &= \{p,c\}, \\ C_{\overline{\lambda}}(\{p,r,c\}) &= \{p,c\}. \end{aligned}$$

In this case, WARPD – and in particular, $\lambda\beta$ – is violated: $p, r \in C_{\lambda}(\{p,r\})$, $p \in C_{\overline{\lambda}}(\{p,r,c\})$, but $r \notin C_{\lambda}(\{p,r,c\})$.

WARPD fails in this case because it does not capture the fact that certain comparisons are easier due to how they are presented. In the case above, the two Rome vacations are identical, except that one costs $100 less. That makes a dominance relationship between these two very easy to perceive. This would not necessarily be true anymore if, for example, both Rome vacations came instead bundled with complicated, difficult to parse assets that nevertheless corresponded to the same r and c.

WARPD does not account for such presentation effects. Instead, is designed to capture situations where eliminating the rational-benchmark lowest-ranked options from the choice correspondence takes less cognition than eliminating alternatives that are ranked higher and thus closer in ranking to the true optimum. ◊

2.3 Cognition-dependent representation

We know that WARP holds if and only if there exists a unique and rational $\succsim$ such that, for all $S \in \mathcal{S}$, $C(S) = \{x \in S : x \succsim y \text{ for all } y \in S\}$. Furthermore, if X is finite, WARP is also equivalent to the existence of a function $u : X \longrightarrow \mathbb{R}$ such that, for all $S \in \mathcal{S}$,

$$C(S) = \{x \in S : \text{ for all } y \in S,\ u(y) - u(x) \leq 0\}. \tag{5}$$

The representation that will be equivalent to WARPD (in Theorem 2.1), on the other hand, must capture the "approximate rationality" of the cognitively-constrained decision-maker. This is indeed going to be a clear feature of the representation in Definition 2.1 below.

Definition 2.1 (Cognition-dependent representation)**.** Given a set of cognition indices $\Lambda \subseteq [0,1]$ with $\overline{\lambda} = 1 \in \Lambda$, the family $\mathcal{C} := \{C_\lambda : \lambda \in \Lambda\}$ admits a cognition-dependent representation if and only if, there exist functions $u : X \longrightarrow \mathbb{R}$ and $\varepsilon : \Lambda \times X \longrightarrow \mathbb{R}_+$ such that, for all $S \in \mathcal{S}$,

$$C_\lambda(S) = \{x \in S : \text{ for all } y \in S,\ u(y) - u(x) \leq \varepsilon(\lambda, x)\}, \tag{6}$$

with (i) for all $x \in X$, $\varepsilon(\overline{\lambda}, x) = 0$, (ii) for all $x \in X$, $\varepsilon(\cdot, x)$ is non-increasing, and (iii) for all $x, y \in X, \lambda \in \Lambda$, $u(x) \geq u(y) \implies u(x) + \varepsilon(\lambda, x) \geq u(y) + \varepsilon(\lambda, y)$. $\Diamond$

The representation in 2.1 allows an item x to be chosen from a set S as long as its rational benchmark utility $u(x)$ is within a threshold $\varepsilon(\lambda; x)$ of the true optimum in S. Furthermore, as the threshold decreases with increasing λ, and hence cognition, the decision-maker's choices become closer to the rational benchmark.

Example 2.3 revisits the scenario presented in the Introduction (Example 1.1) to illustrate the mechanics of the representation in Definition 2.1.

Example 2.3. Recall that Xavier's conclusions at the maximal cognition index are that $a \succ b \succ c \succ d \succ e$, which can be represented by any utility function u satisfying $u(a) > u(b) > u(c) > u(d) > u(e)$.

At λ_0 (the first scenario), where Xavier only has a few seconds to consider each movie, he determines that e is the least preferred option. Using the representation in 2.1, this means that $u(\square) - u(e) > \varepsilon(\lambda_0, e)$ for all $\square \in a, b, c, d$, and for any $\circ$ also in a, b, c, d, $u(\square) - u(\circ) \leq \varepsilon(\lambda_0, \circ)$.

At the higher cognition index $\lambda_1 > \lambda_0$, the thresholds shrink in size. While Xavier cannot have "unnoticed" or "forgotten" that e is his least preferred movie, he can now distinguish that a is superior to both c and d, resulting in $u(a) - u(c) > \varepsilon(\lambda_1, c)$ and $u(a) - u(d) > \varepsilon(\lambda_1, d)$. However, movie b is still perceived as similarly enjoyable to movie a at index λ_1: $u(a) - u(b) \leq \varepsilon(\lambda_1, b)$. Furthermore, movie b is

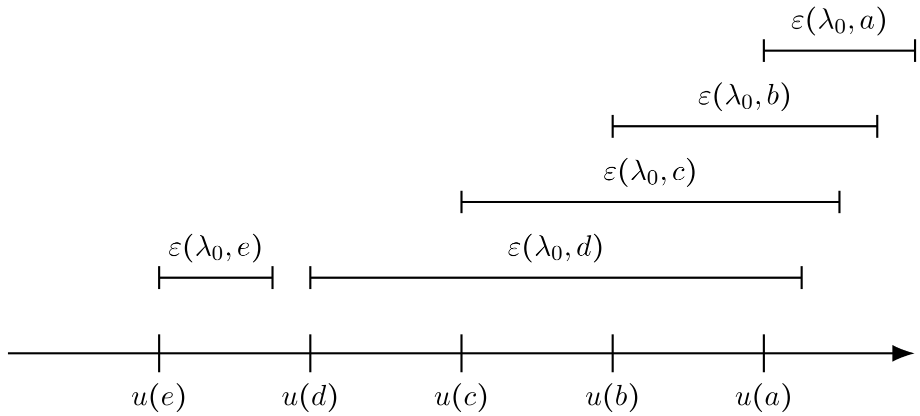

Figure 1: Each alternative's rational benchmark utility is represented in the bottom axis, where we see that $u(a) > u(b) > u(c) > u(d) > u(e)$. The thresholds ε for each of the five alternatives appear above the axis, represented as segments with length corresponding to the threshold magnitude at λ_0. Note that $u(\square) - u(e) > \varepsilon(\lambda_0, e)$ for all $\square \in a, b, c, d$, and for any $\circ$ also in a, b, c, d, $u(\square) - u(\circ) \leq \varepsilon(\lambda_0, \circ)$.

also perceived as similarly enjoyable to both c and d: $u(y) - u(c) \leq \varepsilon(\lambda_1, c)$ and $u(b) - u(d) \leq \varepsilon(\lambda_1, d)$. Therefore, perceived enjoyment is not necessarily transitive at acognition index $\lambda < \overline{\lambda}$, as full rationality is not guaranteed to hold.

Finally, at $\overline{\lambda}$, rationality is restored and the full ranking is revealed as all thresholds collapse to zero. ◊

2.4 Equivalence

The main theorem that relates cognition-dependent choice behavior and utility can now be stated:

Theorem 2.1. *The family $\mathcal{C}$ satisfies WARPD and WOCI if and only if it admits a cognition-dependent representation.*

The full proof of the Theorem above can be found in appendix A.1.

2.5 Discussion: WOCI, WARPD and evidence

Direct evidence for WOCI and WARPD is difficult in large part due to the well-known challenges in observing non-singleton choice correspondences (Bouacida, 2021, Balakrishnan et al., 2022). Nevertheless, there is an indication that the contraction of choice correspondences via cognition occurs in certain circumstances, as evidenced by the experimental results reviewed in this section.

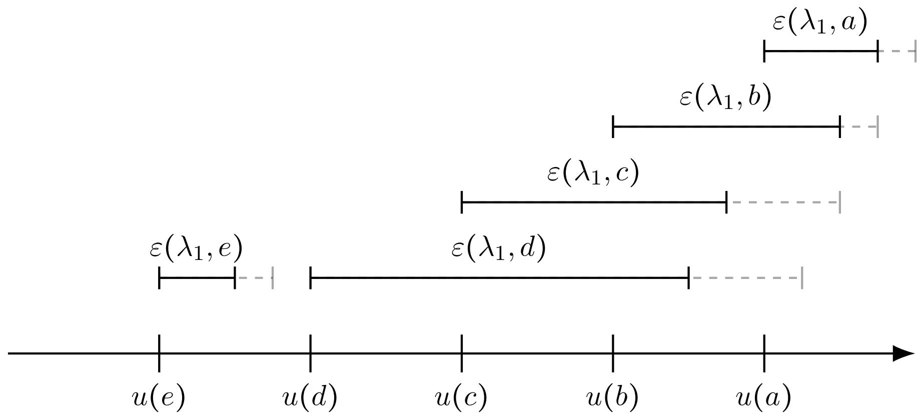

Figure 2: As we move to λ_1, all thresholds shrink in size Note that, now, $u(a) - u(c) > \varepsilon(\lambda_1, c)$ and $u(a) - u(d) > \varepsilon(\lambda_1, d)$ – indicating that Xavier now perceives a strict preference of a over both c and d, in addition to e.

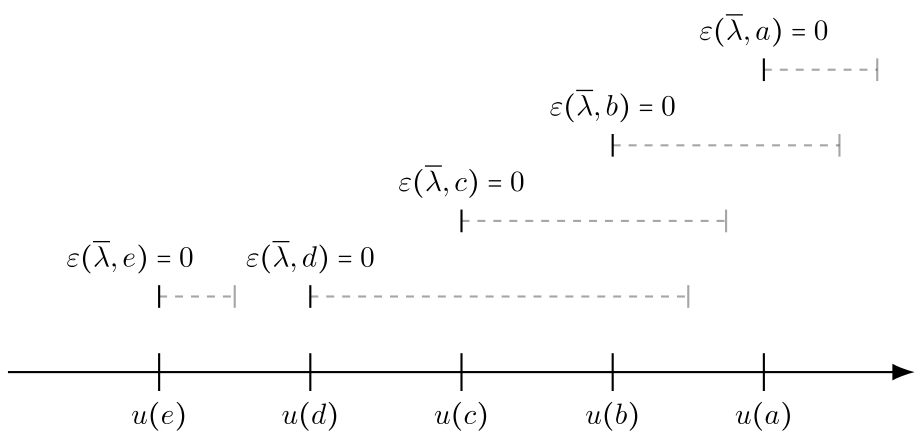

Figure 3: At $\overline{\lambda}$, rationality is restored, as the thresholds collapse to zero.

2.5.1 Information treatments

Bouacida (2021) is concerned with the experimental elicitation of choice correspondences. In his experiment[1], participants were presented with four different paid tasks and had to choose between them. These were an addition task (labeled “1”), a spell-check task (“2”), a memory task (“3”) and a copy task (“4”). Subjects made choices for all subsets of tasks. At the end of the session, participants had then three minutes to earn as much as possible by performing one randomly chosen task from their selected set in one of the choice tasks they had performed previously.

The experiment included three treatments to investigate the influence of the information provided on the size of chosen sets. In the sentence treatment, participants were given a vague description of each one the tasks. In the video treatment, participants first received the sentence treatment, followed by a video that explained each task in more detail, demonstrating the interface and specific instructions. Finally, in the training treatment, participants went through the video treatment, followed by a 1-minute training session on each task before making their choices.

Note that the quantity of information increased across treatments: the sentence treatment provided the least information, followed by the video treatment, and the training treatment offered the most detailed information. The ranking of these information treatments mirrors this paper’s notion of ranked cognition levels from WOCI. Our “identifying assumption” is to interpret “higher cognition” as being “better informed” about one’s true preferences. This interpretation is valid in situations where the information provided includes only details useful for breaking ties between similarly desirable options.[2]

Notably, the experiment does not feature the same individual making decisions under distinct information treatments. Because of this we cannot verify WOCI[3] directly at the level of the individual. In the aggregate, though, we may observe whether a higher information treatment mean, on average, smaller choice correspondences. There is indeed evidence in support of this; in particular, across all choice tasks, the proportion of singleton choice correspondences was 33.47% for the sentence treatment, 45.34% for the video treatment and 46.56% for the training treatment. Similarly, the proportion of completely unselective choice correspondences (i.e. such that all available tasks were selected) was 46.83% for the sentence treatment, 34.99% in the video treatment and 31.57% in the training treatment.

[1]The experiment mentioned in this section was produced in the Laboratoire d’Economie Experimentale de Paris, with the support of two ANR projects: CHOp (ANR-17-CE26-0003) and DynaMITE (ANR-13-BSH1-0010) and Labex OSE (10-LABX-0093). Subjects were recruited using Orsee (Greiner, 2004, 2015); zTree was used for the experimental part (Fischbacher, 2007).

[2]See Appendix A.3 for further discussion on a cognition versus informational approach.

[3]Or WARPD, introduced later, for that matter.

A similar result holds if we consider the number of revealed indifference relations in the data. The data indicates an average of 2.26 indifferences for the sentence treatment, 1.38 for the video treatment and 1.49 for the training treatment.

Bouacida (2021) performs a similar analysis and compares the number of indifferences per treatment type (See Figure 6 in Bouacida (2021)). He also notes that more information overall leads to less indifference, though he observes that the prevalence of indifferences seems to be quite heterogeneous between subjects.

Supplementary tables containing data on the proportion of participants choosing, for each $S \subseteq \{1,2,3,4\}$, all possible $T \subsetneq S$ and its proper subsets are available in appendix A.2.

2.5.2 Transient budget contraction

Ross et al. (2020) reason that, in the face of a financial setback, consumers undergo a prioritization process, in which they think more deeply and carefully about what is it that they value most and least. Consequently, individuals who have dealt with a budget contraction should exhibit refined preferences compared to those who have not experienced such a setback. From an empirical standpoint, one would expect that consumers who have encountered a budget contraction and subsequently had their budget restored to display fewer unique items in their post-contraction choice sets, compared to what they initially had in their pre-contraction choice sets.

In line with this explanation and using the language of current paper, this would mean that a budget contraction, even if transitory, causes consumers to employ higher levels of *cognition* to their choices by bringing the trade-offs they face to the forefront.

To support this claim, Ross et al. (2020) conducted a series of experiments in which participants were tasked with allocating a finite resource (e.g., time, space, or money) across various options before, during, and after an exogenously imposed temporary contraction in the available amount of the finite resource. For instance, in the initial experiment, participants were prompted to envision planning a vacation to Europe and then distribute 21 travel days among 12 cities. Subsequently, the travel time contracted to 7 days before returning to the original 21. The findings indicate that, overall, participants significantly allocated the 21 travel days to fewer cities when performing the distribution post-contraction (8.97 cities on average) compared to the pre-contraction allocation (9.39 cities). Additionally, for most participants, the cities that were cut during the contraction were the ones that most experienced a reduced share in the final allocation. This is suggestive of WOCI: if an alternative gets cut during the budget contraction, then that means the decision-maker realized that it was less valuable than other available options. This realization means that, once the budget is restored, the decision-maker should not choose as much of it as before.

Similar experiments were conducted where participants had to allocate vegetable varieties to planting rows, money to shore excursions while on a cruise and tokens to a variety of Easter candies (See Table 1 in Ross et al. (2020) for a summary of their experimental results). In all cases, post-contraction allocation sets were in general narrower, as WOCI would predict.

3 Examples and applications

This section analyzes two applications of the cognition-dependent representation, highlighting in particular instances where consumers' limited cognition results in distinct forms of price insensitivity.

3.1 Collusive outcomes in Bertrand duopolies

Asch and Seneca (1975, 1976) have noted that low-profitability firms producing undifferentiated goods seem to be more prone to collusive outcomes. The explanation offered by the authors is that unsatisfactory levels of profit may push firms into collusive agreements.

Here we explore a different mechanism through which producers of undifferentiated goods competing solely on prices in a low-profitability sector might induce collusive outcomes. Importantly, what drives the result is consumers' low cognition applied to choice tasks between similarly priced items. Thus, in this case, collusive outcomes may emerge without producers having to engage in a collusion agreement.

Consider two firms $i = 1, 2$, each producing identical units of a good x, which are then sold to a unit mass of consumers. The marginal cost of production is constant and equal to $c > 0$.

Each consumer has a maximum budget of $w > c$ to spend on the good, and will purchase a single unit from only one of the two firms. Let x_i denote the product of firm i. Since the product offerings are undifferentiated, a consumer employing maximal cognition will succeed in identifying and choosing the lowest-priced option:[4]

$$C_{\overline{\lambda}}(\{x_1, x_2\}) = \{x_i, i \in \{1, 2\} : p_i - p_j \leq 0 \text{ for all } j \in \{1, 2\}\}.$$

The setting described above yields a unique equilibrium where both firms set $p_1 = p_2 = c$. Any price set above marginal costs invites each participating firm to marginally undercut the other in hopes of capturing the entire market.

[4] If $p_1 = p_2$, assume the tie is broken using a coin flip.

However, this assumes consumers react to any $\varepsilon > 0$ price differential, no matter how small. This type of behavior is known to be generally false – a rule-of-thumb described in marketing, for example, states that consumers only seem to react to price differentials of around 20% or more (e.g. Henderson Britt (1975), Kamen and Toman (1970)).

Incorporating this observation into the example, consider then that x_i is chosen if and only if $(p_i - p_j) \leq (1-\lambda)p_j$, where $(1-\lambda)$ denotes the proportional price differential necessary to trigger a decisive reaction (e.g. 0.2):

$$C_\lambda(\{x_1, x_2\}) = \{x_i, i \in \{1,2\} : p_i - p_j \leq (1-\lambda)p_j \text{ for all } j \in \{1,2\}\}. \tag{7}$$

Note that $C_\lambda(\{x_1,x_2\}) = \{x_1,x_2\}$ whenever $p_2/(2-\lambda) \leq p_1 \leq (2+\lambda)p_2$. For these values, both options are acceptable choices. In this case, assume consumers pick one by flipping a (fair) coin.[5]

If $\lambda < 1$, $p_1 = p_2 = c$ is no longer an equilibrium: if $p_i = c$, then firm j would prefer to increase their prices slightly, as this would go unnoticed by the consumers.

In fact, if potential margins are sufficiently low, i.e. $w > c \geq w\lambda/(2-\lambda)$, the unique equilibrium is now at $p_1 = p_2 = w$, as the proposition below shows.[6]

Proposition 3.1. *Assume consumers select an alternative by uniformly picking an element of $C_\lambda(\{x_1,x_2\})$. If $(w-c) \leq (2/\lambda)(1-\lambda)c$, then the unique equilibrium is at $p_1 = p_2 = w$.*

The proof can be found in appendix A.4. Intuitively, if margins are sufficiently low, it is not worth it for the competing firms to undercut each other, since marginal price cuts are ineffective in acquiring market share, and larger price cuts would be too costly.

Note that the lower the λ, the higher the potential margins $(w-c)$ are allowed to be while still sustaining the collusive outcome. This too is intuitive. If consumers are less perceptive of price differentials, then undercutting becomes costlier – because an even higher cut is necessary.

3.2 Monopolist pricing and brand loyalty

The example in this section involves adding an inertia behavioral assumption *à la* Bewley (Bewley, 2002). Suppose a monopolist introduces a new product to a

[5] An implicit assumption here is that, if consumers are behaviorally indifferent between x_1 and x_2, then they are also behaviorally indifferent between x_1, x_2 and any lottery that randomizes between x_1 and x_2.

[6] Note that the tie-breaking rule has bite, because whatever profit-sharing occurs when the consumer is indifferent now matters for a measurable range of values. Specifically, if $\alpha < (1-\alpha)$ is the proportion of customers that purchase x_i whenever there is a tie, then $p_1 = p_2 = w$ is still the unique equilibrium if $w \geq c \geq w[(1-\alpha)/\alpha - (1-\lambda)]/(2-\lambda)$, provided $\lambda \leq 1 - (1-2\alpha)/(2\alpha)$.

unit mass of consumers. If these consumers have no previous experience with the product, then they are likely to be hesitant to make a purchase, unless the benefits from doing so clearly outweigh the costs. On the other hand, customers who are accustomed to purchasing a particular product might continue to do so, unless there is a noticeable decrease in quality or an increase in price that makes this status quo too disadvantageous.

Note that inertia plays a role precisely because cognitively-constrained consumers may be behaviorally indifferent.[7] What follows show that this simple behavioral postulate interacts with cognition-dependent choice, and leads to non-monotonicity in the prices.

Formally, consider a monopolist that produces a certain good at zero marginal cost. There is a unit mass of potential customers, indexed by a parameter $v \in [0,1]$. Each consumer has two options available to choose from: they can either `buy` or `not buy`. At a maximal level of cognition, buying the monopolist's product yields utility $u(v, \texttt{buy}) = v - p$, where p is the price selected by the monopolist. By contrast, not buying yields $u(v, \texttt{not buy}) = 0$.

Fix $\Lambda \subseteq [0,1]$, with $1 \in \Lambda$, and let the usual ordering of real numbers also represent the order of cognition indices. Following the cognition-dependent representation of 2.1, let $\varepsilon(\overline{\lambda}, \texttt{buy}) = \varepsilon(\overline{\lambda}, \texttt{not buy}) = 0$ and, for lower cognition, let $\varepsilon(\lambda; \texttt{buy}) = (1-\lambda)^2$ and $\varepsilon(\lambda; \texttt{not buy}) = (1-\lambda)$. Note that the inertia associated with not buying is stronger than the one for buying, since $\varepsilon(\lambda; \texttt{not buy}) > \varepsilon(\lambda; \texttt{buy})$ for all $\lambda \in (0,1)$. We then have that a consumer with value v facing a price of p will have the following choice correspondence:[8]

$$C_\lambda(v,p) = \begin{cases} \{\texttt{buy}, \texttt{not buy}\} & \text{if } -(1-\lambda)^2 \le (v-p) \le (1-\lambda) \\ \{\texttt{buy}\} & \text{if } (v-p) > (1-\lambda) \\ \{\texttt{not buy}\} & \text{if } (v-p) < -(1-\lambda)^2. \end{cases}$$

Whenever $p - (1-\lambda)^2 \le v \le p + (1-\lambda)$, a consumer's choice correspondence contains both alternatives. To break this tie, consider a threshold-type rule: given an exogenous threshold type $\overline{v} \in [0,1]$, a consumer with $-(1-\lambda)^2 \le (v-p) \le (1-\lambda)$ will pick "`buy`" if $v \ge \overline{v}$ and "`not buy`" if $v < \overline{v}$. The idea is to interpret $\overline{v}$ as the lowest type that purchased the product in a past decision. If the product is newly introduced (so that no one has any experience with it), we can think of $\overline{v}$ as 1.

[7]This contrasts with Dean et al. (2017), for example: in their paper, the underlying preferences change to favor the status quo.

[8]In order to maintain property (iii) of the cognition-dependent representation, the thresholds defined would have to depend on whether $v - p \ge 0$ or $v - p < 0$. To fully comply with the representation, it suffices to tweak the threshold definitions as follows: $\varepsilon(\lambda; \texttt{buy}) = \mathbf{1}_{\{v-p\ge 0\}}(1-\lambda)^2$ and $\varepsilon(\lambda; \texttt{not buy}) = \mathbf{1}_{\{v-p\le 0\}}(1-\lambda)$. This change does not affect the choice correspondences, which is why it was omitted from the main text.

Monopolist's problem. The monopolist chooses a price to maximize profits. Let $p^*(\lambda,\bar{v})$ denote the optimal price; it then must solve:

$$p^*(\lambda,\bar{v}) = \text{argmax}_p\, \pi(\lambda,\bar{v};p),$$

where $\pi(\lambda,\bar{v};p)$ is the profit function given by

$$\pi(\lambda,\bar{v};p) = \begin{cases} (\lambda-p)p & \text{if } \overline{v} \geq p+(1-\lambda) \\ (1-\overline{v})p & \text{if } p+(1-\lambda)^2 \leq \overline{v} \leq p-(1-\lambda) \\ (1-p+(1-\lambda)^2)p & \text{if } \overline{v} \leq p-(1-\lambda)^2. \end{cases} \tag{8}$$

It is easy to see that $p^*(1,\overline{v}) = 1/2$ and $\pi(1,\overline{v};1/2) = (1/2)^2 = 1/4$ regardless of $\bar{v}$. That is, if cognition is maximal, consumers act as pure utility maximizers; as a result, the product is sold to the one-half of consumers that value the good above its price of 1/2. These conclusions however change once we allow λ to take values in $(0,1)$.

Proposition 3.2. *For each $\lambda \in (0,1)$, there exist thresholds $\overline{v}_\lambda$ and $\overline{v}^\lambda \in (0,1)$ with $\overline{v}_\lambda < \overline{v}^\lambda$, such that the monopolist's optimal prices $p^*(\lambda,\overline{v})$ satisfy:*

$$p^*(\lambda,\bar{v}) = \begin{cases} \frac{1+(1-\lambda)^2}{2} & \text{if } \overline{v} \in [0,\overline{v}_\lambda], \\ \overline{v}+(1-\lambda)^2 & \text{if } \overline{v} \in (\overline{v}_\lambda,\overline{v}^\lambda), \\ \frac{\lambda}{2} & \text{if } \overline{v} \in [\overline{v}^\lambda,1]. \end{cases} \tag{9}$$

Moreover, the optimal profit $\pi^*(\lambda,\bar{v}) := \pi(\lambda,\bar{v};p^*(\lambda,\bar{v}))$ is continuous in $\bar{v}$ and satisfies:

$$\pi^*(\lambda,\overline{v}) = \begin{cases} \left(\frac{1+(1-\lambda)^2}{2}\right)^2 & \text{if } \overline{v} \in [0,\overline{v}_\lambda], \\ \left(\overline{v}+(1-\lambda)^2\right)(1-\overline{v}) & \text{if } \overline{v} \in (\overline{v}_\lambda,\overline{v}^\lambda), \\ \left(\frac{\lambda}{2}\right)^2 & \text{if } \overline{v} \in [\overline{v}^\lambda,1]. \end{cases} \tag{10}$$

As direct consequences of the proposition above, we have:

Corollary 3.2.1. *The following properties hold:*

- *For $\overline{v} \leq \overline{v}_\lambda$, $p^*(\lambda,\overline{v})$ is constant and above $p^*(1,\overline{v}) = 1/2$; $\pi^*(\lambda,\overline{v})$ is constant and above $\pi^*(1,\overline{v}) = 1/4$.*
- *For $\overline{v} \geq \overline{v}_\lambda$, $p^*(\lambda,\overline{v})$ is constant and below $p^*(1,\overline{v}) = 1/2$; $\pi^*(\lambda,\overline{v})$ is constant and below $\pi^*(1,\overline{v}) = 1/4$.*
- *For $\overline{v}_\lambda < \overline{v} < \overline{v}_\lambda$, $p^*(\lambda,\overline{v})$ is strictly increasing and linear in $\overline{v}$, whereas $\pi^*(\lambda,\overline{v})$ is strictly decreasing in $\overline{v}$.*

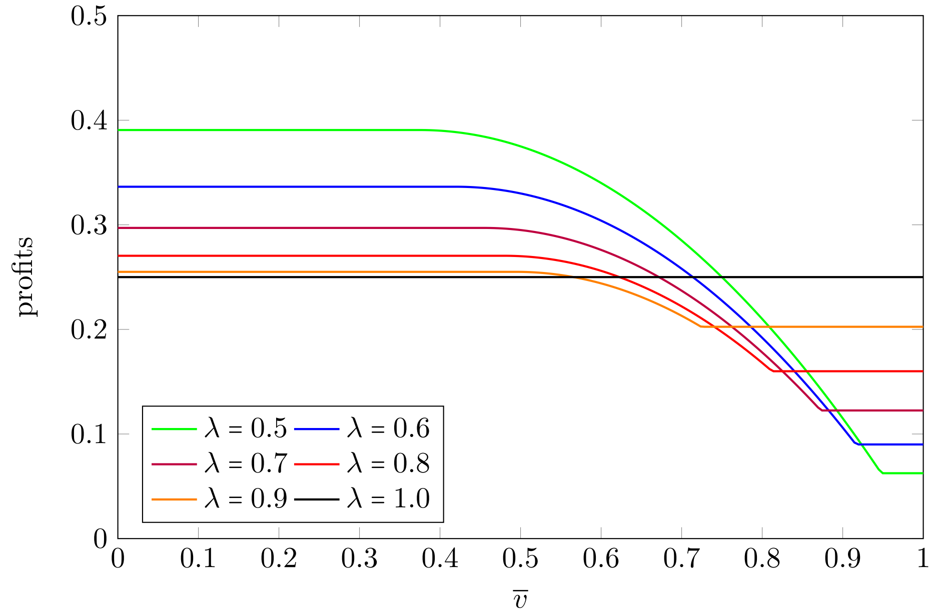

Figure 4: Profits for a monopolist given $\overline{v}$, for different levels of consumers' choice effort λ.

The graphs shown in Figures 4 and 5 display optimal prices and profits as functions of $\overline{v}$, for different levels of consumer cognition λ. If $\overline{v}$ is low, most consumers will be inclined to buy the monopolist's product, even when indifferent. In this case, the monopolist benefits from consumers' low cognition, as price hikes above 1/2 have but a limited negative impact on overall demand. Thus $p^\star(\lambda, \overline{v}) > p^\star(1, \overline{v}) = 1/2$ and $\pi^\star(\lambda, \overline{v}) > \pi^\star(1, \overline{v}) = 1/4$. Conversely, when $\overline{v}$ is high, consumers are less prone to buy the product. This means that the lower the cognition, the larger the necessary price cuts must be to convince consumers to buy: in this case, $p^\star(\lambda, \overline{v}) < p^\star(1, \overline{v}) = 1/2$ and $\pi^\star(\lambda, \overline{v}) < \pi^\star(1, \overline{v}) = 1/4$.

This simple framework provides an explanation as to why a monopolist might choose to offer coupons, discounts, free shipping or other similar marketing techniques, especially when just introducing a product that customers might be unfamiliar with. The model also suggests that, in those cases, the monopolist would prefer consumers' cognition to be high, which could also suggest the employment of informative advertising targeting potential consumers in contexts where they are likely to be more alert.

3.2.1 Multiple periods

The model introduced above can the applied to a dynamic setting where prices set today may affect the inertial option for consumers in subsequent periods. To simplify the discussion, consider the following simplification of the consumers'

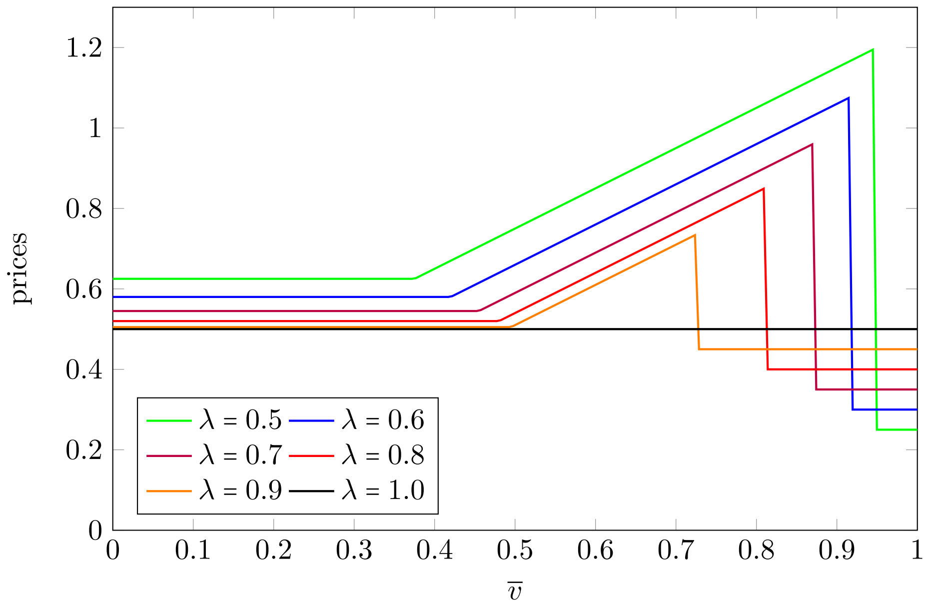

Figure 5: Prices for a monopolist given $\overline{v}$, for different levels of consumers' choice effort λ.

payoff structure from the static setting. As before, let $u(v, \texttt{buy}) = v - p$ and $\varepsilon(\lambda, \texttt{not buy}) = (1-\lambda))$; however, let $\varepsilon(\lambda, \texttt{buy}) = 0$. That is, assume there is no positive buying inertia: consumers stop buying as soon as the surplus turns negative.[9]

Time is discrete and indexed by $t = 1, 2, \ldots, T$. Suppose there are a total of T periods. Consumers will purchase at most one unit of the good per period. The subscript of t added to the variables and parameters indicates the period they are perrtinent to. Specifically, the threshold types should be period-specific and endogenously determined as follows. Let $\bar{v}_1 = 1$ and $\bar{v}_t$ be defined recursively by

$$\bar{v}_t = \begin{cases} p_{t-1} + (1-\lambda) & \text{if } \overline{v}_{t-1} > p_{t-1} + (1-\lambda) \\ \overline{v}_{t-1} & \text{if } p_{t-1} \leq \overline{v}_{t-1} \leq p_{t-1} + (1-\lambda) \\ p_{t-1} & \text{if } \overline{v}_{t-1} < p_{t-1}. \end{cases} \tag{11}$$

i.e. $\bar{v}_t$ is the lowest type that purchased the product at $t-1$.

Similar to before, period t's profits are given by:

$$\pi_t(\lambda, \bar{v}_t, p_t) = \begin{cases} (\lambda - p_t)p_t & \text{if } \overline{v}_t \geq p_t + (1-\lambda) \\ (1-\overline{v})p_t & \text{if } p_t \leq \overline{v}_t \leq p_t + (1-\lambda) \\ (1-p_t)p_t & \text{if } \overline{v}_t \leq p_t. \end{cases} \tag{12}$$

[9]To comply with (iii), in reality let $\varepsilon(\lambda, \texttt{not buy}) = \mathbf{1}_{\{v-p\leq 0\}}(1-\lambda))$.

We use $p_t^*(\lambda)$, $\bar{v}_t^*$ to denote, respectively, the optimal prices and thresholds for the monopolist. We also define the cumulative profits

$$\Pi^*(\lambda) := \sum_{t=1}^{T} \pi_t(\lambda, \bar{v}_t^*, p_t^*(\lambda)).$$

Proposition 3.3. *When $T = 1$, the optimal price is given by $p_1^*(\lambda) = \lambda/2$, and profits are $\Pi^*(\lambda) = \lambda^2/4$.*

When $T \geq 2$, $p_1^*(\lambda) = \lambda - 1/2$ and $p_t^*(\lambda) = 1/2$ for all $t \in \{2, \ldots, T\}$. Total profits are given by $\Pi^*(\lambda) = \lambda(\lambda - 1/2) + (T-1)1/4$.

If $T = 1$, then the solution is identical to the solution of the static setting.[10] When there are subsequent periods, however, $p_1^*(\lambda) = \lambda - 1/2 < \lambda/2$. In both instances, prices fall below the rational benchmark as the monopolist tries to overcome consumer inertia with substantial discounts. When brand loyalty develops over time, however, initial prices decrease even further, as doing so not only attracts current-period customers but also enhances consumers' propensity towards future product purchases.

Note that $p_1^*(\lambda) = \lambda - 1/2$ can be negative if λ is sufficiently low, as the monopolist is willing to trade-off short-term profitability to encourage brand adoption.

Because the thresholds for `buy` are always zero, a price of 1/2 and a per period profit of 1/4 is the best that the monopolist can achieve. By setting $p_1^*(\lambda)$ so low, it manages to maximize per-period profits from period 2 onward.

4 Revealed fuzzy preferences

In this paper, the choice correspondence depends on the cognition of the decision maker at the moment of choice. Consequently, our objective is not only to uncover the decision maker's fundamental preference hierarchy, but also to quantify the ordinal level of cognitive challenge they experience when distinguishing between any two options. These dual aspects of preferences can be effectively represented and analyzed by means of fuzzy sets and fuzzy relations theory.

The theory of fuzzy sets is a field of mathematics pioneered by Zadeh (1965), and with more recent applications to control systems (Nguyen et al., 2019), image processing and pattern recognition (Chi et al., 1996), expert systems (Yager, 1992, Sikchi et al., 2013), among others.

[10] The modification in the thresholds here does not affect the optimal price for $\bar{v}$ high, because only the top part of the indifference region matters for those values.

As this section illustrates, the model introduced in this paper lies within a broader framework where the concept of rational decision making is extended to fuzzy environments.

4.1 Fuzzy set theory and fuzzy relations

Unlike classical set theory, where an element either fully belongs to a set or does not belong at all, fuzzy set theory introduces the notion of partial degrees of membership. This approach in practice allows for the representation of certain types of imprecise or vague information in a structured manner.

The degrees of membership are typically represented by elements in the interval $[0, 1]$. Then, 0 represents complete non-membership, 1 signifies full membership, and values in between express varying degrees of partial membership. As an illustration, consider the set "Tall" for individuals' heights. In classical set theory, one might establish a threshold height, say 5'10", to separate those who are "tall" from those who are not. In contrast, fuzzy set theory suggests that the concept of "tallness" is not inherently binary. It allows for the assignment of partial membership degrees to individuals' heights, indicating their vague level of tallness. For instance, an individual with a height of 6'0" might have a membership degree of 0.7 in the set "Tall," implying a somewhat tall stature, while a person standing at 5'6" might receive a membership degree of 0.3, signifying a less pronounced but nonzero degree of "tallness".

The example above suggests how the membership grades of fuzzy sets can be seen as assessments of the extent to which an element encapsulates the attributes delineated by a set characterized by vagueness or imprecision. In the illustration above, this vagueness arises as a consequence of our linguistic imprecision: the descriptor "tall", in particular, lacks a well-defined mathematical formulation.

The degrees of membership characteristic of fuzzy sets can also have other interpretations. For instance, the degree of membership of an element in a fuzzy set can signify the extent of similarity between the element and the prototype or ideal representation of the set. Other alternative interpretations involve degrees of plausibility, belief and/or truth assessments (Zadeh, 1971, Liu and Kerre, 1998, Shimoda, 2002).

The interpretation most useful to us is however that of **granularity**. It relates to how finely or coarsely the set categorizes elements based on their attributes or characteristics. To illustrate this concept, Figure 6 depicts an identical image of a circle at four distinct resolutions, ranging from the lowest quality (panel 6a) to the highest (panel 6d). It becomes readily apparent that point A lies outside the boundaries of the circle, and point B falls within them. These distinctions remain even when examining the circle's lowest-resolution representation, in panel

6a. In contrast, to establish conclusively that both points C and D exist outside the circle, we must scrutinize the higher-resolution depictions provided in panels 6d and 6c, respectively. Since the discernment of C's position outside the circle requires a more refined level of detail, it follows that C enjoys a higher degree of membership within the circle in comparison to D, although not quite reaching the same extent of membership as that of B.

4.1.1 Fuzzy binary relations

The concept of fuzziness naturally extends to binary relations.

A fuzzy relation is a mathematical construct within the framework of fuzzy set theory that serves as a means to represent and quantify the relationships between elements. Unlike classical binary relations (usually called "crisp"), fuzzy relations accommodate the notion of partial or graded relationships by assigning degrees of membership to pairs of elements, signifying the extent of their association. Typically, fuzzy relations are expressed in the form of matrices or functions, wherein each element or function value corresponds to the degree of association between elements in a given ordered pair.

Definition 4.1 (Fuzzy binary relation)**.** A fuzzy binary relation R is a mapping from $X \times X$ into $V := [0, 1]$. $\Diamond$

As with fuzzy sets, membership values within the range of 0 to 1 can be interpreted in several ways. For example, the degree of relation between two elements can signify the intensity of the relationship. Alternatively, it may represent the degree of similarity between the elements in the related pair or the level of certainty associated with the pair's relationship.

In this paper, the values outputted by the function R signify how challenging it is for the decision maker to perceive a definitive preference for the second input in relation to the first. That is, the value of $R(x, y)$ is an ordinal expression of the difficulty involved in the decision maker's perception of x as worse than y.

The central concept behind this interpretation revolves around the idea that the decision maker possesses underlying tastes, likes and dislikes that adhere to the classical definition of a crisp rational preference relation, denoted by a reflexive, complete and transitive $\succsim \in X \times X$.

However, the decision maker's ability to perceive these preferences may be flawed due to limitations in cognitive processes. As a result, the fuzzy relation R is introduced to quantify her capacity to accurately perceive the relative rankings among the alternatives presented to her.

The subsequent discussion refers to the decision maker's inherent rational preferences $\succsim$ as her *rational benchmark preferences*.

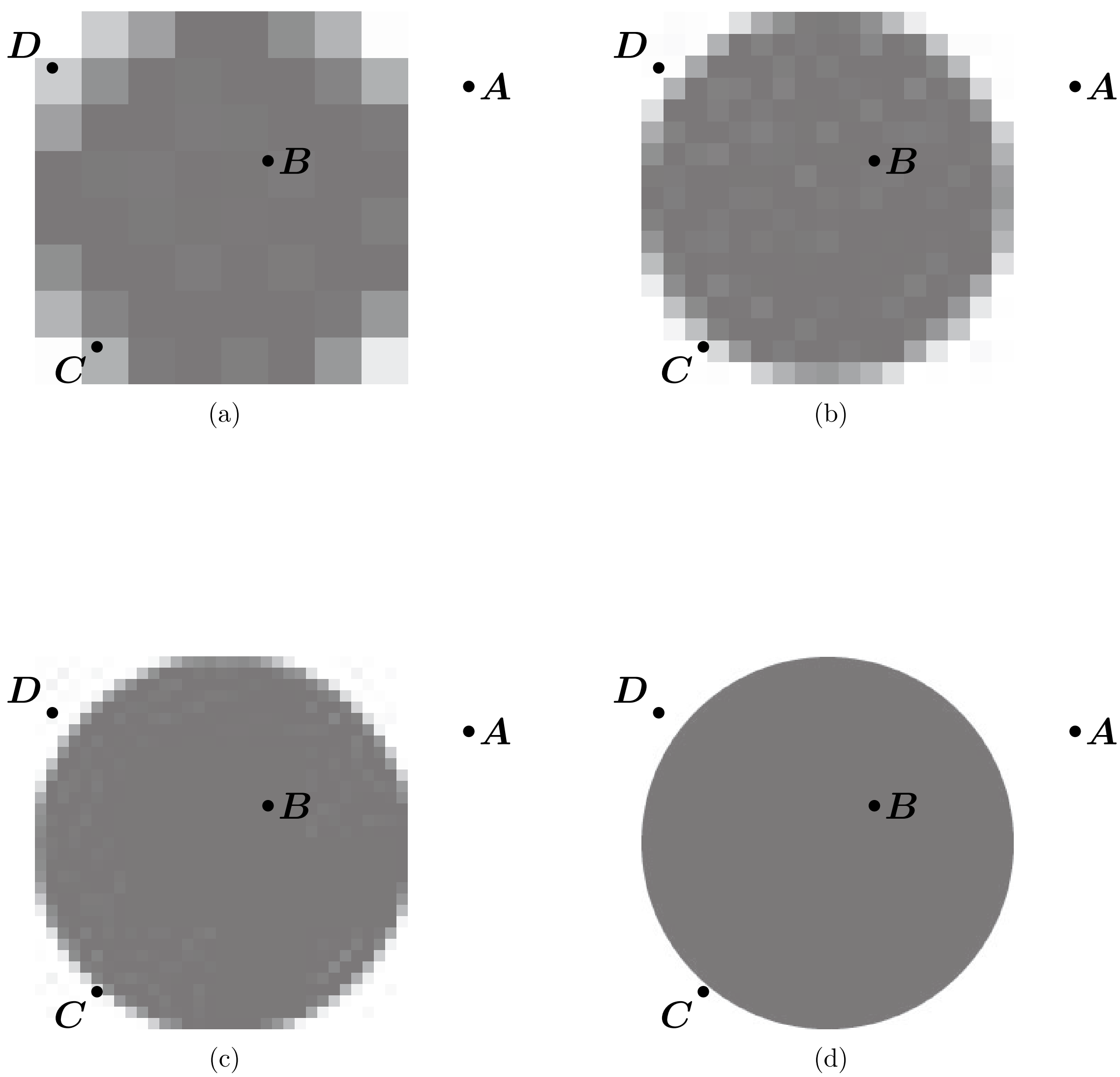

Figure 6: In all panels, B is definitely inside the circle, while A is definitely not. It becomes clear that D also does not belong in the circle in panel b. It is clear that C does not belong in the circle only in panel d.

A crucial assumption is that an erroneous strict rational-benchmark preference is never perceived. In other words, if x is genuinely worse than y according to $\succsim$, then the decision maker will never perceive y as worse than x.

These observations motivate the properties that will be subsequently imposed on R. Firstly, for all elements $x \in X$, $R(x,x) = 1$. This condition conveys that it is impossible, or infinitely difficult, for the decision maker to assert that an element is inferior to itself. Similarly, for any pair of elements x and y in X, either $R(x,y) = 1$ or $R(y,x) = 1$ (or conceivably both in cases of "true" rational-benchmark indifference). These conditions are natural fuzzy counterparts of the reflexivity and completeness properties of crisp binary relations, at least under the interpretation provided.

Definition 4.2 (Fuzzy Reflexivity)**.** A fuzzy relation R on $X \times X$ is **fuzzy reflexive** only if, for all $x \in X$, $R(x,x) = 1$. ◊

Definition 4.3 (Fuzzy Completeness)**.** A fuzzy relation R on $X \times X$ is **fuzzy complete** only if, for all $x, y \in X$, $\max\{R(x,y), R(y,x)\} = 1$. ◊

The matter of transitivity is more nuanced. If the rational benchmark $\succsim$ adheres to transitivity, then it follows that for all elements x, y, and z in X, if $R(x,y) = 1$ and $R(y,z) = 1$, then it must be that $R(x,z) = 1$.

In addition to the above, assume the following extra assumption: preferences between elements that are ranked further apart are perceived with greater ease than preferences between elements that are more closely ranked. In other words, if $x \succsim y \succsim z$, then $R(z,x) \leq \min\{R(z,y), R(y,x)\}$.

Definition 4.4 (Fuzzy Transitivity)**.** A fuzzy relation R on $X \times X$ is **fuzzy transitive** only if, for all $x, y, z \in X$,

$$\max\{R(x,y), R(y,z)\} = 1 \implies R(x,z) \geq \min\{R(x,y), R(y,z)\}.$$

◊

Note the definition above states the fuzzy transitivity property in a slightly different form than the paragraph above explaining it does. The two definitions are equivalent (See Proposition A.3 in appendix A.6 for the proof) when fuzzy completeness holds; the one in Definition 4.4 is mathematically more convenient for deriving results.[11]

[11] It is also a common form of transitivity, called the drastic transitivity, found on the literature on fuzzy relations. The drastic transitivity happens to be the least restrictive form of transitivity based on using t-norms as conjunction operators. See Ovchinnikov (1981) for a reference in these concepts.

The considerations above serve as a clear motivation for the following definition of fuzzy rationality. This concept closely parallels the classical (crisp) rationality concept for non-fuzzy preferences.

Definition 4.5 (Fuzzy rationality)**.** A fuzzy relation R on $X \times X$ is *fuzzy rational* only if it satisfies fuzzy reflexivity, fuzzy completeness and fuzzy transitivity. $\lozenge$

4.1.2 Fuzzy Choice

One of the fundamental results in economic theory establishes that a unique rational $\succsim$ can be revealed from observed choices if, and only if these choices satisfy the Weak Axiom of Revealed Preference (WARP). That is, WARP holds if and only if there exists a unique complete and transitive crisp relation $\succsim$ such that, for all $S \in \mathcal{S}$, $C(S) = \{x \in S : x \succsim y \text{ for all } y \in S\}$. To put it differently, WARP holds if and only if there exists a binary relation $\succsim$ that can rationalize the choice correspondence C.

The goal is to show that similar equivalence between the Weak Axiom of Revealed Preference Difficulty (WARPD) and fuzzy binary relations can be established. More precisely, WARPD holds if and only if there exists a unique *fuzzy* complete and transitive relation R capable of rationalizing the family of cognition-dependent choices.

Before we get into fuzzy rationalizability and the equivalence result, a few definitions are in order. Given a fuzzy relation R, it is possible to establish a corresponding fuzzy choice correspondence, denoted as C^R, induced by R. This fuzzy choice correspondence, for each subset S of X, outputs a fuzzy set $C^R(S)$. Every element x within the set S is assigned a membership value ranging from 0 to 1 (inclusive) in $C^R(S)$. These membership values encapsulate the extent to which each element in S satisfies certain criteria based on the fuzzy relation R, as defined below.

Definition 4.6 (Fuzzy Choice Correspondence)**.** Given a fuzzy relation R on X, the *fuzzy choice correspondence* induced by R is a function $C^R : \mathcal{S} \longrightarrow [0,1]^{\mathcal{S}}$ where

$$C^R(S,x) = \min_{y \in S}\{R(x,y)\}.$$

$\lozenge$

The membership value of x in the fuzzy set $C^R(S)$ is represented by $C^R(S,x)$. [12] The interpretation of these membership values aligns with the granularity interpretation detailed in the beginning of section 4.1. Specifically, if $C^R(S,x) > C^R(S,x')$, then we understand this as meaning that a lower level of discernment

[12] instead of $C^R(S)(x)$, to simplify the notation.

or detail is required to establish that x' is dominated by another alternative in S. Option x may either require a higher level of discernment or be fully undominated, regardless of the discernment or detail applied.

Definition 4.7 (α-cut)**.** Given a fuzzy relation R on X, its associated fuzzy choice correspondence C^R and a real number $\alpha \in [0,1]$, the α-cut of C^R is a (crisp) choice correspondence $C^{R;\alpha} : \mathcal{S} \longrightarrow \mathcal{S}$ where

$$C^{R;\alpha}(S) = \{x \in S : C^R(S,x) \geq \alpha\}. \tag{13}$$

◊

The definition above outlines a way to translate the fuzzy set $C^R(S)$ into a family of crisp sets. This transformation involves the selection of a threshold value $\alpha \in [0,1]$. In this process, only those elements within $C^R(S)$ whose membership degrees surpass α are included in $C^{R;\alpha}(S)$. Conversely, elements with membership degrees below α are excluded from $C^{R;\alpha}(S)$. If we interpret α as a metric of the desired level of granularity, detail, perceptiveness or discernment, measured on a scale ranging from 0 to 1, then the members of $C^{R;\alpha}(S)$ are precisely the elements x of S that remain undominated for that particular level.

4.1.3 Fuzzy rationalizability and equivalence

One might intuit that the measure of granularity represented by the values in $[0,1]$ will eventually be translated into the cognition levels from previous sections. This is indeed the case, as the following definition of fuzzy rationalizability establishes. The idea is to construct a function that maps Λ into $[0,1]$, so that each cognition index λ corresponds to a given level of granularity.

Definition 4.8 (Fuzzy rationalizability)**.** The family $\mathcal{C}$ is fuzzy rationalizable by the fuzzy rational relation R if there exists a function $\alpha : \Lambda \longrightarrow [0,1]$ such that, for all $\lambda \in \Lambda$,

$$C_\lambda(S) = C^{R;\alpha(\lambda)}(S),$$

with $\alpha(\overline{\lambda}) = 1$ for some $\overline{\lambda} \in \Lambda$.

The family $\mathcal{C}$ is **fuzzy rationalizable** if it is fuzzy rationalizable by some fuzzy-rational binary relation R. ◊

Finally, just as WARP is equivalent to crisp rationalizability, we can now establish the equivalence between WARPD and fuzzy rationalizability.

Theorem 4.1. *The following are equivalent.*

(i) The family $\mathcal{C}$ satisfies WARPD and WOCI.

(ii) The family $\mathcal{C}$ admits a cognition-dependent representation.

(iii) The family $\mathcal{C}$ is fuzzy-rationalizable.

5 Additional characterization results, special cases and extensions

In what follows, it will be useful to define the following family of crisp binary relations $\mathcal{R}(R) := \{R_\alpha : \alpha \in A\}$ where, given a fuzzy binary relation R, $A = \text{Im}(R)$ and for all $x, y \in X$, $xR_\alpha y \iff R(x,y) \geq \alpha$.

With some abuse of notation, when the fuzzy binary relation R in question rationalizes $\mathcal{C}$, we shall use R_λ instead of $R_{\alpha(\lambda)}$. In that case, it can be shown that

$$C_\lambda(S) = \{x \in S \,:\, \text{for all } y \in S,\ xR_\lambda y\}.$$

Appendix A.6 contains a thorough discussion of $\mathcal{R}(R)$ and its properties. It also provides an equivalent characterization of fuzzy rationalizability using this family of crisp binary relations.

For this section's purposes, it suffices to note that if R is fuzzy rational, then $\mathcal{R}(R) := \{R_\alpha : \alpha \in A\}$ satisfies:[13]

(i) $xR_{\alpha'}y \implies xR_\alpha y$ for all $\alpha' > \alpha$.

(ii) If xR_1y and yR_1z, then $zR_\alpha x$ implies $zR_\alpha y$ and $yR_\alpha x$.

(iii) R_1 is a weak order.

5.1 Incomplete data

Suppose we observe $\{C_\lambda : \lambda \in \tilde{\Lambda}\}$, where $\tilde{\Lambda} \subset \Lambda$. Previous results still hold as long as $\overline{\lambda} \in \tilde{\Lambda}$. In some situations, however, we might contend with the fact that we cannot observe decision-maker at her fully-rational level of cognition.

To understand the role of $\overline{\lambda}$, consider the following equivalent decomposition of WARPD into three sub-properties.[14]

Proposition 5.1 (WARPD decomposition)**.** *A family $\mathcal{C}$ satisfies WARPD if and only if the three conditions below hold.*

(i) ***Contraction.*** *For all $S, T \in \mathcal{S}$ with $S \subseteq T$, $C_\lambda(T) \cap S \subseteq C_\lambda(S)$.*

(ii) ***Expansion.*** *For all $S \in \mathcal{S}$ and $x \in S$, if $x \in C_\lambda(\{x,y\})$ for all $y \in S$, then $x \in C_\lambda(S)$.*

[13]Proof in A.6.

[14]This decomposition is similar to a decomposition of WARP found in Aleskerov et al. (2007).

(iii) ***No Cognitive Cycles.*** *For all* $x, y, z \in X$*, if either* $x \in C_{\overline{\lambda}}(\{x, y\})$ *and* $y \in C_{\lambda}(\{y, z\})$ *or* $x \in C_{\lambda}(\{x, y\})$ *and* $y \in C_{\overline{\lambda}}(\{y, z\})$*, then* $x \in C_{\lambda}(\{x, z\})$*.*

The proof can be found in appendix A.7.

The contraction property is exactly Sen's α, or IIA, as discussed in section 2.3. Expansion establishes that, if an alternative x is undominated in any pairwise comparison between itself and another member of S, then x must be undominated in S as a whole. Finally, No Cognitive Cycles establishes that some preference cycles can occur, i.e. it is possible for x to not be chosen from $\{x, z\}$ even if x is chosen from $\{x, y\}$ and y is chosen from $\{y, z\}$. The key is that, for this to happen, neither x nor y could have been true optimal choices in their respective sets, but merely acceptable given the level of cognition. This makes it possible for the error to be small between x and y and between y and z, but noticeably larger between x and z, triggering x's exclusion.

It is clear that $\overline{\lambda}$ has an important role in the third property. Without observing it, its validity cannot be determined. If WOCI holds, however, then it is possible to bound what the choices at $\overline{\lambda}$ are allowed to be. To do so, the concept of the transitive core from Nishimura (2018), reproduced below, becomes useful.

Definition 5.1 (Transitive Core at λ)**.** Fix $\mathcal{C}$, rationalizable by the fuzzy binary relation R. The transitive core of R_{λ} is a binary relation $\succsim_{\lambda}$ where, for all $\lambda \in \Lambda$, $x, y \in X$, $x \succsim_{\lambda} y$ if and only if, for all $z \in X$,

(i) $zR_{\lambda}x \implies zR_{\lambda}y$,

(ii) $yR_{\lambda}z \implies xR_{\lambda}z$.

◊

As usual, we use $\sim_{\lambda}$ and $\succ_{\lambda}$ to denote the symmetric and asymmetric parts of $\succsim_{\lambda}$.

The binary relation $\succsim_{\lambda}$ thus defined is complete and transitive – whereas R_{λ} itself might not be. However, if R_{λ} is also complete and transitive (that is, if WARP holds at λ), then it can be shown that $\succsim_{\lambda}$ and R_{λ} coincide (Nishimura, 2018). In particular, this implies that $\succsim_{\overline{\lambda}}$ is equivalent to $R_{\overline{\lambda}}$. To simplify notation, in what follows we write $\succsim$ instead of $\succsim_{\overline{\lambda}}$.

The next proposition shows that $\succsim$ is "embedded" into each $\succsim_{\lambda}$, and thus can be effectively approximated by intersecting all observed transitive cores.

Since binary relations are mathematically subsets of the Cartesian product $X \times X$, we take containment, intersections and unions to have their usual meaning. For example, we say that a binary relation $\gtrdot$ is contained in another binary relation $\rhd$ ($\gtrdot \subseteq \rhd$) if $x \gtrdot y \implies x \rhd y$.

Moreover, if $\blacktriangleright$ is the intersection of a collection B of binary relations (write $\blacktriangleright = \bigcap_{\rhd \in B} \rhd$) we have that $x \blacktriangleright y \iff [x \rhd y$ for all $\rhd \in B]$.

Proposition 5.2. *For any* $\tilde{\Lambda} \subseteq \Lambda$,

$$\succsim \subseteq \bigcap_{\lambda \in \tilde{\Lambda}} \succsim_\lambda .$$

The proof can be found in appendix A.7. The following example illustrates how the intersection of observed transitive cores provides information on the decision maker's rational benchmark preferences.

Example 5.1. Revisiting the movie ranking example of 1.1 and 2.3, suppose we observe only λ_0 and λ_1.

Consider a and b. Note that $a \in C_{\lambda_0}(\{a, z\})$ and $b \in C_{\lambda_0}(\{a, z\})$ for all $z \in \{a, b, c, d, e\}$. Similarly, whenever $z \in C_{\lambda_0}(\{a, z\})$, it is also the case that $z \in C_{\lambda_0}(\{b, z\})$ and vice-versa. Thus, $a \succsim_{\lambda_0} b$ and $b \succsim_{\lambda_0} a$, or $a \sim_{\lambda_0} b$.

A similar analysis yields the following relationships at λ_0:

$$a \sim_{\lambda_0} b \sim_{\lambda_0} c \sim_{\lambda_0} d \succ_{\lambda_0} e.$$

Repeating the exercise for λ_1 yields:

$$a \succ_{\lambda_1} b \succ_{\lambda_1} c \sim_{\lambda_1} d \succ_{\lambda_1} e.$$

By performing the intersection of $\succsim_{\lambda_0}$ and $\succsim_{\lambda_1}$, we obtain a new binary relation $\succsim_*$ where

$$a \succ_* b \succ_* c \sim_* d \succ_* e.$$

We know that in the rational benchmark, $a \succ b \succ c \succ d \succ e$. Indeed, $a \succsim_* b \succsim_* c \succsim_* d \succsim_* e$. Moreover, $b \not\succsim_* a$ implies $b \not\succsim_{\overline{\lambda}} a$, $c \not\succsim_* b$ implies $c \not\succsim_{\overline{\lambda}} b$, and so on; thus $\succsim \subseteq \succsim_*$.

Note that, in this case, $\succsim_*$ is identical to $\succsim_{\lambda_1}$ and $\succsim_{\lambda_1} \subseteq \succsim_{\lambda_0}$. At a glance this might suggest that the transitive cores are nested. This is however not true in general.

To illustrate this, suppose at a cognition index of, say, λ_2 between λ_1 and $\overline{\lambda}$, Xavier realizes that movie b is decisively better than movies c and d. Specifically, he now perceives that both a and b are superior to c, d and e and also that that c, d are better than e. Essentially, his perception is almost identical to the one at λ_1, except he now perceives that b is ranked above c and d.

In this scenario, the transitive core at λ_2 would not strictly rank a and b because no third alternative z interacts with a differently than it does with b:

$$a \sim_{\lambda_2} b \succ_{\lambda_2} c \sim_{\lambda_2} d \succ_{\lambda_2} e$$

In λ_1, Xavier could distinguish that a was better than c, but not that a was better than b or b was better than c. However, because Xavier treated a in the comparison between a and c differently than he treated b in the comparison between b and c (strict preference in the first, indifference in the second), the transitive core knows to "split" the a and b into separate indifference classes, with a ranked above b.

At λ_2, there is no third alternative z for which Xavier behaves distinctly when comparing a to z compared to when comparing b to z. Thus, the transitive core at λ_2 maintains the indifference between a and b. $\Diamond$

5.2 X infinite

Suppose X has infinitely many elements, and endow it with a metric. Fix a fuzzy binary relation $R : X \times X \longrightarrow [0,1]$.

Axiom 4 (Continuity)**.** For all $\alpha \in \text{Im}(R)$ and all sequences $\{x_n\}$ in X with $\lim x_n = x$, (i) if there exists $N \in \mathbb{N}$ such that $R(y, x_n) \geq \alpha$ for all $n \geq N$, then $R(y, x) \geq \alpha$, and (ii) if there exists $N' \in \mathbb{N}$ such that $R(x_n, y)$ for all $n \geq N$, then $R(x, y) \geq \alpha$.

Theorem 5.1. *Suppose X is compact. The family $\mathcal{C}$ satisfies WARPD, WOCI and continuity if and only if it admits a cognition-dependent representation where u is continuous.*

Corollary 5.2.1. *The function u in a cognition-dependent representation is unique up to a monotone transformation. Moreover, given u, each $\varepsilon(\lambda, x)$ is bounded by some $\overline{\varepsilon}(\lambda, x) > 0$ and $\underline{\varepsilon}(\lambda, x) > 0$ satisfying $\underline{\varepsilon}(\lambda, x) \geq \sup_{\lambda' > \lambda} \varepsilon(\lambda', x)$:*

$$\overline{\varepsilon}(\lambda, x) > \varepsilon(\lambda, x) \geq \underline{\varepsilon}(\lambda, x).$$

Moreover, if X is connected, then the threshold function ε is unique given u.

The proofs can be found in Appendix A.7.

5.3 No intransitive indifferences

In Examples 1.1 and 2.3, at λ_1 Xavier placed movie b in two categories: “good” and “okay”. That means that he perceived b as similar to both a (the other “good” movie) and c and/or d (the two other “okay” movies). His “similar perceived enjoyment” relation at λ_1 is thus not transitive. Indeed, we can expect there to be some overlap between categories when the perception of each item’s desirability is not “perfect”.

In other cases, however, overlaps of the type above do not make sense. To illustrate, consider the following example where a decision-maker’s rational benchmark involves lexicographic preferences over attributes.

Example 5.2 (Preferences that are lexicographic in attributes)**.** The hiring manager at a large company is assessing several candidates for a practical job, where experience holds the greatest importance. In cases where candidates possess similar experience levels, their educational qualifications are used to break ties. If ties persist, the manager considers the candidates' soft skills.

Table 1 summarizes the characteristics of the applicants.

Candidate	Experience	Education	Soft Skills
a	High	Low	Medium
b	High	High	Low
c	Medium	High	High
d	High	High	High
e	Low	High	Low

Table 1: Summary of candidate's skills.

The ranking manager is required to provide a ranking of the candidates. At a low level of cognition, say, λ_0, the manager only considers experience levels. Consequently, candidates a, b, and d are ranked together at the top. Candidate c falls into an intermediate category, and candidate e is ranked the lowest. This way, the manager classifies the candidates into three strictly ranked "bins" based on their experience levels alone: high, medium, and low. Each bin contains equally ranked candidates for that level of cognition.

Moving to the higher cognition index λ_1, where educational qualifications are also taken into account, the top bin is further divided into sub-bins. One sub-bin includes candidates b and d, both possessing high experience and high education. The other sub-bin contains candidate a, who has high experience but low education. The remaining bins remain the same since each already contains a single candidate.

Finally, at a high enough level of cognition ($\overline{\lambda}$), the manager evaluates all three attributes, including soft skills. As a result, the top bin is further subdivided. Candidate d emerges alone in the high-experience, high-education, high-soft skills sub-bin. Candidate b follows as a high-experience, high-education candidate with low-soft skills. Candidate a is ranked next, followed by candidate c and then candidate e.

It is important to note that at each step, a complete, transitive ranking is provided. Increasing the level of cognition refines the ranking by breaking down indifference classes into smaller, more detailed components. The absence of overlap between categories occurs due to preferences that are lexicographic in the attributes, and due to the fact that cognition here enables the consideration of attributes in order of importance. ◊

Example 5.3 (Left-digit bias)**.** A potential buyer of a used car has cognition level λ, where they only notice the leftmost digit of the car's mileage counter. The cars being evaluated have mileages ranging from 20,000 to 39,999.

Assuming everything else is equal, lower mileage is considered preferable. At the cognition index λ, the buyer then divides the cars into two categories or "bins": the 20,000s bin and the 30,000s bin. In this division, all cars in the first bin are considered better than any car in the second bin. However, without extra cognition applied to her choices choices, the buyer is unable to establish a strict ranking among the cars within each individual bin.

This changes if the buyer transitions to a new cognition index $\lambda' > \lambda$, where increased effort now allows them to notice the two leftmost digits of the mileage. Then, each of the previous two bins (20,000s and 30,000s) is subdivided into 10 sub-bins. This enables the buyer to further refine their ranking within each bin.

As cognition increases to higher levels and the buyer progressively takes into account the third, fourth, and fifth digits of the mileage counters, they are able to obtain increasingly detailed rankings by partitioning previous indifference classes into smaller categories. ◊

The examples above suggest that, for certain choices, lower cognition generate coarser rankings, as before, but these rankings are complete. In particular, this implies that WARP now holds for each λ instead of only at $\overline{\lambda}$. This is a significant strengthening of WARPD into WARPC (or Weak Axiom of Revealed Preference over Categories) below , which itself maps into a specific strengthening of the utility functional representation, and also of the specific notion of fuzzy transitivity (5.3).

Axiom 5 (Weak Axiom of Revealed Preference over Categories (WARPC))**.** For all $\lambda \in \Lambda$, $x, y \in X$ and $S, T \subseteq \mathcal{S}$ with $\{x, y\} \subseteq S \cap T$,

$$x \in C_\lambda(S), y \in C_\lambda(T) \implies x \in C_\lambda(T) \text{ and } y \in C_\lambda(S). \tag{14}$$

As mentioned, WARPC imposes WARP at each λ. As the examples suggest, this assumption guarantees a complete and transitive ranking at each level of cognition, and indifferences under WARPC are always transitive.

Definition 5.2. Given a set Λ with $\{0, 1\} \subseteq \Lambda \subseteq [0, 1]$, the family $\mathcal{C} := \{C_\lambda : \lambda \in \Lambda\}$ admits a cognition-dependent categorical representation (CDCR) if and only if, there exist functions $u : X \longrightarrow \mathbb{R}$ and $\varepsilon : \Lambda \times X \longrightarrow \mathbb{R}_+$ such that, for all $S \in \mathcal{S}$,

$$C_\lambda(S) = \{x \in S : \text{ for all } y \in S,\, u(y) - u(x) \le \varepsilon(\lambda, x)\}, \tag{15}$$

with (i) for all $x \in X$, $\varepsilon(\overline{\lambda}, x) = 0$, (ii) for all $x \in X$, $\varepsilon(\cdot, x)$ is non-increasing, (iii) for all $x, y \in X, \lambda \in \Lambda$, $u(x) \geq u(y) \implies u(x) + \varepsilon(\lambda, x) \geq u(y) + \varepsilon(\lambda, y)$, and (iv) for all $x, y \in X, \lambda \in \Lambda$, $u(x) > u(y) + \varepsilon(\lambda, y) \implies u(x) > u(z) + \varepsilon(\lambda, z)$ for all z with $u(z) \leq u(y) + \varepsilon(\lambda, y)$. ◊

The utility functional representation is similar to 2.1, except for the extra property of the thresholds ε. This property establishes that, if x is perceived as strictly better than y, then x must be perceived as strictly better than z, for all z in the same category as y.

To wrap up the characterization of WARPC, we modify fuzzy transitivity into strong fuzzy transitivity and prove an analogous equivalence of fuzzy strong rationalizability and WOCI/WARPC as to fuzzy rationalizability and WOCI/WARPD.

Definition 5.3 (Strong Fuzzy Transitivity)**.** A fuzzy relation R on $X \times X$ is **strongly fuzzy transitive**[15] only if, for all $x, y, z \in X$,

$$R(x, z) \geq \min\{R(x, y), R(y, z)\}.$$

◊

It's important to note that strong fuzzy transitivity is almost identical to fuzzy transitivity, with the distinction that $R(x, z) \geq \min\{R(x, y), R(y, z)\}$ holds even when $R(x, y), R(y, z)$ are both less than 1. In other words, the level of detail, discernment, or granularity required for the decision-maker to assert that x is worse than z must be at least as high as the minimum level required to perceive that either x is worse than y or that y is worse than z.

Definition 5.4 (Strong fuzzy rationality)**.** A fuzzy relation R on $X \times X$ is *fuzzy rational* only if it satisfies fuzzy reflexivity, fuzzy completeness and strong fuzzy transitivity. ◊

Definition 5.5 (Strong fuzzy rationalizability)**.** The family $\mathcal{C}$ is strongly fuzzy rationalizable by R (i) R is strongly rational, and (ii) there exists a function $\alpha : \Lambda \longrightarrow [0, 1]$ such that, for all $\lambda \in \Lambda$,

$$C_\lambda(S) = C^{R;\alpha(\lambda)}(S),$$

with $\alpha(\overline{\lambda}) = 1$ for some $\overline{\lambda} \in \Lambda$.

The family $\mathcal{C}$ is **strongly fuzzy-rationalizable** if it is strongly fuzzy rationalizable by some R. ◊

[15]This is also known as the min transitivity – the most restrictive of transitivities that stem from a t-norm. See Ovchinnikov (1981) for a reference on these concepts.

Theorem 5.2. *The following are equivalent.*

(i) The family $\mathcal{C}$ satisfies WARPC and WOCI.

(ii) The family $\mathcal{C}$ admits a cognition-dependent categorical representation.

(iii) The family $\mathcal{C}$ is strongly fuzzy-rationalizable.

The proof can be found in Appendix A.7.

6 Relation to the Literature

Information-based models. The general notion of decision-making under cognitive limitations is and has been the topic of extensive strands of literature that focus on information acquisition and processing constraints.

Despite some thematic similarities between the present work and these different strands, there are some important distinctions. First, the model in this paper specifically allows for intransitive indifference, and hence cycles, for a given cognition level. In contrast, in models such as rational inattention/costly information acquisition (Sims, 2003, Matějka and McKay, 2015, de Oliveira et al., 2017, Caplin et al., 2018, Hébert and Woodford, 2023, Steiner et al., 2017, Chambers et al., 2020, Maćkowiak et al., 2023), costly contemplation (Ergin and Sarver, 2010), DDM (Fehr and Rangel, 2011, Woodford, 2014, Fudenberg et al., 2020), search (Stigler, 1961, Caplin and Dean, 2011, Delaney et al., 2020) and others, conditional on information, preferences are always complete and transitive.

Second, in this paper's model, even if cognition is identified with information (e.g. Section 2.2.1), it can only be used to break ties. In contrast, in information-based models such as the above, strict preference reversals can and do occur.

Semiorders, just-noticeable differences and more on intransitive indifferences. The case of intransitivite indifferences can be traced back to at least Armstrong (1950) and Luce (1956). They were among the first to study situations where a gradual accumulation of changes means local indifference, i.e. at each step, but not globally, i.e. when comparing the starting point with the end point. This phenomenom is attributed to existing limits to our perception of small differences.

In this paper paper, cognition is capable of refining perception and thus shrinking these limits in a particular order according to how difficult certain comparisons are. By contrast, in the semiorder literature, limits to perception are typically assumed to be a fixed, physical constraint (Jamison and Lau, 1973, Fishburn, 1975, Gilboa and Lapson, 1995).

Fuzzy choice and preferences. Attempts to apply techniques from fuzzy set theory to choice problems are relatively scarce, and typically focus on specific mathematical properties of fuzzy binary relations (Ovchinnikov, 1981).

While the present paper, through WOCI and WARPD, is able to recover fuzziness as a property of the revealed preferences, other examples such as Dasgupta and Deb (1991), Dutta (1987), Banerjee (1995) attempt to justify fuzziness as a primitive in terms of preference imprecision, uncertainty or vagueness. The present work seems to be the first to discuss fuzziness as a result of behavioral inputs from choices.

Other related work. Manzini and Mariotti (2007) and Manzini and Mariotti (2012) study choices generated by applying different rationales sequentially. Because the rationales are allowed to have strict disagreements, IIA is violated, which is not the case in the present paper. They also assume that the sequence of rationales persists until a single option remains; thus they do not study when and how indifferences may emerge and what their properties are.

The work most closely related to this study is presented in Tyson (2021). In that work, the author examines decision-makers with cognitive limitations who stochastically perceive only a coarse representation of their preference rankings. The nature of these coarse rankings bears a resemblance to the concept proposed in this paper. However, in Tyson (2021), these coarse rankings are assumed to be complete, meaning that perceived indifferences are always transitive. Additionally, the specific coarse ranking observed by the decision-maker is subject to randomness, following a particular distribution based on the exponential. Tyson (2021) also assumes that the decision-maker selects one element from her random choice correspondence also at random, uniformly. In contrast, this paper's model has choice correspondences that are deterministic given cognition. Moreover, no explicit assumptions about the mechanism through which an alternative is picked from the set of choice correspondences is made.

7 Conclusion

This paper introduces cognition as a crucial mental resource, essential for the detailed articulation of choices and preferences. In the absence of full cognition, decision-makers are only approximate optimizers, in a sense that is made precise by either (i) combination of WOCI/WARPD, (ii) the cognition-dependent representation or (iii) the notion of fuzzy rationalizability.

Applications then showed how consumers' limited cognition can influence competitiveness and pricing decisions.

The model proposed places a prominent emphasis on indifference: as cognitive levels decrease, the prevalence of indifference increases, resulting in larger choice correspondences. In this context, decision-makers employing lower levels of cognition might care very little about which specific option is picked from a multitude of alternatives they find indifferent. That means they can then become more susceptible to behavioral biases and heuristics that influence the final selection. This susceptibility can manifest in various ways, such as inertia (e.g., failing to cancel an auto-renewing subscription), advertising effects (e.g., choosing a brand seen recently on TV or clicking the first ad link in a Google search), and reliance on alphabetical order (e.g., selecting a service provider based on its position in the phone book). Investigating the implications of such behavioral postulates, especially when combined with indifference is a promising avenue for future research.

Methodologically, the paper contributes to the literature on fuzziness. It establishes economic foundations for fuzzy preferences and fuzzy choice, assigning economic significance to degrees of membership and degrees of relationship through a granularity interpretation of fuzziness. While the granularity interpretation is suitable for the contexts considered in the paper, it's important to note that other interpretations of fuzziness could be equally valid and valuable, particularly in regards to decision settings enriched by a secondary dimension, such as similarity, or preference intensity. Studying the applicability of fuzziness in these scenarios could provide some new and valuable insights, and is worth investigating.

References

Aleskerov, F., D. Bouyssou, and B. Monjardet (2007): *Utility maximization, choice and preference*, Springer Science & Business Media.

Armstrong, W. E. (1950): "A Note on the Theory of Consumer's Behaviour," *Oxford Economic Papers*, 2, 119–122.

Asch, P. and J. J. Seneca (1975): "Characteristics of collusive firms," *The Journal of Industrial Economics*, 223–237.

——— (1976): "Is Collusion Profitable?" *The Review of Economics and Statistics*, 58, 1–12.

Balakrishnan, N., E. A. Ok, and P. Ortoleva (2022): "Inference of Choice Correspondences," Working paper.

Banerjee, A. (1995): "Fuzzy choice functions, revealed preference and rationality," *Fuzzy Sets and Systems*, 70, 31–43.

Bewley, T. F. (2002): "Knightian decision theory. Part I," *Decisions in Economics and Finance*, 25, 79–110.

Bouacida, E. (2021): "Identifying Choice Correspondences: A General Method and an Experimental Implementation," Working paper, Lancaster University, Department of Economics.

Caplin, A. and M. Dean (2011): "Search, choice, and revealed preference," *Theoretical Economics*, 6, 19–48.

Caplin, A., M. Dean, and J. Leahy (2018): "Rational Inattention, Optimal Consideration Sets, and Stochastic Choice," *The Review of Economic Studies*, 86, 1061–1094.

Chambers, C. P., C. Liu, and J. Rehbeck (2020): "Costly information acquisition," *Journal of Economic Theory*, 186, 104979.

Chi, Z., H. Yan, and T. Pham (1996): *Fuzzy algorithms: with applications to image processing and pattern recognition*, vol. 10, World Scientific.

Dasgupta, M. and R. Deb (1991): "Fuzzy choice functions," *Social Choice and Welfare*, 8, 171–182.

de Oliveira, H., T. Denti, M. Mihm, and K. Ozbek (2017): "Rationally inattentive preferences and hidden information costs," *Theoretical Economics*, 12, 621–654.

Dean, M., Ö. Kibris, and Y. Masatlioglu (2017): "Limited attention and status quo bias," *Journal of Economic Theory*, 169, 93–127.

Delaney, J., S. Jacobson, and T. Moenig (2020): "Preference discovery," *Experimental Economics*, 23, 694–715.

Dutta, B. (1987): "Fuzzy preferences and social choice," *Mathematical Social Sciences*, 13, 215–229.

Ergin, H. and T. Sarver (2010): "A unique costly contemplation representation," *Econometrica*, 78, 1285–1339.

Fehr, E. and A. Rangel (2011): "Neuroeconomic Foundations of Economic Choice–Recent Advances," *Journal of Economic Perspectives*, 25, 3–30.

Fischbacher, U. (2007): "z-Tree: Zurich toolbox for ready-made economic experiments," *Experimental economics*, 10, 171–178.

Fishburn, P. C. (1975): "Semiorders and choice functions," *Econometrica: Journal of the Econometric Society*, 975–977.

Fudenberg, D., W. Newey, P. Strack, and T. Strzalecki (2020): "Testing the drift-diffusion model," *Proceedings of the National Academy of Sciences*, 117, 33141–33148.

Gilboa, I. and R. Lapson (1995): "Aggregation of semiorders: Intransitive indifference makes a difference," *Economic Theory*, 5, 109–126.

Greiner, B. (2004): "An Online Recruitment System for Economic Experiments," in *Forschung und wissenschaftliches Rechnen 2003*, ed. by V. M. Kurt Kremer, GWDG Bericht 63, Goettingen : Ges. fuer Wiss. Datenverarbeitung, 79–93.

——— (2015): "Subject pool recruitment procedures: organizing experiments with ORSEE," *Journal of the Economic Science Association*, 1, 114–125.

Hébert, B. and M. Woodford (2023): "Rational inattention when decisions take time," *Journal of Economic Theory*, 208, 105612.

Henderson Britt, S. (1975): "How Weber's law can be applied to marketing," *Business Horizons*, 18, 21–29.

Jamison, D. T. and L. J. Lau (1973): "Semiorders and the theory of choice," *Econometrica: Journal of the Econometric Society*, 901–912.

Kamen, J. M. and R. J. Toman (1970): "Psychophysics of Prices," *Journal of Marketing Research*, 7, 27–35.

Kreps, D. (1988): *Notes on the Theory of Choice*, Westview press.

Liu, Y. and E. E. Kerre (1998): "An overview of fuzzy quantifiers. (I). Interpretations," *Fuzzy Sets and Systems*, 95, 1–21.

Luce, R. D. (1956): "Semiorders and a Theory of Utility Discrimination," *Econometrica*, 24, 178–191.

Maćkowiak, B., F. Matějka, and M. Wiederholt (2023): "Rational inattention: A review," *Journal of Economic Literature*, 61, 226–273.

Manzini, P. and M. Mariotti (2007): "Sequentially rationalizable choice," *American Economic Review*, 97, 1824–1839.

——— (2012): "Choice by lexicographic semiorders," *Theoretical Economics*, 7, 1–23.

Matějka, F. and A. McKay (2015): "Rational inattention to discrete choices: A new foundation for the multinomial logit model," *American Economic Review*, 105, 272–298.

Nguyen, A.-T., T. Taniguchi, L. Eciolaza, V. Campos, R. Palhares, and M. Sugeno (2019): "Fuzzy control systems: Past, present and future," *IEEE Computational Intelligence Magazine*, 14, 56–68.

Nishimura, H. (2018): "The transitive core: Inference of welfare from nontransitive preference relations," *Theoretical Economics*, 13, 579–606.

Ovchinnikov, S. V. (1981): "Structure of fuzzy binary relations," *Fuzzy Sets and Systems*, 6, 169–195.

Ross, G. R., M. G. Meloy, and K. A. Carlson (2020): "Preference refinement after a budget contraction," *Journal of Consumer Research*, 47, 412–430.

Shimoda, M. (2002): "A natural interpretation of fuzzy sets and fuzzy relations," *Fuzzy Sets and Systems*, 128, 135–147.

Sikchi, S. S., S. Sikchi, and M. Ali (2013): "Fuzzy expert systems (FES) for medical diagnosis," *International Journal of Computer Applications*, 63.

Sims, C. A. (2003): "Implications of rational inattention," *Journal of monetary Economics*, 50, 665–690.

Steiner, J., C. Stewart, and F. Matějka (2017): "Rational inattention dynamics: Inertia and delay in decision-making," *Econometrica*, 85, 521–553.

Stigler, G. J. (1961): "The economics of information," *Journal of political economy*, 69, 213–225.

Tyson, C. J. (2021): "Exponential Satisficing," *American Economic Journal: Microeconomics*, 13, 439–67.

Woodford, M. (2014): "Stochastic Choice: An Optimizing Neuroeconomic Model," *The American Economic Review*, 104, 495–500.

Yager, R. R. (1992): "Expert systems using fuzzy logic," in *An introduction to fuzzy logic applications in intelligent systems*, Springer, 27–44.

Zadeh, L. (1971): "Similarity relations and fuzzy orderings," *Information Sciences*, 3, 177–200.

Zadeh, L. A. (1965): "Fuzzy sets," *Information and control*, 8, 338–353.

A Appendix

A.1 Proofs and auxiliary results: characterization and main theorem

Proposition A.1 (Ordering Λ)**.** *If the family $\{C_\lambda : \lambda \in \Lambda\}$ satisfies WOCI, then $\geq$ defined as above weakly orders Λ.*

Proof. The relation $\geq$ is a weak order if and only if it is complete and transitive.

Completeness. Fix $\lambda, \lambda' \in \Lambda$. Suppose that $\lambda' \not\geq \lambda$. That is, there exists a set $S \in \mathcal{S}$ such that $C_{\lambda'}(S) \not\subseteq C_\lambda(S)$. By Nestedness, it must be that $C_\lambda(S) \subsetneq C_{\lambda'}(S)$. By Consistency, that means that $C_\lambda(T) \subseteq C_\lambda(T)$ for all $T \in \mathcal{S}$, and thus $\lambda \geq \lambda'$.

Transitivity. Suppose $\lambda \geq \lambda'$ and $\lambda' \geq \lambda''$. Then, for all $S \in \mathcal{S}$, $C_\lambda(S) \subseteq C_{\lambda'}(S) \subseteq C_{\lambda''}(S)$. That means that $\lambda \geq \lambda''$ as well. ■

Proposition A.2. *If the family $\{C_\lambda : \lambda \in \Lambda\}$ satisfies WOCI and X is finite, then there exists a finite number of $\geq$-equivalence classes on Λ.*

Proof. Since X is finite, we can write $\mathcal{S} := \{S_1, \dots, S_n\}$. For each i, the family of sets $\{C_\lambda(S_i) : \lambda \in \Lambda\}$ is nested, in the sense that $C_\lambda(S_i) \subseteq C_{\lambda'}(S_i)$ whenever $\lambda \geq \lambda'$.

Fix i, and consider the function $g_i : \Lambda \longrightarrow C_\lambda(S_i)$. Note that $|\mathrm{Im}\,(g_i)| < \infty$ – after all, S_i is finite, so there are finitely many subsets of S_i. Since each $C_\lambda(S_i)$ is necessarily a subset of S_i, it follows that the image of g_i must be finite.

Let $C_i^1, C_i^2, \dots, C_i^{L_i}$ denote the possible values that g_i can take, and consider the inverse images $\Lambda_i^{\ell_i} := g_i^{-1}(C_i^{\ell_i})$ for $\ell_i \in \{1, \dots, L_i\}$. The collection of inverse images $\Lambda_i = \left\{\Lambda_i^1, \dots, \Lambda_i^{L_i}\right\}$ is a partition of Λ.

Construct the finer partition by intersecting elements of each Λ_i:

$$\tilde{\Lambda} = \{\Lambda_1^{\ell_1} \cap \Lambda_2^{\ell_2} \cap \ldots \cap \Lambda_n^{\ell_n} : \ell_i \in \{1, \dots, L_i\} \text{ for all } i \in \{1, \dots, n\}\}$$

Note that $\tilde{\Lambda}$ is finite. Moreover, for any λ, λ' belonging to an element $\Lambda_1^{\ell_1} \cap \Lambda_2^{\ell_2} \cap \ldots \cap \Lambda_n^{\ell_n}$ of $\tilde{\Lambda}$, by construction $C_\lambda(S_i) = C_{\lambda'}(S_i) = C_i^{\ell_i}$. ■

Proposition 2.2. *WARPD holds if and only if $\mathcal{C}$ satisfies*

(i) ***α (independence of irrelevant alternatives):*** *For all $\lambda \in \Lambda$ $S, T \in \mathcal{S}$ with $S \subseteq T$, and $x \in S$,*

$$x \in C_\lambda(T) \implies x \in C_\lambda(S).$$

(ii) **$\lambda\beta$:** *For all $\lambda \in \Lambda$, $S, T \in \mathcal{S}$ with $S \subseteq T$, $x, y \in C_\lambda(S)$,*

$$y \in C_\lambda(T) \text{ and } [x \in C_{\overline{\lambda}}(S) \text{ or } y \in C_{\overline{\lambda}}(T)] \implies x \in C_\lambda(T).$$

Proof. **WARPD $\implies$ α.** Suppose WARPD holds, and let $S, T \in \mathcal{S}$ with $S \subseteq T$ and $\lambda \in \Lambda$. Let $x \in C_\lambda(T) \cap S$. Because $C_{\overline{\lambda}}(S)$ must be nonempty, there exists $y \in C_{\overline{\lambda}}(S)$. By WARPD, $y \in C_\lambda(S)$.

WARPD $\implies$ $\lambda\beta$. Let $S, T \in \mathcal{S}$ with $S \subseteq T$ and $\lambda \in \Lambda$.

Fix $x, y \in C_\lambda(S)$ and $y \in C_\lambda(T)$. Suppose $x \in C_{\overline{\lambda}}(S)$. By WARPD, $x \in C_\lambda(T)$.

Now suppose instead that $y \in C_{\overline{\lambda}}(T)$. By WARPD, once again we must have $x \in C_\lambda(T)$.

α and $\lambda\beta$ $\implies$ WARPD. Fix $\lambda \in \Lambda$, $S, T \in \mathcal{S}$ with $\{x, y\} \subseteq S \cap T$.

Let $x \in C_{\overline{\lambda}}(S)$ and $y \in C_\lambda(T)$.

From α: $C_{\overline{\lambda}}(S) \cap (S \cap T) \subseteq C_{\overline{\lambda}}(S \cap T)$. Thus, $x \in C_{\overline{\lambda}}(S \cap T)$.

Also from α: $C_\lambda(T) \cap (S \cap T) \subseteq C_\lambda(S \cap T)$. Thus, $y \in C_\lambda(S \cap T)$.

From $\lambda\beta$: $x \in C_{\overline{\lambda}}(\{x\})$ and $x \in C_{\overline{\lambda}}(S)$, implying $x \in C_\lambda(S)$. Similarly, $x \in C_\lambda(S \cap T)$.

From $\lambda\beta$: $x, y \in C_\lambda(S \cap T)$ and $x \in C_\lambda(S)$. Because $x \in C_{\overline{\lambda}}(S)$, we have that $y \in C_\lambda(S)$.

Also from $\lambda\beta$: $x, y \in C_\lambda(S \cap T)$ and $y \in C_\lambda(T)$. Because $x \in C_{\overline{\lambda}}(S \cap T)$, we must have that $x \in C_\lambda(T)$. ∎

Theorem 2.1. *The family $\mathcal{C}$ satisfies WARPD and WOCI if and only if it admits a cognition-dependent representation.*

Proof. **WARPD and WOCI $\implies$ cognition-dependent representation.**

Fix $\mathcal{C}$ satisfying WARPD and WOCI.

With some abuse of notation, redefine Λ using its equivalence classes (according to the weak order $\geq$ implied by WOCI) as its basic elements. This guarantees that Λ is linearly ordered.

Let u be a utility function representing the decision maker's preferences at cognition level $\overline{\lambda}$. We know that such a function exists because WARPD implies WARP at $\overline{\lambda}$.

Define the thresholds as follows. Let $\varepsilon(\lambda, x) := \max_{z \in X}\{u(z) : x \in C_\lambda(\{x, z\})\} - u(x)$. Clearly, $\varepsilon(\lambda, x) \geq 0$ for all x, λ, with $\varepsilon(\overline{\lambda}, x) = 0$.

We first show that, for every $S \in \mathcal{S}$, $x \in C_\lambda(S)$ if and only if $u(y) - u(x) \leq \varepsilon(\lambda, x)$ for all $y \in S$:

$u(y) > u(x) + \varepsilon(\lambda, x) \implies x \notin C_\lambda(S)$**.** Fix $S \in \mathcal{S}$, $\lambda \in \Lambda$, and let $x \in S$. Suppose that there exists $y \in S$ with $u(y) > u(x) + \varepsilon(\lambda, x)$. That means $x \notin C_\lambda(\{x, y\})$. By WOCI, $y \in C_{\overline{\lambda}}(\{x, y\})$. By WARPD, $x \notin C_\lambda(S)$.

$x \notin C_\lambda(S) \implies u(y) > u(x) + \varepsilon(\lambda, x)$**.** suppose that $x \notin C_\lambda(S)$. Then, there exists y such that $x \notin C_\lambda(\{x,y\})$, implying (by WOCI) that $C_{\overline{\lambda}}(\{x,y\}) = C_\lambda(\{x,y\}) = \{y\}$, and thus that $u(y) > u(x)$ (by WARP at $\overline{\lambda}$). By way of contradiction, assume $u(y) \leq u(x) + \varepsilon(\lambda, x)$. Then, there exists $z \in X$ such that $u(z) \geq u(y)$ and $x \in C_\lambda(\{x,y\})$. We have that $z \in C_{\overline{\lambda}}(\{x,y,z\})$ (by WARP at $\overline{\lambda}$); moreover, because $x \in C_\lambda(\{x,z\})$, WARPD implies $x \in C_\lambda(\{x,y,z\})$. Since $y \in C_{\overline{\lambda}}(\{x,y\})$ (again, by WARP at $\overline{\lambda}$), WARPD also then implies that $x \in C_\lambda(\{x,y\})$ – a contradiction. Hence, it must be that $u(y) > u(x) + \varepsilon(\lambda, x)$.

Next, we show that the remaining properties of the cognition-dependent representation hold.

(i) **for all** $x \in X$**,** $\varepsilon(\overline{\lambda}, x) = 0$**.** Evident from the definition of ε, and from the fact that WARPD implies WARP at $\overline{\lambda}$.

(ii) **for all** $x \in X$**,** $\varepsilon(\cdot, x)$ **is non-increasing.** Fix λ, λ' arbitrarily, and suppose $\varepsilon(\lambda, x) > \varepsilon(\lambda', x)$ for some x. Then, there exists a z such that $x \in C_\lambda(\{x,z\})$ but $x \notin C_{\lambda'}(\{x,z\})$. By WOCI, that means that $C_{\lambda'}(\{x,z\}) \subsetneq C_\lambda(\{x,z\})$, and thus $\lambda' > \lambda$.

(iii) **for all** $x, y \in X$**,**$\lambda \in \Lambda$**,** $u(x) \geq u(y)$**, then** $u(x) + \varepsilon(\lambda, x) \geq u(y) + \varepsilon(\lambda, y)$**.** Suppose, by way of contradiction, that $u(x) \geq u(y)$ but $u(x) + \varepsilon(\lambda, x) < u(y) + \varepsilon(\lambda, y)$ for some x, $y \in X$ and $\lambda \in \Lambda$. Then, there exists z_y such that $u(z_y) > u(x) \geq u(y)$, $y \in C_\lambda(\{y, z_y\})$ but $x \notin C_\lambda(\{x, z_y\})$. Since $u(z_y) > u(x) \geq u(y)$, $z_y \in C_{\overline{\lambda}}(\{x,y,z_y\})$. By WARPD, it must then be that $C_\lambda(\{x,y,z_y\}) = \{y, z_y\}$. But because $x \in C_{\overline{\lambda}}(\{x,y\})$ and $y \in C_\lambda(\{x,y,z_y\})$, WARPD also implies that x is in $C_\lambda(\{x,y,z_y\})$, a contradiction.

Cognition-dependent representation $\implies$ WARPD and WOCI.

Suppose the family $\mathcal{C} := \{C_\lambda : \lambda \in \Lambda\}$ admits a cognition-dependent representation. We show that WOCI and WARPD hold.

WOCI holds. Nestedness. For every $\lambda, \lambda' \in \Lambda$, either $\lambda > \lambda'$, $\lambda < \lambda'$ or $\lambda = \lambda'$ holds (since Λ is totally ordered). If $\lambda = \lambda'$, then $\varepsilon(\lambda, \cdot) = \varepsilon(\lambda', \cdot)$, and thus $C_\lambda(S) = C_{\lambda'}(S)$ for all S. Suppose then that $\lambda > \lambda'$ without loss of generality; then $u(y) - u(x) \leq \varepsilon(\lambda, x) \implies u(y) - u(x) \leq \varepsilon(\lambda', x)$. Thus, $C_\lambda(S) \subseteq C_{\lambda'}(S)$ for all S and nestedness holds.

Consistency. Regarding consistency, if $C_\lambda(T) \subsetneq C_{\lambda'}(T)$ for some T, that means that there exists $x, y \in T$ such that $\varepsilon(\lambda', x) > u(y) - u(x) > \varepsilon(\lambda, x)$, so that $\lambda \geq \lambda'$. But then, from the above, we know that $C_\lambda(S) \subseteq C_{\lambda'}(S)$ for all S.

WARPD holds. Let $\lambda \in \Lambda$, $x, y \in X$ and $S, T \in \mathcal{S}$ with $\{x, y\} \subseteq S \cap T$. Let $x \in C_{\overline{\lambda}}(S)$ and $y \in C_\lambda(T)$. We show that $x \in C_\lambda(T)$ and $y \in C_\lambda(S)$.

$x \in C_\lambda(T)$**.** The above implies $u(x) \geq u(z_S)$ for all $z_S \in S$, and $u(y)+\varepsilon(\lambda, y) \geq u(z_T)$ for all $z_T \in T$. Since $u(x) \geq u(y)$, $u(x) + \varepsilon(\lambda, x) \geq u(y) + \varepsilon(\lambda, y)$ (by property (iii) of the cognition-dependent representation), and thus $u(x) + \varepsilon(\lambda, x) \geq u(z_T)$ for all $z_T \in T$, i.e. $x \in C_\lambda(T)$.

$y \in C_\lambda(S)$**.** Since $u(y) + \varepsilon(\lambda, y) \geq u(z_T)$ for all $z_T \in T$ and $u(x) \geq u(z_S)$ for all $z_S \in S$, we have $u(y) + \varepsilon(\lambda, y) \geq u(x) \geq u(z_S)$ for all $z_S \in S$, i.e. $y \in C_\lambda(S)$.

■

A.2 Experimental evidence

For each set S of alternatives and each $T \subsetneq S$, the proportion of subjects that chose T or a proper subset of T should increase with the information treatment, if WOCI holds in the aggregate. However, this expected pattern is not universally observed for all S, T (See Table 2 below for the data on choices from $\{1, 2, 3, 4\}$, when all alternatives are available).

Moving from the sentence to the video treatment seems to have the expected effect (See e.g. Table 2, where the only – very minor – exception is for $T = \{1, 3\}$). The training information treatment seems to cause more frequent reversals – despite a general trend of choice set contraction (in terms of cardinality). This could be due to agent heterogeneity. Another potential explanation is that the information provided in the training treatments goes beyond a few details necessary for breaking ties between similarly attractive options. Instead, the acquired information through practice might be much more substantial and even surprising, fundamentally altering how subjects perceive the problem and hence causing strict reversals in how the options are ranked.

A.3 Information interpretation

Suppose we interpret an increase in cognition as a type of information acquisition, i.e. more cognition reinterpreted as "better informed" in a particular sense.

To perform this translation, let Ω denote the set of states of the world. Assume each state of the world is a possible complete and transitive ranking of alternatives. For example, with three alternatives, there are the following 14 possible states of

$C_\lambda(\{1,2,3,4\})$	sentence	video	training
$\{1\}$	4.55%	3.36%	5.41%
$\{2\}$	3.03%	7.14%	1.35%
$\{3\}$	1.52%	5.46%	1.35%
$\{4\}$	3.03%	5.04%	10.81%
$\{1,2\},\{1\},\{2\}$	10.61%	14.29%	10.81%
$\{1,3\},\{1\},\{3\}$	16.67%	16.39%	12.16%
$\{1,4\},\{1\},\{4\}$	7.58%	13.45%	21.62%
$\{2,3\},\{2\},\{3\}$	4.55%	14.29%	2.7%
$\{2,4\},\{2\},\{4\}$	9.09%	15.55%	27.03%
$\{3,4\},\{3\},\{4\}$	6.06%	15.55%	13.51%
$\{1,2,3\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}$	33.33%	35.29%	20.27%
$\{1,2,4\},\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\}$	16.67%	36.13%	60.81%
$\{1,3,4\},\{1\},\{3\},\{4\},\{1,3\},\{1,4\},\{3,4\}$	31.82%	42.44%	35.14%
$\{2,3,4\},\{2\},\{3\},\{4\},\{2,3\},\{2,4\},\{3,4\}$	27.27%	36.13%	36.49%

Table 2: $S = \{1,2,3,4\}$

the world:

$$\begin{array}{ll} x \succ y \succ z & x \succ z \succ y \\ y \succ x \succ z & y \succ z \succ x \\ z \succ x \succ y & z \succ y \succ x \\ x \sim y \succ z & x \succ y \sim z \\ x \sim z \succ y & y \succ x \sim z \\ y \sim z \succ x & y \succ z \sim x \\ z \succ x \sim y & x \sim y \sim z \end{array}$$

Under absolute ignorance, assume that the agent treats all alternatives as identical, i.e. the decision-maker acts as if she's indifferent. We denote that by $x \square y \square z$, where $\square$ will stand in for "does not know" the relationship between elements.

Now suppose an agent can acquire information regarding the state of the world. Here are the things the DM can learn if $x \succ y \succ z$ is the true state of the world:

- $x \square y$, $y \square z$, $x \square z$

- $x \succ z$, $x \square y$, $y \square z$

- $x \succ z$ $x \succ y$, $y \square z$ or $x \succ z$, $x \square y$, $y \succ z$ (exclusive; only one of these is available, depending on which comparison is easier, x and y or y and z)

- $x \succ y \succ z$

The idea is that the relationship between alternatives that are ranked further apart is easier to gauge than the relationship between alternatives that are closer together.

Importantly, though, the decision-maker does not know this, or rather, the decision-maker does not use this information to derive more information.

For example, if she acquires a signal that tells her that $x \succ z$, $x \square y$, $y \square z$, then if she understood the process by which this signal is generated, then she would know that x and z must be further apart; i.e. it must be that $x \gtrsim y$. From this point, she would probably then choose x, because even if x and y are tied in the true state of the world, she could reason that there is a chance that they are not, making x the superior option.

This is not the approach we take. Instead, the decision-maker's signals is already the condensed, post-processing, potentially flawed interpretation of the signal by the decision-maker has. Thus we treat it as it is. If $x \succ z$, $x \square y$, $y \square z$, then at this point she chooses x when given x and z, but will not be able to pick a single option when given x and y or y and z. That is, we assume that the decision-maker cannot make any more inferences about the true state of the world at that point.

This is a fundamentally different approach from an information acquisition model where the decision-maker uses the information obtained to rationally update her prior, i.e. where she does the maximum amount of inference that she can. In these models, cognitive constraints are usually assumed to be costs to information acquisition, but that is clearly not the same as imposing the inference constraints that we do here.

If we want both approaches to coincide, then intransitive indifferences obviously cannot exist. Given that, suppose $x \succ y \succ z$ is the true state of the world. Then the available signals are

- $x \square y$, $y \square z$, $x \square z$

- $x \succ z$ $x \succ y$, $y \square z$ or $x \succ z$, $x \square y$, $y \succ z$

- $x \succ y \succ z$

Suppose the decision maker receives $x \succ z$ $x \succ y$, $y \square z$. This paper's model predicts that she would pick x over y, x over z but wouldn't be able to pick a single alternative out of y and z. In order for her to have this same behavior while exerting the maximum amount of inference, assume all utilities take values in $\{2,1,0\}$, and that the decision-maker's prior establishes $\Pr(u(w) = j) = 1/3$ for all $w \in \{x,y,z\}$ and all $j \in \{2,1,0\}$. Then, when she learns $x \succ z$ $x \succ y$, $y \square z$, the decision maker realizes that the only possibilities are:

$$\begin{aligned}
&u(x) = 2, u(y) = u(z) = 1 \\
&u(x) = 2, u(y) = u(z) = 0 \\
&u(x) = 1, u(y) = u(z) = 0 \\
&u(x) = 2, u(y) = 1, u(z) = 0 \\
&u(x) = 2, u(y) = 0, u(z) = 1
\end{aligned}$$

then, $\mathbf{E}[u(x)] = 2 \cdot 4/5 + 1 \cdot 1/5 = 9/5$ and $\mathbf{E}[u(y)] = \mathbf{E}[u(z)] = 2/5$, implying the DM at this point prefers x to both y and z, but would be indifferent between y and z.

A.4 Proof: collusive outcomes in Bertrand duopolies

Proposition 3.1. *Assume consumers select an alternative by uniformly picking an element of $C_\lambda(\{x_1,x_2\})$. If $(w-c) \le (2/\lambda)(1-\lambda)c$, then the unique equilibrium is at $p_1 = p_2 = w$.*

Proof. Each firm's demand is given by:

$$D_i(p_1,p_2) = \begin{cases} 0 & \text{if } p_i > p_j(2-\lambda) \\ 1/2 & \text{if } p_j/(2-\lambda) \le p_i \le p_j(2-\lambda) \\ 1 & \text{if } p_i < p_j/(2-\lambda) \end{cases}$$

Profits for firm i then are:

$$\Pi_i(p_1,p_2) = \begin{cases} 0 & \text{if } p_i > p_j(2-\lambda) \\ (p_i - c)/2 & \text{if } p_j/(2-\lambda) \le p_i \le p_j(2-\lambda) \\ p_i - c & \text{if } p_i < p_j/(2-\lambda) \end{cases}$$

We assume that to undercut the competitor's price, a firm needs to discount their own price by a minimum of $k > 0$, where k is sufficiently small.

There are a few cases to consider, depending on the relationship between parameters. Usually, the solution boils down to verifying whether it is preferable to split the market (and thus choose $p_i = \max\{p_j(2-\lambda), w\}$) or to undercut (i.e. choose $p_i = p_j/(2-\lambda) - k$).

Case I: $c(2-\lambda) < w/(2-\lambda)$

- If $p_j \leq c/(2-\lambda)$, any price $p_i \geq c$ is a best-response, as it guarantees zero profits:
$$BR(p_2) = [c, \infty)$$

- If $c/(2-\lambda) < p_j \leq c(2-\lambda)$, then the best-response is to choose $p_i = p_j(2-\lambda)$:
$$BR(p_j) = \{p_j(2-\lambda)\}$$

- If $c(2-\lambda) < p_j \leq w/(2-\lambda)$, then
$$BR(p_j) = \begin{cases} \{p_j(2-\lambda)\} & \text{if } 0 < \lambda \leq 2-\sqrt{2}\sqrt{\frac{w}{c+2k+w}} \text{ or } p_j < \frac{(2-\lambda)(c+2k)}{2-(2-\lambda)^2} \\ \{p_j/(2-\lambda)-k\} & \text{if } 2-\sqrt{2}\sqrt{\frac{w}{c+2k+w}} < \lambda < 1 \text{ and } p_j > \frac{(2-\lambda)(c+2k)}{2-(2-\lambda)^2} \\ \{p_j/(2-\lambda)-k, p_j(2-\lambda)\} & \text{if } 2-\sqrt{2}\sqrt{\frac{w}{c+2k+w}} < \lambda < 1 \text{ and } p_j = \frac{(2-\lambda)(c+2k)}{2-(2-\lambda)^2} \end{cases}$$
where the thresholds for λ and p_j are computed by comparing profits.

- If $w/(2-\lambda) < p_j < w$, then
$$BR(p_j) = \begin{cases} \{w\} & \text{if } 0 < \lambda \leq \frac{2(c+2k)}{c+2k+w} \text{ or} \\ & \frac{2(c+2k)}{c+2k+w} < \lambda < 2-\sqrt{\frac{2w}{c+2k+w}} \text{ and } p_j < \frac{(2-\lambda)(c+2k+w)}{2} \\ \{p_j/(2-\lambda)-k\} & \text{if } 2-\sqrt{2}\sqrt{\frac{w}{c+2k+w}} \leq \lambda < 1 \text{ or} \\ & \frac{2(c+2k)}{c+2k+w} < \lambda < 2-\sqrt{\frac{2w}{c+2k+w}} \text{ and } p_j > \frac{(2-\lambda)(c+2k+w)}{2} \\ \{p_j/(2-\lambda)-k, p_j(2-\lambda)\} & \text{if } \frac{2(c+2k)}{c+2k+w} < \lambda < 2-\sqrt{\frac{2w}{c+2k+w}} \text{ and } p_j = \frac{(2-\lambda)(c+2k+w)}{2} \end{cases}$$

- If $w \leq p_j \leq w(2-\lambda)$, then anything above w is not optimal. The question is whether to choose w and split the market, or to choose $p_j/(2-\lambda)-k$.
$$BR(p_j) = \begin{cases} \{w\} & \text{if } 0 < \lambda < \frac{2(c+2k)}{c+2k+w} \text{ and } p_j < \frac{(2-\lambda)(c+2k+w)}{2} \\ \{p_j/(2-\lambda)-k\} & \text{if } \frac{2(c+2k)}{c+2k+w} \leq \lambda < 1 \text{ or } p_j > \frac{(2-\lambda)(c+2k+w)}{2} \\ \{p_j/(2-\lambda)-k, p_j(2-\lambda)\} & \text{if } 0 < \lambda < \frac{2(c+2k)}{c+2k+w} \text{ and } p_j = \frac{(2-\lambda)(c+2k+w)}{2} \end{cases}$$

- If $p_j > w(2-\lambda)$, then clearly $BR(p_j) = \{w\}$.

We conclude that, in this case, a pure-strategy equilibrium exists if and only if $w \in BR(w)$, i.e. if and only if $0 < \lambda < \frac{2(c+2k)}{c+2k+w}$ (note that $w \leq \frac{(2-\lambda)(c+2k+w)}{2}$ for those values of λ). See Figures 7, 8 and 9 for a depiction of Case I and its three sub-cases.

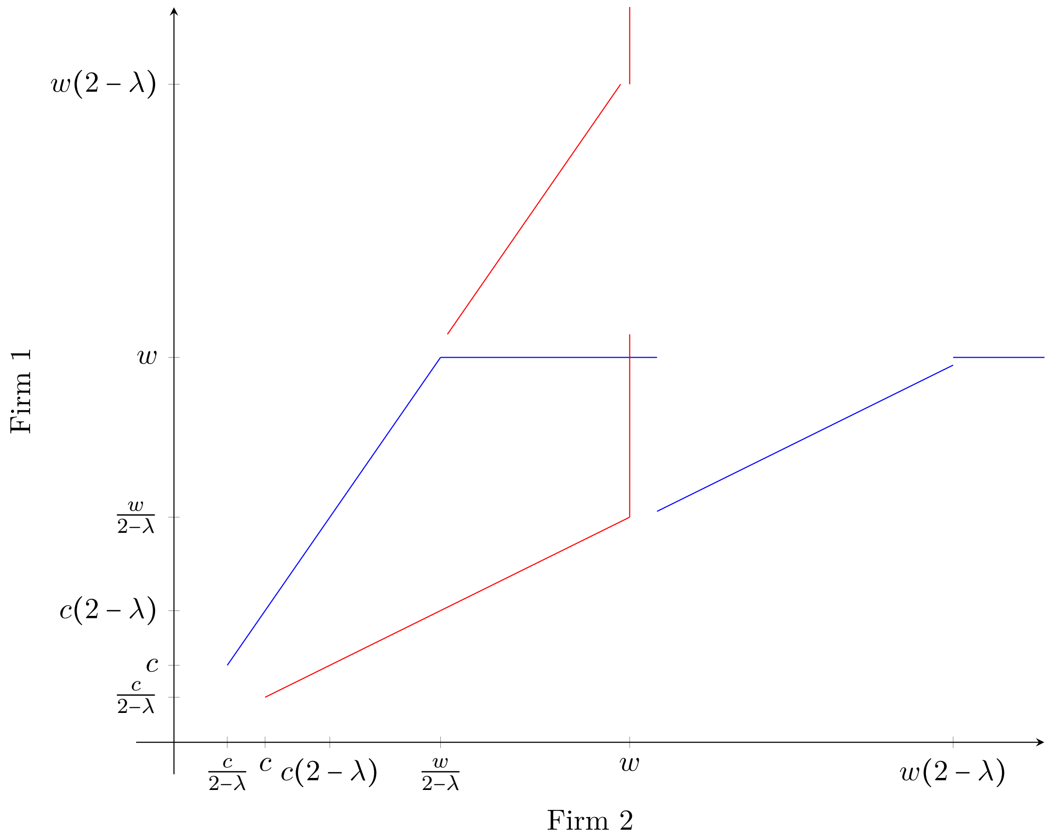

Figure 7: Case I ($c < w/(2-\lambda)^2$), low λ ($\lambda < 2(c+2k)/(c+2k+w)$)

Case II: $c(2-\lambda) \geq w/(2-\lambda) \geq c$ In Cases II and III, where margins are sufficiently low, the unique Nash equilibrium occurs when both firms choose $p_1 = p_2 = w$. The full analysis is similar to the one for Case I:

- $p_j \leq c/(2-\lambda)$
$$BR(p_j) = [c, \infty)$$
- $c/(2-\lambda) < p_j \leq c(2-\lambda)$
$$BR(p_j) = \{p_j(2-\lambda)\}$$
- $w/(2-\lambda) < c(2-\lambda) < p_j \leq w$ The two options are w or $p_j/(2-\lambda) - k$. As it turns out, by comparing profits we obtain
$$BR(p_j) = w.$$

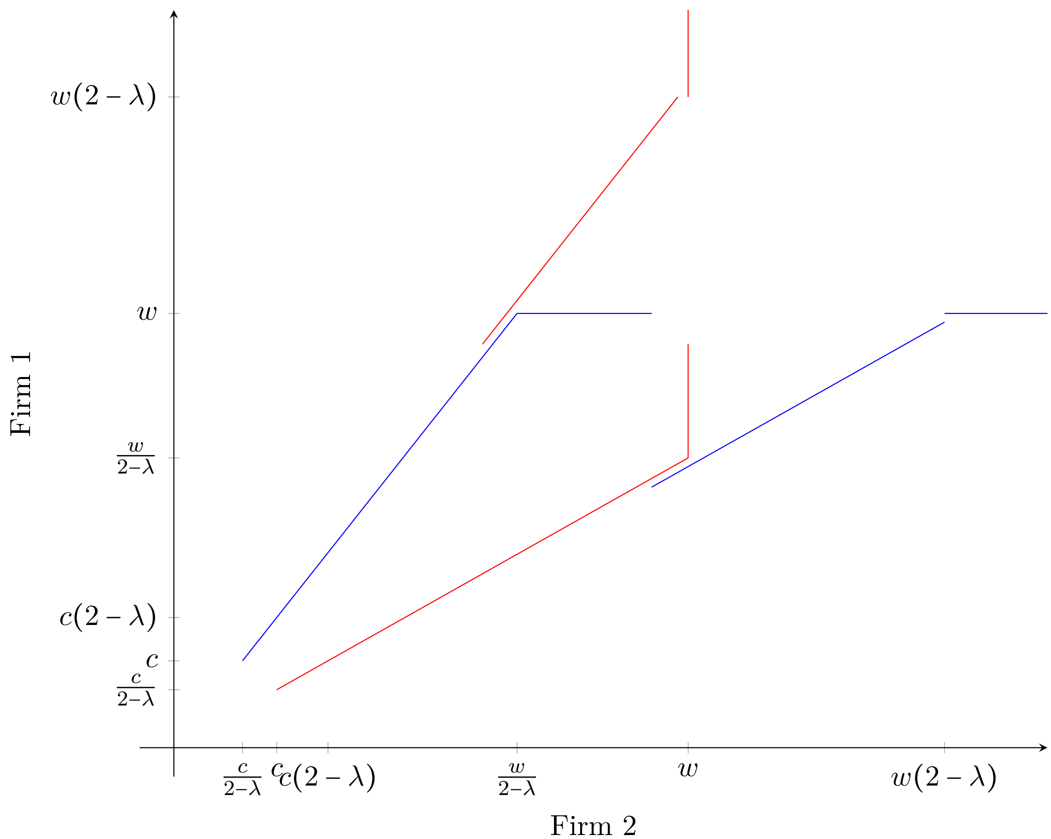

Figure 8: Case I ($c < w/(2-\lambda)^2$), intermediate λ ($2(c+2k)/(c+2k+w) < \lambda < 2-\sqrt{\frac{2w}{c+2k+w}}$)

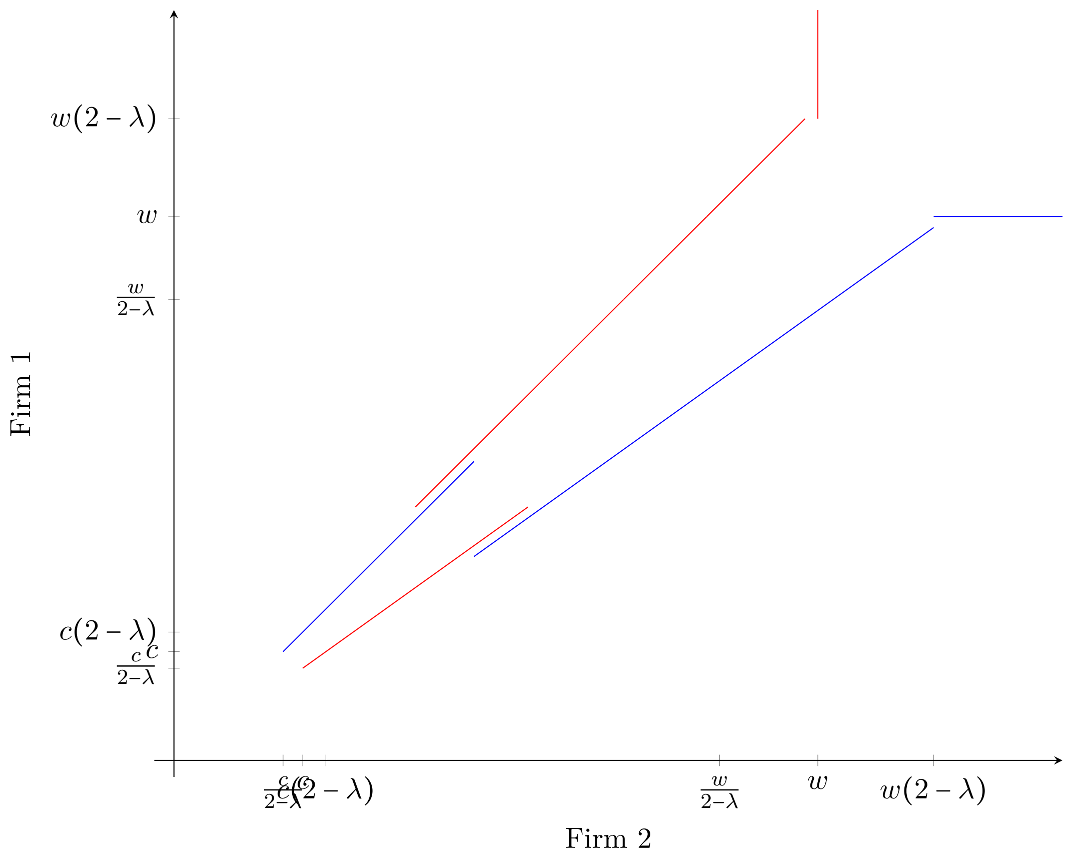

Figure 9: Case I ($c < w/(2-\lambda)^2$), high λ ($\lambda > 2 - \sqrt{\frac{2w}{c+2k+w}}$)

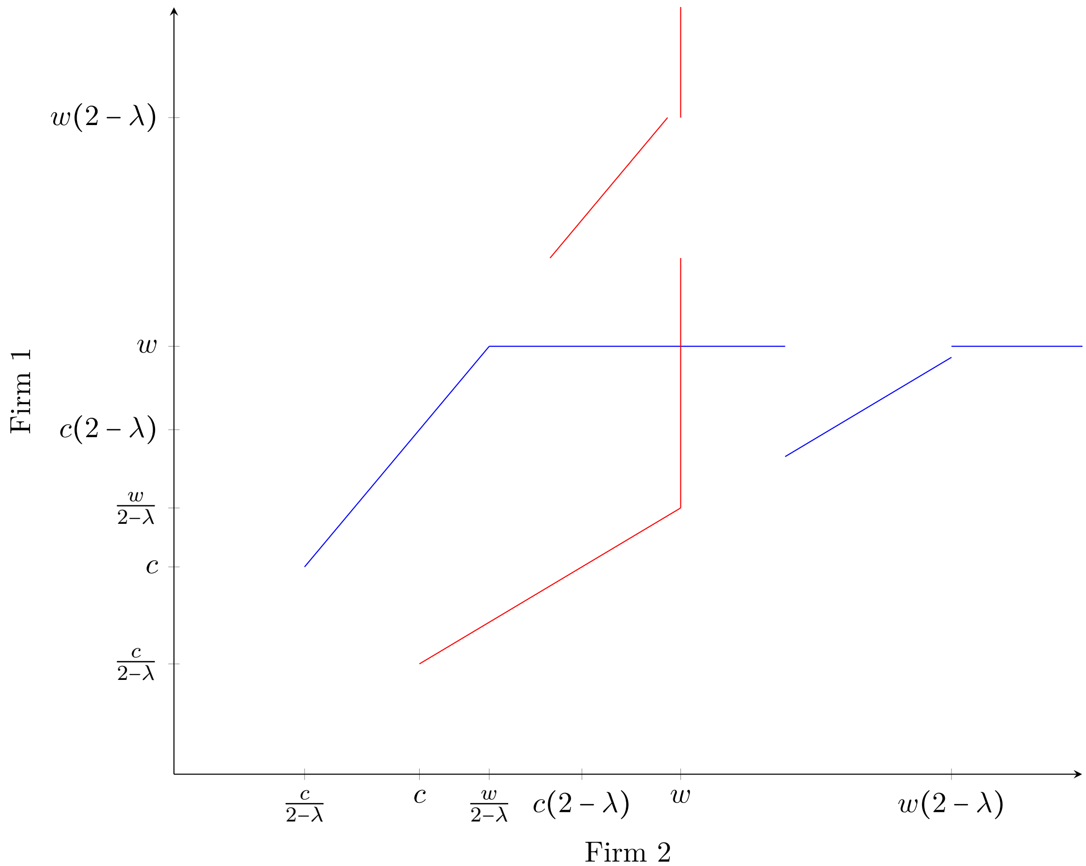

Figure 10: Case II $(w/(2-\lambda) \geq c \geq w/(2-\lambda)^2)$

- $w < p_j \leq w(2-\lambda)$

$$BR(p_j) = \begin{cases} \{w\} & \text{if } p_j < \frac{(2-\lambda)(c+2k+w)}{2} \\ \{p_j/(2-\lambda) - k\} & \text{if } p_j > \frac{(2-\lambda)(c+2k+w)}{2} \\ \{w, p_j/(2-\lambda) - k\} & \text{if } p_j = \frac{(2-\lambda)(c+2k+w)}{2}. \end{cases}$$

- $p_2 > w(2-\lambda)$ In this case,

$$BR(p_j) = w.$$

See Figure 10

Case III: $c(2-\lambda) > w > c > w/(2-\lambda)$

- $p_j \leq c_j/(2-\lambda)$

$$BR(p_j) = [c, \infty)$$

- $c/(2-\lambda) < p_j \leq c$
$$BR(p_j) = \{\min\{p_j(2-\lambda), w\}\}$$

- $c < p_j \leq c(2-\lambda)$ Because $p_j(2-\lambda) > c(2-\lambda) > w$,
$$BR(p_2) = \{w\}$$

- $c(2-\lambda) < p_j \leq w(2-\lambda)$
$$BR(p_j) = \begin{cases} \{w\} & \text{if } p_j < \frac{(2-\lambda)(c+2k+w)}{2} \\ \{p_j/(2-\lambda) - k\} & \text{if } p_j > \frac{(2-\lambda)(c+2k+w)}{2} \\ \{w, p_j/(2-\lambda) - k\} & \text{if } p_j = \frac{(2-\lambda)(c+2k+w)}{2}. \end{cases}$$

- $p_2 > w(2-\lambda)$
  In this case,
$$BR(p_2) = \{w\}.$$

See Figure 11 for a depiction of Case III.

■

A.5 Proofs: monopolist pricing and brand loyalty

Proposition 3.2. *For each $\lambda \in (0,1)$, there exist thresholds $\overline{v}_\lambda$ and $\overline{v}^\lambda \in (0,1)$ with $\overline{v}_\lambda < \overline{v}^\lambda$, such that the monopolist's optimal prices $p^*(\lambda, \overline{v})$ satisfy:*

$$p^*(\lambda, \bar{v}) = \begin{cases} \frac{1+(1-\lambda)^2}{2} & \text{if } \overline{v} \in [0, \overline{v}_\lambda], \\ \overline{v} + (1-\lambda)^2 & \text{if } \overline{v} \in (\overline{v}_\lambda, \overline{v}^\lambda), \\ \frac{\lambda}{2} & \text{if } \overline{v} \in [\overline{v}^\lambda, 1]. \end{cases} \tag{9}$$

Moreover, the optimal profit $\pi^*(\lambda, \bar{v}) := \pi(\lambda, \bar{v}; p^*(\lambda, \bar{v}))$ is continuous in $\bar{v}$ and satisfies:

$$\pi^*(\lambda, \overline{v}) = \begin{cases} \left(\frac{1+(1-\lambda)^2}{2}\right)^2 & \text{if } \overline{v} \in [0, \overline{v}_\lambda], \\ \left(\overline{v} + (1-\lambda)^2\right)(1-\overline{v}) & \text{if } \overline{v} \in (\overline{v}_\lambda, \overline{v}^\lambda), \\ \left(\frac{\lambda}{2}\right)^2 & \text{if } \overline{v} \in [\overline{v}^\lambda, 1]. \end{cases} \tag{10}$$

Proof. Maximizing each piece of the profit function, we get

$$p = \min\{\lambda/2, \overline{v} - (1-\lambda)\}$$

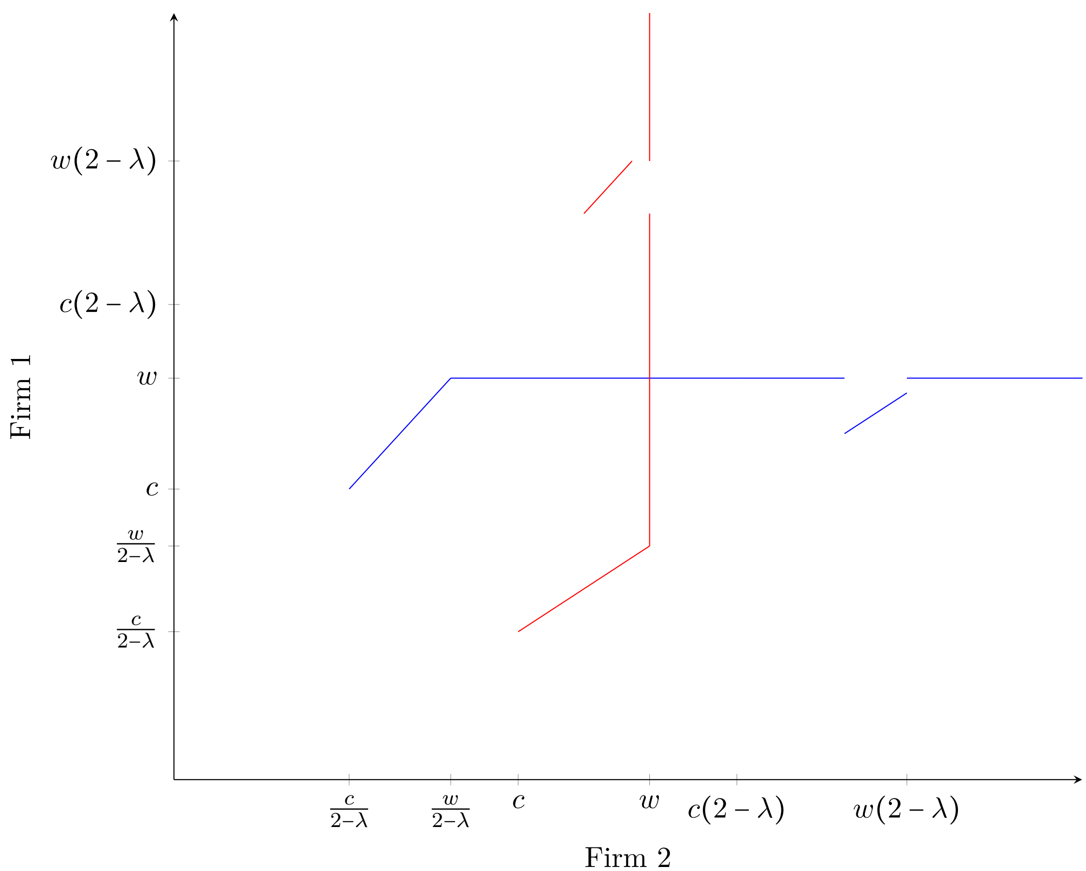

Figure 11: Case III ($c > w/(2-\lambda)$)

for the top part,

$$p = \overline{v} + (1-\lambda)^2$$

for the middle part, and

$$p = \max\{(1+(1-\lambda)^2)/2, \overline{v} + (1-\lambda)^2\}$$

for the bottom part. Using each of these prices, profits are then:

$$\min\{\lambda/2, \overline{v} - (1-\lambda)\}(\lambda - \min\{\lambda/2, \overline{v} - (1-\lambda)\})$$

for the top price,

$$(\overline{v} + (1-\lambda)^2)(1-\overline{v})$$

for the middle price, and

$$\max\{(1+(1-\lambda)^2)/2, \overline{v} + (1-\lambda)^2\}(1 - \max\{(1+(1-\lambda)^2)/2, \overline{v} + (1-\lambda)^2\} + (1-\lambda)^2)$$

for the bottom price.

Now, all that remains is to determine which of the three possibilities above is better for each pair $(\lambda, \overline{v})$. Solving yields:

$$p^*(\lambda, \overline{v}) = \begin{cases} \frac{1+(1-\lambda)^2}{2} & \text{if } \overline{v} \leq \frac{1}{2} - \frac{(1-\lambda)^2}{2} \\ \overline{v} + (1-\lambda)^2 & \text{if } \frac{1}{2} - \frac{(1-\lambda)^2}{2} < \overline{v} < \frac{1}{2}\left(2\lambda - \lambda^2 + \sqrt{\lambda^4 - 4\lambda^3 + 7\lambda^2 - 8\lambda + 4}\right) \\ \frac{\lambda}{2} & \text{if } \overline{v} \geq \frac{1}{2}\left(2\lambda - \lambda^2 + \sqrt{\lambda^4 - 4\lambda^3 + 7\lambda^2 - 8\lambda + 4}\right) \end{cases}$$

with maximal profits

$$\pi^*(\lambda, \overline{v}) = \begin{cases} \left(\frac{1+(1-\lambda)^2}{2}\right)^2 & \text{if } \overline{v}_1 \leq \frac{1}{2} - \frac{(1-\lambda)^2}{2} \\ \left(\overline{v}_1 + (1-\lambda)^2\right)(1-\overline{v}_1) & \text{if } \frac{1}{2} - \frac{(1-\lambda)^2}{2} < \overline{v}_1 < \frac{1}{2}\left(2\lambda - \lambda^2 + \sqrt{\lambda^4 - 4\lambda^3 + 7\lambda^2 - 8\lambda + 4}\right) \\ \left(\frac{\lambda}{2}\right)^2 & \text{if } \overline{v}_1 \geq \frac{1}{2}\left(2\lambda - \lambda^2 + \sqrt{\lambda^4 - 4\lambda^3 + 7\lambda^2 - 8\lambda + 4}\right) \end{cases}$$

so that $\overline{v}_\lambda := \frac{1}{2} - \frac{(1-\lambda)^2}{2}$ and $\overline{v}^\lambda := \frac{1}{2}\left(2\lambda - \lambda^2 + \sqrt{\lambda^4 - 4\lambda^3 + 7\lambda^2 - 8\lambda + 4}\right)$. ■

Proposition 3.3. *When $T = 1$, the optimal price is given by $p_1^*(\lambda) = \lambda/2$, and profits are $\Pi^*(\lambda) = \lambda^2/4$.*

When $T \geq 2$, $p_1^*(\lambda) = \lambda - 1/2$ and $p_t^*(\lambda) = 1/2$ for all $t \in \{2, \ldots, T\}$. Total profits are given by $\Pi^*(\lambda) = \lambda(\lambda - 1/2) + (T-1)1/4$.

Proof. Consider the case of $T = 2$.

At $t = T$, the solution follows:

$$p_t^*(\lambda, \overline{v}_t) = \begin{cases} \frac{\lambda}{2} & \text{if } \bar{v} \geq \bar{v}^\lambda \\ \overline{v}_t & \text{if } \frac{1}{2} < \overline{v} < \bar{v}^\lambda \\ \frac{1}{2} & \text{if } \overline{v} \leq \frac{1}{2} \end{cases}$$

where $\bar{v}^\lambda = \frac{1}{2}\left(1 + \sqrt{1 - \lambda^2}\right)$.

with profit

$$\pi_t^*(\lambda, \overline{v}_t) = \begin{cases} \frac{\lambda^2}{4} & \text{if } \bar{v}_t \geq \bar{v}_t^\lambda \\ \overline{v}_t(1 - \overline{v}_t) & \text{if } \frac{1}{2} < \overline{v}_t < \bar{v}_t^\lambda \\ \frac{1}{4} & \text{if } \overline{v}_t \leq \frac{1}{2} \end{cases}$$

Since at $t = 1$, $\bar{v}_1 = 1$, we have that $\bar{v}_2 = \min\{p_{t-1} + (1 - \lambda), 1\}$. Hence we can write the optimal second-period prices as a function of the first-period prices:

$$p_t^*(\lambda, p_{t-1}) = \begin{cases} \frac{\lambda}{2} & \text{if } p_{t-1} + (1 - \lambda) \geq \bar{v}_t^\lambda \\ \overline{v}_t & \text{if } \frac{1}{2} < p_{t-1} + (1 - \lambda) < \bar{v}^\lambda \\ \frac{1}{2} & \text{if } p_{t-1} + (1 - \lambda) \leq \frac{1}{2} \end{cases}$$

Thus, at $t = 1$, the monopolist maximizes the following combined profit function:

$$\Pi(\lambda, \bar{v}_1; p_1) = \begin{cases} \lambda^2/4 & \text{if } p_1 + (1 - \lambda) \geq 1 \\ (\lambda - p_1)p_1 + \lambda^2/4 & \text{if } 1 \geq p_1 + (1 - \lambda) \geq \bar{v}^\lambda \\ (\lambda - p_1)p_1 + (\lambda - p_1)(p_1 + (1 - \lambda)) & \text{if } \bar{v}^\lambda > p_1 + (1 - \lambda) > 1/2 \\ (\lambda - p_1)p_1 + 1/4 & \text{if } p_1 + (1 - \lambda) \leq 1/2 \end{cases} \tag{16}$$

Maximizing the expression above yields $p_1^* = \lambda - 1/2$. Note that p_1^* can very well be negative, if $\lambda < 1/2$; this is worth it because it drives down $\bar{v}_2$.

If $T > 2$, then the same strategy should be optimal. Choosing an identical p_1 as in the case above ensures that the profits are as high as possible in the first two periods. For the remaining periods, the firm can then pick $p_t(\lambda) = 1/2$ and ensure maximum per-period profits at every subsequent t. Thus, there is no better strategy. ∎

A.6 Proofs and auxiliary results: revealed fuzzy preferences

Proposition A.3. *If R is a fuzzy complete relation, the following two properties are equivalent.*

(i) For all $x, y, z \in X$, $\max\{R(x,y), R(y,z)\} = 1 \implies R(x,z) \geq \min\{R(x,y), R(y,z)\}$.

(ii) For all $x, y, z \in X$, $\min\{R(x,y), R(y,z)\} = 1 \implies R(z,x) \leq \min\{R(z,y), R(y,x)\}$.

Proof. Suppose (i) holds. Fix $x, y, z \in X$ such that $\min\{R(x,y), R(y,z)\} = 1$. Then, $R(z,y) \geq \min\{R(z,x), R(x,y)\} = R(z,x)$. Similarly, $R(y,x) \geq \min\{R(y,z), R(z,x)\} = R(z,x)$. Putting both conclusions together, we get $R(z,x) \leq \min\{R(z,y), R(y,x)\}$. Note that this direction does not require fuzzy completeness.

Now suppose (ii) holds. Fix $x, y, z \in X$ such that $\max\{R(x,y), R(y,z)\} = 1$.

If $R(x,z) = 1$, then there's nothing to prove. Then consider, in turns, that exactly one of $R(x,y), R(y,z)$ is equal to 1, and that $R(x,z) < 1$.

Suppose $R(x,y) = 1$, $R(y,z) < 1$. Fuzzy completeness gives us that $R(z,x) = 1$. Then, $R(y,z) \leq \min\{R(y,x), R(x,z)\}$. This implies $R(x,z) \geq R(y,z) = \min\{R(x,y), R(y,z)\}$.

Now suppose that $R(x,y) < 1$ and $R(y,z) = 1$. Fuzzy completeness gives us $R(y,x) = 1$. Then, $R(x,y) \leq \min\{R(x,z), R(z,y)\}$. This implies $R(x,z) \geq R(x,y) = \min\{R(x,y), R(y,z)\}$. ■

Theorem 4.1. *The following are equivalent.*

(i) The family $\mathcal{C}$ satisfies WARPD and WOCI.

(ii) The family $\mathcal{C}$ admits a cognition-dependent representation.

(iii) The family $\mathcal{C}$ is fuzzy-rationalizable.

Proof. The equivalence between (i) and (ii) has already been established, so it suffices to show that (i) and (iii) are equivalent.

Suppose WARPD and WOCI hold, and take Λ to be a linear order with maximum $\overline{\lambda}$. Let $\alpha : \Lambda \longrightarrow (0,1]$ be any strictly increasing function with $\alpha(\overline{\lambda}) = 1$.

Let $R(x,y) = \max\{\alpha(\lambda) : x \in C_\lambda(\{x,y\})\}$.

We must show that R is fuzzy rational and that, for all S, $C_\lambda(S) = C^{R;\alpha(\lambda)}(S)$.

Because each C_λ is nonempty, $C_{\overline{\lambda}}(\{x,y\})$ contains either x, or y, or both. Then, $R(x,y) = 1$ or $R(y,x) = 1$, i.e. R is fuzzy complete.

Now suppose $R(x,y) = 1$ and $R(y,z) = 1$ and consider $\lambda = \max\{\mu \in \Lambda : z \in C_\mu(\{x,z\})\}$. Note that $R(z,x) = \alpha(\lambda)$, since α is strictly increasing. Successive applications of WARPD yield: $z \in C_\lambda(\{x,y,z\}) \implies z \in C_\lambda(\{y,z\}) \implies y \in C_\lambda(\{x,y,z\}) \implies y \in C_\lambda(\{x,y\})$. That means that $R(y,x) \geq \alpha(\lambda)$ and $R(z,y) \geq \alpha(\lambda)$ both hold, and thus R is fuzzy transitive.

Finally, we show that $C_\lambda(S) = C^{R;\alpha(\lambda)}(S)$ for every $S \in \mathcal{S}$. Let $x \in C_\lambda(S)$. By WARPD, $x \in C_\lambda(\{x,y\})$ for all $y \in S$. Thus, $R(x,y) \geq \alpha(\lambda)$ for all $y \in S$. But then, $x \in C^{R;\alpha(\lambda)}(S)$.

Conversely, suppose $x \in C^{R;\alpha(\lambda)}(S)$. Then, $R(x,y) \geq \alpha(\lambda)$ for all $y \in S$. That means $x \in C_\lambda(\{x,y\})$ for all $y \in S$. By WARPD, we have that $x \in C_\lambda(S)$.

Now assume that (iii) holds. We show that WARPD and WOCI hold.

Let Λ be ordered according to α. That is, say that $\lambda' > \lambda$ whenever $\alpha(\lambda') > \alpha(\lambda)$. WOCI clearly holds since the α-cuts are nested. Moreover, $\alpha(\overline{\lambda}) = 1$ for some $\overline{\lambda}$; since $1 \geq \alpha(\lambda)$ for all λ, $\overline{\lambda} \geq \lambda$ for all λ.

For WARPD, fix $x, y \in S \cap T$ with $x \in C_{\overline{\lambda}}(S)$ and $y \in C_\lambda(T)$.

Suppose $x \notin C_\lambda(T)$. Then, there exists $z_T \in T$ such that $R(x, z_T) < \alpha(\lambda)$. By fuzzy completeness, $R(z_T, x) = 1$. Since $x \in C_{\overline{\lambda}}(S)$, $R(x,y) \geq \alpha(\overline{\lambda}) = 1 \implies R(x,y) = 1$.

Then, by fuzzy transitivity, $R(y, z_T) \leq R(x, z_T) < \alpha(\lambda)$; thus $y \not\in nC_\lambda(T)$, a contradiction. Thus, $x \in C_\lambda(T)$.

Now suppose $y \notin C_\lambda(S)$. That is, there exists $z_S \in S$ such that $R(y, z_S) < \alpha(\lambda)$. That means $R(z_S, y) = 1$, by fuzzy completeness. Moreover, $R(x, z_S) = 1$, since $x \in C_{\overline{\lambda}}(S)$. By fuzzy transitivity, we have that $R(y,x) \leq R(y, z_S) < \alpha(\lambda)$. But then y couldn't be in $C_\lambda(T)$, a contradiction. ■

A.7 Proofs: Extensions

A.7.1 Fuzzy rationalizability: an alternative characterization

Instead of dealing with a fuzzy rational preference relation R, one can equivalently characterize a system of nested crisp binary relations. Let A be a finite subset of $[0,1]$, with $1 \in A$, and consider the following definition.

Definition A.1 (System of nested semiorders (SNS))**.** A family $\{R_\alpha : \alpha \in A\}$ of (crisp) binary relations a system of nested semiorders if:

(i) $xR_{\alpha'}y \implies xR_\alpha y$ for all $\alpha' > \alpha$.

(ii) If xR_1y and yR_1z, then $zR_\alpha x$ implies $zR_\alpha y$ and $yR_\alpha x$.

(iii) R_1 is a weak order.

◊

A family $\{R_\alpha : \alpha \in A\}$ is a system of nested *semiorders* because each R_α is, in fact, a semiorder, as the following proposition demonstrates.

Proposition A.4. *If $\{R_\alpha : \alpha \in A\}$ is a SNS, then for each $\alpha \in A$, R_α is a semiorder.*

Proof. **Partial order:** Completeness and reflexivity is a direct consequence of (iii) and (i). If $z \not R_\alpha y$ and $y \not R_\alpha x$, then by (iii) $z \not R_\alpha x$.

Semitransitivity: if $z \not R_\alpha y$ and $y \not R_\alpha x$, then if $w R_\alpha x$ we have two cases. First, if $w R_1 x$, then we have that $w R_1 x$ and $x R_1 z$. By (ii), because $z \not R_\alpha x$, it must be that $z \not R_\alpha w$.

Second, if $x R_1 w$, then it must be that $x R_1 w$ and $w R_1 y$ (Otherwise, $w R_\alpha x$ would imply $y R_\alpha x$, by (iii)). Thus, $w R_1 y$ and $y R_1 z$, so since $z \not R_\alpha y$, it must be that $z \not R_\alpha w$.

Interval order: Finally, we show that the interval order property holds. Suppose $x \not R_\alpha y$ and $z \not R_\alpha w$. If $x R_\alpha w$, then we have two cases.

First, if $x R_1 w$, then $y R_1 x$, $x R_1 w$ and $w R_1 z$, so $x \not R_\alpha z$, Because $y R_1 x$ and $x R_1 z$, this implies $z \not R_\alpha y$.

Second, if $w R_1 x$, then because $x R_\alpha w$ it must be that $y R_1 w$. Since $w R_1 z$, we must then have that $z \not R_\alpha y$. ∎

Proposition A.5. *The family* $\{R_\alpha : \alpha \in A\}$ *is an SNS if and only if the function* $R : X \times X \longrightarrow [0,1]$ *defined by*

$$R(x,y) := \max\{\alpha \in A : x R_\alpha y\}, \quad \textit{for all } x, y \in X \tag{17}$$

is fuzzy rational.

Proof. Let $\{R_\alpha : \alpha \in A\}$ be a system of nested semiorders.

since $x R_\alpha y \iff R(x,y) \geq \alpha$, it must be that $R(x,y) = \max\{\alpha \in A : x R_\alpha y\}$

Since R_1 is a weak order, $x R_1 y$ or $y R_1 x$ for all $x, y \in X$. Thus, $\max\{R(x,y), R(y,x)\} = 1$.

Now suppose $R(x,y) = 1$ or $R(y,z) = 1$ for some $x, y, z \in X$. Let us consider a few cases.

If both $R(x,y) = 1$ and $R(y,z) = 1$ then $x R_1 y$ and $y R_1 z$, implying $x R_1 z$ (by (iii)), and thus $R(x,z) = 1$.

If $R(x,y) = 1$ and $R(y,z) < 1$, then it must be that $R(z,y) = 1$. If $R(x,z) = 1$, then there's nothing left to prove. If $R(z,x) = 1$, then $z R_1 x$ and $x R_1 y$. For $\alpha = R(y,z)$, we have that $y R_\alpha z$. That means $y R_\alpha x$ and $x R_\alpha z$, implying $R(x,z) \geq \alpha = R(y,z) = \min\{R(x,y), R(y,z)\}$.

If $R(x,y) < 1$ and $R(y,z) = 1$, then it must be that $R(y,x) = 1$. If $R(x,z) = 1$, then there's nothing left to prove. If $R(z,x) = 1$, then $y R_1 z$ and $z R_1 x$. For $\alpha = R(x,y)$, we have that $x R_\alpha y$. That means $x R_\alpha z$ and $z R_\alpha y$, implying $R(x,z) \geq \alpha = R(x,y) = \min\{R(x,y), R(y,z)\}$.

Now fix $R : X \times X \longrightarrow [0,1]$ fuzzy rational, and let $x R_\alpha y \iff R(x,y) \geq \alpha$. The first property holds trivially.

If xR_1y and yR_1z, then $R(x,y) = R(y,z) = 1$. Then, $R(z,x) \leq \min\{R(z,y), R(y,x)\}$. So whenever $zR_\alpha x$, $zR_\alpha y$ and $yR_\alpha x$.

Finally, $\max\{R(x,y), R(y,x)\} = 1$ implies xR_1y or yR_1x, i.e. R_1 is complete. Moreover, xR_1y and $yR_1z \iff R(x,y) = R(y,z) = 1$, which in turn implies $R(x,z) = 1 \iff xR_1z$, so R_1 is transitive. Thus, R_1 is a weak order. ∎

The above implies that fuzzy rationalizability could be alternatively characterized in terms of an SNS, since the above equivalence implies

Corollary A.5.1.

$$C^{R;\alpha}(S) = \{x \in S \,:\, \textit{for all } y \in S,\ xR_\alpha y\}.$$

A.7.2 Incomplete data: proofs

Proposition 5.1 (WARPD decomposition)**.** *A family $\mathcal{C}$ satisfies WARPD if and only if the three conditions below hold.*

(i) ***Contraction.*** *For all $S, T \in \mathcal{S}$ with $S \subseteq T$, $C_\lambda(T) \cap S \subseteq C_\lambda(S)$.*

(ii) ***Expansion.*** *For all $S \in \mathcal{S}$ and $x \in S$, if $x \in C_\lambda(\{x,y\})$ for all $y \in S$, then $x \in C_\lambda(S)$.*

(iii) ***No Cognitive Cycles.*** *For all $x, y, z \in X$, if either $x \in C_{\overline{\lambda}}(\{x,y\})$ and $y \in C_\lambda(\{y,z\})$ or $x \in C_\lambda(\{x,y\})$ and $y \in C_{\overline{\lambda}}(\{y,z\})$, then $x \in C_\lambda(\{x,z\})$.*

Proof. **WARPD $\implies$ (i):**

Suppose $x \in C_\lambda(T) \cap S$. Let $z \in C_{\overline{\lambda}}(S)$. By WARPD, since $x, z \in S \cap T$, $x \in C_\lambda(T)$ and $z \in C_{\overline{\lambda}}(S)$, we must have that $x \in C_\lambda(S)$.

WARPD $\implies$ (ii):

Suppose $x \in S$, and $x \in C_\lambda(\{x,y\})$ for all $y \in S$. Let $z \in C_{\overline{\lambda}}(S)$. By WARPD, since $x, z \in \{x,z\} \cap S$, $x \in C_\lambda(\{x,z\})$ and $z \in C_{\overline{\lambda}}(S)$, we must have that $x \in C_\lambda(S)$.

WARPD $\implies$ (iii):

Suppose $x \in C_\lambda(\{x,y\})$ and $y \in C_{\overline{\lambda}}(\{y,z\})$. Note that WARPD implies $y \in C_\lambda(\{y,z\})$ as well. Now consider $\{x,y,z\}$. If $x \in C_{\overline{\lambda}}(\{x,y,z\})$, then WARPD guarantees that $x \in C_{\overline{\lambda}}(\{x,z\})$ and thus $x \in C_\lambda(\{x,z\})$. If $y \in C_{\overline{\lambda}}(\{x,y,z\})$, then by WARPD $x \in C_\lambda(\{x,y,z\})$. Then, even if $z \in C_{\overline{\lambda}}(\{x,z\})$ (and not x), WARPD guarantees that $x \in C_\lambda(\{x,z\})$. Finally, if $z \in C_{\overline{\lambda}}(\{x,y,z\})$, then $y \in C_{\overline{\lambda}}(\{x,y,z\})$ as well (by WARPD, since $y \in C_{\overline{\lambda}}(\{y,z\})$). Since $x \in C_\lambda(\{x,y\})$, this means that $x \in C_\lambda(\{x,y,z\})$. Then, even if $z \in C_{\overline{\lambda}}(\{x,z\})$ (and not x), WARPD guarantees that $x \in C_\lambda(\{x,z\})$.

Similarly, suppose now $x \in C_{\overline{\lambda}}(\{x,y\})$ and $y \in C_\lambda(\{y,z\})$. Now consider $\{x,y,z\}$. If $x \in C_{\overline{\lambda}}(\{x,y,z\})$, then WARPD guarantees that $x \in C_{\overline{\lambda}}(\{x,z\})$ and thus $x \in C_\lambda(\{x,z\})$. If $y \in C_{\overline{\lambda}}(\{x,y,z\})$, then by WARPD $x \in C_{\overline{\lambda}}(\{x,y,z\})$, thus $x \in C_{\overline{\lambda}}(\{x,z\})$ and $x \in C_\lambda(\{x,z\})$. Finally, if $z \in C_{\overline{\lambda}}(\{x,y,z\})$, then $y \in C_\lambda(\{x,y,z\})$ as well and thus $x \in C_\lambda(\{x,y,z\})$. Then, even if $z \in C_{\overline{\lambda}}(\{x,z\})$ (and not x), WARPD still guarantees that $x \in C_\lambda(\{x,z\})$.

(i)+(ii)+(iii) $\implies$ WARPD:

Let $\{x,y\} \subseteq S \cap T$ with $x \in C_{\overline{\lambda}}(S)$ and $y \in C_\lambda(T)$.

By Contraction, $x \in C_{\overline{\lambda}}(\{x,y\})$ and $y \in C_\lambda(\{y,z\})$ for all $z \in T$. By No Cycles, $x \in C_\lambda(\{x,z\})$ for all $z \in T$. By Expansion, $x \in C_\lambda(T)$.

Moreover, by Contraction, $x \in C_{\overline{\lambda}}(\{x,t\})$ for all $t \in S$ and $y \in C_\lambda(\{x,y\})$. By No Cycles, $y \in C_\lambda(\{y,t\})$ for all $t \in S$. By Expansion, $y \in C_\lambda(S)$. ∎

A.7.3 X infinite: proofs

Theorem 5.1. *Suppose X is compact. The family $\mathcal{C}$ satisfies WARPD, WOCI and continuity if and only if it admits a cognition-dependent representation where u is continuous.*

Proof. The continuity of u is a standard for regular rational preferences; see e.g. Kreps (1988).

The rest of the proof goes through basically without any changes. The only thing to note is that, in setting

$$\varepsilon(\lambda,x) := \max_{z\in X}\{u(z) : x \in C_\lambda(\{x,z\})\} - u(x)$$

We have that $\{z \in X : x \in C_\lambda(\{x,z\})\}$ is a closed subset of X (and thus compact) if R_λ as defined is continuous. Then, the continuity of u ensures its maximum is attained within the set. Thus ε remains well-defined. ∎

Corollary 5.2.1. *The function u in a cognition-dependent representation is unique up to a monotone transformation. Moreover, given u, each $\varepsilon(\lambda,x)$ is bounded by some $\overline{\varepsilon}(\lambda,x) > 0$ and $\underline{\varepsilon}(\lambda,x) > 0$ satisfying $\underline{\varepsilon}(\lambda,x) \geq \sup_{\lambda'>\lambda}\varepsilon(\lambda',x)$:*

$$\overline{\varepsilon}(\lambda,x) > \varepsilon(\lambda,x) \geq \underline{\varepsilon}(\lambda,x).$$

Moreover, if X is connected, then the threshold function ε is unique given u.

Proof. u is unique up to a monotone transformation and $\inf\{u(z) : x \not R_\lambda z\} - u(x) > \varepsilon(\lambda,x) \geq \max\{\sup_{\lambda'>\lambda}\varepsilon(\lambda',x), \max\{u(z) : xR_\lambda z\}\} - u(x)$.

If, given u, for every $\tilde{u} \in (u(z), u(z'))$ for all z, z' in X, there exists $\tilde{z}$ such that $u(\tilde{z}) = \tilde{u}$, then $\varepsilon(\lambda,x)$ is unique given u (a sufficient condition for this is for X to be connected). ∎

A.7.4 No intransitive indifferences: proofs

Theorem 5.2. *The following are equivalent.*

(i) The family $\mathcal{C}$ satisfies WARPC and WOCI.

(ii) The family $\mathcal{C}$ admits a cognition-dependent categorical representation.

(iii) The family $\mathcal{C}$ is strongly fuzzy-rationalizable.

Proof. **WARPC and WOCI $\implies$ CDCR**

Suppose $u(x) > u(y) + \varepsilon(\lambda, y)$ and fix z with $u(z) \in [u(y), u(y) + \varepsilon(\lambda, y)]$. Since $y \in C_\lambda(\{y, z_y\})$ and $z_y \in C_\lambda(\{z_y, z\})$, $y \in C_\lambda(\{z, z_y, y\})$, implying $y \in C_\lambda(y, z)$.

By WARPC, if $z \in C_\lambda(\{x, z\})$ and $y \in C_\lambda(\{y, z\})$, then $y \in C_\lambda(\{x, y\})$, a contradiction.

CDCR $\implies$ WARPC and WOCI.

WARPC holds. Let $\lambda \in \Lambda$, $x, y \in X$ and $S, T \in \mathcal{S}$ with $\{x, y\} \subseteq S \cap T$. Let $x \in C_\lambda(S)$ and $y \in C_\lambda(T)$. We show that $x \in C_\lambda(T)$ and $y \in C_\lambda(S)$.

$x \in C_\lambda(T)$. Since $x \in C_\lambda(S)$, $u(x) + \varepsilon(\lambda, x) \geq u(y)$. Suppose there exists $z_T \in T$ such that $u(z_T) > u(x) + \varepsilon(\lambda, x)$. By property (iv), $u(z_T) > u(y) + \varepsilon(\lambda, y)$, a contradiction with the fact that $y \in C_\lambda(T)$.

$y \in C_\lambda(S)$. Identical argument as above, with y and T instead of x and S.

■