# Applying 'no distortion at the top' for solving a bidimensional screening model.

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**Abstract**

In this article, we show how the use of the 'no distortion at the top' property, in addition to optimality conditions from Araujo, Vieira, and Calagua (2022, Economic Theory, 1-26), allows to solving the model of optimal regulation of a firm that has private information about its cost and demand functions, introduced by Lewis and Sappington (1988, The RAND Journal of Economics, 438-457).

**Keywords:** two-dimensional screening, no distortion at the top, regulation of a monopoly, exclusion.

JEL Classification: D82, L51, C69.

## 1 Introduction

Screening problems arise in several economic situations, including optimal taxation, nonlinear pricing, regulation of monopolists, and auctions. Most of the literature has modeled the private information of the agent by a parameter in one dimension. However, in many situations, it is required to consider multiple dimensions to model the agent's characteristics accurately.

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In one-dimensional models, the Spence-Mirrlees or single-crossing condition allows order the types by their marginal valuation for the principal's instrument. In multiple dimensions, the lack of an exogenous order among types makes it difficult to solve the problem. Thus, compared to one-dimensional models, there are few examples with analytical solution. We refer to Laffont, Maskin, and Rochet [1987], Basov [2001], and Araujo, Vieira, and Calagua [2022a], who have developed different techniques to obtain the solution.

One of the first screening models with bidimensional types was proposed by Lewis and Sappington [1988]. The authors modeled the regulatory policy of a monopolist firm with private information about both costs and demand, where the contract consists of the unit price to be charged and the subsidy received from the government. Lewis and Sappington's analysis led to a solution that Armstrong [1999] noted was incorrect. However, Armstrong did not provide the correct solution. Additionally, Armstrong [1999] conjectured that in the optimal contract, a positive mass of types should be excluded, like in the nonlinear pricing setting (see Armstrong [1996]). That is, if the firm's parameters belong to a certain region of positive measure, the firm will select a contract in which there will be no production because at the given price there will be no demand. Nevertheless, Calagua [2023] obtained the numerical solution of this model, and shows that such an exclusion feature is not present.

In this work we obtain the correct analytical solution of the model of Lewis and Sappington [1988]. The analysis is based on the necessary optimality conditions from Araujo, Vieira, and Calagua [2022a], which are also sufficient for optimality as Perez Vilcarromero [2022] has proved. Although one of the main assumptions of Araujo, Vieira, and Calagua [2022a] is not valid, we can manage this issue by using the 'no distortion at the top' property. The solution obtained confirms the numerical approximation from Calagua [2023]. The relevant conclusions are, as mentioned, that it is not optimal to exclude a positive mass of agents, and that price is almost always above marginal cost.

The 'no distortion at the top' property refers that the type of agent that gets the maximum allocation has no distortion with respect to the allocation with complete information. In one dimension, single-crossing is sufficient for this property but is not necessary (see Vagliasindi [1996] for a model of finite types). Araujo and Moreira [2010] analyzed a model without single-crossing and concluded that there is no distortion at the middle. There are also examples in Schottmüller [2015] and Araujo, Vieira, and Parra [2022b] without single-crossing where can be no distortion at the top, and examples with distortion for all the types. In the two-dimensional model we consider, the prevalent assumption is that single-crossing holds in each parameter direction. Whit this assumption, Araujo, Vieira, and Calagua [2022a] and Perez Vilcarromero [2022] showed that the dimensionality of the optimal allocation can be reduced to one, and therefore the 'no distortion at the top' property is expected to hold[1].

The plan of the paper is as follows: Section 2 describes the regulation model proposed by Lewis and Sappington [1988]. Section 3 summarizes the optimality conditions of Araujo, Vieira, and Calagua [2022a], extended by Perez Vilcarromero [2022], for a general formulation of a screening problem. Section 4 is devoted to obtaining the solution of the model. Conclusions are given in Section 5.

# 2 Lewis and Sappington's model

In the framework of the regulation of a monopolistic company, Lewis and Sappington [1988] considered that the demand for the firm's product $q = Q(p, a)$ and the costs of producing output $q$, $C(q, b)$, involve the firm's private information parameters $(a, b)$ distributed over $\Theta = [\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}]$ according to a strictly positive density function $\rho(a, b)$.

The regulator offers the firm a menu of unit prices $p$ and corresponding subsidy $t$ conforming to the firm's type is revealed. The profit of the firm of type $(a, b)$ is $p\,Q(p, a) - C(Q(p, a), b) + t$. The profit reservation level is type independent and normalized at zero. It is assumed that the regulator can ensure that the firm serves all demand at the established prices. The regulator's objective function is the expected consumer surplus net of the transfer to the firm

$$\int_{\underline{a}}^{\overline{a}} \int_{\underline{b}}^{\overline{b}} \{\Pi(Q(p(a, b), a), a) - p(a, b)Q(p(a, b), a) - t(a, b)\}\rho(a, b)db\, da \quad (1)$$

where $\Pi(Q, a) = \int_0^Q P(\xi, a)d\xi$, and $P(\cdot)$ denotes the inverse demand curve.

The regulator's problem is to design the menu of contracts $(p(a, b), t(a, b))$

[1] A general result of 'no distortion at the top' as a necessary condition in multiple dimensions is under construction.

to maximize (1) subject to individual rationality

$$p(a,b)Q(p(a,b),a) - C(Q(p(a,b),a),b) + t(a,b) \geq 0$$

and incentive compatibility constraints

$$p(a,b)Q(p(a,b),a) - C(Q(p(a,b),a),b) + t(a,b) \geq \\ p(\widehat{a},\widehat{b})Q(p(\widehat{a},\widehat{b}),a) - C(Q(p(\widehat{a},\widehat{b}),a),b) + t(\widehat{a},\widehat{b})$$

Lewis and Sappington derived a solution for the particular example

$$Q(p,a) = \alpha - p + a \quad , \quad C(q,b) = K + (c_0 + b)q \tag{2}$$

with $\alpha, K$ and $c_0$ positive constants and a uniform distribution over $\Theta = [0,1]^2$. However, in Armstrong [1999], the author noted that Lewis and Sappington's solution for this example was incorrect, but without solving it correctly.

In order to fit the formulation of the screening problem as in Araujo, Vieira, and Calagua [2022a] and Perez Vilcarromero [2022] to apply the optimality conditions, we introduce the following change of variables:

$$\begin{aligned} v(p,a,b) &:= pQ(p,a) - C(Q(p,a),b) \\ H(p,a) &:= pQ(p,a) - \Pi(Q(p,a),a) \\ V(a,b) &:= v(p(a,b),a,b) + t(a,b) \end{aligned}$$

Now, the regulator's problem can be written as (notice that the new variable $V$ is the firm's profit)

$$\max_{p(\cdot),V(\cdot)} \int_{\underline{a}}^{\overline{a}} \int_{\underline{b}}^{\overline{b}} \{v(p(a,b),a,b) - H(p(a,b),a) - V(a,b)\}\rho(a,b)db\,da \tag{P}$$

subject to

(IR) $V(a,b) \geq 0 \qquad \forall\ (a,b) \in \Theta$

(IC) $V(a,b) - V(\widehat{a},\widehat{b}) \geq v(p(\widehat{a},\widehat{b}),a,b) - v(p(\widehat{a},\widehat{b}),\widehat{a},\widehat{b}) \qquad \forall\ (a,b),(\widehat{a},\widehat{b}) \in \Theta$

# 3 Optimality conditions

We briefly expose of the methodology in Perez Vilcarromero [2022], that extends Araujo, Vieira, and Calagua [2022a], to obtain optimality conditions. Consider the screening problem (P) for a general density $\rho$ and $\Theta = [0,1]^2$.

The basic assumptions are

1. the constant signs of $v_a$ and $v_b$
2. the constant signs of $v_{pa}$ and $v_{pb}$ (single-crossing in each direction).

The first assumption is used to eliminate IR constraints and express the objective function just in terms of the instrument variable. This assumption is not guaranteed in Lewis and Sappington's model. However, we only need $V(a,b)$ expressed in terms of the minimum value of $V$ reached at some corner, say

$$V(a,b) = V(1,1) + \int_1^b V_b(a,\xi)d\xi + \int_1^a V_a(\xi,1)d\xi \tag{3}$$

Integrating (3) over the type set, using the envelope theorem[2] and integration by parts we obtain :

$$\begin{aligned}\int_0^1\int_0^1 V(a,b)\rho(a,b)dadb =& V(1,1) - \int_0^1\int_0^1 F_2(a,b)v_b(p(a,b).a.b)dadb \\ & - \int_0^1 v_a(p(a,1),a,1)\Big(\int_0^1 F_1(a,b)db\Big)da\end{aligned}$$

where

$$F_1(a,b) = \int_0^a \rho(\xi,b)d\xi \quad , \quad F_2(a,b)\int_0^b \rho(a,\xi)d\xi$$

This will be useful if we know a priori that $V$ is minimized at such corner, which justifies the assumption of the constant signs of $v_a$ and $v_b$ because, by the envelope theorem, we can know the increasing directions of $V$. Thus, assigning $V(1,1) = 0$ is optimal and guarantees that all IR constraints are satisfied.

[2]The incentive compatibility constraints can be expressed as $V(a,b)$ being the optimal value of the problem

$$\max_{(\widehat{a},\widehat{b})\in[0,1]^2}\{v(p(\widehat{a},\widehat{b}),a,b) + t(\widehat{a},\widehat{b})\}$$

in which $(a,b)$ is the solution. From the envelope theorem we have

$$V_a(a,b) = v_a(p(a,b),a,b) \quad , \quad V_b(a,b) = v_b(p(a,b),a,b)$$

Therefore, the objective on the screening problem (P) can be written as

$$\int_0^1 \int_0^1 G(p(a,b),a,b)dadb + \int_0^1 v_a(p(a,1),a,1)\Big(\int_0^1 F_1(a,b)db\Big)da \tag{4}$$

where

$$G(p,a,b) = \Big(v(p,a,b) - H(p,a) + \frac{F_2(a,b)}{\rho(a,b)} v_b(p,a,b)\Big)\rho(a,b)$$

On the other hand, from the incentive constraints, the following PDE must be satisfied:

$$\frac{-v_{pb}}{v_{pa}} p_a + p_b = 0 \tag{5}$$

The idea is to find the optimal value of $p$ over the norteast frontier $\Gamma = \{0\} \times [0,1] \cup [0,1] \times 1$, denoted by $\phi$[3]. For this purpose, consider the initial value problem

$$\frac{-v_{pb}}{v_{pa}} p_a + p_b = 0$$
$$p_{|\Gamma} = \phi(r)$$

and solve it by using the method of characteristics, which allows for a reparametrization of the type set in terms of the contour lines of $p$. Specifically, following the method of characteristics, define $a(r,s), b(r,s))$ as the solution of

$$\begin{aligned} a_s(r,s) &= -\frac{v_{pb}}{v_{pa}}(\phi(r), a(r,s), b(r,s)) & a(r,0) &= \alpha(r) \\ b_s(r,s) &= 1 & b(r,0) &= \beta(r) \end{aligned} \tag{6}$$

where $(\alpha(r), \beta(r)) = (0, 2r)$ for $r \in [0, 1/2]$ and $(\alpha(r), \beta(r)) = (2r-1, 1)$ for $r \in [1/2, 1]$ describes the east frontier and the north frontier, respectively.

The method provides a change of variables $(r,s) \to (a(r,s), b(r,s))$ such that $p(a(r,s), b(r,s)) = \phi(r)$. That is, the solution of system (6) are the contour curves of $p(\cdot,\cdot)$ parametrized by $s \in [0, U(\phi(r), r)]$.

[3]As in Araujo, Vieira, and Calagua [2022a] and Perez Vilcarromero [2022], the frontier to considerer could be different.

Observe that the tangent vector of the contour curve $(a(r,s), b(r,s))$ are given by $(-\frac{v_{pb}}{v_{pa}}(\phi(r), a(r,s), b(r,s)), 1)$.

The change of variables to re-express the objective leads to the following equivalent problem:

$$\max_{\phi(\cdot)} \int_0^1 \Big( \int_0^{U(\phi(r),r)} G(\phi(r), a(r,s), b(r,s)) \Big| \frac{\partial(a,b)}{\partial(r,s)} \Big| ds + 2v_a(\phi(r), 2r-1, 1)T(r)\mathbb{1}_{[\frac{1}{2},1]} \Big) dr$$

$$\text{s.t. } \ \phi(\cdot) \text{ is nonnegative and nondecreasing}$$

where $T(r) = \int_0^1 F_1(2r-1, b)db$, and $U(\phi(r), r)$ denotes the upper limit of $s$
.

Thus, from the Euler equations, the optimal $\phi(\cdot)$ must satisfy

$$\int_0^{U(\phi(r),r)} \frac{G_p}{v_{pa}}(\phi(r), a(r,s), b(r,s))ds = 0 \qquad \text{for } r \in [0, 1/2] \qquad (7)$$

$$\int_0^{U(\phi(r),r)} \frac{G_p}{v_{pa}}(\phi(r), a(r,s), b(r,s))ds = -T(r) \qquad \text{for } r \in [1/2, 1] \qquad (8)$$

# 4 Solving the Lewis and Sappington's model

We focus on solving the problem (P) with uniform distribution of types over $\Theta = [0,1]^2$, and $Q(p,a)$ and $C(q,b)$ defined as in (2):

$$Q(p,a) = \alpha - p + a \quad , \quad C(q,b) = K + (c_0 + b)q$$

Thus we have:

$$v(p,a,b) = -p^2 + (a + b + \alpha + c_0)p - (\alpha + a)(c_0 + b) - K$$

$$\Pi(Q(p,a), a) = (\alpha + a)(\alpha - p + a) - \frac{(\alpha - p + a)^2}{2}$$

In view of $v_{pa} = v_{pb} = 1$, the assumption 1) is satisfied. This implies that $p(\cdot,\cdot)$ is nondecreasing in $a$ and $b$. Therefore, the '*top type*' is $(1,1)$, that gets the maximum price: $p(1,1) \geq p(a,b)$ for all $(a,b) \in [0,1]^2$.

The assumption 2) is not fully satisfied. From the envelope theorem we have

$$\begin{aligned} V_a(a,b) =& v_a(p(a,b),a,b) = p(a,b) - c_0 - b \\ V_b(a,b) =& v_b(p(a,b),a,b) = p(a,b) - \alpha - a = -Q(p(a,b),a) \end{aligned}$$

thus, we can be sure that $V_b \leq 0$ (because $Q \geq 0$). However, the sign of $V_a$ is not a priori determined.

Recall that assumption 2) were needed to eliminate the IR constraints since it provides information at which point the function $V$ reaches the minimum. We can ensure this by the following result, in which the 'no distortion at the top' property is used.

**Proposition 4.1.** *The firm's profit $V$ is nonincreasing over $[0,1] \times \{1\}$*

*Proof: For no distortion at the top, the price allocated to type $(1,1)$ must maximize the social surplus in the scenario of complete information, in which the utility of the regulator is $\Pi(Q(p,1),1) - pQ(1,1) - t$ and utility of the monopolist is $pQ(1,1) - C(Q(p,1),1+t$. Then $p(1,1)$ must be the solution of*

$$\max_{p>0}\{\Pi(Q(p,1),1) - C(Q(p,1),1)\}$$

*or*

$$\max_{p>0}\{(\alpha + 1 - c_0 - 1)(\alpha + 1 - p) - \frac{(\alpha + 1 - p)^2}{2} - K\}$$

*Thus we obtain*

$$p(1,1) = c_0 + 1 \tag{9}$$

*On the other hand, over the north frontier of the type set $[0,1]^2$ we have*

$$V_a(a,1) = p(a,1) - c_0 - 1 \quad \forall\ a \in [0,1] \tag{10}$$

*Since $p(1,1) = c_0 + 1$ is the maximum price, then*

$$c_0 + 1 \geq p(a,1) \quad \forall\ a \in [0,1] \tag{11}$$

*Combining (10) and (11) we obtain $V_a(a,1) \leq 0\ \forall\ a \in [0,1]$.* $\square$

Therefore, $V$ reaches the minimum at $(1,1)$. This guarantees the expression in (3) and the validity to apply the optimality conditions.

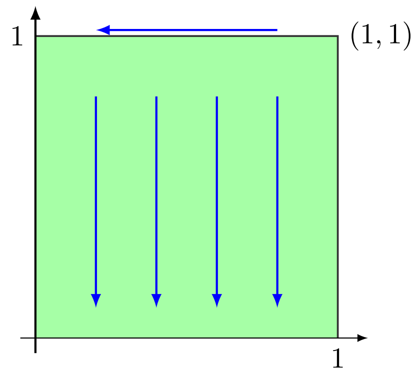



Figure 1: Arrows represent the nondecreasing direction of the firm's profit $V$. While $V$ weakly increases vertically over all the type set, horizontally it is only over the north frontier as shown in Proposition 4.1.

The remaining analysis is quite simple. Due to the tangent vector of the contour curves of an implementable $p(\cdot,\cdot)$ is constant $(a_s(r,s), b_s(r,s)) = (-1,1)$, the contour curves are lines with slope equal to -1.

The solution of system (6) is given by

$$\begin{array}{lll} a(r,s) = -s & b(r,s) = s + 2r & \text{for } r \in [0,1/2] \\ a(r,s) = 2r - 1 - s & b(r,s) = s + 1 & \text{for } r \in [1/2,1] \end{array}$$

Besides, by uniform distribution $\rho(a,b) = 1$, then $F_1(a,b) = a$ and $F_2(a,b) = b$. Thus,

$$\begin{aligned} G(p,a,b) &= v(p,a,b) - H(p,a) + \frac{F_2(a,b)}{\rho(a,b)} v_b(p,a,b) \\ &= -p^2 + (a + b + \alpha + c_0)p - (\alpha + a)(c_0 + b) - K \\ &+ \frac{(\alpha + a - p)^2}{2} + b(p - \alpha - a) \end{aligned}$$

Therefore

$$\frac{G_p}{v_{pa}}(p,a,b) = c_0 + 2b - p$$

For $r \in [0,1/2]$, to determine $U = U(\phi(r), r)$, note that the contour line $(a(r,s), b(r,s)) = (-s, s + 2r)$ intersects the south frontier when $b(r,U) = 0$, that is $U(\phi(r), r) = -2r$. By applying (7) we have

$$\int_0^{-2r} \Big( c_0 + 2(s + 2r) - \phi(r) \Big) ds = 0$$

thus we obtain

$$\phi(r) = c_0 + 2r \quad \text{for } r \in [0, 1/2] \tag{12}$$

For $r \in [1/2, 1]$, there is no need to use (8). This is because for $r = 1/2$ the optimal value assigned to type $(0, 1)$ is

$$\phi(1/2) = p(0, 1) = c_0 + 1 = p(1, 1) = \phi(1)$$

Recall that the value of $p(1, 1)$ was determined in (9). Therefore, since $\phi$ is nondecreasing over $[1/2, 1]$, we must have

$$\phi(r) = c_0 + 1 \quad \text{for } r \in [1/2, 1] \tag{13}$$

Returning to the original variables, due to $a + b = 2r$, we obtain that optimal prices are given by

$$p(a, b) = \begin{cases} c_0 + a + b & \text{if } a + b \leq 1 \\ c_0 + 1 & \text{if } a + b > 1 \end{cases} \tag{14}$$

With this, the regulator's objective takes the value: $(c_0 - \alpha)^2/2 + (c_0 - \alpha)/2 - K + 5/24$, the optimal subsidy is given by

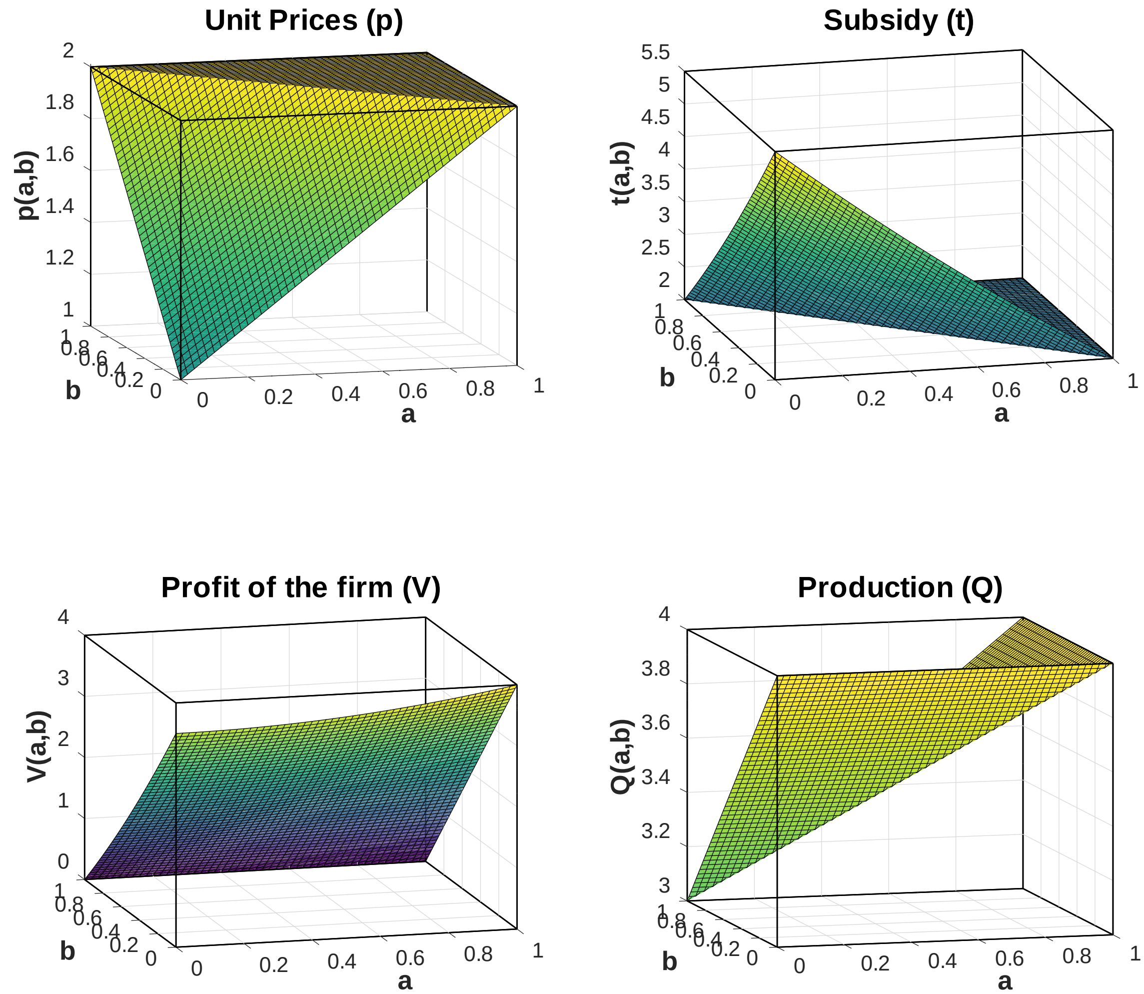
$$t(a, b) = \begin{cases} \dfrac{(a+b)^2}{2} - (\alpha - c_0)(a + b) + \alpha - c_0 - \dfrac{1}{2} + K & \text{if } a + b \leq 1 \\ K & \text{if } a + b > 1 \end{cases} \tag{15}$$

and the firm's profit is:

$$V(a, b) = \begin{cases} \dfrac{a^2 + b^2 - 1}{2} + (1 - b)(\alpha - c_0) & \text{if } a + b \leq 1 \\ (a + \alpha - c_0 - 1)(1 - b) & \text{if } a + b > 1 \end{cases} \tag{16}$$

Next, we provide the graphs of optimal prices $p(\cdot, \cdot)$, subsidy $t(\cdot, \cdot)$, firm's profit $V(\cdot, \cdot)$, production $Q(\cdot, \cdot)$, and the difference between price and marginal cost $p(\cdot, \cdot) - C_q(Q(\cdot, \cdot), \cdot)$ for the case $c_0 = 1$, $\alpha = 5$, and $K = 2$ [4].

[4] Calagua [2023] shows the graphs of the numerical solution for the same parameter values.


Unit Prices (p)
p(a,b)
2
1.8
1.6
1.4
1.2
1
b
a
0
0.2
0.4
0.6
0.8
1
Subsidy (t)
t(a,b)
5.5
5
4.5
4
3.5
3
2.5
2
Profit of the firm (V)
V(a,b)
4
3
2
1
0
Production (Q)
Q(a,b)
4
3.8
3.6
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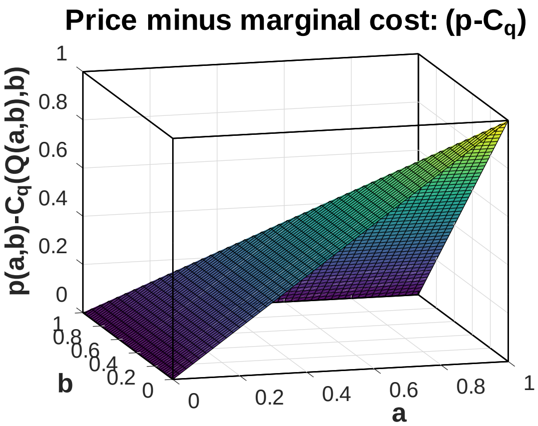
3.2
3




# Conclusions

The solution obtained in (14) and (15) confirms the conclusions from the numerical solutions given in Calagua [2023]. The most relevant are:

1. At the optimum, the firm produces a positive quantity regardless of its type.

2. The regulator induces the firm to price above marginal costs for almost all $(a, b)$ types rather than $a = 0$ or $b = 1$ (i.e., such types with the a priori lowest demand or such types with the highest costs).

3. The type of firm with the highest cost parameter receives zero profit.

4. By the optimal firm's profit $V$, there is no exclusion of a positive mass of types.

5. In the region of types $(a, b)$ with $a + b \leq 1$, the optimal price equals the *adjusted marginal cost* (AMC), defined as

$$AMC(s) = \frac{\int_0^s C_q(Q(p(\tilde{a}, s - \tilde{a}), \tilde{a}), s - \tilde{a})\rho(\tilde{a}, s - \tilde{a})d\tilde{a}}{\int_0^s \rho(\tilde{a}, s - \tilde{a})d\tilde{a}} + \frac{\int_0^s F(\tilde{a}, s - \tilde{a})d\tilde{a}}{\int_0^s \rho(\tilde{a}, s - \tilde{a})d\tilde{a}}$$

where $s$ is such that $a + b = s \leq 1$, and $F(a, b) = \int_0^b \rho(a, \tilde{b})\, d\tilde{b}$.

6. In the region of types $(a, b)$ with $a + b > 1$, bunching is observed in price $p(a, b) = c_0 + 1$, and subsidy $t(a, b) = K$ (which is the firm's fixed cost). This subsidy feature in half of the realizations aligns with the traditional policy regulation approach used in situations without information asymmetry.

Points 3) and 4) are supported by the expression given in (16). Note that, when $\alpha$ is sufficiently large relative to $c_0$[5] (as presumed in Lewis and Sappington [1988]), the firm's profit $V$ equals zero if and only if $b = 1$.

The most important conclusion is the no-exclusion feature in the optimal contract. This is contrary to Armstrong's conjecture and to Barelli, Basov, Bugarin, and King [2014] result, which extended the Armstrong [1996] theorem of generic desirability of exclusion. However, Barelli, Basov, Bugarin, and King [2014] have considered prices to be a subset of $[c_0 + 1, \alpha]$ for applying their theorem, and this is not the case.

A detailed discussion of why the result in Armstrong [1996] about the optimality of exclusion cannot be applied in this regulation model can be found in Calagua [2023]. A rough explanation is that, while Armstrong [1996] result was given for a monopolist as the principal that could obtain more income from customers still in the market versus zero penalties from customers excluded, the regulator has to assume a negative penalty when excluding a type of firm.

About the point 5), Lewis and Sappington [1988] introduced the previous definition of $AMC(s)$ as the sum of the expected marginal costs given $s$ and the bidimensional version of the inverse hazard rate. They also provided the following interpretation of the second term in $AMC(s)$: "*Intuitively, this term captures the optimal mark-up of price above expected marginal cost. The mark-up balances the expected losses from inefficiently low output with the expected gains from reduced information rents that accrue to the firm because of its private knowledge.*"

Baron and Myerson [1982] obtained a conclusion related to '*price equals adjusted marginal cost*' in their analysis of a model where the regulator faces uncertainty solely about the firm's cost function. According to their findings, at the optimum, prices exceed marginal costs for all cost realizations except the lowest one.

[5] It is enough that $\alpha > c_0 + 1$.

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