Conditional Expected Option Returns*

Caio Almeida[†a], **Fousseni Chabi-Yo**[‡b], and **Gustavo Freire**[§c]

[a]Princeton University
[b]University of Massachusetts in Amherst
[c]Nova School of Business and Economics

March 10, 2026
Preliminary draft: Please do not circulate

Abstract

This paper extends option-implied formulas of the expected market return to obtain conditional expected option returns. Relative to existing pricing kernel specifications, we show that allowing for heterogeneous preferences towards left and right tails is key to accurately fit realized call and put option returns. Using our time series of conditional expected option returns, we find that: alphas with respect to conditional expected market returns are strongly significant, where both calls and puts have negative betas; compensation for upside, downside and variance risk is higher during economic downturns and high-volatility periods; and there is a strong factor structure in option returns, dominated by level and moneyness-slope factors.

Keywords: conditional expectations, option-implied, option returns, pricing kernel, risk premium.

JEL Classification: G11, G12, G13.

*We would like to thank seminar participants at the University of Massachusetts in Amherst for useful comments and suggestions.

†calmeida@princeton.edu, Department of Economics, Princeton University.

‡chabiyofatwork@gmail.com, Isenberg School of Management, University of Massachusetts in Amherst.

§gustavo.freire@novasbe.pt, Finance Department, Nova School of Business and Economics.

1 Introduction

Option markets have grown substantially in size and importance over recent decades. In the U.S., open interest has increased more than tenfold since the early 2000s, with demand for S&P 500 options now representing roughly one-third of total shares in the market index (Farago, Khapko, and Ornthanalai, 2021). Options are no longer niche hedging instruments but central venues for price discovery, risk transfer, and the formation of market expectations. Their flexibility allows investors, from large institutions to a growing share of retail participants (Bryzgalova, Pavlova, and Sikorskaya, 2023), to obtain targeted exposure to specific sources of risk in the underlying asset, such as volatility, downside, and upside risks. Understanding compensation for bearing these risks is therefore crucial for connecting derivatives markets to broader questions in asset pricing.

This risk compensation is encoded in expected option returns. Estimating expected option returns, however, is notoriously challenging, as realized option returns are highly nonnormal and sensitive to rare events (Broadie, Chernov, and Johannes, 2009). As a result, the literature has largely focused on unconditional average option returns over long samples (Bakshi, Madan, and Panayotov, 2010; Coval and Shumway, 2001) or on conditional averages obtained by partitioning the sample according to observable state variables (Almeida and Freire, 2022). While informative, these approaches provide limited insight into the time-varying dynamics of expected option returns.

In this paper, we address this challenge by developing a novel methodology to obtain conditional expected option returns at each point in time without imposing assumptions on the underlying asset price dynamics. We extend option-implied formulas for the expected market return, pioneered by Martin (2017) and Chabi-Yo and Loudis (2020), to derive analogous formulas that recover expected returns for options. Our approach allows for a detailed examination of the evolution of expected option returns across strikes and maturities, revealing new stylized facts about the pricing of risk in option markets.

Option-implied formulas for the expected market return consist of three main ingredients: (1) conditional risk-neutral expectations implied by observed option prices; (2) a specification for the stochastic discount factor (SDF) translating risk-neutral expectations into real-world, physical ones; and (3) the payoff of interest, given by the market return. The seminal contribution of Martin (2017) shows that, under log utility, the expected market return is given by the risk-neutral variance of the market index.[1] Extensions of this approach have focused on generalizing (2). Working with a Taylor expansion of marginal utility, Chabi-Yo and Loudis (2020) obtain the expected market return as a combination of risk-neutral variance, skewness and kurtosis. Tetlock, McCoy, and Shah (2024) consider a similar specification but account for time variation in the coefficients of the risk-neutral moments. Chabi-Yo, Gourier, and Langlois (2025) introduce additional terms related to intertemporal hedging needs.

Our first contribution is to generalize (3) to option payoffs. It is easy to see how this is a generalization: the market index is equivalent to a call option with zero strike price. We provide option-implied formulas for expected option returns for all SDF specifications discussed above. These formulas are similar to those for the market return, but depend instead on truncated versions of risk-neutral moments, where the truncation is determined by the option strike. We then compare, across strikes and maturities, the conditional expected option returns to realized ones using different tests, such as predictive regressions. We find that these specifications are not enough to fully characterize option returns. In particular, none of them can match the negative average returns of out-of-the-money calls, which is related to a U-shaped pricing kernel as a function of market returns (Bakshi et al., 2010), i.e., compensation for upside risk.

Our second contribution is to generalize (2) with an SDF specification embedding the properties of Chabi-Yo and Loudis (2020), Tetlock et al. (2024) and Chabi-Yo et al.

[1] Log utility is required for risk-neutral variance to be an exact measure of the expected market return, while under more general conditions, it serves as a lower bound.

(2025), while allowing for different preference parameters for the left and right tails of market returns. This provides the additional flexibility needed to match expected returns of both put and call options. In essence, we combine the Taylor expansion of marginal utility across different regions of the return space in the spirit of Chabi-Yo and Loudis (2024) with the intertemporal hedging needs of Chabi-Yo et al. (2025). We show that this specification yields expected option returns that come much closer to reproducing observed patterns in realized option returns.

Our third contribution is to use our preferred specification for conditional expected option returns to document new stylized facts about the pricing of risk in option markets. We first address the question of whether index options are nonredundant or too expensive relative to the market. Previous evidence, which relied on realized option returns, is subject to unreliable statistical inference (Broadie et al., 2009). By regressing conditional expected option returns on conditional expected market returns, which are not prone to this critique, we find highly significant alphas, confirming the common notion that options are nonredundant. Importantly, both puts and calls have negative market betas. While this is usually expected for puts, it is at first glance surprising for calls. The reason is that compensation for upside risk makes expected call returns more negative when expected market returns are higher.

We then analyze the cyclicality of compensation for upside, downside, and variance risk by studying the dynamics of conditional expected returns on calls, puts, and straddles, respectively. We document that compensation for these risks is higher during economic crises and periods of elevated volatility, as reflected in highly negative expected returns. Finally, we characterize the factor structure of conditional expected option returns using principal component analysis (PCA). We find that the factor structure of puts is stronger than that of calls, and that two factors generally suffice to explain most of the variation in option returns: a level factor and a moneyness-slope factor, resembling a volatility and a tail factor, respectively.

The remainder of the paper is organized as follows. After a brief discussion of the related literature, Section 2 derives option-implied formulas for expected option returns under prior SDF specifications from the literature. Section 3 introduces the new SDF specification we propose and the corresponding expected option returns. Section 4 presents the data we use and contains our empirical analysis. Section 5 concludes the paper. Appendices A, B, and C provide theoretical results supporting the paper, while Appendix D discusses implementation details and Appendix E collects figures and tables.

1.1 Related literature

Our paper is mainly related to three strands of the literature. The first strand studies index option returns. The expected return of options written on the market is intrinsically difficult to measure: nonlinear payoffs and rare tail realizations generate noisy sample means and unreliable inference (Broadie et al., 2009). Existing literature therefore focuses primarily on unconditional expected returns on options. Coval and Shumway (2001), Bakshi et al. (2010), and Chaudhuri and Schroder (2015) link expected option returns and their variation across moneyness to the shape of the pricing kernel as a function of market returns, which Almeida and Freire (2022) show is related to demand in the option market. Bakshi and Kapadia (2003) and Baele, Driessen, Ebert, Londono, and Spalt (2019) examine the relation between expected option returns and the variance risk premium. Driessen and Maenhout (2007), Santa-Clara and Saretto (2009), and Bondarenko (2014) document that puts are expensive in terms of highly negative expected returns. Jones (2006), Büchner and Kelly (2022), and Fournier, Jacobs, and Orłowski (2024) use factor models to study the dynamics of option returns. By using our new measure of conditional expected option returns, we provide new stylized facts about the pricing of risk in option markets.

While almost all existing evidence on expected option returns is unconditional, a separate, second strand of the literature uses option prices to infer conditional expected returns on the underlying market or stock (Back, Crotty, and Kazempour, 2022; Bollerslev and Todorov, 2011; Chabi-Yo, Dim, and Vilkov, 2023; Chabi-Yo et al., 2025; Chabi-Yo and Loudis, 2020; Kadan and Tang, 2020; Martin, 2017; Martin and Wagner, 2019; Schneider, 2019; Tetlock et al., 2024). We extend this literature along two dimensions: we consider option payoffs instead of the underlying return, and generalize risk preferences to account for both intertemporal hedging needs and heterogeneous preferences towards the left and right tails of the return distribution.

A related, third strand of the literature exploits the forward-looking information in option prices to recover, more generally, conditional physical probabilities and risk preferences. Ross (2015) provides theoretical conditions under which it is possible to uniquely recover these objects directly from the risk-neutral distribution, which Jackwerth and Menner (2020) show does not work well empirically. Schneider and Trojani (2019) work, instead, with physical probabilities minimizing variance while satisfying a set of economic constraints. Kim (2023) and Schreindorfer and Sichert (2025) use statistical methods to estimate the conditional pricing kernel. Beason and Schreindorfer (2022) and Chabi-Yo and Loudis (2024) decompose the unconditional and conditional equity premium, respectively, into different regions of the return space. Pazarbasi, Schneider, and Vilkov (2024) recover the dispersion among all possible beliefs consistent with observed option prices and economic assumptions. We are the first to recover conditional expected option returns without making assumptions about the underlying asset price dynamics.[2]

[2]For parametric models with known characteristic functions, expected option returns can be computed analytically (Broadie et al., 2009; Johannes, Kaeck, Seeger, and Shah, 2024; Rubinstein, 1984).

2 Option-implied formulas of expected option returns

In this section, we describe our first contribution, which is to derive option-implied formulas for conditional expected option returns under prior SDF specifications from the literature. Based on these formulas, we provide preliminary evidence on how expected option returns compare to realized ones across different strikes and maturities.

2.1 Expected option returns implied by prior SDF specifications

As discussed in the Introduction, option-implied formulas for expected returns consist of three main ingredients: (1) conditional risk-neutral expectations implied by observed option prices; (2) a specification for the SDF translating risk-neutral expectations into physical ones; and (3) the payoff of interest. While the literature has mostly focused on the market return as the payoff of interest, we consider a more general class of payoffs: option returns. This is a generalization as the market index is equivalent to an index call option with zero strike price. In this section, we derive formulas for expected option returns implied by existing SDF specifications.

Let $m_{t\to T_1}$ denote the SDF between time t and maturity T_1, and let $\mathbb{E}_t$ and $\mathbb{E}_t^*$ denote expectations under the physical and risk-neutral measures, respectively. Let S_t denote the price of the underlying asset at time t, and consider European call and put options with strike K_0 and maturity T_1, with time-t prices $C_t(K_0)$ and $P_t(K_0)$. The gross returns from time t to T_1 on the call and put options are given by:

$$R^c_{t\to T_1} = \frac{(S_{T_1} - K_0)^+}{C_t(K_0)}, \qquad R^p_{t\to T_1} = \frac{(K_0 - S_{T_1})^+}{P_t(K_0)}. \tag{2.1}$$

In the theory that follows, we focus on calls, as expressions for puts are analogous. Under no-arbitrage conditions, physical expectations can be expressed in terms of risk-neutral quantities using the change of measure implied by the SDF. Accordingly, the expected return on the call option can be written as:

$$\begin{aligned}\mathbb{E}_t\left[R^c_{t\to T_1}\right] &= \mathbb{E}_t\left[\frac{m_{t\to T_1}}{\mathbb{E}_t m_{t\to T_1}}\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\frac{(S_{T_1}-K_0)^+}{C_t(K_0)}\right]\\ &= \mathbb{E}^*_t\left[\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\frac{(S_{T_1}-K_0)^+}{C_t(K_0)}\right] \qquad (2.2)\\ &= \frac{S_t}{C_t(K_0)}\mathbb{E}^*_t\left[\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\left(R_{M,t\to T_1}-k_0\right)^+\right], \qquad (2.3)\end{aligned}$$

where $R_{M,t\to T_1} \equiv S_{T_1}/S_t$ denotes the gross return on the underlying asset, and $k_0 \equiv K_0/S_t$ is the normalized strike.

Option-implied formulas for expected option returns will depend, therefore, on the specification assumed for the SDF. Recent studies employ various inverse SDF formulations to recover the equity risk premium, i.e., the expected market return. The seminal work of Martin (2017) assumes an investor with log utility who is fully invested in the market, which implies an inverse SDF of the form:

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{R_{M,t\to T_1}}{R_{f,t\to T_1}}, \qquad (2.4)$$

where $R_{f,t\to T_1}$ is the risk-free rate from t to T_1. The expected return on the call option thus becomes:

$$\mathbb{E}_t R^c_{t\to T_1} = \frac{S_t}{C_t\left[K_0\right]R_{f,t\to T_1}}\mathbb{E}^*_t\left[R_{M,t\to T_1}\left(R_{M,t\to T_1}-k_0\right)^+\right]. \qquad (2.5)$$

If $k_0 = 0$, the formula boils down to the expected market return formula of Martin (2017).

Chabi-Yo and Loudis (2020) provide a more general SDF specification by considering a Taylor expansion of the unknown marginal utility of a representative investor, yielding the following expression for the inverse SDF:

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\sum_{i=0}^{3} \frac{a_{i,t}}{R^i_{f,t\to T_1}} \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^i}{\sum_{i=0}^{3} \frac{a_{i,t}}{R^i_{f,t\to T_1}} \mathbb{E}^*_t \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^i}, \tag{2.6}$$

which depends on coefficients related to preferences parameters:

$$a_{0,t} = 1,\ a_{1,t} = \frac{1}{\tau_t},\ a_{2,t} = \frac{1-\rho_t}{\tau_t^2},\ a_{2,t} = \frac{1-2\rho_t+\kappa_t}{\tau_t^3}, \tag{2.7}$$

where τ_t, ρ_t and κ_t denote risk tolerance (the inverse of relative risk aversion), skewness tolerance and kurtosis tolerance, respectively. The expected call return is, in this case:

$$\mathbb{E}_t R^c_{t\to T_1} = \frac{S_t}{C_t\left[K_0\right]} \frac{\sum_{i=0}^{3} \frac{a_{i,t}}{R^i_{f,t\to T_1}} \mathbb{E}^*_t \left[\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^i \left(R_{M,t\to T_1} - k_0\right)^+\right]}{\sum_{i=0}^{3} \frac{a_{i,t}}{R^i_{f,t\to T_1}} \mathbb{E}^*_t \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^i}. \tag{2.8}$$

That is, it depends on truncated risk-neutral moments, where the truncation depends on the option strike.

Tetlock et al. (2024) assume a closely related inverse SDF motivated by a marginal investor trading in the market and portfolios of options on the market replicating higher-order moments of the market return:

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = 1 + \sum_{k=1}^{K} \frac{1}{R_{f,t\to T_1}} \omega_{k,t} \left(\widetilde{R}^k_{M,t\to T_1} - c_k\right), \tag{2.9}$$

where $c_k = \mathbb{E}^*_t \left(\widetilde{R}^k_{M,t\to T_1}\right)$ and $\widetilde{R}_{M,t\to T_1} = R_{M,t\to T_1} - R_{f,t\to T_1}$. The expected return on the call option is then:

$$\mathbb{E}_t R^c_{t\to T_1} = \frac{S_t}{C_t\left[K_0\right]} \left\{ \begin{array}{c} \mathbb{E}^*_t \left(R_{M,t\to T_1} - k_0\right)^+ \\ + \sum_{k=1}^{K} \frac{1}{R_{f,t\to T_1}} \omega_{k,t} \left(\mathbb{E}^*_t \left(\widetilde{R}^k_{M,t\to T_1} - c_k \right) \left(R_{M,t\to T_1} - k_0\right)^+ \right) \end{array} \right\}. \quad (2.10)$$

While the coefficients $a_{i,t}$ in the SDF specification of Chabi-Yo and Loudis (2020) are constrained by economic theory, the portfolio weights $\omega_{k,t}$ in the approach by Tetlock et al. (2024) are left unspecified and estimated from the data. Specifically, Chabi-Yo and Loudis (2020) set preference parameters to constant traditional values, while Tetlock et al. (2024) estimate time-varying weights to match a measure of the variance risk premium in an expanding-window basis.

Finally, while Martin (2017), Chabi-Yo and Loudis (2020), and Tetlock et al. (2024) work with preferences within a one-period framework, Chabi-Yo et al. (2025) extend the analysis to a multi-period setting that incorporates intertemporal hedging motives. In this case, considering two periods (T_1, the maturity of the option, and $T_N > T_1$, the representative agent's investment horizon), the inverse SDF, when a second-order expansion of the inverse marginal utility is used, takes the form:

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\sum_{i=0}^{2} \frac{a_{i,t}}{R^i_{f,t\to T_1}} \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^i + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}} \mathbb{M}^{*(2)}_{T_1\to T_N}}{\sum_{i=0}^{2} \frac{a_{i,t}}{R^i_{f,t\to T_1}} \mathbb{E}^*_t \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^i + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}} \mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N}}, \quad (2.11)$$

where the coefficients $a_{i,t}$ are given by (2.7), $R_{f,T_1\to T_N}$ is the risk-free rate from T_1 to T_N, and $\mathbb{M}^{*(i)}_{T_1\to T_N} = \mathbb{E}^*_{T_1} \left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^i$ is the future risk-neutral moment of the market. This specification implies the following call expected return:

$$\mathbb{E}_t R^c_{t\to T_1} = \frac{S_t}{C_t[K_0]} \frac{\left\{ \begin{array}{c} \sum_{i=0}^{2} \frac{a_{i,t}}{R^i_{f,t\to T_1}} \mathbb{E}^*_t \left[\left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^i \left(R_{M,t\to T_1} - k_0\right)^+ \right] \\ + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}} \mathbb{E}^*_t \left[\mathbb{M}^{*(2)}_{T_1\to T_N} \left(R_{M,t\to T_1} - k_0\right)^+ \right] \end{array} \right\}}{\sum_{i=0}^{2} \frac{a_{i,t}}{R^i_{f,t\to T_1}} \mathbb{E}^*_t \left(R_{M,t\to T_1} - R_{f,t\to T_1}\right)^i + \frac{a_{2,t}}{R^2_{f,T_1\to T_N}} \mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N}} \tag{2.12}$$

The expected option return formulas in (2.5), (2.8), (2.10), and (2.12) depend exclusively on risk-neutral quantities at time t, which can be computed from contemporaneously observed option prices.[3] In Appendix C, we derive closed-form expressions for these quantities using the spanning results of Carr and Madan (2001). As a result, expected option returns can be computed at time t using only option prices observed at that time.

In the next subsection, we evaluate the extent to which the different SDF specifications above imply expected option returns that are informative about realized option returns using a simple diagnostic test. Specifically, average conditional expected option returns should align closely with average realized option returns over the sample, across different strikes k_0 and maturities T_1. It is important to note, however, that these specifications were primarily proposed to recover the equity risk premium rather than expected option returns. Accordingly, any limitations in explaining the cross-sectional patterns of option returns should not be interpreted as criticisms of the original contributions of these papers.

2.2 Preliminary evidence

For a given strike and maturity, the average over the sample of the conditional expected option return should be statistically indistinguishable from the unconditional expected return obtained by averaging realized option returns. We now test to what extent

[3] The only risk-neutral quantity that does not depend on information at time t is the future risk-neutral moment $\mathbb{M}^{*(2)}_{T_1\to T_N}$ in (2.12). Chabi-Yo et al. (2025) circumvent this issue by assuming it to be a quadratic function of the excess market return from t to T_1, such that $\mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N}$ can then be computed with option prices at time t. We also adopt this assumption throughout the paper.

this holds for conditional expected option returns implied by the different SDF specifications available in the literature. We consider options across fixed deltas and fixed maturities. Using deltas instead of strikes ensures that moneyness is measured in a way that naturally adjusts for changes in volatility and time to maturity. Section 4.1 details the data we use and the relevant implementation details.

Therefore, for each day in the sample, we compute conditional expected returns for calls and puts implied by (i) the log-utility benchmark of Martin (2017), (ii) the Taylor expansion approach of Chabi-Yo and Loudis (2020) (CL), (iii) the time-varying specification of Tetlock et al. (2024) (TMS), and (iv) the intertemporal hedging extension of Chabi-Yo et al. (2025) (CGL).[4] We then average these quantities over the sample and compare them to average realized option returns. We compute 95% confidence intervals from a block-bootstrap applied to realized option returns. If average conditional expected option returns are inside the confidence intervals, they are statistically indistinguishable from the average realized option returns.

Figure 1 reports the results for maturities of one, three, and six months. Focusing first on call options in the upper panels, it can be seen that existing approaches fit in-the-money calls well. This is consistent with the fact that these SDF specifications have been shown to yield accurate measures of the expected return on the market, which can be seen as a deep in-the-money call. However, across all specifications, expected out-of-the-money call returns are either positive or close to zero, in contrast with the strongly negative realized average returns documented in the literature. This indicates that the existing SDF specifications do not generate sufficient compensation for upside risk, i.e., they do not generate U-shaped SDFs (Bakshi et al., 2010).

The lower panels of Figure 1 display the evidence for put options. In this case, all spec-

[4]For CL and CGL, we consider the second order expansion and use the preference parameters in Chabi-Yo and Loudis (2020): $\tau = 1$ and $\rho = 2$. For TMS, we thank the authors for making their time-varying weights available to us.

ifications generate negative average put returns that decrease for lower absolute deltas, which is qualitatively in line with average realized put returns. That is, all approaches imply an SDF with compensation for downside risk. Quantitatively, however, the fit is not perfect for one-month options. Specifically, Martin and CL (CGL) generate average put returns that are not negative enough (too negative). For three-month and six-month puts, all methods lie within the confidence intervals.

It is also possible to analyze implications for compensation for variance risk by using straddles, which represent long positions in volatility. A straddle can be viewed as a portfolio of a call and a put with the same strike and maturity. Its return can be computed from call and put returns through:

$$R^{\text{str}}_{t\to T}(K) = \frac{P_t(K)}{P_t(K) + C_t(K)}\, R^{p}_{t\to T}(K) + \frac{C_t(K)}{P_t(K) + C_t(K)}\, R^{c}_{t\to T}(K). \tag{2.13}$$

As a result, conditional expectations for call and put returns obtained from each SDF specification immediately imply conditional expectations for straddle returns. We then compare these model-implied expected straddle returns to realized straddle returns using the same set of diagnostics as above.

Figure 2 reports the results for the three maturities. The only specification delivering expected returns that are on average always statistically indistinguishable from realized returns is CGL. The approaches based on Martin and CL do not generate enough compensation for variance risk at the one-month and three-month horizons, with expected returns that are not negative enough. The same is true for TMS for one-month options.

In sum, the main challenge for existing approaches is to reproduce the negative average returns of out-of-the-money calls, which requires an SDF that assigns high marginal utility weight to extreme positive market outcomes. The specification that comes closer to matching average call returns is CGL. While alternative preference parameters could improve the fit for calls by making the SDF more volatile, this would come at the expense of making average put returns even more negative. Hence, to be able to accurately fit both puts and calls at the same time, we propose a new SDF specification that extends CGL to allow for heterogeneous preferences for the left and right tails of market returns.

3 A new SDF specification for expected option returns

In this section, we describe our second contribution: providing a new SDF specification to compute expected option returns. Specifically, to afford more flexibility in reproducing expected returns of both puts and calls, we extend the specification of Chabi-Yo et al. (2025), which is the most general one from the previous section, to allow for heterogeneous preferences towards the left and right tails of the market return distribution. This essentially combines their specification with the approach of Chabi-Yo and Loudis (2024) of Taylor expanding the marginal utility in different regions of the underlying return space, which they used to decompose the equity premium into distinct components.

To do so, we begin by considering the following two-period return: $R_{M,t\to T_1}R_{f,T_1\to T_N}$, where $R_{M,t\to T_1}$ denotes the market return from time t to T_1 and $R_{f,T_1\to T_N}$ is the risk-free return from time T_1 to T_N. Let $m_{t\to T_1}$ denote the SDF from time t to T_1 and $m_{t\to T_N}$ the SDF from t to T_N. Under no-arbitrage conditions, the physical expectation can be expressed in terms of risk-neutral quantities:

$$\begin{aligned}\mathbb{E}_t\left[R_{M,t\to T_1}R_{f,T_1\to T_N}\right] &= \mathbb{E}_t^*\left[\frac{\mathbb{E}_t m_{t\to T_N}}{m_{t\to T_N}}R_{M,t\to T_1}R_{f,T_1\to T_N}\right]\\ &= R_{f,T_1\to T_N}\mathbb{E}_t^*\left[R_{M,t\to T_1}\mathbb{E}_{T_1}^*\left[\frac{\mathbb{E}_t m_{t\to T_N}}{m_{t\to T_N}}\right]\right]. \qquad (3.1)\end{aligned}$$

Using a similar approach, we obtain the following:

$$\mathbb{E}_t\left[R_{M,t\to T_1}R_{f,T_1\to T_N}\right] = R_{f,T_1\to T_N}\mathbb{E}_t\left[R_{M,t\to T_1}\right] = R_{f,T_1\to T_N}\mathbb{E}_t^*\left[R_{M,t\to T_1}\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}\right]. \quad (3.2)$$

Equating expressions (3.1) and (3.2), and invoking the uniqueness of the SDF yields the key relationship:

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \mathbb{E}^*_{T_1}\left[\frac{\mathbb{E}_t m_{t\to T_N}}{m_{t\to T_N}}\right]. \tag{3.3}$$

Equation (3.3) shows that, without arbitrage, the inverse SDF from time t to T_1 can be expressed as the expectation at time T_1 of the inverse SDF from t to T_N. Next, consider a representative agent who maximizes the expected utility of the terminal wealth:

$$\max_{\omega_t} \mathbb{E}_t \max_{\omega_{T_1}} \mathbb{E}_{T_1} \max_{\omega_{T_2}} ... \mathbb{E}_{T_{N-1}} \max_{\omega_{T_{N-1}}} u\left[W_{t\to T_N}\right],$$

subject to the wealth dynamics:

$$\begin{aligned} W_{t\to T_N} &= W_{t\to T_1}\left(R_{f,T_1\to T_N} + \omega_{T_1}^\intercal\left(\mathbf{R}_{T_1\to T_N} - R_{f,T_1\to T_N}\mathbf{1}\right)\right), \\ W_{t\to T_1} &= W_t\left(R_{f,t\to T_1} + \omega_t^\intercal\left(\mathbf{R}_{t\to T_1} - R_{f,t\to T_1}\mathbf{1}\right)\right). \end{aligned}$$

Assume that the market return serves as a proxy for the return on aggregate wealth. The first-order conditions at t imply that the SDF $m_{t\to T_N}$ is proportional to the marginal utility of the terminal wealth:

$$m_{t\to T_N} \propto u'\left[W_{t\to T_N}\right],$$

which implies:

$$\frac{\mathbb{E}_t m_{t\to T_N}}{m_{t\to T_N}} = \frac{1/u'\left[W_{t\to T_N}\right]}{\mathbb{E}^*_t\left[1/u'\left[W_{t\to T_N}\right]\right]}.$$

Hence, the inverse SDF in equation (3.3) can be rewritten as:

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{\mathbb{E}^*_{T_1} 1/u'\left[W_{t\to T_N}\right]}{\mathbb{E}^*_t\left[1/u'\left[W_{t\to T_N}\right]\right]}.$$

Since the exact functional form of $u\left[.\right]$ is unknown, we assume that the standard regularity conditions on the utility are satisfied and introduce the following notation for tractability:

$$x = R_{M,t\to T_1},\ y = R_{M,T_1\to T_N},\ x_0 = R_{f,t\to T_1}, \text{ and } y_0 = R_{f,T_1\to T_N}. \tag{3.4}$$

We define:

$$h[x,y] = \frac{1}{u'[W_t xy]},$$

and consider the decomposition:

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \frac{h[x,y]}{\mathbb{E}_t^*(h[x,y])} 1_{x<x_0} + \frac{h[x,y]}{\mathbb{E}_t^*(h[x,y])} 1_{x>x_0}. \tag{3.5}$$

Our goal is to derive the expected return at time t of options with payoffs determined at time T_1. The option payout depends on the distributions of x. For instance, defining $R_{M,t\to T_1} = \frac{S_{T_1}}{St}$, a call option with strike K_0 has a payoff of $(S_{T_1} - K)^+ = S_t(x - k_0)^+$, where k_0 denotes moneyness, while a put option has a payoff of $(K - S_{T_1})^+ = S_t(k_0 - x)^+$.

Although the inverse SDF in equation (3.3) implicitly depends on y through the conditional expectation at time T_1, the option payoff depends solely on x. This motivates the partitioning equation (3.5) with respect to x, rather than y. Partitioning (3.5) into two regions also facilitates a Taylor series expansion of $h[x,y]$ in each region without explicitly specifying the utility function. Intuitively, expanding each term of the decomposition (3.5) in the two regions produces a more accurate local approximation than expanding $h[x,y]$ globally, particularly when option payoffs are highly nonlinear in various regions of x. We, therefore, write:

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \sum_{s\in\{d,u\}} \frac{\mathbb{E}_t m^{(s)}_{t\to T_1}}{m^{(s)}_{t\to T_1}} \text{ with } \frac{\mathbb{E}_t m^{(s)}_{t\to T_1}}{m^{(s)}_{t\to T_1}} = \frac{\mathbb{E}^*_{T_1}[f_s[x,y]]}{\mathbb{E}_t^*\left(\mathbb{E}^*_{T_1}[f_s[x,y]]\right)} 1_{A_s},$$

with

$$A_d = \{x < x_0\} \text{ and } A_u = \{x > x_0\},$$

where

$$f_s[x,y] = \frac{u^{'}[W_t x_s y_0]}{u^{'}[W_t x y]}.$$

Notice that the inverse marginal utility is continuous at x_0 and admits a derivative at x_0. This implies two conditions that must be satisfied:

$$\left(\frac{\mathbb{E}_t m^{(u)}_{t\to T_1}}{m^{(u)}_{t\to T_1}}\right)_{x=x_0} = \left(\frac{\mathbb{E}_t m^{(d)}_{t\to T_1}}{m^{(d)}_{t\to T_1}}\right)_{x=x_0} \text{ and,} \tag{3.6}$$

$$\left\{\frac{\partial}{\partial x}\left(\frac{\mathbb{E}_t m^{(u)}_{t\to T_1}}{m^{(u)}_{t\to T_1}}\right)\right\}_{x=x_0} = \left\{\frac{\partial}{\partial x}\left(\frac{\mathbb{E}_t m^{(d)}_{t\to T_1}}{m^{(d)}_{t\to T_1}}\right)\right\}_{x=x_0}. \tag{3.7}$$

In theory, nothing prevents us from partitioning the set of the range of values x into more than two regions. While doing so theoretically provides more accuracy of the approximation, it introduces additional complexity into the final results and poses challenges when implementing the theory. This motivates us to stick to two regions. We continue the development of the theory by first defining the following preference parameters:

$$\frac{1}{\tau^s_t} = -\frac{W_t x_s y_0 u^{''}[W_t x_s y_0]}{u^{'}[W_t x_s y_0]},\ \rho^s_t = \frac{1}{2}\frac{u^{'''}[W_t x_s y_0]\, u^{'}[W_t x_s y_0]}{\left(u^{''}[W_t x_s y_0]\right)^2},\ \kappa^s_t = \frac{1}{3!}\frac{u^{''''}[W_t x_s y_0]\left(u^{'}[W_t x_s y_0]\right)^2}{\left(u^{''}[W_t x_s y_0]\right)^3}.$$

These parameters are evaluated at the terminal wealth $W_{t\to T_N} = W_t x_s y_0$. It is important to note that they are not evaluated at random terminal wealth. Knowledge of these parameters alone is not sufficient to recover the full functional form of the utility. Here $\frac{1}{\tau^s_t}$ denotes the risk aversion, ρ^s_t captures the preference for skewness and κ^s_t reflects the preference for kurtosis. For tractability, we further denote:

$$a_{0,0,s} = 1,\ a_{1,0,s} = \frac{1}{\tau^s_t},\ a_{2,0,s} = \frac{(1-\rho^s_t)}{(\tau^s_t)^2},\ a_{3,0,s} = \frac{(1-2\rho^s_t+\kappa^s_t)}{(\tau^s_t)^3}, \tag{3.8}$$

and

$$a_{0,1,s} = \frac{1}{\tau_t^s},\ a_{0,2,s} = \frac{(1-\rho_t^s)}{(\tau_t^s)^2},\ a_{0,3,s} = \frac{(1-2\rho_t^s+\kappa_t^s)}{(\tau_t^s)^3}, \tag{3.9}$$

and

$$a_{1,2,s} = 2a_{2,0,s} + 3a_{3,0,s}.$$

We also define the excess market return in each region as $R_{s,M,t\to T_1} = R_{M,t\to T_1} - x_s$, and the future risk neutral moment of the market as $\mathbb{M}^{*(i)}_{T_1\to T_N} = \mathbb{E}^*_{T_1}\left(R_{M,T_1\to T_N} - R_{f,T_1\to T_N}\right)^i$. Our next goal is to fully characterize the inverse SDF without taking a stand on the functional form of the utility and then use the inverse SDF to provide an expression for the call and put option expected returns in terms of risk-neutral quantities under no-arbitrage conditions. To do so, we rely on Taylor expansion series, where the coefficients defined in (3.8) and (3.9) are loaded in various terms of the expansion of the inverse marginal utility.

We perform a second-order Taylor expansion series of the inverse marginal utility $f^{[s]}[x,y]$ around $(x,y) = (x_s, y_0)$ and show that its expectation at time T_1 produces:

$$\mathbb{E}^*_{T_1} f^{[s]}[x,y] = \sum_{i=0}^{2} \frac{a_{i,0,s}}{x_s^i} R^i_{s,M,t\to T_1} + \frac{a_{0,2,s}}{y_0^2}\mathbb{M}^{*(2)}_{T_1\to T_N}. \tag{3.10}$$

The proof of (3.10) is in Appendix A. We then replace this expression in (3.3)-(3.5) and show the following result.

Proposition 3.1. *Under no-arbitrage conditions, a second-order Taylor expansion series of the inverse marginal utility in each region produces an inverse SDF (3.3) where*

$$\frac{\mathbb{E}_t m^{(s)}_{t\to T_1}}{m^{(s)}_{t\to T_1}} = \frac{\sum_{i=0}^{2} \frac{a_{i,0,s}}{x_s^i} R^i_{s,M,t\to T_1} 1_{A_s} + \frac{a_{0,2,s}}{y_0^2} 1_{A_s} \mathbb{M}^{*(2)}_{T_1\to T_N}}{\sum_{i=0}^{2} \frac{a_{i,0,s}}{x_s^i} \mathbb{E}^*_t R^i_{s,M,t\to T_1} + \frac{a_{0,2,s}}{y_0^2}\mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N}} \ \textit{for } s \in \{u,d\},$$

where $a_{i,j,s}$ are defined in (3.8)-(3.9), y_0 is the return on the risk-free asset defined in

(3.4) and $R_{s,M,t\to T_1} = R_{M,t\to T_1} - x_s$.

Now, equipped with the inverse SDF in Proposition 3.1, we show how to express the expected return on options in terms of risk-neutral quantities. We start by considering a call option with strike K_0 and maturity T_1. Under no-arbitrage conditions, the expected return on the call option is given by (2.3). We then replace the expression of the inverse SDF in (2.3) and show:

Proposition 3.2. *Under no-arbitrage conditions, the expected return on a call option with maturity* T_1 *and strike* K_0 *is*

$$\mathbb{E}_t R^c_{t\to T_1} = \frac{S_t}{C_t\left[K_0\right]}\left(\frac{\mathbb{ECN}^u\left[k_0\right]}{\mathbb{ED}^u_2} + \frac{\mathbb{ECN}^d\left[k_0\right]}{\mathbb{ED}^d_2}\right), \tag{3.11}$$

with

$$\mathbb{ECN}^s\left[k_0\right] = \sum_{j=0}^{2}\tfrac{a_{j,0,s}}{x_s^j}\mathbb{E}^*_t\left[R^j_{s,M,t\to T_1}\left(R_{M,t\to T_1}-k_0\right)^+ 1_{A_s}\right] + \tfrac{a_{0,2,s}}{y_0^2}\mathbb{E}^*_t\left[\left(R_{M,t\to T_1}-k_0\right)^+ 1_{A_s}\mathbb{M}^{*(2)}_{T_1\to T_N}\right],$$

and

$$\mathbb{ED}^s_2 = \sum_{i=0}^{2}\tfrac{a_{i,0,s}}{x_s^i}\mathbb{E}^*_t R^i_{s,M,t\to T_1} + \tfrac{a_{0,2,s}}{y_0^2}\mathbb{E}^*_t\mathbb{M}^{*(2)}_{T_1\to T_N}. \tag{3.12}$$

Proposition 3.11 decomposes the expected return on a call option into two risk premium components. Compensation for exposure to the upside region through upside-specific risk factors that appear in the SDF and compensation for exposure to the downside region through downside-specific risk factors. All compensations are functions of risk-neutral quantities. They can be recovered from a cross-section of option prices at each time providing a measure of the expected excess return of the call option with maturity T_1 in real time. An additional key advantage of this measure is that from these compensations, we observe that as the investment horizon T_N changes, the expected return of the call option changes. This highlights the fact that, given a fixed option maturity

T_1, the compensation required by investors to hold the call option until maturity differs between investors with different investment horizons T_N. In a similar fashion, we show:

Proposition 3.3. *Under no-arbitrage conditions, the expected return on a put option with maturity T_1 and strike K_0 is*

$$\mathbb{E}_t R^p_{t\to T_1} = \frac{S_t}{P_t\left[K_0\right]}\left(\frac{\mathbb{EPN}^u\left[k_0\right]}{\mathbb{ED}_2^u} + \frac{\mathbb{EPN}^d\left[k_0\right]}{\mathbb{ED}_2^d}\right),$$

with

$$\mathbb{EPN}^s\left[k_0\right] = \sum_{j=0}^{2}\frac{a_{j,0,s}}{x_u^j}\mathbb{E}_t^*\left[R^j_{u,M,t\to T_1}\left(k_0 - R_{M,t\to T_1}\right)^+ 1_{A_s}\right] + \frac{a_{0,2,s}}{y_0^2}\mathbb{E}_t^*\left[\left(k_0 - R_{M,t\to T_1}\right)^+ 1_{A_s}\mathbb{M}^{*(2)}_{T_1\to T_N}\right],$$

where $\mathbb{ED}_2^s$ is defined in (3.12).

Similarly to the expected return on call options, the expected excess return on a put option can be decomposed into risk premium components that reflect a compensation for an exposure to the two regions under consideration. For a fixed T_1, a change in the investment horizon T_N leads to a change in the expected return of the put options, which implies that investors with different investment horizons require different compensations for holding the put option until maturity T_1 when T_N varies. Closed-form expressions of the risk-neutral quantities appearing above are given in Appendix C. While we focus on a second-order approximation of the inverse marginal utility, Appendix B provides the formulas based on a third-order approximation.

4 Main empirical analysis

4.1 Data description and implementation details

Our proxy for the aggregate stock market return is the return on the S&P 500 index, sourced from OptionMetrics. For the risk-free rate, we use the 3-month Treasury bill rate provided by the Federal Reserve Economic Data (FRED) database. We further obtain the S&P 500 option quotes from OptionMetrics. More specifically, our raw option data consists of end-of-day bid and ask quotes, strike, expiration date and volume for each option for the sample period ranging from January 4, 1996 to February 28, 2023. As standard in the literature, we use the mid-point of the bid and ask quotes as the option price and apply a handful of filters to the raw data. Observations with non-positive bid, with implied volatility greater than 300%, and violating the usual no-arbitrage conditions are dropped. To calculate the time to expiration of each option, we take into account if the contract settlement is at the open or close. Since the S&P 500 index typically pays a dividend, we estimate the dividend yield for each maturity from the put-call parity relation, using the pair of call and put options that are closest to at-the-money.

In Appendix D, we describe how we interpolate and extrapolate from the raw option data to obtain synthetic option prices for the same deltas and maturities each day, which are then used to compute the risk-neutral quantities as in Appendix C. We consider absolute deltas from 0.2 to 0.8 with 0.05 increments, encompassing OTM (0.2 to 0.35), ATM (0.4 to 0.6) and ITM (0.65 to 0.8) options. Working with deltas instead of strikes guarantees that options cover a comprehensive range of strikes that adjusts with the option maturity and the volatility on a given day. For maturities T_1, we take one month, three months and six months ($T_1 = 30, 90, 180$), while we fix T_N to be one year ($T_N = 365$).

4.2 Diagnostic tests of expected option returns

Motivated by the limitations of prior SDF specifications in matching realized option returns on average, we introduce a more flexible SDF specification that allows for different preference parameters in the left and right tails of the market return distribution. The specification combines the higher-moment structure of Chabi-Yo and Loudis (2020), the intertemporal hedging component of Chabi-Yo et al. (2025), and the expansion of marginal utility across different regions of the return space of Chabi-Yo and Loudis (2024). We now assess how well this specification approximates realized option returns with the same diagnostic tests of Section 2.2.

We keep risk tolerance fixed to $\tau^d = \tau^u = 1$ and consider three sets of parameters for skewness preferences: (i) $\rho^d = \rho^u = 2$, (ii) $\rho^d = 3$ and $\rho^u = 4$, (iii) and $\rho^d = 3$ and $\rho^u = 6$. Specification (i) is directly comparable to the preference parameters used in the existing SDF specifications in the literature. Specification (ii) increases both parameters, and disproportionately more the right tail one. Finally, specification (iii) keeps ρ^d fixed relative to (ii) while it increases ρ^u, allowing us to illustrate how this will essentially affect only the expected returns of calls. Potential extensions include estimating these parameters and allowing them to be time-varying.

Figure 3 compares average conditional expected option returns implied by our method with average realized ones. Specification (i) suffers from the same limitation as the existing specifications in Section 2.2: it fails to generate average call returns that decrease with the strike and eventually become negative. In other words, it does not imply compensation for upside risk. For puts, average returns are indistinguishable from realized ones, albeit less negative in general.

Increasing ρ^d to 3 with specification (ii) makes average expected put returns much closer to the data. At the same time, setting ρ^u to 4 also brings average expected call returns qualitatively closer to realized ones, with this specification being able to generate the decreasing pattern with respect to the strike. However, average returns are still not negative enough for the most OTM calls. Increasing ρ^u further to 6 substantially improves the quantitative fit, with average expected call returns that are nearly always indistinguishable from average realized returns. Importantly, increasing ρ^u has limited impact on expected put returns, since ρ^d was kept fixed. This illustrates the flexibility of our approach to fit both expected call and put returns at the same time.

Figure 4 further shows results for straddles. As can be seen, apart from specification (i), our approach always yields average expected straddle returns that are consistent with average realized ones. This indicates that these specifications imply realistic compensations not only for downside and upside risk, but also variance risk. To be conservative in the analysis that follows, we consider (ii) as our baseline specification and use it for the remaining results in the paper.

4.3 Predictive regressions

While the results so far are informative, they ignore time-series information by focusing directly on unconditional averages. In this subsection, we implement predictive regressions in which realized option payoffs are regressed on model-implied expected payoffs, separately for calls, puts, and straddles across moneyness groups and maturities. We focus on option payoffs rather than returns due to their better behaved dynamics. Specifically, for a given SDF specification, we regress $(R_{M,t\to T_1} - k_0)^+$ on $\mathbb{E}_t^*\left[\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}(R_{M,t\to T_1} - k_0)^+\right]$. Note that this is equivalent to working with deleveraged option returns rather than raw returns, which has been common in recent literature modeling the time series of option returns (Büchner and Kelly, 2022; Fournier et al., 2024).[5] A well-specified model implies $\alpha = 0$, $\beta > 0$, and $\beta = 1$. For ease of exposition, we focus on the one-month maturity and equally-weighted portfolios of options within three moneyness categories: OTM,

[5] To see that, note that $\frac{C_t(K_0)}{S_t} R^c_{t\to T_1} = \frac{C_t(K_0)}{S_t}\frac{(S_{T_1} - K_0)^+}{C_t(K_0)} = (R_{M,t\to T_1} - k_0)^+$.

ATM and ITM, as defined in Section 4.1.

Tables 1 and 2 evaluate the ability of the different SDF specifications to explain the time series variation in option payoffs. For each approach and moneyness group, we regress the realized option payoff over the next 30 days on the model-implied conditional expectation of the same payoff. For call payoffs (Table 1), all models deliver intercepts that are economically small and statistically insignificant across moneyness groups. In addition, the slope coefficients are positive and statistically significant in all cases, implying that the models capture meaningful time series variation in realized call payoffs.

However, the tests of $\beta = 1$ reveal an important limitation of the benchmark SDF specifications. For OTM calls, the Martin, CL, and TMS specifications reject $\beta = 1$ at conventional levels, with estimated slopes substantially below one. This pattern indicates that these models overestimate realized call payoffs, consistent with implied expected call returns that are too high. By contrast, our specification (ACF) delivers a slope closer to one for OTM calls and does not reject $\beta = 1$, suggesting that the additional flexibility in the right-tail component improves the fit.

For put payoffs (Table 2), the intercepts remain close to zero for all models, although the CGL specification produces a statistically significant intercept for OTM puts. Across most moneyness groups, the slope coefficients are positive and statistically significant, indicating that the models capture a nontrivial share of the variation in realized put payoffs. The explanatory power increases with moneyness, as reflected in higher R^2 for ITM puts. Unlike for calls, the tests of $\beta = 1$ generally do not reject the null for the CL and TMS specifications. In particular, our method delivers the slope coefficients that are closest to one.

Overall, the predictive regressions indicate that all SDF specifications generate expected payoffs that are directionally consistent with realized outcomes, as reflected in significant slope coefficients and small intercepts. The main differences arise in the right tail: prior SDF specifications tend to deliver slopes below one for OTM calls, whereas our specification substantially improves this dimension while maintaining the strongest fit for put payoffs.

4.4 Nonredundancy of index options

We then revisit the question of whether index options are redundant assets relative to the market index. Prior evidence is largely based on realized option returns, which are highly nonnormal and dominated by rare events, making statistical inference difficult (Broadie et al., 2009). Our framework offers a complementary approach: it allows us to study nonredundancy using conditional expected returns rather than realized returns.

Table 3 reports regressions of model-implied conditional expected option returns on model-implied conditional expected market returns at the 30-day horizon. This exercise evaluates whether time variation in expected market returns can account for the time variation in expected option returns within each SDF specification. We focus on ATM options for ease of exposition.

Across all panels, the market return is strongly related to expected option returns, as reflected in large and statistically significant slope coefficients. However, the intercepts are also highly significant in all cases, implying economically meaningful alphas. This confirms that index options are nonredundant with respect to the underlying market, even when expected returns are measured conditionally rather than through realized option returns.

Panel A shows that for ATM calls, the benchmark specifications (M, CL, TMS, and CGL) deliver large positive market betas, consistent with the view that expected call returns comove positively with expected market returns. In contrast, the ACF specification implies a negative market beta for ATM calls. This finding reflects that, under a U-shaped SDF, higher expected market returns coincide with higher compensation for upside risk, which makes expected call returns more negative.

Panel B shows that ATM put expected returns have strongly negative market betas across all specifications, as expected. The explanatory power is high, with R^2 values generally above 60% and reaching nearly 90% under ACF. Panel C shows analogous results for ATM straddles, with negative betas across all models, consistent with the interpretation of straddles as exposure to variance risk.

This result provides strong evidence that index options are nonredundant. Unlike realized-return evidence, these conclusions are not driven by a small number of extreme events, since the dependent variable is a model-implied conditional expectation rather than an ex-post realized payoff. Importantly, both calls and puts have negative market betas. The negative beta for puts is expected, as put payoffs increase in market downturns. The negative beta for calls is less intuitive at first glance, since call payoffs are increasing in the underlying index.

The key to understanding this finding is the U-shaped pricing kernel implied by the data. When the right tail of the market return distribution carries a high price of risk, call options deliver payoffs in states that are expensive under the SDF. As a result, their expected returns can be more negative precisely when expected market returns are higher. In other words, calls load negatively on the component of expected market returns that reflects compensation for upside tail risk.

4.5 Cyclicality of upside, downside, and variance compensation

We next study the cyclicality of compensation for different sources of risk by analyzing how conditional expected returns on calls, puts, and straddles depend on conditional risk-neutral moments extracted from option prices. We interpret calls as primarily loading on upside tail risk, puts on downside tail risk, and straddles on variance risk. Table 4 reports results for regressing expected option returns on risk-neutral moments, again focusing on ATM options for the one-month horizon. The results highlight that risk-neutral higher moments contain substantial information about expected option returns, but the economic interpretation depends strongly on the SDF specification.

Panel A shows that risk-neutral variance is highly significant to explain expected call returns. However, its sign differs across specifications. In M, CL, TMS, and CGL, β_2 is positive, whereas in ACF it is significantly negative. This sign reversal implies that under ACF, expected call returns become more negative when risk-neutral volatility is high, consistent with the U-shaped pricing kernel captured by this specification. Risk-neutral skewness is also statistically significant, and enters with a negative sign in TMS, CGL, and ACF. This indicates that call expected returns are more negative when risk-neutral skewness is higher. Risk-neutral kurtosis enters negatively in all models but becomes insignificant under ACF. Finally, the explanatory power differs substantially: while M and CL deliver very high R^2 values, TMS and CGL exhibit intermediate fit, and the R^2 is much lower for ACF, suggesting that risk-neutral moments capture only a small fraction of the time-variation in expected call returns under this specification.

Panel B shows that risk-neutral variance is significantly negative in M, CL, TMS, and ACF, implying that expected put returns are more negative when risk-neutral volatility is high. In contrast, the variance coefficient is insignificant under CGL. Risk-neutral skewness is always significant and enters with a positive sign, indicating that expected put returns become more negative when skewness is more negative. Risk-neutral kurtosis is positive and significant for M and CL but becomes statistically weak in TMS, CGL, and ACF. Overall, the R^2 values remain high for M and CL, moderate for TMS and ACF, and noticeably lower for CGL.

Panel C reveals that risk-neutral variance plays a limited role in explained expected straddle returns for most specifications, but is strongly significant for ACF with a negative sign. This implies that expected straddle returns become more negative when risk-neutral volatility is high, consistent with a larger compensation for variance risk in volatile states. Risk-neutral skewness is significant in all models and enters with a positive sign, indicating that expected straddle returns become more negative when skewness is more negative. Risk-neutral kurtosis is mostly insignificant. The R^2 values are broadly similar across specifications, typically around 60–75%.

Figures 5–7 illustrate the time series dynamics of conditional expected option returns implied by our preferred SDF specification. Figure 5 shows that conditional expected call returns exhibit strong heterogeneity across moneyness. Expected returns on OTM calls become substantially more negative during crisis periods, while expected returns on ITM calls increase. This pattern implies that compensation for upside risk is countercyclical and rises during crises. Importantly, this result contrasts with implications of more restrictive SDF specifications such as Martin, which imply that expected call returns increase uniformly across strikes in crisis periods. Our results therefore suggest that properly capturing the pricing of upside-tail states is essential for matching the dynamics of call expected returns.

Figure 6 shows that expected put returns decline sharply during crisis periods for all moneyness groups. This indicates that the compensation for downside risk is strongly countercyclical: investors require a larger premium to bear downside exposure precisely when aggregate risk is elevated. The broad comovement across OTM, ATM, and ITM puts suggests that crisis periods affect downside risk premia in a pervasive way, rather than only through deep tail insurance.

Finally, Figure 7 plots conditional expected returns on ATM straddles, which represent direct exposure to variance risk. Expected straddle returns become more negative during crisis periods, implying that compensation for variance risk is also countercyclical. This evidence complements the patterns observed for puts and calls, and shows that option-implied compensation for variance risk is highest precisely in high-volatility states.

4.6 Factor structure of conditional expected option returns

Finally, we examine the factor structure of conditional expected option returns across the option surface. We perform principal component analysis (PCA) on panels of conditional expected returns across moneyness and maturities, separately for calls, puts, and straddles. Figures 8–13 summarize the principal component structure of conditional expected option returns implied by our preferred SDF specification. The results reveal a strong low-dimensional factor structure, consistent with the view that the dynamics of option risk premia are driven by a small number of aggregate risk factors.

Figure 8 shows that conditional expected call returns exhibit a strong factor structure, with roughly 95% of the total variation explained by the first three principal components. Figure 9 plots the first three components. The first component closely resembles a variance-related factor, increasing sharply in periods of market stress (up to a sign normalization, since PCA eigenvectors are identified only up to sign). The second component appears more closely related to jump or tail-risk conditions, while the third component is comparatively noisier.

The loadings in Figure 10 provide additional economic interpretation. The first principal component loads positively on OTM calls and negatively on ITM calls across maturities, consistent with a moneyness-slope factor. The second component loads in the same direction across moneyness and resembles a level factor. Finally, the third component is primarily associated with maturity differences, consistent with a maturity factor.

The factor structure is even stronger for puts. Figure 11 shows that the first two principal components explain almost all variation in conditional expected put returns. Figure 12 indicates that the first component captures the dominant time-series movements in expected put returns and appears related to aggregate variance and tail-risk conditions. Higher-order components are substantially noisier and explain comparatively little additional variation. The loadings in Figure 13 show that the first put component loads positively on all puts but assigns larger weights to OTM puts, indicating that it combines both a level and a moneyness-slope dimension. The second component is primarily a maturity factor, and the third component resembles curvature across maturities, though its incremental explanatory power is limited.

Overall, these results show that conditional expected option returns admit a parsimonious factor representation, with a stronger factor structure for puts relative to calls. This is consistent with the fact that there is substantial heterogeneity in the dynamics of expected call returns across moneyness, as seen in Figure 5. For both calls and puts, the dominant factors resemble a volatility-related component and a tail-related component, with maturity variation playing a secondary role.

5 Conclusion

This paper develops a new option-implied methodology to recover conditional expected option returns at each point in time, without imposing assumptions on the underlying asset price dynamics. Building on the option-implied expected return framework of Martin (2017) and subsequent extensions, we generalize these formulas from the market payoff to option payoffs across strikes and maturities. This yields closed-form expressions for expected call, put, and straddle returns under a range of prominent SDF specifications, in which conditional physical expectations are recovered from observed option prices and risk-neutral moments.

Our first set of results shows that existing SDF specifications, while successful for the market return, do not fully characterize option returns. In particular, none of the benchmark models can reproduce the negative average expected returns of out-of-the-money calls, highlighting the importance of a U-shaped pricing kernel and compensation for upside risk. Motivated by this failure, we propose a more flexible SDF specification that retains the key features of the literature, such as preferences for higher moments and intertemporal hedging needs, while allowing for different preference parameters in the left and right tails of the market return distribution. This extension substantially improves the ability of the model to match the cross-section and time-series behavior of expected returns on both calls and puts.

Using our preferred specification, we document several new stylized facts about the pricing of risk in option markets. First, regressing conditional expected option returns on conditional expected market returns yields large and statistically significant alphas, confirming that index options are nonredundant even when inference is based on conditional expectations rather than realized returns. Second, both calls and puts exhibit negative market betas, consistent with a U-shaped pricing kernel in which upside risk commands compensation. Third, expected returns on calls, puts, and straddles are strongly countercyclical: compensation for upside, downside, and variance risk rises sharply during crises and periods of elevated volatility, as reflected in more negative expected option returns. Finally, a principal component analysis reveals a simple factor structure for conditional expected option returns, with two dominant factors corresponding to a level component and a moneyness-slope component, resembling volatility and tail factors.

Overall, our results highlight that derivatives markets contain rich information about the time-varying price of risk that is not fully captured by existing approaches focusing on the market return alone. By providing a unified option-implied framework for conditional expected returns across the entire option surface, this paper offers new tools for empirical asset pricing and opens the door to further work on the joint dynamics of equity and option risk premia, the macroeconomic determinants of tail compensation, and the role of heterogeneous preferences in shaping option markets.

References

Almeida, C., and Freire, G. (2022). Demand in the option market and the pricing kernel. *SSRN Working Paper*.

Back, K., Crotty, K., and Kazempour, S. M. (2022). Validity, tightness, and forecasting power of risk premium bounds. *Journal of Financial Economics*, *144*(3), 732-760.

Baele, L., Driessen, J., Ebert, S., Londono, J. M., and Spalt, O. G. (2019). Cumulative prospect theory, option returns, and the variance premium. *Review of Financial Studies*, *32*(9), 3667-3723.

Bakshi, G., and Kapadia, N. (2003). Delta-hedged gains and the negative market volatility risk premium. *Review of Financial Studies*, *16*(2), 527-566.

Bakshi, G., Madan, D., and Panayotov, G. (2010). Returns of claims on the upside and the viability of U-shaped pricing kernels. *Journal of Financial Economics*, *97*(1), 130-154.

Beason, T., and Schreindorfer, D. (2022). Dissecting the equity premium. *Journal of Political Economy*, *130*(8), 2203-2222.

Bollerslev, T., and Todorov, V. (2011). Tails, fears, and risk premia. *The Journal of Finance*, *66*(6), 2165–2211.

Bondarenko, O. (2014). Why are put options so expensive? *The Quarterly Journal of Finance*, *4*(1), 1450005.

Broadie, M., Chernov, M., and Johannes, M. (2009). Understanding index option returns. *The Review of Financial Studies*, *22*(11), 4493-4529.

Bryzgalova, S., Pavlova, A., and Sikorskaya, T. (2023). Retail trading in options and the rise of the big three wholesalers. *Journal of Finance*, *78*(6), 3465–3514.

Büchner, M., and Kelly, B. (2022). A factor model for option returns. *Journal of Financial Economics*, *143*(3), 1140-1161.

Carr, P., and Madan, D. (2001). Optimal positioning in derivative securities. *Quantitative*

Finance, *1*(1), 19–37.

Chabi-Yo, F., Dim, C., and Vilkov, G. (2023). Generalized bounds on the conditional expected excess return on individual stocks. *Management Science*, *69*(2), 922-939.

Chabi-Yo, F., Gourier, E., and Langlois, H. (2025). Option-implied risk premia with intertemporal hedging. *SSRN Working Paper*.

Chabi-Yo, F., and Loudis, J. (2020). The conditional expected market return. *Journal of Financial Economics*, *137*(3), 752-786.

Chabi-Yo, F., and Loudis, J. A. (2024). A decomposition of conditional risk premia and implications for representative agent models. *Management Science*, *70*(10), 6804-6834.

Chaudhuri, R., and Schroder, M. (2015). Monotonicity of the stochastic discount factor and expected option returns. *The Review of Financial Studies*, *28*(5), 1462-1505.

Coval, J. D., and Shumway, T. (2001). Expected option returns. *The Journal of Finance*, *56*(3), 983-1009.

Driessen, J., and Maenhout, P. J. (2007). An empirical perspective on option pricing anomalies. *Review of Finance*, *11*(4), 561-603.

Farago, A., Khapko, M., and Ornthanalai, C. (2021). Asymmetries and the market for put options. *Working paper, University of Toronto*.

Fournier, M., Jacobs, K., and Orłowski, P. (2024). Modeling conditional factor risk premia implied by index option returns. *The Journal of Finance*, *79*(3), 2289-2338.

Gatheral, J. (2004). A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives. *Presentation at Global Derivatives*.

Jackwerth, J. C., and Menner, M. (2020). Does the Ross recovery theorem work empirically? *Journal of Financial Economics*, *137*(3), 723-739.

Johannes, M., Kaeck, A., Seeger, N., and Shah, N. (2024). Expected 1dte option returns.

SSRN Working Paper.

Jones, C. S. (2006). A nonlinear factor analysis of s&p 500 index option returns. *The Journal of Finance*, *61*(5), 2325-2363.

Kadan, O., and Tang, X. (2020). A bound on expected stock returns. *The Review of Financial Studies*, *33*(4), 1565-1617.

Kim, H. J. (2023). Characterizing the conditional pricing kernel: A new approach. *SSRN Working Paper*.

Martin, I. (2017). What is the expected return on the market? *The Quarterly Journal of Economics*, *132*(1), 367–433.

Martin, I., and Wagner, C. (2019). What is the expected return on a stock? *The Journal of Finance*, *74*(4), 1887-1929.

Pazarbasi, A., Schneider, P., and Vilkov, G. (2024). Dispersion of beliefs bounds: Sentimental recovery. *Management Science*, *70*(12), 8284-8300.

Ross, S. (2015). The recovery theorem. *The Journal of Finance*, *70*(2), 615-648.

Rubinstein, M. (1984). A simple formula for the expected rate of return of an option over a finite holding period. *The Journal of Finance*, *39*(5), 1503-1509.

Santa-Clara, P., and Saretto, A. (2009). Option strategies: Good deals and margin calls. *Journal of Financial Markets*, *12*(3), 391-417.

Schneider, P. (2019). An anatomy of the market return. *Journal of Financial Economics*, *132*(2), 325-350.

Schneider, P., and Trojani, F. (2019). (almost) model-free recovery. *The Journal of Finance*, *74*(1), 323–370.

Schreindorfer, D., and Sichert, T. (2025). Conditional risk and the pricing kernel. *Journal of Financial Economics*, *171*(C).

Tetlock, P. C., McCoy, J., and Shah, N. (2024). The implied equity premium. *SSRN Working Paper*.

A Proof of Equations (3.10) and (B.1)

Consider the function:

$$f_s[x,y] = \frac{u'[W_t x_s y_0]}{u'[W_t xy]}.$$

- Let's perform a second-order Taylor expansion-series of $f^{[s]}[x,y]$ around $(x,y) = (x_s, y_0)$:

$$\begin{aligned} f^{[s]}[x,y] &= 1 + \sum_{i=1}^{2} \frac{1}{i!} \left\{ \frac{\partial^i f^{[s]}[x,y]}{\partial x} \right\}_{(x,y)=(x_s,y_0)} (x - x_s)^i \\ &+ \sum_{i=1}^{2} \frac{1}{i!} \left\{ \frac{\partial^i f^{[s]}[x,y]}{\partial^i y} \right\}_{(x,y)=(x_s,y_0)} (y - y_0)^i \\ &+ \left\{ \frac{\partial^2 f^{[s]}[x,y]}{\partial x \partial y} \right\}_{(x,y)=(x_s,y_0)} (x - x_s)(y - y_0). \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}^*_{T_1} f^{[s]}[x,y] &= 1 + \sum_{i=1}^{2} \frac{1}{i!} \left\{ \frac{\partial^i f^{[s]}[x,y]}{\partial x} \right\}_{(x,y)=(x_s,y_0)} \mathbb{E}^*_{T_1} (x - x_s)^i \\ &+ \sum_{i=1}^{2} \frac{1}{i!} \left\{ \frac{\partial^i f^{[s]}[x,y]}{\partial^i y} \right\}_{(x,y)=(x_s,y_0)} \mathbb{E}^*_{T_1} (y - y_0)^i \\ &+ \left\{ \frac{\partial^2 f^{[s]}[x,y]}{\partial x \partial y} \right\}_{(x,y)=(x_s,y_0)} \mathbb{E}^*_{T_1} (x - x_s)(y - y_0). \end{aligned}$$

Since $\mathbb{E}^*_{T_1}(x - x_s)(y - y_0) = (x - x_s)\mathbb{E}^*_{T_1}(y - y_0) = 0$ because $\mathbb{E}^*_{T_1}(y - y_0) = 0$, it follows that

$$\begin{aligned} \mathbb{E}^*_{T_1} f^{[s]}[x,y] &= 1 + \sum_{i=1}^{2} \frac{1}{i!} \left\{ \frac{\partial^i f^{[s]}[x,y]}{\partial x} \right\}_{(x,y)=(x_s,y_0)} \mathbb{E}^*_{T_1} (x - x_s)^i \\ &+ \frac{1}{2!} \left\{ \frac{\partial^2 f^{[s]}[x,y]}{\partial^2 y} \right\}_{(x,y)=(x_s,y_0)} \mathbb{E}^*_{T_1} (y - y_0)^2 \end{aligned}$$

which simplifies to

$$\mathbb{E}_{T_1}^* f^{[s]}[x,y] = \sum_{i=0}^{2} \frac{a_{i,0,s}}{x_s^i} (x - x_s)^i + \frac{a_{0,2,s}}{y_0^2} \mathbb{E}_{T_1}^* (y - y_0)^2$$

where

$$a_{0,0,s} = 1,\ a_{1,0,s} = \frac{1}{\tau_t^s},\ a_{2,0,s} = \frac{(1-\rho_t^s)}{(\tau_t^s)^2},\ a_{0,2,s} = \frac{(1-\rho_t^s)}{(\tau_t^s)^2} \tag{A.1}$$

- Let's perform a third-order Taylor expansion-series of $f^{[s]}[x,y]$ around $(x,y) = (x_s, y_0)$:

$$\begin{aligned} f^{[s]}[x,y] &= 1 + \sum_{i=1}^{3} \frac{1}{i!} \left\{ \frac{\partial^i f^{[s]}[x,y]}{\partial x} \right\}_{(x,y)=(x_s,y_0)} (x - x_s)^i \\ &+ \sum_{i=1}^{3} \frac{1}{i!} \left\{ \frac{\partial^i f^{[s]}[x,y]}{\partial^i y} \right\}_{(x,y)=(x_s,y_0)} (y - y_0)^i \\ &+ \left\{ \frac{\partial^2 f^{[s]}[x,y]}{\partial x \partial y} \right\}_{(x,y)=(x_s,y_0)} (x - x_s)(y - y_0) \\ &+ \frac{3}{3!} \left\{ \frac{\partial^3 f^{[s]}[x,y]}{\partial x \partial^2 y} \right\}_{(x,y)=(x_s,y_0)} (x - x_s)(y - y_0)^2 \\ &+ \frac{3}{3!} \left\{ \frac{\partial^3 f^{[s]}[x,y]}{\partial^2 x \partial y} \right\}_{(x,y)=(x_s,y_0)} (x - x_s)^2 (y - y_0) \end{aligned}$$

Thus the expectation under the risk neutral world of $f^{[s]}[x,y]$ at T_1 is

$$\begin{aligned}
\mathbb{E}_{T_1}^{*} f^{[s]}[x,y] &= 1+\sum_{i=1}^{3}\frac{1}{i!}\left\{\frac{\partial^{i} f^{[s]}[x,y]}{\partial x}\right\}_{(x,y)=(x_s,y_0)}\mathbb{E}_{T_1}^{*}(x-x_s)^{i} \\
&+\sum_{i=1}^{3}\frac{1}{i!}\left\{\frac{\partial^{i} f^{[s]}[x,y]}{\partial^{i} y}\right\}_{(x,y)=(x_s,y_0)}\mathbb{E}_{T_1}^{*}(y-y_0)^{i} \\
&+\left\{\frac{\partial^{2} f^{[s]}[x,y]}{\partial x\partial y}\right\}_{(x,y)=(x_s,y_0)}(x-x_s)\,\mathbb{E}_{T_1}^{*}(y-y_0) \\
&+\frac{3}{3!}\left\{\frac{\partial^{3} f^{[s]}[x,y]}{\partial x\partial^{2} y}\right\}_{(x,y)=(x_s,y_0)}(x-x_s)\,\mathbb{E}_{T_1}^{*}(y-y_0)^{2} \\
&+\frac{3}{3!}\left\{\frac{\partial^{3} f^{[s]}[x,y]}{\partial^{2} x\partial y}\right\}_{(x,y)=(x_s,y_0)}(x-x_s)^{2}\,\mathbb{E}_{T_1}^{*}(y-y_0)
\end{aligned}$$

Since $\mathbb{E}_{T_1}^{*}(y-y_0)=0$, this expression simplifies to

$$\begin{aligned}
\mathbb{E}_{T_1}^{*} f^{[s]}[x,y] &= 1+\sum_{i=1}^{3}\frac{1}{i!}\left\{\frac{\partial^{i} f^{[s]}[x,y]}{\partial x}\right\}_{(x,y)=(x_s,y_0)}\mathbb{E}_{T_1}^{*}(x-x_s)^{i} \\
&+\sum_{i=2}^{3}\frac{1}{i!}\left\{\frac{\partial^{i} f^{[s]}[x,y]}{\partial^{i} y}\right\}_{(x,y)=(x_s,y_0)}\mathbb{E}_{T_1}^{*}(y-y_0)^{i} \\
&+\frac{3}{3!}\left\{\frac{\partial^{3} f^{[s]}[x,y]}{\partial x\partial^{2} y}\right\}_{(x,y)=(x_s,y_0)}(x-x_s)\,\mathbb{E}_{T_1}^{*}(y-y_0)^{2}
\end{aligned}$$

This expression finally simplifies to:

$$\mathbb{E}_{T_1}^{*} f^{[s]}[x,y]=\sum_{i=0}^{3}\frac{a_{i,0,s}}{x_s^{i}}(x-x_s)^{i}+\sum_{i=2}^{3}\frac{a_{0,i,s}}{y_0^{i}}\mathbb{E}_{T_1}^{*}(y-y_0)^{i}+\frac{a_{1,2,s}}{y_0^{2}x_s}(x-x_s)\,\mathbb{E}_{T_1}^{*}(y-y_0)^{2}$$

where

$$a_{0,0,s}=1,\ a_{1,0,s}=\frac{1}{\tau_t^{s}},\ a_{2,0,s}=\frac{(1-\rho_t^{s})}{(\tau_t^{s})^{2}},\ a_{3,0,s}=\frac{(1-2\rho_t^{s}+\kappa_t^{s})}{(\tau_t^{s})^{3}} \tag{A.2}$$

and

$$a_{0,1,s}=\frac{1}{\tau_t^{s}},\ a_{0,2,s}=\frac{(1-\rho_t^{s})}{(\tau_t^{s})^{2}},\ a_{0,3,s}=\frac{(1-2\rho_t^{s}+\kappa_t^{s})}{(\tau_t^{s})^{3}} \tag{A.3}$$

and

$$a_{1,2,s} = 2a_{2,0,s} + a_{3,0,s} \tag{A.4}$$

B Third-order approximation of the inverse marginal utility

In this section, we provide expressions for the expected return on call and put options when we go beyond the second-order expansion of the inverse marginal utility. More specifically, we perform a third-order Taylor expansion series of $f^{[s]}[x,y]$ around $(x,y) = (x_s, y_0)$ and show that its expectation at time T_1 produces:

$$\mathbb{E}^*_{T_1} f^{[s]}[x,y] = \sum_{i=0}^{3} \frac{a_{i,0,s}}{x_s^i} R^i_{s,M,t\to T_1} + \sum_{i=2}^{3} \frac{a_{0,i,s}}{y_0^i} \mathbb{M}^{*(i)}_{T_1\to T_N} + \frac{a_{1,2,s}}{y_0^2 x_s} R_{s,M,t\to T_1} \mathbb{M}^{*(2)}_{T_1\to T_N}. \tag{B.1}$$

The proof of (B.1) is in Appendix A. We then replace (B.1) in (3.3) and show:

Proposition B.1. *Under no-arbitrage conditions, a third Taylor expansion series of the inverse marginal utility in each region produces an inverse SDF of the form (3.3) where*

$$\frac{\mathbb{E}_t m^{(s)}_{t\to T_1}}{m^{(s)}_{t\to T_1}} = \frac{\sum_{i=0}^{3} \frac{a_{i,0,s}}{x_s^i} R^i_{u,M,t\to T_1} 1_{A_s} + \sum_{i=2}^{3} \frac{a_{0,i,s}}{y_0^i} 1_{A_s} \mathbb{M}^{*(i)}_{T_1\to T_N} + \frac{a_{1,2,s}}{y_0^2 x_s} R_{s,M,t\to T_1} 1_{A_s} \mathbb{M}^{*(2)}_{T_1\to T_N}}{\sum_{i=0}^{3} \frac{a_{i,0,s}}{x_s^i} \mathbb{E}^*_t R^i_{s,M,t\to T_1} + \sum_{i=2}^{3} \frac{a_{0,i,s}}{y_0^i} \mathbb{E}^*_t \mathbb{M}^{*(i)}_{T_1\to T_N} + \frac{a_{1,2,s}}{y_0^2 x_s} \mathbb{E}^*_t \left(R_{s,M,t\to T_1} \mathbb{M}^{*(2)}_{T_1\to T_N} \right)} \quad \textit{for } s \in \{u,d\}.$$

We then exploit Proposition B.1 and show the following results:

Proposition B.2. *Under no-arbitrage conditions, the expected return on a call option with maturity T_1 and strike K_0 is*

$$\mathbb{E}_t R^c_{t\to T_1} = \frac{S_t}{C_t[K_0]} \left(\frac{\mathbb{ECN}^u[k_0]}{\mathbb{ED}^u_3} + \frac{\mathbb{ECN}^d[k_0]}{\mathbb{ED}^d_3} \right)$$

where

$$\mathbb{ECN}^s[k_0] = \left\{ \begin{array}{c} \sum_{j=0}^{3} \frac{a_{j,0,s}}{x_s^j} \mathbb{E}_t^* \left[R_{s,M,t\to T_1}^j \left(R_{M,t\to T_1} - k_0 \right)^+ 1_{A_s} \right] \\ + \sum_{j=2}^{3} \frac{a_{0,j,s}}{y_0^j} \mathbb{E}_t^* \left[\left(R_{M,t\to T_1} - k_0 \right)^+ 1_{A_s} \mathbb{M}_{T_1\to T_N}^{*(j)} \right] \\ + \frac{a_{1,2,s}}{y_0^2 x_u} \mathbb{E}_t^* \left[R_{s,M,t\to T_1} \left(R_{M,t\to T_1} - k_0 \right)^+ 1_{A_s} \mathbb{M}_{T_1\to T_N}^{*(2)} \right] \end{array} \right\}$$

and

$$\mathbb{ED}_3^s = \sum_{i=0}^{3} \frac{a_{i,0,s}}{x_s^i} \mathbb{E}_t^* R_{s,M,t\to T_1}^i + \sum_{i=2}^{3} \frac{a_{0,i,s}}{y_0^i} \mathbb{E}_t^* \mathbb{M}_{T_1\to T_N}^{*(i)} + \frac{a_{1,2,s}}{y_0^2 x_s} \mathbb{E}_t^* \left(R_{s,M,t\to T_1} \mathbb{M}_{T_1\to T_N}^{*(2)} \right) \quad \text{(B.2)}$$

for $s \in \{u, d\}$.

Although Proposition B.2 decomposes the expected return on call options into compensation for exposure to the two regions, it shows the incremental contribution of the third-order Taylor expansion series of the inverse marginal utility on the expected return on call options when comparing it with the expected return on call options implied by the second-order expansion series of the inverse marginal utility.

Next, we derive the compensation required by investors to hold a put option until maturity.

Proposition B.3. *Under no-arbitrage conditions, the expected return on a put option with maturity* T_1 *and strike* K_0 *is*

$$\mathbb{E}_t R_{t\to T_1}^p = \frac{S_t}{P_t[K_0]} \left(\frac{\mathbb{EPN}^u[k_0]}{\mathbb{ED}_3^u} + \frac{\mathbb{EPN}^d[k_0]}{\mathbb{ED}_3^d} \right)$$

where

$$\mathbb{EPN}^s[k_0] = \left\{ \begin{array}{c} \sum_{j=0}^{3} \frac{a_{j,0,s}}{x_s^j} \mathbb{E}_t^* \left[R_{s,M,t\to T_1}^j \left(k_0 - R_{M,t\to T_1} \right)^+ 1_{A_s} \right] + \\ \sum_{j=2}^{3} \frac{a_{0,j,s}}{y_0^j} \mathbb{E}_t^* \left[\left(k_0 - R_{M,t\to T_1} \right)^+ 1_{A_s} \mathbb{M}_{T_1\to T_N}^{*(j)} \right] \\ + \frac{a_{1,2,s}}{y_0^2 x_s} \mathbb{E}_t^* \left[R_{s,M,t\to T_1} \left(k_0 - R_{M,t\to T_1} \right)^+ 1_{A_s} \mathbb{M}_{T_1\to T_N}^{*(2)} \right] \end{array} \right\}$$

for $s \in \{u, d\}$ *and* $\mathbb{ED}_3^s$ *is defined in (B.2).*

C Computation of the risk-neutral quantities

C.1 First subset of risk-neutral quantities

In this subsection, we derive closed-form expressions, under the risk-neutral measure, for a family of building-block expectations that will later enter our option-implied expansions. These objects combine (i) powers of the underlying $(x - x_d)^j$ or $(x - x_u)^j$ (ii) in-the-money payoffs from calls or puts, and (iii) indicator functions that restrict the domain to regions above or below a barrier such as x_0. Formally, we are interested in

$$\mathbb{E}_t^* \left[(x - x_u)^j (k_0 - x)^+ 1_{x > x_0} \right] \tag{C.1}$$

$$\mathbb{E}_t^* \left[(x - x_d)^j (k_0 - x)^+ 1_{x < x_0} \right] \tag{C.2}$$

$$\mathbb{E}_t^* \left[(x - x_u)^j (x - k_0)^+ 1_{x > x_0} \right] \tag{C.3}$$

$$\mathbb{E}_t^* \left[(x - x_d)^j (x - k_0)^+ 1_{x < x_0} \right] \tag{C.4}$$

$$\mathbb{E}_t^* \left[(x - x_u)^j \right] \tag{C.5}$$

$$\mathbb{E}_t^* \left[(x - x_d)^j \right] \tag{C.6}$$

These expectations can all be written in terms of traded option prices observed at time t. This establishes a direct and model-free link between higher-order moments and cross-moments of the underlying and observable option prices. For notational tractability, we introduce the following shorthand:

$$\begin{aligned}
\mathbb{MC}_{d,t}^{*(n)} &= \mathbb{E}_t^* \left[(x - x_d)^n (x - k_0)^+ 1_{x < x_0} \right] && \text{(C.7)} \\
\mathbb{MC}_{u,t}^{*(n)} &= \mathbb{E}_t^* \left[(x - x_u)^n (x - k_0)^+ 1_{x > x_0} \right] \\
\mathbb{M}_{d,t}^{*(n)} &= \mathbb{E}_t^* \left[(x - x_d)^n \right] \\
\mathbb{M}_{u,t}^{*(n)} &= \mathbb{E}_t^* \left[(x - x_u)^n \right] && \text{(C.8)}
\end{aligned}$$

and, analogously, for the put-style objects

$$\begin{aligned}
\mathbb{MP}_{d,t}^{*(n)} &= \mathbb{E}_t^*\left[(x-x_d)^n (k_0-x)^+ 1_{x<x_0}\right] && \text{(C.9)}\\
\mathbb{MP}_{u,t}^{*(n)} &= \mathbb{E}_t^*\left[(x-x_u)^n (k_0-x)^+ 1_{x>x_0}\right] &&\\
\mathbb{M}_{d,t}^{*(n)} &= \mathbb{E}_t^*\left[(x-x_d)^n\right] &&\\
\mathbb{M}_{u,t}^{*(n)} &= \mathbb{E}_t^*\left[(x-x_u)^n\right] && \text{(C.10)}
\end{aligned}$$

These quantities will be fundamental primitives in our decomposition of conditional expected excess returns.

Case: $n = 0$. We begin with the benchmark case n = 0, which already illustrates how payoffs restricted to events like $\{x < x_0\}$ or $\{x > x_0\}$ can be written in terms of option prices with suitably chosen strikes. For the call-region objects, we have

$$\begin{aligned}
\mathbb{MC}_{d,t}^{*(0)} &= \mathbb{E}_t^*\left[(x-k_0)^+ 1_{x<x_0}\right]\\
&= \mathbb{E}_t^*\left[(x-k_0)^+\right] - \mathbb{E}_t^*\left[(x-k_0)^+ 1_{x>x_0}\right]\\
&= \mathbb{E}_t^*\left[(x-k_0)^+\right] - \mathbb{E}_t^*\left[(x-k_0)\, 1_{x>k_0} 1_{x>x_0}\right]\\
&= \mathbb{E}_t^*\left[(x-k_0)\, 1_{x>k_0}\right] - \mathbb{E}_t^*\left[(x-k_0)\, 1_{x>\max(k_0,x_0)}\right]\\
&= \mathbb{E}_t^*\,(x-k_0)^+ - \mathbb{E}_t^*\left[(x-\max(k_0,x_0))\, 1_{x>\max(k_0,x_0)}\right]\\
&\quad - (\max(k_0,x_0)-k_0)\, \mathbb{E}_t^*\left[1_{x>\max(k_0,x_0)}\right]\\
&= \mathbb{E}_t^*\,(x-k_0)^+ - \mathbb{E}_t^*\left[(x-\max(k_0,x_0))^+\right]\\
&\quad - (\max(k_0,x_0)-k_0)\, \mathbb{P}_t^*\left[x>\max(k_0,x_0)\right]\\
&= \frac{R_{f,t\to T_1}}{S_t} C_t\left[k_0 S_t\right] - \frac{R_{f,t\to T_1}}{S_t} C_t\left[S_t \max(k_0,x_0)\right]\\
&\quad - (\max(k_0,x_0)-k_0)\, \mathbb{P}_t^*\left[x>\max(k_0,x_0)\right]
\end{aligned}$$

Similarly

$$
\begin{aligned}
\mathbb{MC}_{u,t}^{*(0)} &= \mathbb{E}_t^* \left[(x - k_0)^+ 1_{x > x_0} \right] \\
&= \mathbb{E}_t^* \left[(x - k_0)^+ \right] - \mathbb{E}_t^* \left[(x - k_0)^+ 1_{x < x_0} \right] \\
&= \frac{R_{f,t \to T_1}}{S_t} C_t \left[k_0 S_t \right] - \mathbb{MC}_{d,t}^{*(n)}
\end{aligned}
$$

For the pure moments

$$
\begin{aligned}
\mathbb{M}_{d,t}^{*(0)} &= 1 \\
\mathbb{M}_{u,t}^{*(0)} &= 1
\end{aligned}
$$

We now perform the analogous decomposition for the put-region objects. This yields

$$
\begin{aligned}
\mathbb{MP}_{d,t}^{*(0)} &= \mathbb{E}_t^* \left[(k_0 - x)^+ 1_{x < x_0} \right] \\
&= \mathbb{E}_t^* \left[(k_0 - x) 1_{x < k_0} 1_{x < x_0} \right] \\
&= \mathbb{E}_t^* \left[(k_0 - x) 1_{x < \min(k_0, x_0)} \right] \\
&= \mathbb{E}_t^* \left[(k_0 - \min(k_0, x_0) + \min(k_0, x_0) - x) 1_{x < \min(k_0, x_0)} \right] \\
&= \mathbb{E}_t^* \left[(k_0 - \min(k_0, x_0)) 1_{x < \min(k_0, x_0)} \right] \\
&\quad + \mathbb{E}_t^* \left[(\min(k_0, x_0) - x) 1_{x < \min(k_0, x_0)} \right] \\
&= (k_0 - \min(k_0, x_0)) \mathbb{P}_t^* \left[x < \min(k_0, x_0) \right] + \frac{R_{f,t \to T_1} P_t \left[S_t \min(k_0, x_0) \right]}{S_t}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{MP}_{u,t}^{*(0)} &= \mathbb{E}_t^* \left[(k_0 - x)^+ 1_{x > x_0} \right] \\
&= \mathbb{E}_t^* \left[(k_0 - x)^+ \right] - \mathbb{E}_t^* \left[(k_0 - x)^+ 1_{x < x_0} \right] \\
&= \frac{R_{f,t \to T_1} P_t \left[S_t k_0 \right]}{S_t} - \mathbb{MP}_{d,t}^{*(0)}
\end{aligned}
$$

Again, we have:

$$\begin{aligned} \mathbb{M}_{d,t}^{*(0)} &= 1 \\ \mathbb{M}_{u,t}^{*(0)} &= 1 \end{aligned} \tag{C.11}$$

C.1.1 Computation of $\mathbb{M}_{d,t}^{*(n)}$ and $\mathbb{M}_{u,t}^{*(n)}$ when n$\neq 0$

We next consider higher-order moments under the risk-neutral measure. These moments admit convenient integral representations in terms of out-of-the-money option prices.

$$\mathbb{M}_{d,t}^{*(n)} = \mathbb{E}_t^* \left[(x - x_d)^n \right] \text{ and } \mathbb{M}_{u,t}^{*(n)} = \mathbb{E}_t^* \left[(x - x_u)^n \right] \tag{C.12}$$

Observe that

$$(x - x_d)^n = \left(\frac{S_{T_1}}{S_t} - x_d \right)^n$$

Standard spanning arguments (Carr and Madan, 2001) then imply

$$\mathbb{E}_t^* (x - x_d)^n = \frac{R_{f,t \to T_1} n (n-1)}{S_t^2} \left\{ \int_{x_d}^{\infty} \left(\frac{K}{S_t} - x_d \right)^{n-2} C_t [K] \, dK + \int_0^{x_d} \left(\frac{K}{S_t} - x_d \right)^{n-2} P_t [K] \, dK \right\}$$

and

$$\mathbb{E}_t^* (x - x_u)^n = \frac{R_{f,t \to T_1} n (n-1)}{S_t^2} \left\{ \int_{x_u}^{\infty} \left(\frac{K}{S_t} - x_u \right)^{n-2} C_t [K] \, dK + \int_0^{x_u} \left(\frac{K}{S_t} - x_u \right)^{n-2} P_t [K] \, dK \right\}$$

These formulas show that all central moments of x around arbitrary reference points x_x or x_u can be synthesized from a strip of calls and puts.

C.1.2 Computation of $\mathbb{MC}^{*(n)}_{d,t}$, $\mathbb{MC}^{*(n)}_{u,t}$, $\mathbb{MP}^{*(n)}_{d,t}$ and $\mathbb{MP}^{*(n)}_{u,t}$ when n$\neq 0$

We now turn to the mixed terms that combine $(x - x_d)^n$ or $(x - x_u)^n$ with in-the-money call or put payoffs, restricted to regions above or below x_0. These appear when we expand nonlinear pricing kernels around multiple reference states. First, note that

$$\begin{aligned}
\mathbb{MC}^{*(n)}_{d,t} &= \mathbb{E}^*_t \left(x - x_d\right)^n \left(x - k_0\right)^+ 1_{x<x_0} \\
&= \mathbb{E}^*_t \left(x - x_d\right)^n \left(x - k_0\right) 1_{x<x_0} 1_{x>k_0} \\
&= \mathbb{E}^*_t \left(x - x_d\right)^n \left(x - k_0\right) \left(1 - 1_{x>x_0}\right) 1_{x>k_0} \\
&= \mathbb{E}^*_t \left[\left(x - x_d\right)^n \left(x - k_0\right) 1_{x>k_0}\right] - \mathbb{E}^*_t \left[\left(x - x_d\right)^n \left(x - k_0\right) 1_{x>x_0} 1_{x>k_0}\right] \\
&= \mathbb{E}^*_t \left[\left(x - x_d\right)^n \left(x - k_0\right) 1_{x>k_0}\right] - \mathbb{E}^*_t \left[\left(x - x_d\right)^n \left(x - k_0\right) 1_{x>\max(x_0,k_0)}\right]
\end{aligned}$$

and

$$\mathbb{MC}^{*(n)}_{u,t} = \mathbb{E}^*_t \left(x - x_d\right)^n \left(x - k_0\right)^+ 1_{x>x_0} = \mathbb{E}^*_t \left(x - x_d\right)^n \left(x - k_0\right) 1_{x>\max(x_0,k_0)}$$

The key object we now need to characterize is the general truncated moments

$$\mathbb{E}^*_t \left[G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha}\right]$$

where for notational convenience we define

$$G^{(n)}\left[S_{T_1}\right] = \left(\frac{S_{T_1}}{S_t} - x_d\right)^n \left(\frac{S_{T_1}}{S_t} - k_0\right)$$

The goal is to obtain an option-based expression for this expectation for an arbitrary threshold α. To proceed, we write $G^{(n)}[.]$ in a (risk-neutral) spanning form.

$$\begin{aligned}
G^{(n)}[S_{T_1}] &= G^{(n)}[S_t R_{f,t\to T_1}] + G_S^{(n)}[S_t R_{f,t\to T_1}](S_{T_1} - S_t R_{f,t\to T_1}) \\
&\quad + \int_{S_t R_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}[K](S_{T_1} - K)^+ dK \\
&\quad + \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K](K - S_{T_1})^+ dK
\end{aligned}$$

Multiplying by $1_{\{x>\alpha\}} = 1_{\{S_{T_1}>\alpha S_t\}}$ and reorganizing indicator functions yields, after a sequence of algebraic steps, and

$$\begin{aligned}
G^{(n)}[S_{T_1}]\, 1_{x>\alpha} &= G^{(n)}[S_t R_{f,t\to T_1}]\, 1_{S_{T_1}>\alpha S_t} + G_S^{(n)}[S_t R_{f,t\to T_1}](S_{T_1} - S_t R_{f,t\to T_1})\, 1_{S_{T_1}>\alpha S_t} \\
&\quad + \int_{S_t R_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}[K](S_{T_1} - K)^+ 1_{S_{T_1}>\alpha S_t} dK \\
&\quad + \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K](K - S_{T_1})^+ 1_{S_{T_1}>\alpha S_t} dK
\end{aligned}$$

which simplifies to

$$\begin{aligned}
G^{(n)}[S_{T_1}]\, 1_{x>\alpha} &= G^{(n)}[S_t R_{f,t\to T_1}]\, 1_{S_{T_1}>\alpha S_t} + G_S^{(n)}[S_t R_{f,t\to T_1}](S_{T_1} - S_t R_{f,t\to T_1})\, 1_{S_{T_1}>\alpha S_t} \\
&\quad + \int_{S_t R_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}[K](S_{T_1} - K)\, 1_{S_{T_1}>\max(\alpha S_t, K)} dK \\
&\quad + \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K](K - S_{T_1})\left(1 - 1_{S_{T_1}>K}\right) 1_{S_{T_1}>\alpha S_t} dK
\end{aligned}$$

and

$$\begin{aligned}
G^{(n)}[S_{T_1}]\, 1_{x>\alpha} &= G^{(n)}[S_t R_{f,t\to T_1}]\, 1_{S_{T_1}>\alpha S_t} + G_S^{(n)}[S_t R_{f,t\to T_1}](S_{T_1} - S_t R_{f,t\to T_1})\, 1_{S_{T_1}>\alpha S_t} \\
&\quad + \int_{S_t R_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}[K](S_{T_1} - K)\, 1_{S_{T_1}>\max(\alpha S_t, K)} dK \\
&\quad + \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K](K - S_{T_1})\left(1_{S_{T_1}>\alpha S_t} - 1_{S_{T_1}>\max(\alpha S_t, K)}\right) dK
\end{aligned}$$

and

$$\begin{aligned}
G^{(n)}\left[S_{T_1}\right]1_{x>\alpha} &= G^{(n)}\left[S_tR_{f,t\to T_1}\right]1_{S_{T_1}>\alpha S_t} + G_S^{(n)}\left[S_tR_{f,t\to T_1}\right]\left(S_{T_1}-S_tR_{f,t\to T_1}\right)1_{S_{T_1}>\alpha S_t} \\
&+ \int_{S_tR_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}\left[K\right]\left(S_{T_1}-K\right)1_{S_{T_1}>\max(\alpha S_t,K)}dK \\
&+ \int_0^{S_tR_{f,t\to T_1}} G_{SS}^{(n)}\left[K\right]\left(K-S_{T_1}\right)1_{S_{T_1}>\alpha S_t}dK \\
&- \int_0^{S_tR_{f,t\to T_1}} G_{SS}^{(n)}\left[K\right]\left(K-S_{T_1}\right)1_{S_{T_1}>\max(\alpha S_t,K)}dK
\end{aligned}$$

and

$$\begin{aligned}
G^{(n)}\left[S_{T_1}\right]1_{x>\alpha} &= G^{(n)}\left[S_tR_{f,t\to T_1}\right]1_{S_{T_1}>\alpha S_t} + G_S^{(n)}\left[S_tR_{f,t\to T_1}\right]\left(S_{T_1}-S_tR_{f,t\to T_1}\right)1_{S_{T_1}>\alpha S_t} \\
&+ \int_{S_tR_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}\left[K\right]\left(S_{T_1}-K\right)1_{S_{T_1}>\max(\alpha S_t,K)}dK \\
&+ \int_0^{S_tR_{f,t\to T_1}} G_{SS}^{(n)}\left[K\right]\left(K-S_{T_1}\right)1_{S_{T_1}>\alpha S_t}dK \\
&+ \int_0^{S_tR_{f,t\to T_1}} G_{SS}^{(n)}\left[K\right]\left(S_{T_1}-K\right)1_{S_{T_1}>\max(\alpha S_t,K)}dK \qquad \text{(C.13)}
\end{aligned}$$

If $\alpha < R_{f,t\to T_1}$, (C.13) simplifies to

$$\begin{aligned}
& G^{(n)}\left[S_{T_1}\right]1_{x>\alpha} \\
=\ & G^{(n)}\left[S_tR_{f,t\to T_1}\right]1_{S_{T_1}>\alpha S_t} + G_S^{(n)}\left[S_tR_{f,t\to T_1}\right]\left(S_{T_1}-S_tR_{f,t\to T_1}\right)1_{S_{T_1}>\alpha S_t} \\
& + \int_{S_tR_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}\left[K\right]\left(S_{T_1}-K\right)1_{S_{T_1}>\max(\alpha S_t,K)}dK + \int_0^{S_tR_{f,t\to T_1}} G_{SS}^{(n)}\left[K\right]\left(K-S_{T_1}\right)1_{S_{T_1}>\alpha S_t}dK \\
& + \int_0^{S_t\alpha} G_{SS}^{(n)}\left[K\right]\left(S_{T_1}-K\right)1_{S_{T_1}>\max(\alpha S_t,K)}dK + \int_{S_t\alpha}^{S_tR_{f,t\to T_1}} G_{SS}^{(n)}\left[K\right]\left(S_{T_1}-K\right)1_{S_{T_1}>\max(\alpha S_t,K)}dK
\end{aligned}$$

and

$$
\begin{aligned}
& G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha} \\
= \; & G^{(n)}\left[S_t R_{f,t\to T_1}\right] 1_{S_{T_1}>\alpha S_t} + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(S_{T_1} - S_t R_{f,t\to T_1}\right) 1_{S_{T_1}>\alpha S_t} \\
& + \int_{S_t R_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}[K]\left(S_{T_1}-K\right) 1_{S_{T_1}>K} dK + \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(K - S_{T_1}\right) 1_{S_{T_1}>\alpha S_t} dK \\
& + \int_0^{S_t\alpha} G_{SS}^{(n)}[K]\left(S_{T_1}-K\right) 1_{S_{T_1}>\alpha S_t} dK + \int_{S_t\alpha}^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(S_{T_1}-K\right) 1_{S_{T_1}>K} dK
\end{aligned}
$$

and

$$
\begin{aligned}
& G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha} \\
= \; & G^{(n)}\left[S_t R_{f,t\to T_1}\right] 1_{S_{T_1}>\alpha S_t} + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(S_{T_1} - \alpha S_t\right) 1_{S_{T_1}>\alpha S_t} \\
& + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(\alpha S_t - S_t R_{f,t\to T_1}\right) 1_{S_{T_1}>\alpha S_t} \\
& + \int_{S_t R_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}[K]\left(S_{T_1}-K\right)^{+} dK + \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(K - \alpha S_t\right) 1_{S_{T_1}>\alpha S_t} dK \\
& + \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(\alpha S_t - S_{T_1}\right) 1_{S_{T_1}>\alpha S_t} dK \\
& + \int_0^{S_t\alpha} G_{SS}^{(n)}[K]\left(S_{T_1}-\alpha S_t\right) 1_{S_{T_1}>\alpha S_t} dK + \int_0^{S_t\alpha} G_{SS}^{(n)}[K]\left(\alpha S_t - K\right) 1_{S_{T_1}>\alpha S_t} dK \\
& + \int_{S_t\alpha}^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(S_{T_1}-K\right)^{+} dK
\end{aligned}
$$

and

$$\begin{aligned}
& G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha} \\
= \;& G^{(n)}\left[S_t R_{f,t\to T_1}\right] 1_{S_{T_1}>\alpha S_t} + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(S_{T_1}-\alpha S_t\right)^+ \\
& + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(\alpha S_t - S_t R_{f,t\to T_1}\right) 1_{S_{T_1}>\alpha S_t} \\
& + \int_{S_t R_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}[K]\left(S_{T_1}-K\right)^+ dK + \left(1_{S_{T_1}>\alpha S_t}\right)\int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(K-\alpha S_t\right) dK \\
& - \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(S_{T_1}-\alpha S_t\right)^+ dK \\
& + \int_0^{S_t \alpha} G_{SS}^{(n)}[K]\left(S_{T_1}-\alpha S_t\right)^+ dK + \left(1_{S_{T_1}>\alpha S_t}\right)\int_0^{S_t \alpha} G_{SS}^{(n)}[K]\left(\alpha S_t - K\right) dK \\
& + \int_{S_t \alpha}^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(S_{T_1}-K\right)^+ dK
\end{aligned}$$

Taking $\mathbb{E}_t^*[.]$ on both sides of this expression gives an expression entirely in terms of option prices and risk-neutral probabilities:

$$\begin{aligned}
& \mathbb{E}_t^*\left[G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha}\right] \\
= \;& G^{(n)}\left[S_t R_{f,t\to T_1}\right] \mathbb{E}_t^* 1_{S_{T_1}>\alpha S_t} + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right] \mathbb{E}_t^*\left(S_{T_1}-\alpha S_t\right)^+ \\
& + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(\alpha S_t - S_t R_{f,t\to T_1}\right) \mathbb{E}_t^* 1_{S_{T_1}>\alpha S_t} \\
& + \int_{S_t R_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}[K]\, \mathbb{E}_t^*\left(S_{T_1}-K\right)^+ dK + \mathbb{E}_t^*\left(1_{S_{T_1}>\alpha S_t}\right)\int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(K-\alpha S_t\right) dK \\
& - \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\, \mathbb{E}_t^*\left(S_{T_1}-\alpha S_t\right)^+ dK \\
& + \int_0^{S_t \alpha} G_{SS}^{(n)}[K]\, \mathbb{E}_t^*\left(S_{T_1}-\alpha S_t\right)^+ dK + \mathbb{E}_t^*\left(1_{S_{T_1}>\alpha S_t}\right)\int_0^{S_t \alpha} G_{SS}^{(n)}[K]\left(\alpha S_t - K\right) dK \\
& + \int_{S_t \alpha}^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\, \mathbb{E}_t^*\left(S_{T_1}-K\right)^+ dK
\end{aligned} \tag{C.14}$$

and

$$
\begin{aligned}
& \mathbb{E}_t^*\left[G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha}\right] \\
= \; & G^{(n)}\left[S_t R_{f,t\to T_1}\right] \mathbb{P}_t^*\left[S_{T_1}>\alpha S_t\right]+G_S^{(n)}\left[S_t R_{f,t\to T_1}\right] R_{f,t\to T_1} C_t\left[\alpha S_t\right] \\
& +G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(\alpha S_t-S_t R_{f,t\to T_1}\right) \mathbb{P}_t^*\left[S_{T_1}>\alpha S_t\right] \\
& +\int_{S_t R_{f,t\to T_1}}^{\infty} G_{SS}^{(n)}[K] R_{f,t\to T_1} C_t[K] dK+\mathbb{P}_t^*\left[S_{T_1}>\alpha S_t\right] \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(K-\alpha S_t\right) dK \\
& -\int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K] R_{f,t\to T_1} C_t\left[\alpha S_t\right] dK+\int_0^{S_t \alpha} G_{SS}^{(n)}[K] R_{f,t\to T_1} C_t\left[\alpha S_t\right] dK \\
& +\mathbb{P}_t^*\left[S_{T_1}>\alpha S_t\right] \int_0^{S_t \alpha} G_{SS}^{(n)}[K]\left(\alpha S_t-K\right) dK+\int_{S_t \alpha}^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K] R_{f,t\to T_1} C_t[K] dK
\end{aligned} \tag{C.15}
$$

Now, assume that $\alpha > R_{f,t\to T_1}$. (C.13) simplifies to

$$
\begin{aligned}
G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha} = \; & G^{(n)}\left[S_t R_{f,t\to T_1}\right] 1_{S_{T_1}>\alpha S_t}+G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(S_{T_1}-S_t R_{f,t\to T_1}\right) 1_{S_{T_1}>\alpha S_t} \\
& +\int_{S_t R_{f,t\to T_1}}^{\alpha S_t} G_{SS}^{(n)}[K]\left(S_{T_1}-K\right) 1_{S_{T_1}>\max(\alpha S_t, K)} dK \\
& +\int_{\alpha S_t}^{\infty} G_{SS}^{(n)}[K]\left(S_{T_1}-K\right) 1_{S_{T_1}>\max(\alpha S_t, K)} dK \\
& +\int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(K-S_{T_1}\right) 1_{S_{T_1}>\alpha S_t} dK \\
& +\int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(S_{T_1}-K\right) 1_{S_{T_1}>\alpha S_t} dK
\end{aligned} \tag{C.16}
$$

This simplifies to

$$
\begin{aligned}
& G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha} \\
= \; & G^{(n)}\left[S_t R_{f,t\to T_1}\right] 1_{S_{T_1}>\alpha S_t} + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(S_{T_1} - S_t R_{f,t\to T_1}\right) 1_{S_{T_1}>\alpha S_t} \\
& + \int_{S_t R_{f,t\to T_1}}^{\alpha S_t} G_{SS}^{(n)}[K]\left(S_{T_1} - K\right) 1_{S_{T_1}>\alpha S_t} dK \\
& + \int_{\alpha S_t}^{\infty} G_{SS}^{(n)}[K]\left(S_{T_1} - K\right) 1_{S_{T_1}>K} dK \\
& + \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(K - S_{T_1}\right) 1_{S_{T_1}>\alpha S_t} dK \\
& + \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(S_{T_1} - K\right) 1_{S_{T_1}>\alpha S_t} dK
\end{aligned}
\tag{C.17}
$$

and

$$
\begin{aligned}
& G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha} \\
= \; & G^{(n)}\left[S_t R_{f,t\to T_1}\right] 1_{S_{T_1}>\alpha S_t} + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(S_{T_1} - S_t R_{f,t\to T_1}\right) 1_{S_{T_1}>\alpha S_t} \\
& + \int_0^{\alpha S_t} G_{SS}^{(n)}[K]\left(S_{T_1} - K\right) 1_{S_{T_1}>\alpha S_t} dK \\
& + \int_{\alpha S_t}^{\infty} G_{SS}^{(n)}[K]\left(S_{T_1} - K\right)^+ dK \\
& + \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(K - S_{T_1}\right) 1_{S_{T_1}>\alpha S_t} dK
\end{aligned}
$$

and

$$
\begin{aligned}
& G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha} \\
= \; & G^{(n)}\left[S_t R_{f,t\to T_1}\right] 1_{S_{T_1}>\alpha S_t} \\
& + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(S_{T_1}-\alpha S_t\right)^{+} + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(\alpha S_t - S_t R_{f,t\to T_1}\right) 1_{S_{T_1}>\alpha S_t} \\
& + \int_0^{\alpha S_t} G_{SS}^{(n)}[K]\left(S_{T_1}-\alpha S_t\right)^{+} dK + \left(1_{S_{T_1}>\alpha S_t}\right) \int_0^{\alpha S_t} G_{SS}^{(n)}[K]\left(\alpha S_t - K\right) dK \\
& + \int_{\alpha S_t}^{\infty} G_{SS}^{(n)}[K]\left(S_{T_1}-K\right)^{+} dK \\
& + \left(1_{S_{T_1}>\alpha S_t}\right) \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(K-\alpha S_t\right) dK - \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(S_{T_1}-\alpha S_t\right)^{+} dK
\end{aligned}
$$

We then take the expectation under the risk-neutral measure of this quantity to obtain

$$
\begin{aligned}
& \mathbb{E}_t^* G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha} \\
= \; & G^{(n)}\left[S_t R_{f,t\to T_1}\right] \mathbb{E}_t^* 1_{S_{T_1}>\alpha S_t} \\
& + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right] \mathbb{E}_t^*\left(S_{T_1}-\alpha S_t\right)^{+} + G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(\alpha S_t - S_t R_{f,t\to T_1}\right) \mathbb{E}_t^* 1_{S_{T_1}>\alpha S_t} \\
& + \int_0^{\alpha S_t} G_{SS}^{(n)}[K] \mathbb{E}_t^*\left(S_{T_1}-\alpha S_t\right)^{+} dK + \mathbb{E}_t^*\left(1_{S_{T_1}>\alpha S_t}\right) \int_0^{\alpha S_t} G_{SS}^{(n)}[K]\left(\alpha S_t - K\right) dK \\
& + \int_{\alpha S_t}^{\infty} G_{SS}^{(n)}[K] \mathbb{E}_t^*\left(S_{T_1}-K\right)^{+} dK \\
& + \mathbb{E}_t^*\left(1_{S_{T_1}>\alpha S_t}\right) \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(K-\alpha S_t\right) dK - \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K] \mathbb{E}_t^*\left(S_{T_1}-\alpha S_t\right)^{+} dK
\end{aligned}
$$

which simplifies to

$$\begin{aligned}
& \mathbb{E}_t^* G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha} \\
= & \mathbb{E}_t^* G^{(n)}\left[S_{T_1}\right] 1_{x>\alpha} \\
= & G^{(n)}\left[S_t R_{f,t\to T_1}\right] \mathbb{P}_t^*\left[S_{T_1}>\alpha S_t\right] \\
& +G_S^{(n)}\left[S_t R_{f,t\to T_1}\right] R_{f,t\to T_1} C_t\left[\alpha S_t\right]+G_S^{(n)}\left[S_t R_{f,t\to T_1}\right]\left(\alpha S_t-S_t R_{f,t\to T_1}\right) \mathbb{P}_t^*\left[S_{T_1}>\alpha S_t\right] \\
& +R_{f,t\to T_1} C_t\left[\alpha S_t\right] \int_0^{\alpha S_t} G_{SS}^{(n)}[K]\, dK+\mathbb{P}_t^*\left[S_{T_1}>\alpha S_t\right] \int_0^{\alpha S_t} G_{SS}^{(n)}[K]\left(\alpha S_t-K\right) dK \\
& +R_{f,t\to T_1} \int_{\alpha S_t}^{\infty} G_{SS}^{(n)}[K]\, C_t[K]\, dK+\mathbb{P}_t^*\left[S_{T_1}>\alpha S_t\right] \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\left(K-\alpha S_t\right) dK \\
& -R_{f,t\to T_1} C_t\left[\alpha S_t\right] \int_0^{S_t R_{f,t\to T_1}} G_{SS}^{(n)}[K]\, dK
\end{aligned} \tag{C.18}$$

To compute

$$\begin{aligned}
\mathbb{MP}_{d,t}^{*(n)} &= \mathbb{E}_t^*\left[\left(x-x_d\right)^n\left(k_0-x\right)^+ 1_{x<x_0}\right] \qquad \text{(C.19)} \\
\mathbb{MP}_{u,t}^{*(n)} &= \mathbb{E}_t^*\left[\left(x-x_u\right)^n\left(k_0-x\right)^+ 1_{x>x_0}\right]
\end{aligned}$$

notice that

$$\begin{aligned}
\mathbb{MP}_{d,t}^{*(n)} &= \mathbb{E}_t^*\left[\left(x-x_d\right)^n\left(k_0-x\right) 1_{x<k_0} 1_{x<x_0}\right] \\
&= -\mathbb{E}_t^*\left[\left(x-x_d\right)^n\left(x-k_0\right) 1_{x<k_0} 1_{x<x_0}\right] \\
&= -\mathbb{E}_t^*\left[\left(x-x_d\right)^n\left(x-k_0\right)\left(1-1_{x>k_0}\right) 1_{x<x_0}\right] \\
&= -\mathbb{E}_t^*\left[\left(x-x_d\right)^n\left(x-k_0\right) 1_{x<x_0}\right] \\
&\quad +\mathbb{E}_t^*\left[\left(x-x_d\right)^n\left(x-k_0\right)^+ 1_{x<x_0}\right] \\
&= -\mathbb{E}_t^*\left[\left(x-x_d\right)^n\left(x-k_0\right) 1_{x<x_0}\right]+\mathbb{MC}_{d,t}^{*(n)} \\
&= \mathbb{MC}_{d,t}^{*(n)}-\mathbb{E}_t^*\left[\left(x-x_d\right)^n\left(x-k_0\right)\right] \\
&\quad +\mathbb{E}_t^*\left[\left(x-x_d\right)^n\left(x-k_0\right) 1_{x>x_0}\right]
\end{aligned}$$

Thus

$$\mathbb{MP}_{d,t}^{*(n)} = \mathbb{MC}_{d,t}^{*(n)} - \mathbb{E}_t^* \left[(x - x_d)^n (x - k_0) \right] + \mathbb{E}_t^* \left[(x - x_d)^n (x - k_0) 1_{x > x_0} \right] \tag{C.20}$$

We have already provided an expression for $\mathbb{E}_t^* \left[(x - x_d)^n (x - k_0) 1_{x > x_0} \right]$ (see (C.15) and (C.18)). Next

$$\begin{aligned}
\mathbb{MP}_{u,t}^{*(n)} &= \mathbb{E}_t^* \left[(x - x_u)^n (k_0 - x) 1_{x < k_0} 1_{x > x_0} \right] \\
&= \mathbb{E}_t^* \left[(x - x_u)^n (k_0 - x) (1 - 1_{x > k_0}) 1_{x > x_0} \right] \\
&= \mathbb{E}_t^* \left[(x - x_u)^n (k_0 - x) 1_{x > x_0} \right] - \mathbb{E}_t^* \left[(x - x_u)^n (k_0 - x) 1_{x > k_0} 1_{x > x_0} \right] \\
&= \mathbb{E}_t^* \left[(x - x_u)^n (k_0 - x) 1_{x > x_0} \right] - \mathbb{E}_t^* \left[(x - x_u)^n (k_0 - x) 1_{x > \max(k_0, x_0)} \right] \\
&= -\mathbb{E}_t^* \left[(x - x_u)^n (x - k_0) 1_{x > x_0} \right] + \mathbb{E}_t^* \left[(x - x_u)^n (x - k_0) 1_{x > \max(k_0, x_0)} \right]
\end{aligned}$$

In turn, these expectations can be reduced to the general formulas in (C.15) and (C.18), so we do not need any additional spanning identities beyond those we have already established.

C.1.3 Computation of $\mathbb{E}_t^* \left[(x - x_d)^n (x - k_0) \right]$ when n$\neq 0$

Set

$$H^{(n)} [S_{T_1}] = \left(\frac{S_{T_1}}{S_t} - x_d \right)^n \left(\frac{S_{T_1}}{S_t} - k_0 \right)$$

Observe that

$$\mathbb{E}_t^* H^{(n)} [S_{T_1}] = \int_{x_d}^{\infty} G_{SS}^{(n)} [K] R_{f,t \to T_1} C_t [K] dK + \int_0^{x_d} G_{SS}^{(n)} [K] R_{f,t \to T_1} P_t [K] dK$$

C.1.4 Computation of $\mathbb{COV}_t^*\left[(x-x_0)^2,(x-x_d)^n 1_{x<x_0}\right]$ and $\mathbb{COV}_t^*\left[(x-x_0)^2,(x-x_u)^n 1_{x>x_0}\right]$

$$\begin{aligned}\mathbb{COV}_t^*\left[(x-x_0)^2,(x-x_d)^n 1_{x<x_0}\right] &= \mathbb{E}_t^*\left[(x-x_0)^2(x-x_d)^n 1_{x<x_0}\right] \\ &\quad -\left(\mathbb{E}_t^*(x-x_0)^2\right)\left(\mathbb{E}_t^*\left[(x-x_d)^n 1_{x<x_0}\right]\right) \end{aligned} \tag{C.21}$$

and

$$\begin{aligned}\mathbb{COV}_t^*\left[(x-x_0)^2,(x-x_u)^n 1_{x>x_0}\right] &= \mathbb{E}_t^*\left[(x-x_0)^2(x-x_u)^n 1_{x>x_0}\right] \\ &\quad -\left(\mathbb{E}_t^*(x-x_0)^2\right)\left(\mathbb{E}_t^*(x-x_u)^n 1_{x>x_0}\right) \end{aligned} \tag{C.22}$$

We just need to find expressions for $\mathbb{E}_t^*\left[(x-x_0)^2(x-x_d)^n 1_{x<x_0}\right]$ and $\left(\mathbb{E}_t^*\left[(x-x_d)^n 1_{x<x_0}\right]\right)$ because

$$\mathbb{E}_t^*\left[(x-x_0)^2(x-x_u)^n 1_{x>x_0}\right]=\mathbb{E}_t^*\left[(x-x_0)^2(x-x_u)^n\right]-\mathbb{E}_t^*\left[(x-x_0)^2(x-x_u)^n 1_{x<x_0}\right]$$

and

$$\mathbb{E}_t^*\left[(x-x_d)^n 1_{x>x_0}\right]=\mathbb{E}_t^*(x-x_d)^n-\left(\mathbb{E}_t^*\left[(x-x_d)^n 1_{x<x_0}\right]\right)$$

Expressions $\mathbb{E}_t^*\left[(x-x_0)^2(x-x_d)^n 1_{x<x_0}\right]$ and $\left(\mathbb{E}_t^*\left[(x-x_d)^n 1_{x<x_0}\right]\right)$ can be found using a general formula provided in (C.15) and (C.18).

C.1.5 Alternative approach

For completeness, we present an equivalent route to the same objects that emphasizes conditioning on barrier events such as $x>\delta$. We note that

$$\mathbb{MC}^{*(n)}_{d,t} = \mathbb{E}^*_t\left[(x-x_d)^n (x-k_0)^+ 1_{x<x_0}\right] \quad \text{(C.23)}$$

$$\mathbb{MC}^{*(n)}_{u,t} = \mathbb{E}^*_t\left[(x-x_u)^n (x-k_0)^+ 1_{x>x_0}\right] \quad \text{(C.24)}$$

and

$$\mathbb{MP}^{*(n)}_{d,t} = \mathbb{E}^*_t\left[(x-x_d)^n (k_0-x)^+ 1_{x<x_0}\right] \quad \text{(C.25)}$$

$$\mathbb{MP}^{*(n)}_{u,t} = \mathbb{E}^*_t\left[(x-x_u)^n (k_0-x)^+ 1_{x>x_0}\right] \quad \text{(C.26)}$$

Notice that

$$\begin{aligned}
\mathbb{MC}^{*(n)}_{d,t} &= \mathbb{E}^*_t\left[(x-x_d)^n (x-k_0)\, 1_{x>k_0}\right] - \mathbb{E}^*_t\left[(x-x_d)^n (x-k_0)\, 1_{x>\max(x_0,k_0)}\right] \\
\mathbb{MC}^{*(n)}_{u,t} &= \mathbb{E}^*_t\,(x-x_d)^n (x-k_0)\, 1_{x>\max(x_0,k_0)}
\end{aligned}$$

Let's find in general the formula

$$\mathbb{E}^*_t\,(x-x_d)^n (x-k_0)\, 1_{x>\delta}$$

where δ is given. To proceed, define

$$G\left[S_T\right] = \left(\frac{S_T}{S_t} - x_d\right)^n \left(\frac{S_T}{S_t} - k_0\right).$$

We then have

$$\mathbb{E}^*_t\,(x-x_d)^n (x-k_0)\, 1_{x>\delta} = \mathbb{E}^*_t\left[G\left[S_T\right] 1_{S_T>S_t\delta}\right].$$

The spanning formula (Carr and Madan, 2001) allows to write

$$G\left[S_T\right] = G\left[S_t\delta\right] + G_S\left[S_t\delta\right]\left(S_T - S_t\delta\right) + \int_{S_t\delta}^{\infty} G_{SS}\left[K\right]\left(S_T - K\right)^+ dK + \int_0^{S_t\delta} G_{SS}\left[K\right]\left(K - S_T\right)^+ dK$$

and

$$\begin{aligned} G\left[S_T\right] 1_{x>\delta} &= G\left[S_T\right] 1_{S_T>S_t\delta} \\ &= G\left[S_t\delta\right] 1_{S_T>S_t\delta} + G_S\left[S_t\delta\right]\left(S_T - S_t\delta\right) 1_{S_T>S_t\delta} \\ &\quad + \int_{S_t\delta}^{\infty} G_{SS}\left[K\right]\left(S_T - K\right)^+ 1_{S_T>S_t\delta} dK + \int_0^{S_t\delta} G_{SS}\left[K\right]\left(K - S_T\right)^+ 1_{S_T>S_t\delta} dK \end{aligned}$$

and

$$\begin{aligned} G\left[S_T\right] 1_{x>\delta} &= G\left[S_t\delta\right] 1_{S_T>S_t\delta} + G_S\left[S_t\delta\right]\left(S_T - S_t\delta\right) 1_{S_T>S_t\delta} \\ &\quad + \int_{S_t\delta}^{\infty} G_{SS}\left[K\right]\left(S_T - K\right) 1_{S_T>\max(S_t\delta,K)} dK \\ &\quad + \int_0^{S_t\delta} G_{SS}\left[K\right]\left(K - S_T\right) 1_{S_T<K} 1_{S_T>S_t\delta} dK \end{aligned}$$

and

$$\begin{aligned} G\left[S_T\right] 1_{x>\delta} &= G\left[S_t\delta\right] 1_{S_T>S_t\delta} + G_S\left[S_t\delta\right]\left(S_T - S_t\delta\right) 1_{S_T>S_t\delta} \\ &\quad + \int_{S_t\delta}^{\infty} G_{SS}\left[K\right]\left(S_T - K\right) 1_{S_T>K} dK \end{aligned}$$

Hence:

$$\begin{aligned} \mathbb{E}_t^*\left[G\left[S_T\right] 1_{x>\delta}\right] &= G\left[S_t\delta\right] \mathbb{E}_t^*\left[1_{S_T>S_t\delta}\right] + G_S\left[S_t\delta\right] \mathbb{E}_t^*\left[\left(S_T - S_t\delta\right) 1_{S_T>S_t\delta}\right] \\ &\quad + \int_{S_t\delta}^{\infty} G_{SS}\left[K\right] \mathbb{E}_t^*\left[\left(S_T - K\right) 1_{S_T>K}\right] dK \end{aligned}$$

Thus

$$\mathbb{E}_t^* \left[G\left[S_T\right] 1_{x>\delta}\right] = G\left[S_t\delta\right] \mathbb{P}_t^* \left[S_T > S_t\delta\right] + G_S\left[S_t\delta\right] R_{f,t\to T_1} C_t\left[S_t\delta\right] + R_{f,t\to T_1} \int_{S_t\delta}^{\infty} G_{SS}\left[K\right] C_t\left[K\right] dK$$

C.2 Second subset of risk-neutral quantities

We now extend the previous analysis to the joint (x, y) environment, where y represents the continuation payoff (e.g., a market return from T_1 to T_N). Our goal is to characterize expectations in which the x-leg is multiplied by conditional moments of y given information at T_1. These terms are essential for capturing dynamic, multi-period pricing effects. We are interested in the following family

$$\mathbb{E}_t^* \left[(x - x_u)^j (k_0 - x)^+ 1_{x>x_0} \mathbb{E}_{T_1}^* (y - y_0)^{k-j}\right] \tag{C.27}$$

$$\mathbb{E}_t^* \left[(x - x_d)^j (k_0 - x)^+ 1_{x<x_0} \mathbb{E}_{T_1}^* (y - y_0)^{k-j}\right] \tag{C.28}$$

$$\mathbb{E}_t^* \left[(x - x_u)^j (x - k_0)^+ 1_{x>x_0} \mathbb{E}_{T_1}^* (y - y_0)^{k-j}\right] \tag{C.29}$$

$$\mathbb{E}_t^* \left[(x - x_d)^j (x - k_0)^+ 1_{x<x_0} \mathbb{E}_{T_1}^* (y - y_0)^{k-j}\right] \tag{C.30}$$

$$\mathbb{E}_t^* \left[(x - x_u)^j \mathbb{E}_{T_1}^* (y - y_0)^{k-j}\right] \tag{C.31}$$

$$\mathbb{E}_t^* \left[(x - x_d)^j \mathbb{E}_{T_1}^* (y - y_0)^{k-j}\right] \tag{C.32}$$

We will refer to these as the "second set" of risk-neutral quantities because they couple the one-period option-implied structure of x with higher-horizon information transmitted through y. When $k - j = 0$, these expressions collapse to the objects in the first subset. The novel case is $k - j \neq 0$, in which we must compute conditional moments of y under the T_1-forward risk-neutral measure. We begin with the binomial expansion:

$$\begin{aligned}(y-y_0)^n &= \sum_{i=0}^{n}\frac{n!\,(-1)^{n-i}}{(n-i)!i!}y^i y_0^{n-i}\\ &= \sum_{i=0}^{n}\frac{n!\,(-1)^{n-i}}{(n-i)!i!}\left(y^i-y_0^i+y_0^i\right)y_0^{n-i}\\ &= \sum_{i=0}^{n}\frac{n!\,(-1)^{n-i}}{(n-i)!i!}\left(y^i-y_0^i\right)y_0^{n-i}+\sum_{i=0}^{n}\frac{n!\,(-1)^{n-i}}{(n-i)!i!}y_0^n.\end{aligned}$$

Take the expectation at time T_1 :

$$\begin{aligned}\mathbb{E}^*_{T_1}(y-y_0)^n &= \sum_{i=0}^{n}\frac{n!\,(-1)^{n-i}\,y_0^{n-i}}{(n-i)!i!}\left(\mathbb{E}^*_{T_1}y^i-y_0^i\right)+\sum_{i=0}^{n}\frac{n!\,(-1)^{n-i}}{(n-i)!i!}y_0^n\\ &= \sum_{i=0}^{n}\frac{n!\,(-1)^{n-i}\,y_0^{n-i}}{(n-i)!i!}\left(\mathbb{E}^*_{T_1}y^i-y_0^i\right).\end{aligned} \tag{C.33}$$

where

$$\sum_{i=0}^{n}\frac{n!\,(-1)^{n-i}}{(n-i)!i!}y_0^n=0.$$

We assume that $\mathbb{E}^*_{T_1}y^i-y_0^i$ can be written in terms of the (piecewise) deviations of x from x_0

$$\mathbb{E}^*_{T_1}y^i-y_0^i=\theta_{i,t}\left(\left(x^i-x_0^i\right)1_{x>x_0}+\left(x_0^i-x^i\right)1_{x<x_0}\right) \tag{C.34}$$

Thus

$$\theta_{i,t}=\frac{\mathbb{E}^*_t\left[x^iy^i\right]-y_0^i\mathbb{E}^*_t\left[x^i\right]}{\mathbb{E}^*_t\left[x^i\left(x^i-x_0^i\right)1_{x>x_0}\right]+\mathbb{E}^*_t\left[x^i\left(x_0^i-x^i\right)1_{x<x_0}\right]}$$

Observe that

$$\mathbb{E}^*_t\left[x^iy^i\right]=\mathbb{E}^*_t\left[R^i_{M,t\to T_1}R^i_{M,T_1\to T_N}\right]=\mathbb{E}^*_t\left[R^i_{M,t\to T_N}\right] \tag{C.35}$$

Therefore to compute quantities (C.27)-(C.32), we need:

$$\begin{aligned}
&\mathbb{E}_t^*\left[(x-x_d)^j(k_0-x)^+ 1_{x<x_0}\mathbb{E}_{T_1}^*(y-y_0)^{k-j}\right]\\
=&\sum_{i=0}^{k-j}\frac{(k-j)!(-1)^{(k-j)-i}y_0^{(k-j)-i}}{(k-j-i)!i!}\mathbb{E}_t^*\left[(x-x_d)^j(k_0-x)^+ 1_{x<x_0}\left(\mathbb{E}_{T_1}^* y^i-y_0^i\right)\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}_t^*\left[(x-x_d)^j(x-k_0)^+ 1_{x<x_0}\mathbb{E}_{T_1}^*(y-y_0)^{k-j}\right]\\
=&\sum_{i=0}^{k-j}\frac{(k-j)!(-1)^{(k-j)-i}y_0^{(k-j)-i}}{(k-j-i)!i!}\mathbb{E}_t^*\left[(x-x_d)^j(x-k_0)^+ 1_{x<x_0}\left(\mathbb{E}_{T_1}^* y^i-y_0^i\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_t^*\left[(x-x_d)^j\mathbb{E}_{T_1}^*(y-y_0)^{k-j}\right]\\
=&\sum_{i=0}^{k-j}\frac{(k-j)!(-1)^{(k-j)-i}y_0^{(k-j)-i}}{(k-j-i)!i!}\mathbb{E}_t^*\left[(x-x_d)^j\left(\mathbb{E}_{T_1}^* y^i-y_0^i\right)\right]
\end{aligned}$$

As a result

$$\begin{aligned}
&\mathbb{E}_t^*\left[(x-x_u)^j(k_0-x)^+ 1_{x>x_0}\mathbb{E}_{T_1}^*(y-y_0)^{k-j}\right]\\
=&\sum_{i=0}^{k-j}\frac{(k-j)!(-1)^{(k-j)-i}y_0^{(k-j)-i}}{(k-j-i)!i!}\mathbb{E}_t^*\left[(x-x_u)^j(k_0-x)^+ 1_{x>x_0}\left(\mathbb{E}_{T_1}^* y^i-y_0^i\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_t^*\left[(x-x_u)^j(x-k_0)^+ 1_{x>x_0}\mathbb{E}_{T_1}^*(y-y_0)^{k-j}\right]\\
=&\sum_{i=0}^{k-j}\frac{(k-j)!(-1)^{(k-j)-i}y_0^{(k-j)-i}}{(k-j-i)!i!}\mathbb{E}_t^*\left[(x-x_u)^j(x-k_0)^+ 1_{x>x_0}\left(\mathbb{E}_{T_1}^* y^i-y_0^i\right)\right]
\end{aligned}$$

Next, using the binomial expansion, it follows that:

$$
\begin{aligned}
& \mathbb{E}_t^* \left[(x - x_u)^j \, \mathbb{E}_{T_1}^* \, (y - y_0)^{k-j} \right] \\
= & \sum_{i=0}^{k-j} \frac{(k-j)! \, (-1)^{(k-j)-i} \, y_0^{(k-j)-i}}{(k-j-i)! i!} \mathbb{E}_t^* \left[(x - x_u)^j \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right].
\end{aligned}
$$

Computing

$$
\begin{aligned}
& \mathbb{E}_t^* \left[(x - x_d)^j \, (k_0 - x)^+ \, 1_{x<x_0} \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right] \\
& \mathbb{E}_t^* \left[(x - x_d)^j \, (x - k_0)^+ \, 1_{x<x_0} \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right] \\
& \mathbb{E}_t^* \left[(x - x_u)^j \, (k_0 - x)^+ \, 1_{x>x_0} \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right] \\
& \mathbb{E}_t^* \left[(x - x_u)^j \, (x - k_0)^+ \, 1_{x>x_0} \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right] \\
& \mathbb{E}_t^* \left[(x - x_d)^j \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right] \\
& \mathbb{E}_t^* \left[(x - x_u)^j \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right].
\end{aligned}
$$

Using (C.34), it appears that

$$
\mathbb{E}_t^* \left[(x - x_d)^j \, (k_0 - x)^+ \, 1_{x<x_0} \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right] = \theta_{i,t} \mathbb{E}_t^* \left[(x - x_d)^j \, (k_0 - x)^+ \left(x_0^i - x^i \right) 1_{x<x_0} \right] \tag{C.36}
$$

$$
\mathbb{E}_t^* \left[(x - x_d)^j \, (x - k_0)^+ \, 1_{x<x_0} \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right] = \theta_{i,t} \mathbb{E}_t^* \left[(x - x_d)^j \, (x - k_0)^+ \left(x_0^i - x^i \right) 1_{x<x_0} \right] \tag{C.37}
$$

$$
\mathbb{E}_t^* \left[(x - x_u)^j \, (k_0 - x)^+ \, 1_{x>x_0} \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right] = \theta_{i,t} \mathbb{E}_t^* \left[(x - x_u)^j \, (k_0 - x)^+ \left(x^i - x_0^i \right) 1_{x>x_0} \right] \tag{C.38}
$$

$$
\mathbb{E}_t^* \left[(x - x_u)^j \, (x - k_0)^+ \, 1_{x>x_0} \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right] = \theta_{i,t} \mathbb{E}_t^* \left[(x - x_u)^j \, (x - k_0)^+ \left(x^i - x_0^i \right) 1_{x>x_0} \right] \tag{C.39}
$$

$$
\mathbb{E}_t^* \left[(x - x_d)^j \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right] = \theta_{i,t} \begin{pmatrix} \mathbb{E}_t^* \, (x - x_d)^j \, (x^i - x_0^i) \, 1_{x>x_0} \\ + \mathbb{E}_t^* \, (x - x_d)^j \, (x_0^i - x^i) \, 1_{x<x_0} \end{pmatrix} \tag{C.40}
$$

$$
\mathbb{E}_t^* \left[(x - x_u)^j \left(\mathbb{E}_{T_1}^* y^i - y_0^i \right) \right] = \theta_{i,t} \begin{pmatrix} \mathbb{E}_t^* \, (x - x_u)^j \, (x^i - x_0^i) \, 1_{x>x_0} \\ + \mathbb{E}_t^* \, (x - x_u)^j \, (x_0^i - x^i) \, 1_{x<x_0} \end{pmatrix} \tag{C.41}
$$

Observe that:

$$\begin{aligned}
\mathbb{E}_t^*\left[(x-x_u)^j (k_0-x)^+ \left(x^i - x_0^i\right) 1_{x>x_0}\right] &= \mathbb{E}_t^*\left[(x-x_u)^j (k_0-x) 1_{x<k_0} \left(x^i - x_0^i\right) 1_{x>x_0}\right] \\
&= \mathbb{E}_t^*\left[(x-x_u)^j (k_0-x) (1-1_{x>k_0}) \left(x^i - x_0^i\right) 1_{x>x_0}\right] \\
&= \mathbb{E}_t^*\left[(x-x_u)^j (k_0-x) \left(x^i - x_0^i\right) 1_{x>x_0}\right] \\
&\quad +\mathbb{E}_t^*\left[(x-x_u)^j (x-k_0)^+ \left(x^i - x_0^i\right) 1_{x>x_0}\right] \\
&= \mathbb{E}_t^*\left[(x-x_u)^j (k_0-x) \left(x^i - x_0^i\right) 1_{x>x_0}\right] \\
&\quad +\mathbb{E}_t^*\left[(x-x_u)^j (x-k_0) \left(x^i - x_0^i\right) 1_{x>\max(x_0,k_0)}\right]
\end{aligned}$$

The risk neutral expectation

$$\begin{aligned}
& \mathbb{E}_t^*\left[(x-x_d)^j (k_0-x)^+ \left(x_0^i - x^i\right) 1_{x<x_0}\right] \\
= & \mathbb{E}_t^*\left[(x-x_d)^j (k_0-x) 1_{x<k_0} \left(x_0^i - x^i\right) 1_{x<x_0}\right] \\
= & \mathbb{E}_t^*\left[(x-x_d)^j (k_0-x) (1-1_{x>k_0}) \left(x_0^i - x^i\right) 1_{x<x_0}\right] \\
= & \mathbb{E}_t^*\left[(x-x_d)^j (k_0-x) \left(x_0^i - x^i\right) 1_{x<x_0}\right] \\
& +\mathbb{E}_t^*\left[(x-x_d)^j (x-k_0) 1_{x>k_0} \left(x_0^i - x^i\right) 1_{x<x_0}\right] \\
= & \mathbb{E}_t^*\left[(x-x_d)^j (k_0-x) \left(x_0^i - x^i\right)\right] \\
& -\mathbb{E}_t^*\left[(x-x_d)^j (k_0-x) \left(x_0^i - x^i\right) 1_{x>x_0}\right] \\
& +\mathbb{E}_t^*\left[(x-x_d)^j (x-k_0) 1_{x>k_0} \left(x_0^i - x^i\right)\right] \\
& -\mathbb{E}_t^*\left[(x-x_d)^j (x-k_0) 1_{x>k_0} \left(x_0^i - x^i\right) 1_{x>x_0}\right]
\end{aligned}$$

simplifies to

$$
\begin{aligned}
& \mathbb{E}_t^* \left[(x - x_d)^j \, (k_0 - x)^+ \left(x_0^i - x^i \right) 1_{x < x_0} \right] \\
= \; & \mathbb{E}_t^* \left[(x - x_d)^j \, (k_0 - x) \left(x_0^i - x^i \right) \right] \\
& - \mathbb{E}_t^* \left[(x - x_d)^j \, (k_0 - x) \left(x_0^i - x^i \right) 1_{x > x_0} \right] \\
& + \mathbb{E}_t^* \left[(x - x_d)^j \, (x - k_0) \left(x_0^i - x^i \right) 1_{x > k_0} \right] \\
& - \mathbb{E}_t^* \left[(x - x_d)^j \, (x - k_0) \left(x_0^i - x^i \right) 1_{x > \max(x_0, k_0)} \right]
\end{aligned}
$$

Expressions for each of these terms in terms of option prices have already been provided. In summary, both the first and second subsets of risk-neutral quantities are spanned by traded options. The first subset involves only x and payoff truncations; the second subset introduces the continuation component y through $\mathbb{E}_{T_1}^* (y - y_0)^n$, but after suitable algebra these too reduce to expressions involving option-implied quantities and risk-neutral probabilities that are observable at time t. This completes the construction.

D Computation of option prices for fixed deltas and maturities

The Carr and Madan (2001) formula requires a continuum of options across strikes. However, in practice we only observe a discrete set of strikes that does not cover the whole range of moneyness. For this reason, it is necessary to interpolate and extrapolate observed option prices. To do so, we follow the standard practice in the literature of converting option prices to implied volatilities (IVs) using the Black-Scholes formula, fitting an interpolant to them, using the interpolant to generate IVs for a fine grid of strikes, translating IVs back to option prices, and computing Carr-Madan over the fine grid of strikes.[6] Only out-of-the-money options are used to fit the interpolant, as they are more liquid than in-the-money options, which should contain redundant information by put-call parity.

We fit the IV curve across strikes using the parsimonious Stochastic Volatility Inspired (SVI) method of Gatheral (2004). This method has also been used by Beason and Schreindorfer (2022) and combines reliable interpolation of the IV curve with well-behaved extrapolation for extreme moneyness levels. More specifically, the SVI describes the square of IV with the function:

$$\sigma_{BS}^2(k) = a + b\left\{\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right\}, \tag{D.1}$$

where $k = \log(K/S_t)$ is the log-moneyness and a, b, ρ, m and σ are parameters.[7] We fit (D.1) to the cross-section of observed IVs for a given maturity and estimate the parameters by minimizing the IV mean squared error with a constrained nonlinear programming solver.[8]

The maturities available on a given day of our sample vary over time, while our goal is to obtain option prices for the same set of maturities for any day. For each such day in our sample, we proceed as follows. We first select the two neighboring maturities to the maturity of interest and apply the procedure described above to interpolate and extrapolate the IV curve on a fine grid of strikes.[9] Then, from the two IV curves, we linearly interpolate along the maturity dimension the IV to the desired maturity.[10] Finally, we

[6] This approach does not assume that the Black-Scholes model is valid. Rather, the Black-Scholes formula is simply used as a one-to-one mapping between option prices and IVs. This is done because fitting the IV curve is much easier than fitting option prices.

[7] More specifically, a controls the IV level, b the IV slope, ρ the asymmetry of the IV slope for negative and positive k, m the horizontal location of the IV curve, and σ the ATM curvature of the IV curve.

[8] The SVI is well defined for $a \in \mathbb{R}$, $b \geq 0$, $|\rho| \leq 1$, $m \in \mathbb{R}$, $\sigma > 0$ and $a + b\,\sigma\sqrt{1-\rho^2} \geq 0$. We impose these constraints in the optimization, with two small modifications: we replace $a + b\,\sigma\sqrt{1-\rho^2} \geq 0$ with the slightly stronger restriction $a \geq 0$, which yields better behaved extrapolations for the right tail, and we impose $\sigma \geq 0.05$, which helps discipline the IV ATM curvature.

[9] The SVI method accurately fits the observed IVs, with an average R^2 of 98.9%.

[10] In the unusual case that there is no observed maturity above or below the maturity of interest on

back out option prices from the resulting IV curve.

E Figures and tables

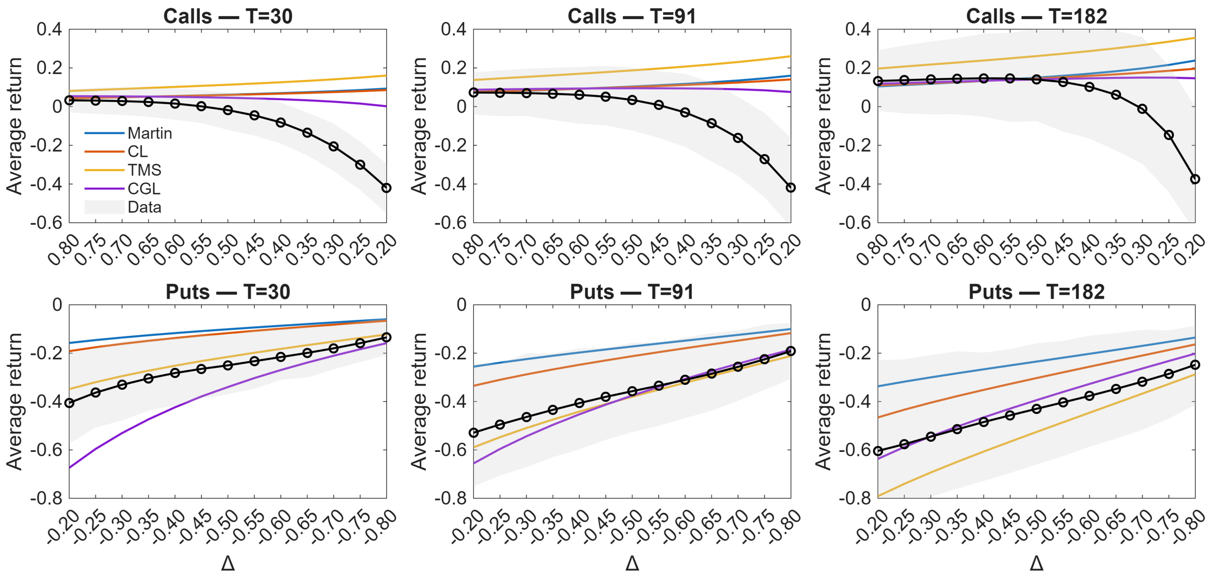

Figure 1: **Realized vs. conditional expected option returns implied by prior SDF specifications.** This figure plots in the upper (lower) panels the average realized call (put) option returns across deltas and maturities together with 95% confidence intervals based on a block-bootstrap with 2000 replications and block length equal to the option maturity. Corresponding averages of the conditional expected call (put) option returns implied by different prior SDF specifications are also plotted. Returns are in excess of the risk-free rate. The sample ranges from January 4, 1996 to February 28, 2023.

a given day, we extrapolate the IV curve based on the two maturities closest to from any direction.

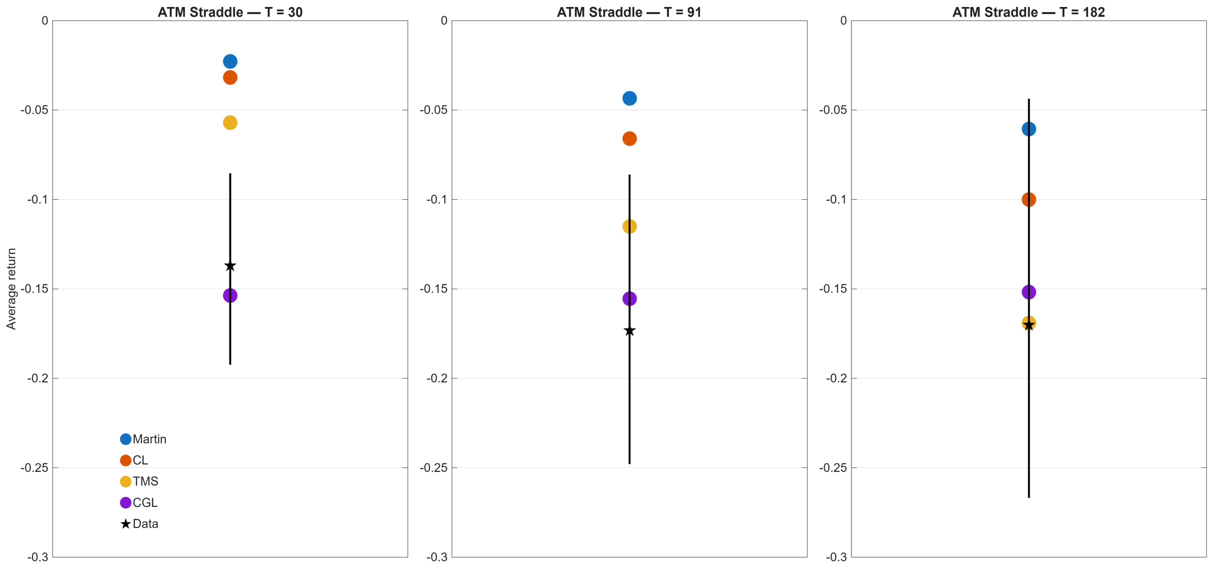

Figure 2: **Realized vs. conditional expected straddle returns implied by prior SDF specifications.** This figure plots the average realized ATM straddle returns across maturities together with 95% confidence intervals based on a block-bootstrap with 2000 replications and block length equal to the option maturity. Corresponding averages of the conditional expected straddle returns implied by different prior SDF specifications are also plotted. Returns are in excess of the risk-free rate. The sample ranges from January 4, 1996 to February 28, 2023.

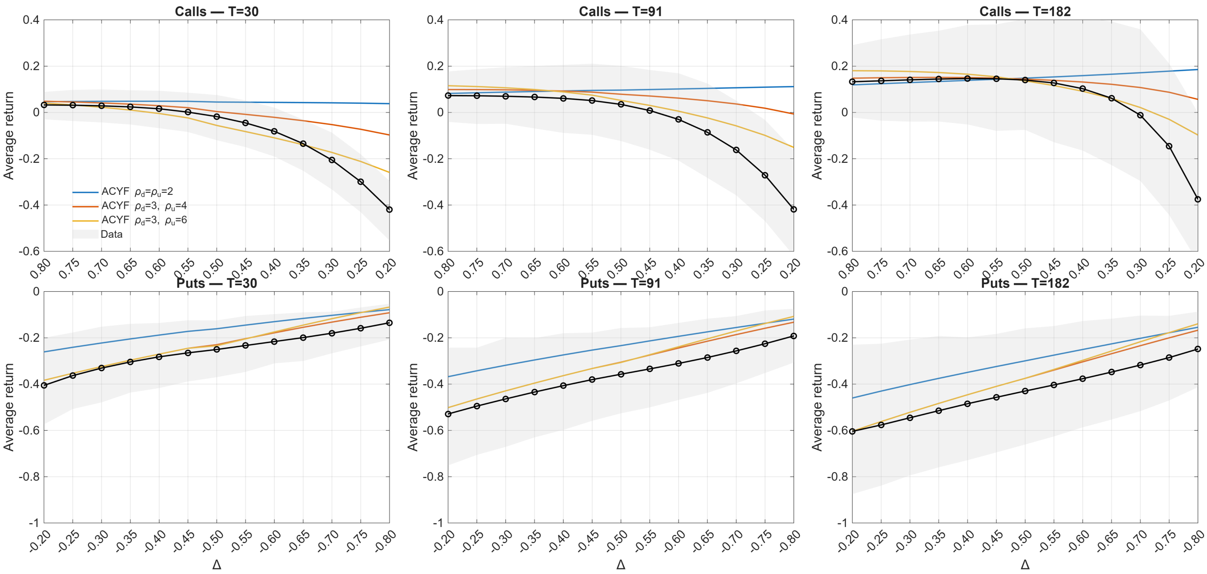

Figure 3: **Realized vs. conditional expected option returns implied by our SDF specification.** This figure plots in the upper (lower) panels the average realized call (put) option returns across deltas and maturities together with 95% confidence intervals based on a block-bootstrap with 2000 replications and block length equal to the option maturity. Corresponding averages of the conditional expected call (put) option returns implied by our SDF specification (for three sets of parameters indicated in the legend) are also plotted. Returns are in excess of the risk-free rate. The sample ranges from January 4, 1996 to February 28, 2023.

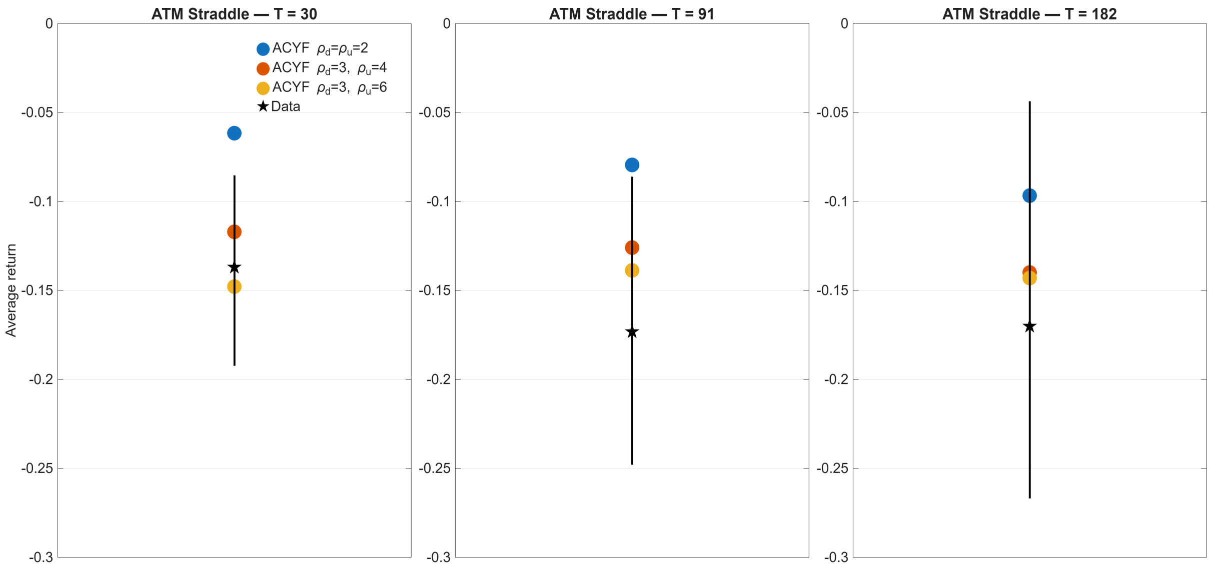

Figure 4: **Realized vs. conditional expected straddle returns implied by our SDF specification.** This figure plots the average realized ATM straddle returns across maturities together with 95% confidence intervals based on a block-bootstrap with 2000 replications and block length equal to the option maturity. Corresponding averages of the conditional expected straddle returns implied by our SDF specification (for three sets of parameters indicated in the legend) are also plotted. Returns are in excess of the risk-free rate. The sample ranges from January 4, 1996 to February 28, 2023.

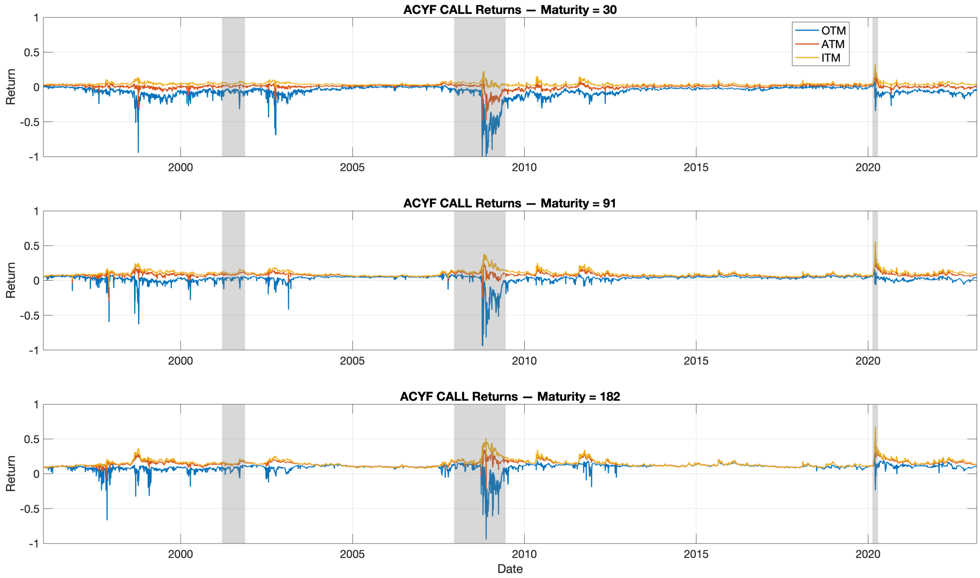

Figure 5: **Conditional expected call returns over time.** This figure plots the time series of conditional expected excess returns on call options implied by our preferred SDF specification. We report expected returns for three moneyness groups: out-of-the-money (OTM), at-the-money (ATM), and in-the-money (ITM). The sample ranges from January 4, 1996 to February 28, 2023.

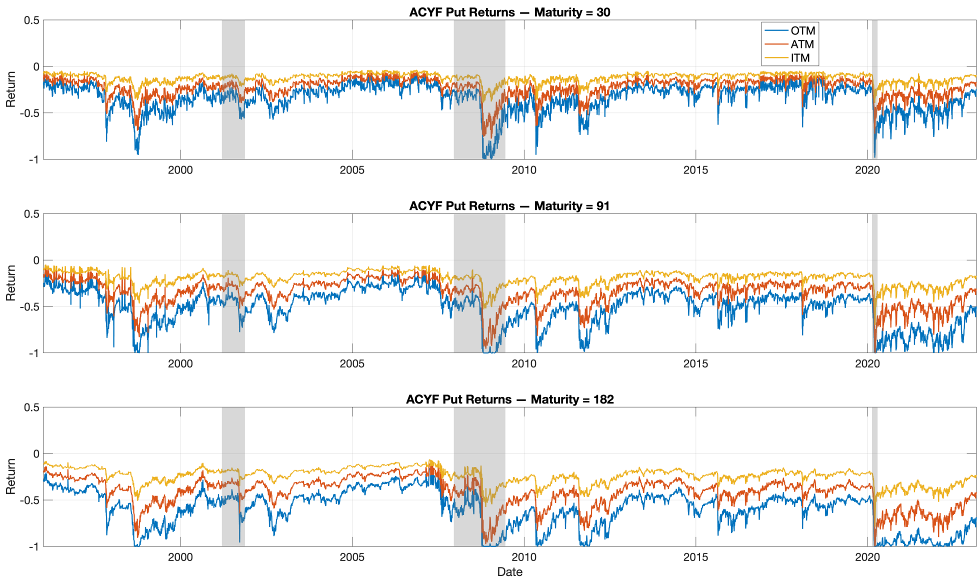

Figure 6: **Conditional expected put returns over time.** This figure plots the time series of conditional expected excess returns on put options implied by our preferred SDF specification. We report expected returns for three moneyness groups: out-of-the-money (OTM), at-the-money (ATM), and in-the-money (ITM). The sample ranges from January 4, 1996 to February 28, 2023.

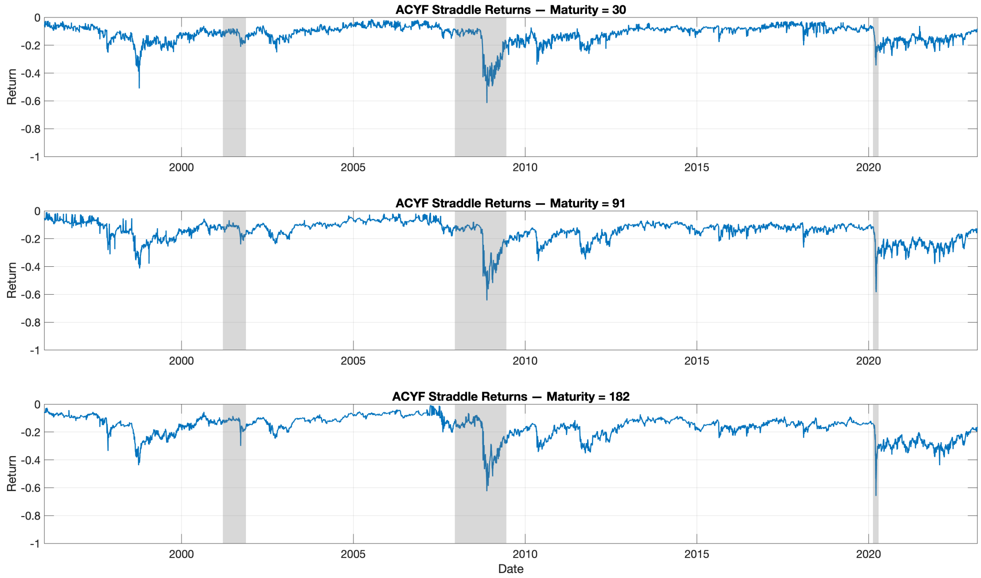

Figure 7: **Conditional expected straddle returns over time.** This figure plots the time series of conditional expected excess returns on ATM straddles implied by our preferred SDF specification. The sample ranges from January 4, 1996 to February 28, 2023.

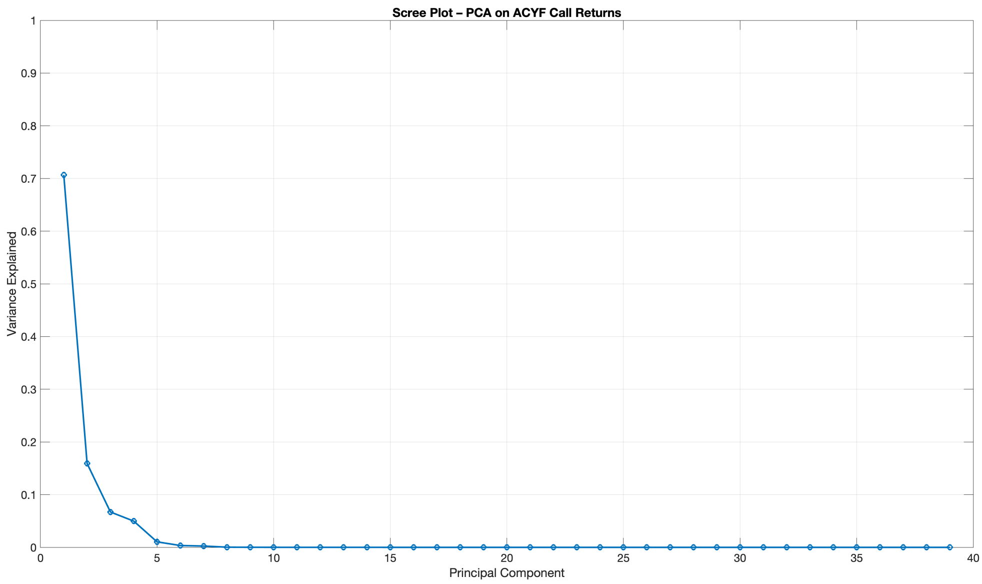

Figure 8: **PCA scree plot for conditional expected call returns.** This figure reports the fraction of total variation in conditional expected call returns explained by successive principal components. The results indicate a strong factor structure, with approximately 95% of the variation explained by the first three principal components. The sample ranges from January 4, 1996 to February 28, 2023.

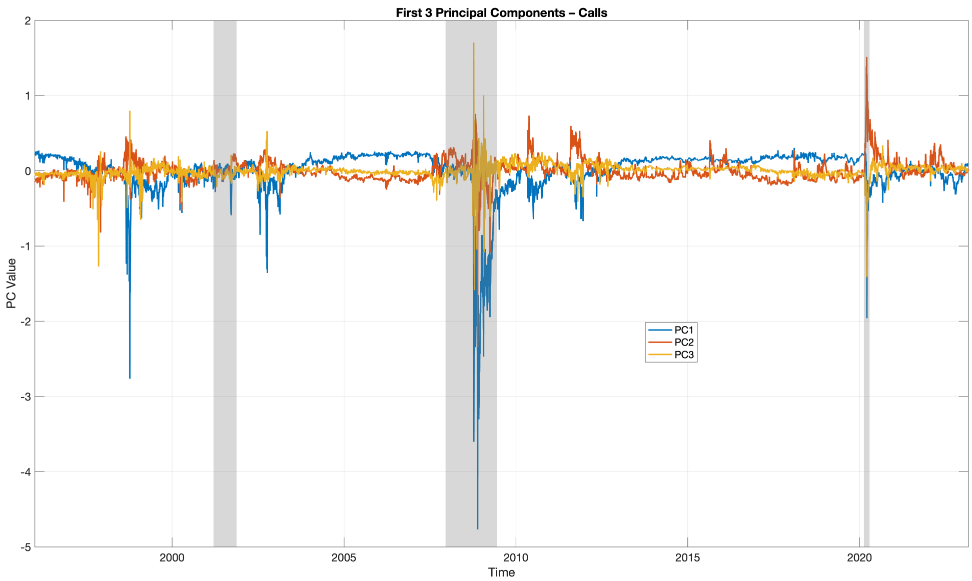

Figure 9: **First three principal components for conditional expected call returns.** This figure plots the time series of the first three principal components extracted from the panel of conditional expected call returns across moneyness and maturities. The first principal component is closely related to variance risk (up to an arbitrary sign normalization), while the second component resembles a jump or tail-risk factor. The third component captures additional variation but is noticeably noisier. The sample ranges from January 4, 1996 to February 28, 2023.

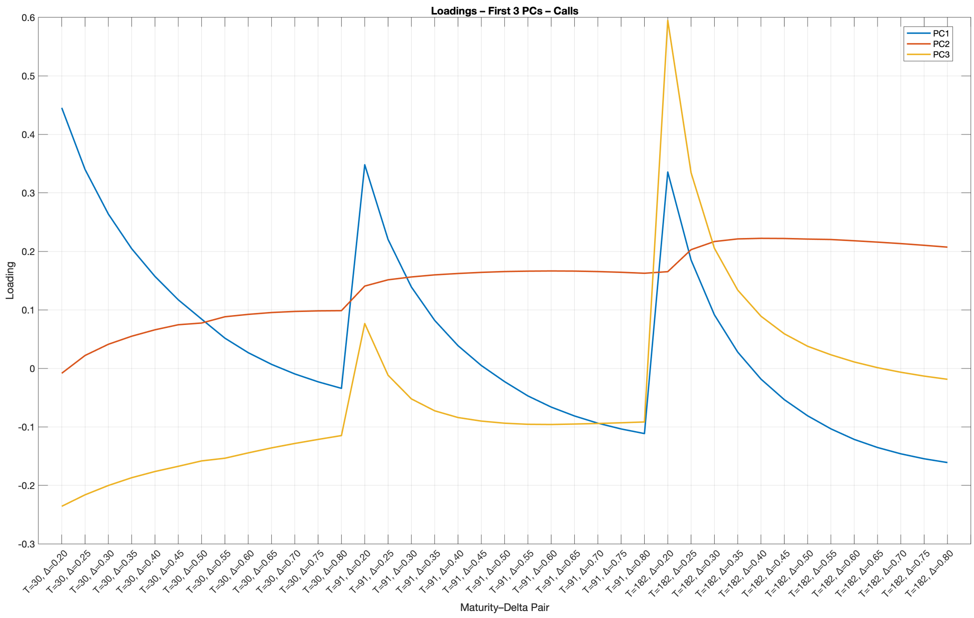

Figure 10: **Loadings of the first three principal components for conditional expected call returns.** This figure reports the loadings of the first three principal components across moneyness and maturities. The first component primarily resembles a moneyness-slope factor, loading positively on OTM calls and negatively on ITM calls across maturities. The second component behaves like a level factor, shifting expected call returns across strikes in the same direction. The third component is most closely related to the maturity dimension and can be interpreted as a maturity factor. The sample ranges from January 4, 1996 to February 28, 2023.

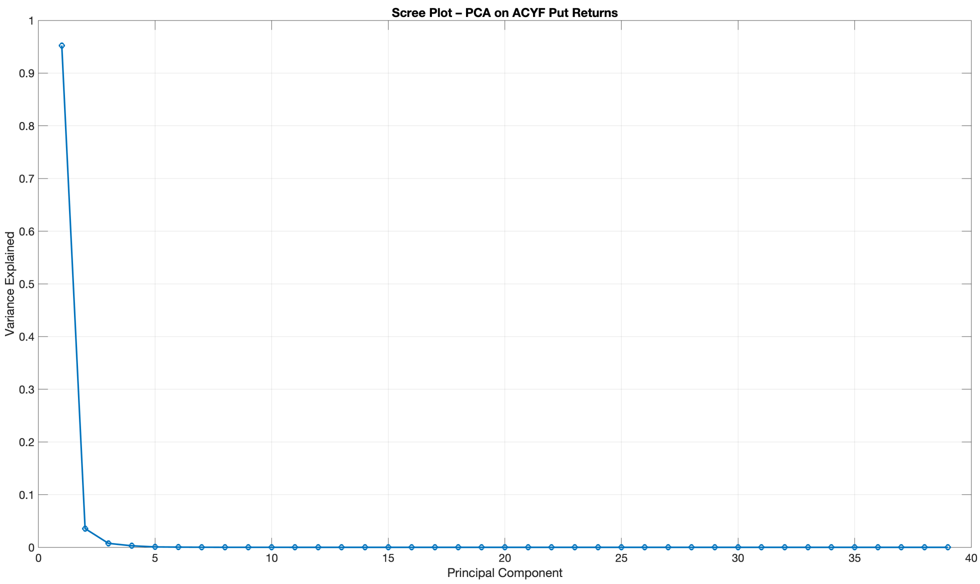

Figure 11: **PCA scree plot for conditional expected put returns.** This figure reports the fraction of total variation in conditional expected put returns explained by successive principal components. The results indicate an even stronger factor structure than for calls: the first two principal components explain almost all of the variation in expected put returns. The sample ranges from January 4, 1996 to February 28, 2023.

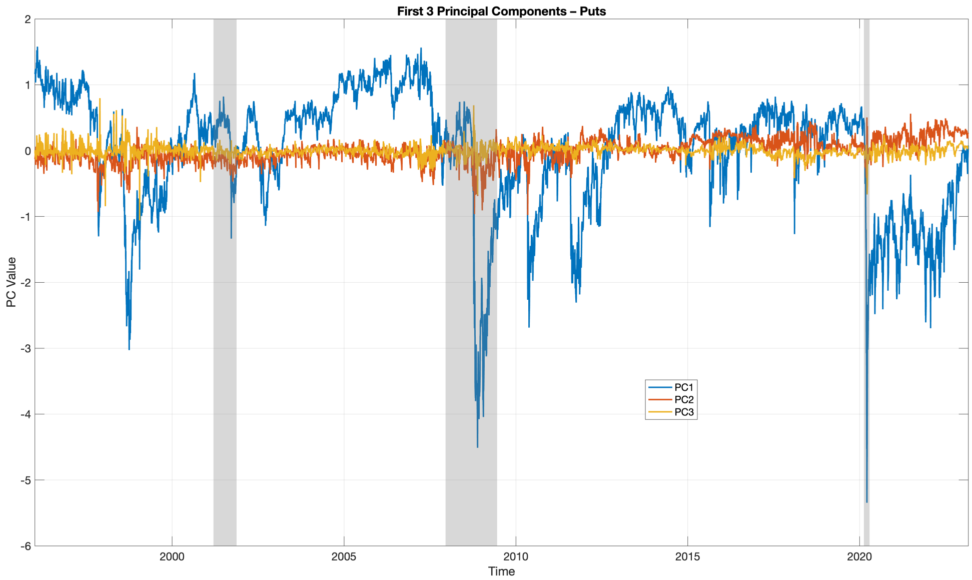

Figure 12: **First three principal components for conditional expected put returns.** This figure plots the time series of the first three principal components extracted from the panel of conditional expected put returns across moneyness and maturities. The first component is strongly related to aggregate variance and tail-risk conditions (and captures most of the variation). The second and third components explain substantially less variation and are comparatively noisier. The sample ranges from January 4, 1996 to February 28, 2023.

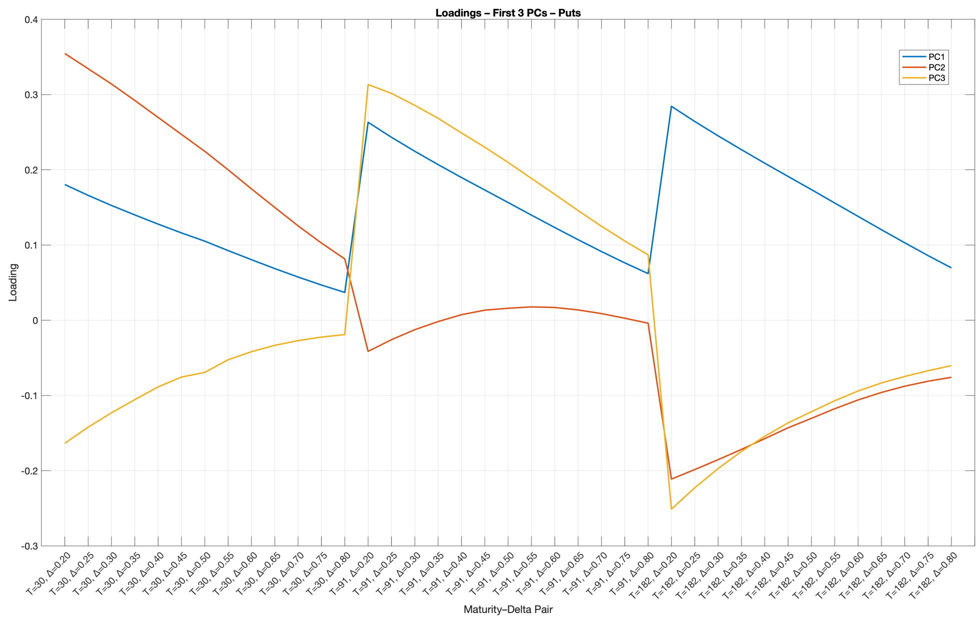

Figure 13: **Loadings of the first three principal components for conditional expected put returns.** This figure reports the loadings of the first three principal components across moneyness and maturities. The first component resembles a combination of a level and slope factor, loading positively on all puts but with larger weights on OTM puts. The second component primarily captures maturity variation and can be interpreted as a maturity factor. The third component displays curvature in the maturity dimension, although it explains relatively little additional variation. The sample ranges from January 4, 1996 to February 28, 2023.

Table 1: Predictive regressions of call option payoffs ($T = 30$)

Model	**Group**	α	t($\alpha = 0$)	β	t($\beta = 0$)	t($\beta = 1$)	$\mathbf{R}^2$ **(%)**
M	OTM	0.00	0.61	0.57	2.65	-2.00	2.34
M	ATM	0.00	0.35	0.85	4.34	-0.74	9.61
M	ITM	0.00	0.20	0.95	6.52	-0.32	19.90
CL	OTM	0.00	0.53	0.59	2.71	-1.89	2.33
CL	ATM	0.00	0.31	0.86	4.40	-0.70	9.63
CL	ITM	0.00	0.20	0.95	6.55	-0.33	19.91
TMS	OTM	0.00	0.76	0.51	2.56	-2.46	2.45
TMS	ATM	0.00	0.71	0.75	4.21	-1.37	9.88
TMS	ITM	0.00	0.72	0.85	6.28	-1.13	20.13
CGL	OTM	0.00	0.30	0.67	2.87	-1.38	2.08
CGL	ATM	0.00	0.18	0.90	4.50	-0.49	9.54
CGL	ITM	0.00	0.23	0.94	6.58	-0.41	19.80
ACF	OTM	0.00	0.41	0.73	3.22	-1.17	1.40
ACF	ATM	-0.00	-0.22	1.02	4.87	0.09	9.47
ACF	ITM	0.00	0.15	0.96	6.67	-0.25	19.77

Notes: This table reports predictive regressions of realized call option payoffs on model-implied conditional expectations for the same horizon. Specifically, for each moneyness group, we estimate

$$\max(R_{M,t\to t+30} - K/S_t, 0) = \alpha + \beta\, \mathbb{E}_t^{\text{Model}}[\max(R_{M,t\to t+30} - K/S_t, 0)] + \varepsilon_{t\to t+30}.$$

Newey–West standard errors are computed using 21 lags. A well-specified model implies $\alpha = 0$, $\beta > 0$, and $\beta = 1$. The sample ranges from January 4, 1996 to February 28, 2023.

Table 2: Predictive regressions of put option payoffs ($T = 30$)

Model	**Group**	α	t($\alpha = 0$)	β	t($\beta = 0$)	t($\beta = 1$)	$\mathbf{R}^2$ **(%)**
M	OTM	0.00	1.13	0.47	1.79	-2.02	0.56
M	ATM	0.00	0.69	0.68	2.92	-1.35	2.49
M	ITM	0.00	0.28	0.85	4.74	-0.83	8.81
CL	OTM	0.00	0.73	0.57	1.81	-1.38	0.61
CL	ATM	0.00	0.48	0.74	2.93	-1.04	2.50
CL	ITM	0.00	0.15	0.88	4.75	-0.63	8.79
TMS	OTM	0.00	1.06	0.54	1.33	-1.14	0.30
TMS	ATM	0.00	0.18	0.92	2.93	-0.24	2.04
TMS	ITM	-0.00	-0.51	1.09	4.93	0.41	8.45
CGL	OTM	0.01	3.45	0.28	0.70	-1.83	0.14
CGL	ATM	0.00	0.98	0.81	2.10	-0.49	1.45
CGL	ITM	-0.00	-0.04	1.06	4.62	0.26	7.98
ACF	OTM	0.00	0.21	0.90	1.40	-0.15	0.62
ACF	ATM	-0.00	-0.10	1.02	2.79	0.05	2.33
ACF	ITM	-0.00	-0.07	0.97	4.75	-0.15	8.63

Notes: This table reports predictive regressions of realized put option payoffs on model-implied conditional expectations for the same horizon. Specifically, for each moneyness group, we estimate

$$\max(K/S_t - R_{M,t\to t+30}, 0) = \alpha + \beta\, \mathbb{E}_t^{\text{Model}}[\max(K/S_t - R_{M,t\to t+30}, 0)] + \varepsilon_{t\to t+30}.$$

Newey–West standard errors are computed using 21 lags. A well-specified model implies $\alpha = 0$, $\beta > 0$, and $\beta = 1$. The sample ranges from January 4, 1996 to February 28, 2023.

Table 3: Regressions of expected option returns on expected market returns ($T = 30$)

Statistic	M	CL	TMS	CGL	ACF
Panel A: ATM Calls					
α	0.04	0.04	0.07	0.03	0.02
t($\alpha = 0$)	28.25	29.16	24.36	24.60	4.09
β	6.29	4.99	5.53	1.67	-1.59
t($\beta = 0$)	16.28	14.27	13.54	10.08	-2.48
R^2 (%)	87.73	85.30	81.99	59.45	13.48
Panel B: ATM Puts					
α	-0.07	-0.08	-0.14	-0.21	-0.12
t($\alpha = 0$)	-29.17	-26.97	-27.04	-17.24	-19.91
β	-8.33	-9.63	-9.50	-14.97	-11.83
t($\beta = 0$)	-12.14	-12.88	-13.45	-9.95	-15.92
R^2 (%)	80.02	81.37	78.43	62.30	87.85
Panel C: ATM Straddles					
α	-0.02	-0.02	-0.04	-0.09	-0.05
t($\alpha = 0$)	-20.93	-20.34	-22.03	-14.78	-12.92
β	-1.86	-3.12	-2.79	-7.35	-7.19
t($\beta = 0$)	-8.64	-13.24	-12.99	-10.00	-14.06
R^2 (%)	42.35	65.06	66.07	61.75	83.29

Notes: This table reports regressions of conditional expected option returns on conditional expected market returns, using a 30-day horizon. Panel A reports results for ATM calls, Panel B for ATM puts, and Panel C for ATM straddles. The regression specification is

$$\mathbb{E}_t^{\text{Model}} R^j_{t\to t+30} = \alpha + \beta\, \mathbb{E}_t^{\text{Model}} R_{M,t\to t+30} + \varepsilon_{t\to t+30},$$

$j \in \{c, p, str\}$. Newey–West standard errors are computed using 21 lags. For the ACF specification, $\mathbb{E}_t^{\text{Model}} R_{M,t\to t+30}$ is taken from the CGL expected market return. The sample ranges from January 4, 1996 to February 28, 2023.

Table 4: Expected option returns and risk-neutral moments (ATM, $T = 30$ days)

	M	CL	TMS	CGL	ACF
Panel A: ATM Call					
α	0.03	0.03	0.06	0.03	0.02
t($\alpha = 0$)	34.81	37.69	16.15	23.96	4.55
β_2	11.52	10.51	12.82	1.78	-7.67
t($\beta_2 = 0$)	28.61	28.10	5.78	3.53	-3.42
β_3	18.97	10.98	-17.14	-24.69	-42.92
t($\beta_3 = 0$)	10.71	7.04	-2.04	-4.41	-3.65
β_4	-24.82	-28.11	-20.46	-10.59	-3.81
t($\beta_4 = 0$)	-10.60	-12.35	-2.28	-1.68	-0.31
R^2 (%)	0.95	0.96	0.68	0.67	0.24
Panel B: ATM Put					
α	-0.06	-0.07	-0.14	-0.27	-0.14
t($\alpha = 0$)	-32.38	-29.51	-26.78	-22.49	-22.29
β_2	-9.60	-10.33	-8.79	2.72	-15.63
t($\beta_2 = 0$)	-12.25	-9.67	-2.53	0.38	-4.60
β_3	17.58	32.14	98.59	220.13	100.74
t($\beta_3 = 0$)	1.92	2.62	3.93	3.30	2.94
β_4	26.17	24.55	-10.70	-0.48	45.30
t($\beta_4 = 0$)	2.78	2.04	-0.50	-0.01	1.54
R^2 (%)	0.86	0.86	0.72	0.45	0.71
Panel C: ATM Straddle					
α	-0.02	-0.02	-0.04	-0.12	-0.06
t($\alpha = 0$)	-23.17	-21.28	-25.35	-20.11	-15.05
β_2	0.06	-0.79	0.72	1.24	-12.55
t($\beta_2 = 0$)	0.15	-1.44	0.67	0.34	-5.16
β_3	18.08	22.26	44.33	103.15	33.54
t($\beta_3 = 0$)	4.12	3.75	4.43	3.21	1.77
β_4	0.93	-1.47	-17.29	-4.16	24.25
t($\beta_4 = 0$)	0.23	-0.28	-2.00	-0.17	1.63
R^2 (%)	0.64	0.75	0.72	0.43	0.65

Notes: This table reports regressions of conditional expected option returns on conditional risk-neutral moments computed from option prices. For each model, we estimate

$$\mathbb{E}_t^{\text{Model}} R^j_{t\to t+30} = \alpha + \beta_2\, \mathbb{M}^2_{t\to t+30} + \beta_3\, \mathbb{M}^3_{t\to t+30} + \beta_4\, \mathbb{M}^4_{t\to t+30} + \varepsilon_{t\to t+30},$$

$j \in \{c, p, str\}$. The regressors $\mathbb{M}^2$, $\mathbb{M}^3$, and $\mathbb{M}^4$ denote the conditional risk-neutral variance, skewness, and kurtosis over the next 30 days. Reported t-statistics are based on Newey–West standard errors with 21 lags. The sample ranges from January 4, 1996 to February 28, 2023.

Online Appendix to

Conditional Expected Option Returns

Caio Almeida[†] **Fousseni Chabi-Yo**[‡] **Gustavo Freire**[§]

March 10, 2026

Abstract

This online appendix collects additional results supporting the main paper.

[†]E-mail: calmeida@princeton.edu, Department of Economics, Princeton University.

[‡]E-mail: chabiyofatwork@gmail.com, Isenberg School of Management, University of Massachusetts in Amherst.

[§]E-mail: gustavo.freire@novasbe.pt, Finance Department, Nova School of Business and Economics.

OA.1 Partition in 3 regions

Partition the positive real line indexed by x into three regions:

$$A_d = \{x < \underline{x}_0\}, \; A_c = \{\underline{x}_0 < x < \overline{x}_0\}, \text{ and } A_u = \{x > \overline{x}_0\}$$

such that

$$A_d \cup A_c \cup A_u = \mathbb{R}^+$$

The inverse SDF can be written as

$$\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} = \sum_{s\in\{d,c,u\}} \frac{\mathbb{E}_t m^{(s)}_{t\to T_1}}{m^{(s)}_{t\to T_1}} \text{ with } \frac{\mathbb{E}_t m^{(s)}_{t\to T_1}}{m^{(s)}_{t\to T_1}} = \frac{\mathbb{E}^*_{T_1}\left[f_s\left[x,y\right]\right]}{\mathbb{E}^*_t\left(\mathbb{E}^*_{T_1}\left[f_s\left[x,y\right]\right]\right)} 1_{A_s}$$

OA.1.1 Second order expansion series

Using a second order expansion-series of $f_s\left[x,y\right]$ around $(x,y) = (x_s, y_0)$, we obtain

$$\mathbb{E}^*_{T_1} f_s\left[x,y\right] = \sum_{i=0}^{2} \frac{a_{i,0,s}}{x^i_s} R^i_{s,M,t\to T_1} + \frac{a_{0,2,s}}{y^2_0} \mathbb{M}^{*(2)}_{T_1\to T_N}$$

where $R_{s,M,t\to T_1} = R_{M,t\to T_1} - x_s$. Thus, for $s \in \{u,d\}$, the inverse SDF takes the form

$$\frac{\mathbb{E}_t m^{(s)}_{t\to T_1}}{m^{(s)}_{t\to T_1}} = \frac{\sum_{i=0}^{2} \frac{a_{i,0,s}}{x^i_s} R^i_{s,M,t\to T_1} 1_{A_s} + \frac{a_{0,2,s}}{y^2_0} 1_{A_s} \mathbb{M}^{*(2)}_{T_1\to T_N}}{\sum_{i=0}^{2} \frac{a_{i,0,s}}{x^i_s} \mathbb{E}^*_t R^i_{s,M,t\to T_1} + \frac{a_{0,2,s}}{y^2_0} \mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N}}.$$

The expected return on call option with a strike K_0 is

$$\mathbb{E}_t R^c_{t\to T_1} = \mathbb{E}^*_t\left[\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}} R^c_{t\to T_1}\right] = \frac{S_t}{C_t\left[K_0\right]} \sum_{s\in\{d,c,u\}} \mathbb{E}^*_t\left[\frac{\mathbb{E}_t m^{(s)}_{t\to T_1}}{m^{(s)}_{t\to T_1}} \frac{\left(S_{T_1} - K_0\right)^+}{S_t}\right]$$

Hence.

$$
\begin{aligned}
& \mathbb{E}_t R^c_{t\to T_1} \\
= & \frac{S_t}{C_t\left[K_0\right]} \sum_{s\in\{d,c,u\}} \mathbb{E}^*_t\left[\left\{\frac{\sum_{i=0}^{2} \frac{a_{i,0,s}}{x^i_s} R^i_{s,M,t\to T_1} 1_{A_s} + \frac{a_{0,2,s}}{y^2_0} 1_{A_s} \mathbb{M}^{*(2)}_{T_1\to T_N}}{\sum_{i=0}^{2} \frac{a_{i,0,s}}{x^i_s} \mathbb{E}^*_t R^i_{s,M,t\to T_1} + \frac{a_{0,2,s}}{y^2_0} \mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N}}\right\} \left(R_{M,t\to T_1} - k_0\right)^+\right]
\end{aligned}
$$

This simplifies to

$$
\mathbb{E}_t R^c_{t\to T_1} = \frac{S_t}{C_t\left[K_0\right]} \sum_{s\in\{d,c,u\}} \frac{\left\{\begin{array}{c} \sum_{i=0}^{2} \frac{a_{i,0,s}}{x^i_s} \mathbb{E}^*_t\left(R^i_{s,M,t\to T_1}\left(R_{M,t\to T_1} - k_0\right)^+ 1_{A_s}\right) \\ +\frac{a_{0,2,s}}{y^2_0} \mathbb{E}^*_t\left(1_{A_s}\left(R_{M,t\to T_1} - k_0\right)^+ \mathbb{M}^{*(2)}_{T_1\to T_N}\right) \end{array}\right\}}{\left\{\sum_{i=0}^{2} \frac{a_{i,0,s}}{x^i_s} \mathbb{E}^*_t R^i_{s,M,t\to T_1} + \frac{a_{0,2,s}}{y^2_0} \mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N}\right\}}
$$

Similarly

$$
\mathbb{E}_t R^p_{t\to T_1} = \frac{S_t}{P_t\left[K_0\right]} \sum_{s\in\{d,c,u\}} \frac{\left\{\begin{array}{c} \sum_{i=0}^{2} \frac{a_{i,0,s}}{x^i_s} \mathbb{E}^*_t\left(R^i_{s,M,t\to T_1}\left(k_0 - R_{M,t\to T_1}\right)^+ 1_{A_s}\right) \\ +\frac{a_{0,2,s}}{y^2_0} \mathbb{E}^*_t\left(1_{A_s}\left(k_0 - R_{M,t\to T_1}\right)^+ \mathbb{M}^{*(2)}_{T_1\to T_N}\right) \end{array}\right\}}{\left\{\sum_{i=0}^{2} \frac{a_{i,0,s}}{x^i_s} \mathbb{E}^*_t R^i_{s,M,t\to T_1} + \frac{a_{0,2,s}}{y^2_0} \mathbb{E}^*_t \mathbb{M}^{*(2)}_{T_1\to T_N}\right\}}
$$

The quantities

$$
\mathbb{E}^*_t\left(R^i_{d,M,t\to T_1}\left(k_0 - R_{M,t\to T_1}\right)^+ 1_{A_d}\right) \text{ and } \mathbb{E}^*_t\left(R^i_{u,M,t\to T_1}\left(k_0 - R_{M,t\to T_1}\right)^+ 1_{A_u}\right)
$$

follows directly from the previous formula. The central region term satisfies

$$
\begin{aligned}
\mathbb{E}^*_t\left(R^i_{c,M,t\to T_1}\left(k_0 - R_{M,t\to T_1}\right)^+ 1_{A_c}\right) &= \mathbb{E}^*_t\left(R^i_{c,M,t\to T_1}\left(k_0 - R_{M,t\to T_1}\right)^+\right) \\
&\quad -\mathbb{E}^*_t\left(R^i_{c,M,t\to T_1}\left(k_0 - R_{M,t\to T_1}\right)^+ 1_{A_u}\right) \\
&\quad -\mathbb{E}^*_t\left(R^i_{c,M,t\to T_1}\left(k_0 - R_{M,t\to T_1}\right)^+ 1_{A_d}\right)
\end{aligned}
$$

and similarly

$$\begin{aligned}\mathbb{E}_t^*\left(1_{A_c}\left(k_0-R_{M,t\to T_1}\right)^+\mathbb{M}^{*(2)}_{T_1\to T_N}\right) &= \mathbb{E}_t^*\left(\left(k_0-R_{M,t\to T_1}\right)^+\mathbb{M}^{*(2)}_{T_1\to T_N}\right)\\ &\quad -\mathbb{E}_t^*\left(1_{A_u}\left(k_0-R_{M,t\to T_1}\right)^+\mathbb{M}^{*(2)}_{T_1\to T_N}\right)\\ &\quad -\mathbb{E}_t^*\left(1_{A_d}\left(k_0-R_{M,t\to T_1}\right)^+\mathbb{M}^{*(2)}_{T_1\to T_N}\right)\end{aligned}$$

OA.1.2 Third order expansion series

Using a third order expansion-series of $f_s[x,y]$ around $(x,y)=(x_s,y_0)$, we obtain

$$\mathbb{E}_{T_1}^* f_s[x,y]=\sum_{i=0}^{3}\frac{a_{i,0,s}}{x_s^i}R^i_{s,M,t\to T_1}+\sum_{i=2}^{3}\frac{a_{0,i,s}}{y_0^2}\mathbb{M}^{*(i)}_{T_1\to T_N}+\frac{a_{1,2,s}}{y_0^2 x_s}R_{s,M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_N}$$

where $R_{s,M,t\to T_1}=R_{M,t\to T_1}-x_s$. Thus, for $s\in\{u,d\}$, the inverse SDF takes the form

$$\frac{\mathbb{E}_t m^{(s)}_{t\to T_1}}{m^{(s)}_{t\to T_1}}=\frac{\sum_{i=0}^{3}\frac{a_{i,0,s}}{x_s^i}1_{A_s}R^i_{s,M,t\to T_1}+\sum_{i=2}^{3}\frac{a_{0,i,s}}{y_0^2}1_{A_s}\mathbb{M}^{*(i)}_{T_1\to T_N}+\frac{a_{1,2,s}}{y_0^2x_s}1_{A_s}R_{s,M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_N}}{\sum_{i=0}^{3}\frac{a_{i,0,s}}{x_s^i}\mathbb{E}_t^*R^i_{s,M,t\to T_1}+\sum_{i=2}^{3}\frac{a_{0,i,s}}{y_0^2}\mathbb{E}_t^*\mathbb{M}^{*(i)}_{T_1\to T_N}+\frac{a_{1,2,s}}{y_0^2x_s}\mathbb{E}_t^*\left(R_{s,M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_N}\right)}.$$

The expected return on call option with a strike K_0 is

$$\mathbb{E}_t R^c_{t\to T_1}=\mathbb{E}_t^*\left[\frac{\mathbb{E}_t m_{t\to T_1}}{m_{t\to T_1}}R^c_{t\to T_1}\right]=\frac{S_t}{C_t[K_0]}\sum_{s\in\{d,c,u\}}\mathbb{E}_t^*\left[\frac{\mathbb{E}_t m^{(s)}_{t\to T_1}}{m^{(s)}_{t\to T_1}}\frac{(S_{T_1}-K_0)^+}{S_t}\right]$$

Hence.

$$\mathbb{E}_t R^c_{t\to T_1}=\frac{S_t}{C_t[K_0]}\sum_{s\in\{d,c,u\}}\mathbb{E}_t^*\left[\frac{\left(\begin{array}{c}\sum_{i=0}^{3}\frac{a_{i,0,s}}{x_s^i}1_{A_s}R^i_{s,M,t\to T_1}+\sum_{i=2}^{3}\frac{a_{0,i,s}}{y_0^2}1_{A_s}\mathbb{M}^{*(i)}_{T_1\to T_N}\\+\frac{a_{1,2,s}}{y_0^2x_s}1_{A_s}R_{s,M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_N}\end{array}\right)}{\left(\begin{array}{c}\sum_{i=0}^{3}\frac{a_{i,0,s}}{x_s^i}\mathbb{E}_t^*R^i_{s,M,t\to T_1}+\sum_{i=2}^{3}\frac{a_{0,i,s}}{y_0^2}\mathbb{E}_t^*\mathbb{M}^{*(i)}_{T_1\to T_N}\\+\frac{a_{1,2,s}}{y_0^2x_s}\mathbb{E}_t^*\left(R_{s,M,t\to T_1}\mathbb{M}^{*(2)}_{T_1\to T_N}\right)\end{array}\right)}\left(R_{M,t\to T_1}-k_0\right)^+\right]$$

This simplifies to

$$\mathbb{E}_t R^c_{t\to T_1} = \frac{S_t}{C_t\left[K_0\right]} \sum_{s\in\{d,c,u\}} \frac{\mathbb{ECN}^s_t}{\mathbb{ED}^s_{t,3}}$$

with

$$\mathbb{ED}^s_3 = \sum_{i=0}^{3} \frac{a_{i,0,s}}{x^i_s} \mathbb{E}^*_t R^i_{s,M,t\to T_1} + \sum_{i=2}^{3} \frac{a_{0,i,s}}{y_0^2} \mathbb{E}^*_t \mathbb{M}^{*(i)}_{T_1\to T_N} + \frac{a_{1,2,s}}{y_0^2 x_s} \mathbb{E}^*_t \left(R_{s,M,t\to T_1} \mathbb{M}^{*(2)}_{T_1\to T_N}\right)$$

and

$$\begin{aligned}
\mathbb{ECN}^s &= \sum_{i=0}^{3} \frac{a_{i,0,s}}{x^i_s} \mathbb{E}^*_t \left[1_{A_s} \left(R_{M,t\to T_1} - k_0\right)^+ R^i_{s,M,t\to T_1}\right] \\
&+ \sum_{i=2}^{3} \frac{a_{0,i,s}}{y_0^2} \mathbb{E}^*_t \left[1_{A_s} \left(R_{M,t\to T_1} - k_0\right)^+ \mathbb{M}^{*(i)}_{T_1\to T_N}\right] \\
&+ \frac{a_{1,2,s}}{y_0^2 x_s} \mathbb{E}^*_t \left[1_{A_s} R_{s,M,t\to T_1} \left(R_{M,t\to T_1} - k_0\right)^+ \mathbb{M}^{*(2)}_{T_1\to T_N}\right] \qquad \text{(OA.1.1)}
\end{aligned}$$

Notice that each risk neutral quantities in (OA.1.1) can be computed for $s = u, d$. When $s = c$

$$\begin{aligned}
\mathbb{E}^*_t \left[1_{A_c} \left(R_{M,t\to T_1} - k_0\right)^+ R^i_{c,M,t\to T_1}\right] &= \mathbb{E}^*_t \left[\left(R_{M,t\to T_1} - k_0\right)^+ R^i_{c,M,t\to T_1}\right] \\
&- \mathbb{E}^*_t \left[1_{A_u} \left(R_{M,t\to T_1} - k_0\right)^+ R^i_{c,M,t\to T_1}\right] \\
&- \mathbb{E}^*_t \left[1_{A_d} \left(R_{M,t\to T_1} - k_0\right)^+ R^i_{c,M,t\to T_1}\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^*_t \left[1_{A_c} \left(R_{M,t\to T_1} - k_0\right)^+ \mathbb{M}^{*(i)}_{T_1\to T_N}\right] &= \mathbb{E}^*_t \left[\left(R_{M,t\to T_1} - k_0\right)^+ \mathbb{M}^{*(i)}_{T_1\to T_N}\right] \\
&- \mathbb{E}^*_t \left[1_{A_u} \left(R_{M,t\to T_1} - k_0\right)^+ \mathbb{M}^{*(i)}_{T_1\to T_N}\right] \\
&- \mathbb{E}^*_t \left[1_{A_d} \left(R_{M,t\to T_1} - k_0\right)^+ \mathbb{M}^{*(i)}_{T_1\to T_N}\right]
\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_t^*\left[1_{A_c} R_{c,M,t\to T_1}\left(R_{M,t\to T_1}-k_0\right)^+\mathbb{M}_{T_1\to T_N}^{*(2)}\right] &= \mathbb{E}_t^*\left[R_{c,M,t\to T_1}\left(R_{M,t\to T_1}-k_0\right)^+\mathbb{M}_{T_1\to T_N}^{*(2)}\right]\\ &\quad -\mathbb{E}_t^*\left[1_{A_u} R_{c,M,t\to T_1}\left(R_{M,t\to T_1}-k_0\right)^+\mathbb{M}_{T_1\to T_N}^{*(2)}\right]\\ &\quad -\mathbb{E}_t^*\left[1_{A_c} R_{c,M,t\to T_1}\left(R_{M,t\to T_1}-k_0\right)^+\mathbb{M}_{T_1\to T_N}^{*(2)}\right]\end{aligned}$$