Bonds and Stocks Returns Comovements in Brazil: Are they Different from those in the US?

Renan Cardoso* Márcio Garcia†

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Abstract

There is a flourishing literature trying to explain the behaviour of US treasury bond returns comoving positively with US stock returns before the 2000's, and working as hedge assets since then. In Brazil, for the last 18 years, we find that the comovement of treasury bonds and stocks returns has been positive and volatile. Inflation being procyclical or countercyclical and the monetary policy showed up relevant for explaining this behavior, as pointed by the literature for the US. But, considering Brazil as an emerging market, we find that movements in country risk also played a considerable role by making Brazilian bonds and stocks prices move in the same direction, generating more positive bond to stock betas[1]. Controlling for country risk, we find periods that a positive correlation between inflation and output indeed would have generated negative bond-stock betas, as was predicted by Campbell, Pflueger and Viceira (2020). Since a positive beta of an asset imply a positive risk premium, by estimating a term structure model we find that this contributed with 1 percentage point in the term spread of bonds.

Keywords: Bond to stock beta; Country risk; Inflation; Emerging Market.

JEL Classification: E43; G12.

*PUC-Rio. E-mail: renan.mcardoso@hotmail.com

†PUC-Rio. E-mail: mgarcia@econ.puc-rio.br

[1] The beta of regressing bonds returns on stock returns.

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1 Introduction

Drawing from the Capital Asset Pricing Model (CAPM), the beta of an asset measures the relation between its returns with the stock market returns. Positive bond to stock betas means that bonds are risky: they co-move together with the stock market. For example, when stock prices are going down, bond prices are also going down. Conversely, if the beta is negative, bond prices are going up when stock prices are going down, so bonds can rebalance a portfolio by providing a hedge against equity risk. Since treasury bonds and stocks are the two main assets in the Brazilian market as the Table 1 below shows, the bond-stock beta is a crucial variable for taking into consideration for portfolio decisions.

Table 1: Brazilian Market Portfolio Weights, 2005–15.

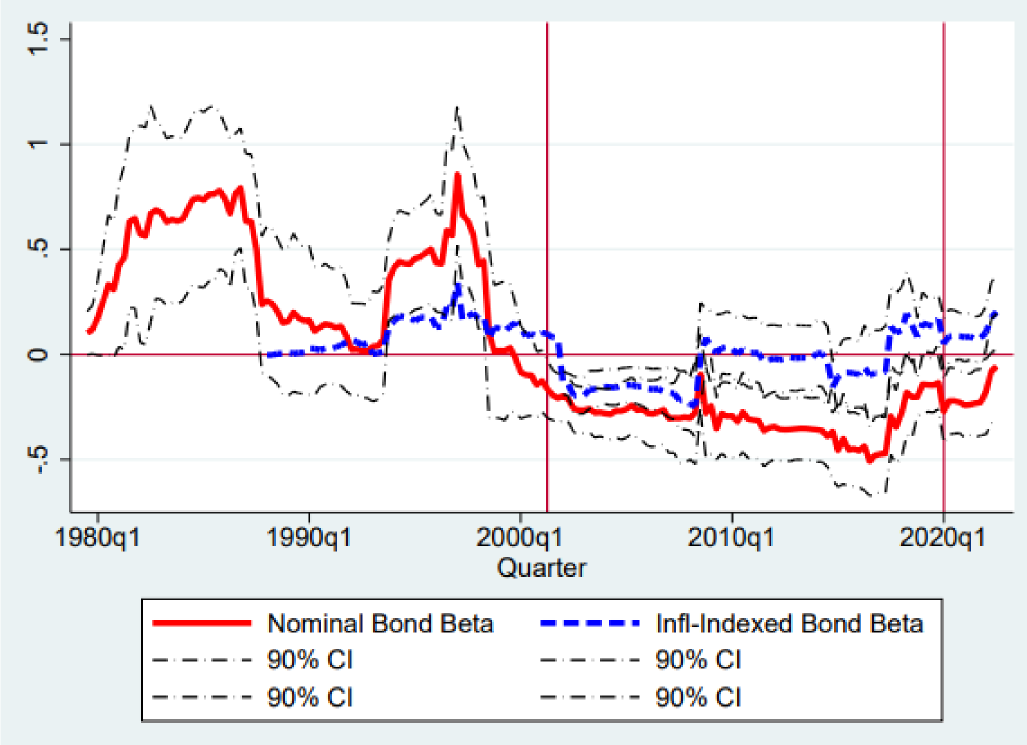
Year (%)	Equity (%)	Private Equity (%)	Gov. Bonds (%)	Corporate Bonds (%)	Bank Funding (%)	Agribusiness (%)	Real Estate (%)	Credit Bonds
2005	48.03	0.10	35.00	3.66	12.31	0.00	0.68	0.22
2006	48.73	0.28	34.35	4.95	10.92	0.02	0.47	0.28
2007	57.08	0.50	28.20	4.90	8.46	0.11	0.35	0.41
2008	36.96	0.42	33.94	7.17	19.77	0.14	0.73	0.87
2009	46.84	0.50	27.99	5.97	17.02	0.22	0.85	0.61
2010	45.94	0.76	28.63	6.23	16.24	0.27	1.33	0.61
2011	40.46	1.23	31.30	7.22	16.44	0.50	2.12	0.71
2012	40.50	1.41	30.71	8.30	14.76	1.00	2.42	0.91
2013	37.43	1.93	31.32	9.25	13.99	1.91	3.25	0.93
2014	33.44	2.29	32.42	10.34	13.76	2.30	4.52	0.93
2015	26.85	2.41	37.04	10.58	13.92	2.85	5.10	1.25

Source: Tessari and Meyer-Cirkel (2017).

Long-term bonds yields can be decomposed in the expected average of future short-term rates and a term premium. The term premium reflects a compensation for the investor carrying an asset for a long period instead of keeping continuously buying many short-term treasury bills. In buying a long-term bond, the investor is more exposed to fluctuations of inflation and interest rates. Considering this, Clarida (2019), in a speech as the US Federal Reserve Vice Chair, said that: "*since the late 1990s, bond returns tend to be high and positive when stock returns are low and negative so that nominal bonds have been a valuable outright hedge against equity risk. As such, we would expect the equilibrium yield on bonds to be lower than otherwise, as investors should bid up their price to reflect their value as a hedge against equity risk (relative to their value when the bond beta to stocks was positive). According to Campbell, Sunderam, and Viceira (2017), the hedging value of nominal bonds with a negative beta to stocks could substantially lower* *the equilibrium term premium on bonds.*" As is clear in Clarida's speech, if the bond beta is positive, bonds yields input a risk premium for holding it. If the beta is negative, the hedging property counterbalances the term premium, decreasing it, and even turning it negative. A positive bond beta, then, has important consequences for the economy, as generating higher yields, which means higher cost of borrowing for the Government, and also an additional component affecting the course of the monetary policy. So it is a relevant indicator for policy makers as well.

In the US, the bond-stock beta was positive in the 1980's and 1990's, as shown in Figure 1, but from the 2000's and on has became negative, making the US treasury bond a safe asset. According to Campbell, Pflueger and Viceira (2020), in the 1980's the bond betas were positive because of the stagflation. They consider nominal bonds in their analysis. So, after the bond is issued, if inflation increases, the real return of the nominal bond falls. If the bondholder tries to sell this bond, the price in the secondary markets will also have dropped. Conversely, if inflation decreases, bond prices increase. Looking for stocks, their prices intrinsically represent expected profits of firms. If the economy is heaten, with considerable output growth, firms expect higher profits and the stock prices increase. So, the stagflation in that period of the US was driving the bond prices down due to inflation, and the stocks down due to the recession.

Figure 1: Rolling Treasury Bond-Stock Betas in the USA (Pflueger (2023)).



Source: Pflueger (2023) shows "*betas from regressing quarterly ten-year Treasury bond excess returns onto quarterly US equity excess returns over five-year rolling windows for the period 1979.Q4-2022.Q3. Quarterly excess returns are in excess of three-month T-bills. Prior to 1999, I replace US Treasury Inflation Protected (TIPS) returns with UK ten-year linker returns. Bond excess returns are computed from changes in yields. Zero-coupon yield curves from Gürkaynak, Sack and Wright (2006, 2008) and the Bank of England. Vertical lines indicate 2001.Q2 and the start of the pandemic 2020.Q1.*"

Since the 2000's the bond-stock betas there turned negative because growth moments were accompanied by increase in inflation. So the kind of inflation is a crucial aspect. With procyclical inflation as in the US 2000's, economic growth makes stock prices go up, and the inflation generated depresses bond prices. Analogously, recessions followed by fall in inflation make bond prices go up and stock prices go down. So, since the 2000's, US bonds are safe and hedge the US stock market. Once they became good hedges, their price also increases in recessions because they perform well in bad moments. Analogously, when the bonds were risky, their price used to fall even more in bad moments of high marginal utility and high risk aversion.

Campbell, Pflueger and Viceira (2020) departure from Campbell and Cochrane (1999)'s habit persistence model to a very similar specification. By showing that the process for habit persistence is consistent with an Euler equation in terms of output gap[2] as in the

[2]Output gap equals log real output minus log potential (steady state) real output.

standard New Keynesian model. They couple it with reduced-form, loglinear dynamics for inflation and Federal Funds rate to close the model and allow for pricing the bonds and stocks. By estimating the parameters of the equations for macroeconomic dynamics separately for data before and after 2001Q2[3], using the Simulated Method of Moments and not using data of asset prices, they match the empirical correlations: if the correlation between inflation and the output gap is negative (positive), bond betas are positive (negative). Furthermore, time-varying risk aversion amplifies the positive (negative) co-movement of bonds and stocks in the period before (after) 2001Q2.

Pflueger (2023) also argues that a quickly responsive inflation-focused monetary policy in the 1980's US was crucial for the positive bond-stock beta. By suddenly raising the real interest rate, the Federal Reserve generated higher bond yields - lower bond prices - and took the economy into a recession driving down stock prices. In the US 2000's a more inertial output-focused monetary policy contributed for allowing procyclical inflation and negative bond-stock beta. She builds counterfactuals augmenting Campbell, Pflueger and Viceira (2020)'s model into a structural New Keynesian asset-pricing model with a NK Phillips Curve, a Taylor Rule, supply, demand and monetary policy shocks. She shows that, even in the US 1980's featuring volatile supply shocks, if the monetary policy was not quickly responsive and inflation-focused, the bond-stock betas would be negative. So supply shocks and a responsive monetary policy are both necessary for positive beta.

In addition to the papers already mentioned, Chernov, Lochstoer and Song (2021) working on a model with Epstein-Zin preferences explain changes in bond-stock correlation by movements in transitory and permanent components of consumption. Swanson (2021) works with Epstein-Zin preferences as well, in a New Keynesian model, and by modelling also defaultable bonds, he shows that they present behavior closer to stocks than non-defaultable bonds. Song (2017) works on a long-run risk model with recursive preferences focusing on monetary policy and inflation dynamics regimes switching to explain the bond-stock correlation. Li, Zha, Zhang and Zhou (2022) explains the signs in the correlation with investment and technology shocks in different monetary and fiscal regimes.

On this paper, we study the determinants of the bond to stock beta in Brazil and the implications for risk premium in the term structure. We document that the bond to stock beta has been positive for the last 18 years in Brazil. Section 2 employs different econometrics approaches for understanding the determinants of the bond to stock beta. We employ regressions with several variables and controls for inspecting what variables are correlated with bonds and stocks returns. We also employ a Canonical Correlation Analysis for analyzing the correlations of all variables in a single framework. The cyclical-

[3] By using a Quandt Likelihood Ratio (QLR) test, they show that this date was when the correlation between inflation and output gap switched sign in the USA.

ity of inflation is important, but also the country risk dynamics of an emerging economics plays a relevant role. On section 3, we estimate the model from Campbell, Sunderam, and Viceira (2017) in order to account for the bond risk premia in Brazil, given their positive beta. And finally, on section 4, we propose a modification of Campbell, Sunderam, and Viceira (2017)'s model augmented with country-risk.

In contrast to Campbell, Pflueger and Viceira (2020), Campbell, Sunderam, and Viceira (2017) do not use consumption-based habit formation preferences, and choose to work with a more tractable term structure model with reduced-form dynamics for the stochastic discount factor (SDF), the real interest rate and the inflation, in order to analyze the trends from the data. We chose to estimate Campbell, Sunderam, and Viceira (2017) model with Brazilian data on this paper, because it a is less structured model that fits better different patterns from different countries. With this model, we are able to decompose the long term bonds into the real interest rate and breakeven inflation components, and measure how the covariance of these components with the stochastic discount factor contributed for higher bond yields. We account for an increase in 1 percentage point in the yield spread of 10-year nominal bonds in relation to 3-month bills, because of the volatility of the bond components and for covarying with the SDF.[4]

2 Empirical Analysis

2.1 Data

For Brazilian nominal bonds data, there is the possibility of using the ID x fixed rate swap (*swap DI x pré*) provided by B3 – Brazil Stock Exchange and Over-the-Counter Market - through the *rb3* R package, or using the IDkA index provided by Anbima. The "IDkA-Pré" for nominal bonds is an index made with LTNs and NTN-Fs that incorporates the returns of the bonds keeping the duration constant over time[5]. It is available with

[4]We also estimated with Brazilian data from 2006 to 2023 the Campbell, Pflueger and Viceira (2020) model. The model says that the nominal and real bond-stock beta in the period should be -0.22 and 0.10, while empirically these numbers are of 0.26 and 0.13, respectively. It is a model designed for the US considering the comovements of inflation, output and interest rate without considering any country risk or default probability. So our key takeaway from this is that the Brazilian nominal term structure inputs emerging market risk premia components that comove with stocks. But since it is a model designed for the US to match moments as high equity premium, we arrived at unrealistic moments for the assets. Also for the US they have two reasonable big periods with clear macro scenarios, namely before and after 2000. For Brazil the sample period is smaller with volatile changes in the output, inflation and interest rates comovements, making the translation into one first and second moment difficult to interpret.

[5]"The Constant Duration Indexes are generated from a initial theoretical application of 1000 monetary units at time (t), in the synthetic asset (n vertice of the Term Structure of Interest Rates), which is sold at the immediate following working day (t+1), by the rate of the n-1 Term Structure of Interest Rates,

maturities of 3M, 1Y, 2Y, 3Y, 4Y and 5Y. For inflation-indexed bonds, there is no liquid market for swaps, so we use the "IDkA-IPCA" for inflation-indexed bonds, build with NTN-Bs for maturities of 2, 3, 5, 10, 15, 20 and 30 years. It works similarly as the "IDkA-Pré" but the variation of the inflation (given by the IPCA) is always incorporated to the index/price. The ID x fixed rate swap starts in Jan/2004 and the IDkA starts in Jan/2006. By construction, the IDkA gives the best representation of a measure for bonds returns. The data series with the IDkA-Pré and the ID x fixed rate swap are very similar (see Figure 20). So for comparability purposes between bonds and attachability with the best representation of returns, we use the IDkA index.

Let x = ln(X). All the returns are annual. Let $p_t^{(n)}$ be the log price of a bond in time t with maturity n. $i_t^{(n)}$ is the log yield, then

$$i_t^{(n)} = -\frac{1}{n} p_t^{(n)}.$$

As described by Anbima, the IDkA ($IDkA_t$) would evolve as

$$IDkA_{t+1} = IDkA_t . \frac{P_{t+1}^{n-1}}{P_t^n} \rightarrow ln(\frac{IDkA_{t+1}}{IDkA_t}) = p_{t+1}^{n-1} - p_t^n,$$

and $r_{t+1}^n = p_{t+1}^{n-1} - p_t^n$ is exactly the log of the ex-post return of holding the asset. Equivalently, using the swap rates, the returns are calculated as $r_{t+1}^n = -(n-1).y_{t+1}^{n-1} + n.y_t^n$.

We focus on *zero-cost portfolios*, as in Campbell, Pflueger and Viceira (2020) by taking the excess returns. For the returns series see Figure 21. The idea is that "by focusing on excess returns, we net out inflation and the level of interest rates, so we focus directly on real risk premia in the nominal term structure" (Cochrane and Piazzesi (2005)). The excess log return is $rx_{t+1}^n = r_{t+1}^n - i_t^{(1)}$, which is the return in excess of the 1-year bill return[6]. For $i_t^{(1)}$ we compute it from the time series "Taxa de juros prefixada - estrutura a termo - LTN - 12 meses - (% a.a.)" from Ipeadata. We focus on 5-Y bonds as in Campbell, Pflueger and Viceira (2020) and because is the IDkA-Pré maximum maturity available. In the Appendix we provide the Figure 23 comparing with the 10 year bond. The movements are the same, just with higher variability for the 10 year bond, so both reach similar qualitative results.

The return of the stock market r_t^s is given by the difference of the log of the Ibovespa, collected from Thomson Reuters. The stock excess return is $rx_t^s = r_t^s - i_t^{(1)}$.

generating a new financial value to be reinvested by the rate of the n vertice at the same date." (Own translation.) Further details on the methodology on https://www.anbima.com.br/pt_br/informar/precos-e-indices/indices/idka.htm

[6]It is very similar to computing excess returns over the accumulated interbank rate (CDI), see Figure 22.

Figure 2: Bonds and Stocks Excess Returns

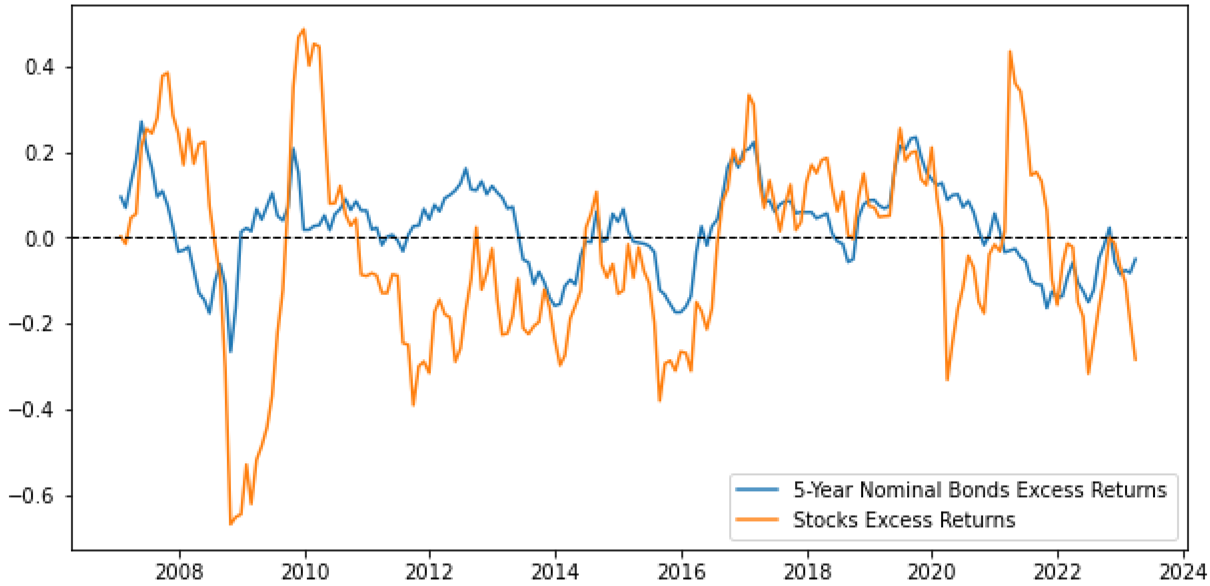
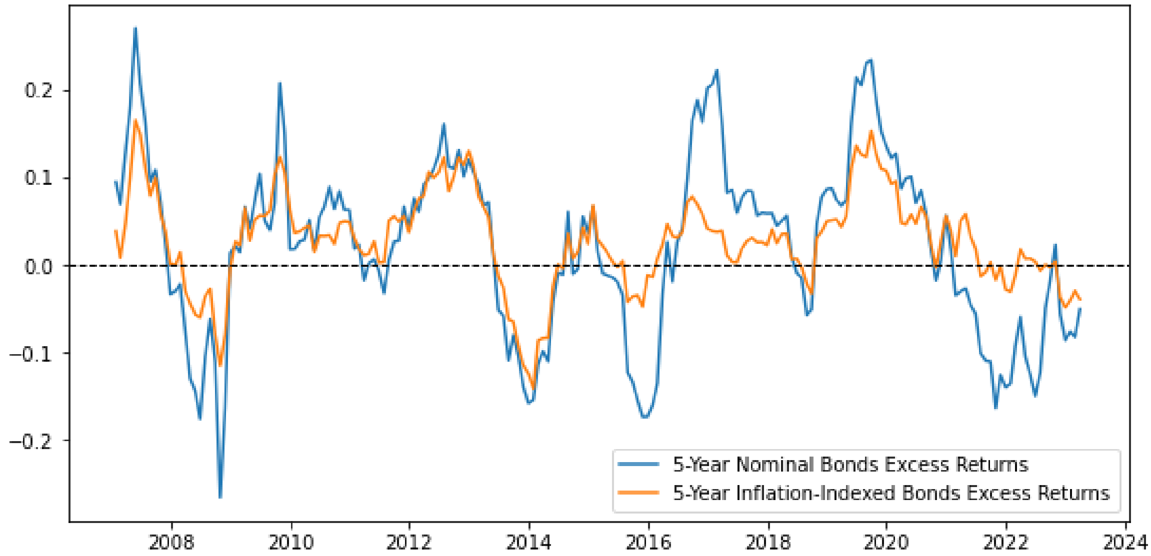


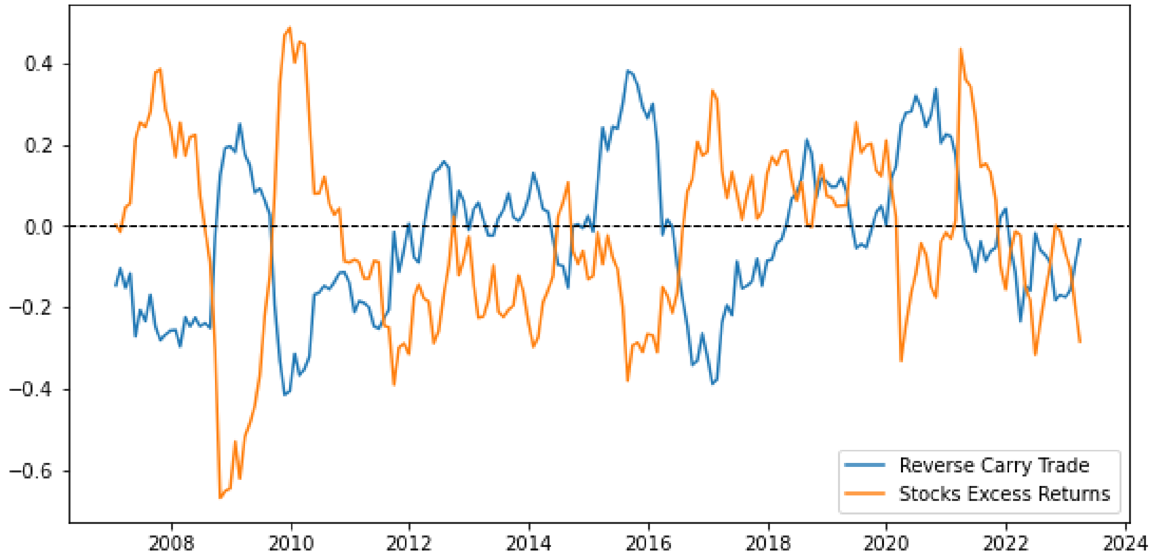
Figure 3: Nominal and Inflation-Indexed Bond Excess Returns



Let e_t be the log of the nominal exchange rate given by the price of one US Dollar in Brazilian Reals. Then the return on "reverse" carry trade[7] of a Brazilian investing in the US is given by $rx^{rct}_{t+1} = i^{(1)*}_t - i^{(1)}_t + \Delta e_{t+1}$. For the 1-year US bond $i^{(1)*}_t$ we use the "Market Yield on U.S. Treasury Securities at 1-Year Constant Maturity, Quoted on an Investment Basis" by the FRED.

[7]Usually the carry trade is an operation of buying a bond in a high interest rate country and financing it in a low interest rate country. The "Reverse" stands for doing the opposite buying US Treasury bonds.

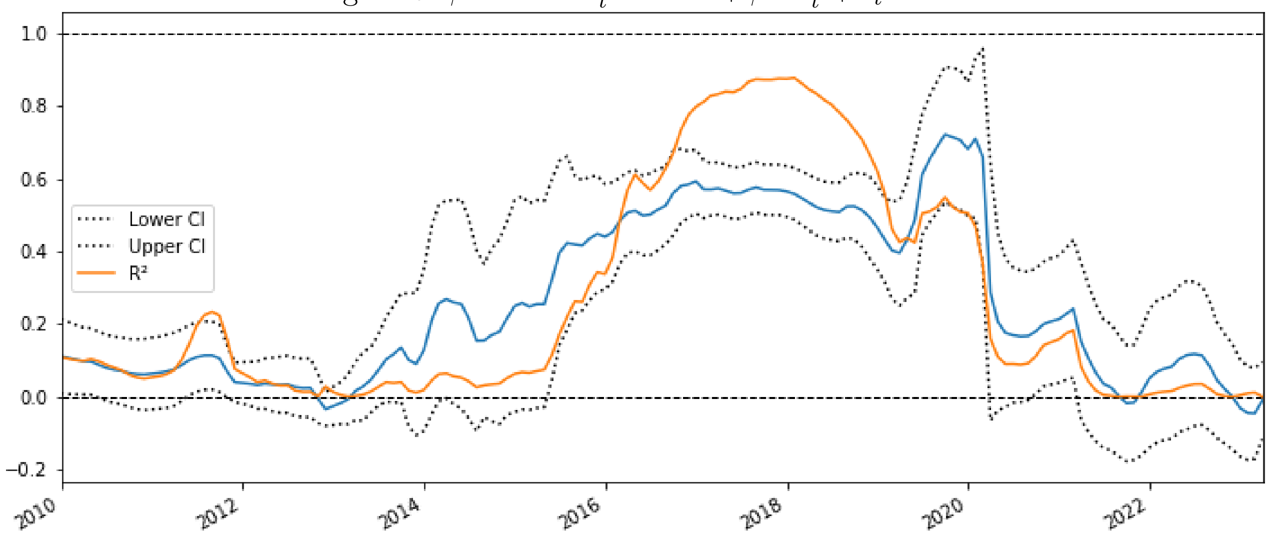
Figure 4: Reverse Carry Trade and Stocks Excess Returns



2.2 Bond Betas

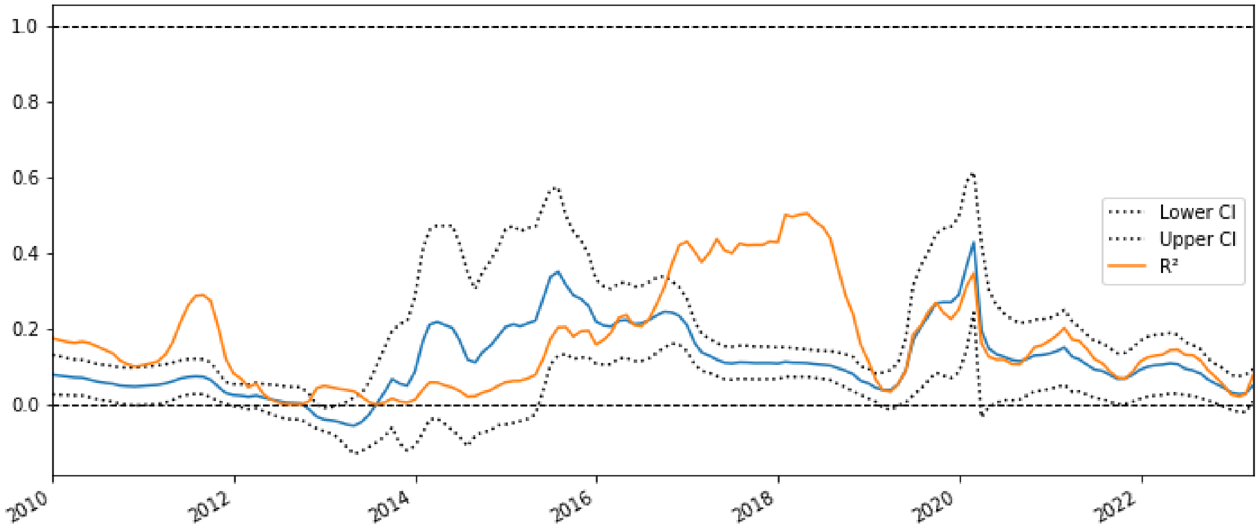
We run Rolling OLS, that is OLS regressions on moving windows. Here we use a window of 3 years (36 observations for each regression). Estimates covariance matrices calculated using Newey-West correction method with 18 lags for autocorrelation and heteroscedasticity issues. We compute the nominal bond-stock betas, by regressing nominal bonds excess returns into stocks excess returns, and plot with a 95% confidence interval and the respective R^2s (Figure 5). In the Appendix, we also plot the betas using the swap starting in 2004, see Figure 24. From 2015 to beginning of 2020, the series are highly positively correlated. From 2006 to 2014, and after 2020, they are slightly positively correlated.

Figure 5: βs from $rx_t^{Pre5} = \alpha + \beta.rx_t^s + \epsilon_t$



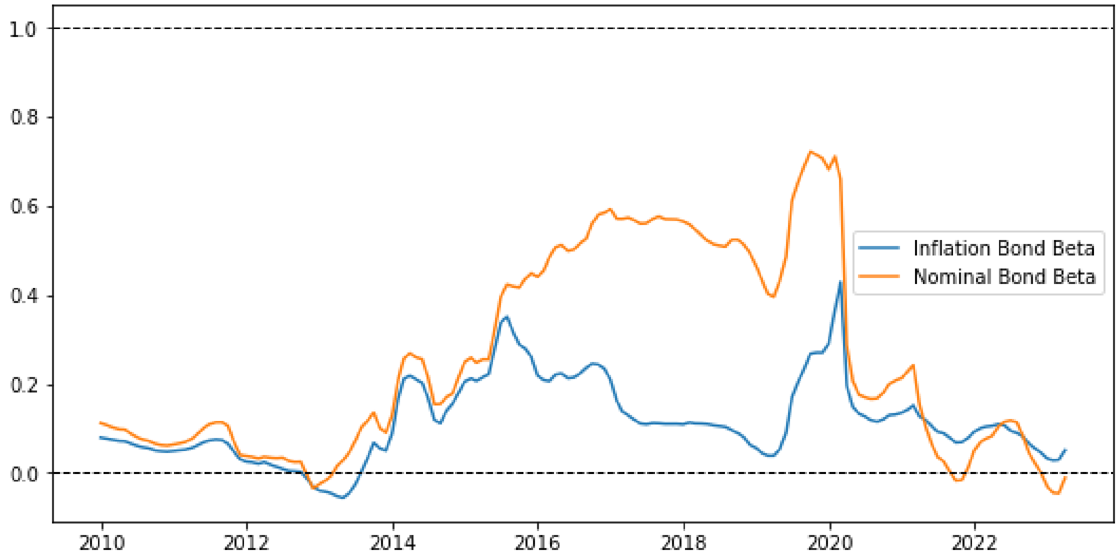
We also run the rolling regressions $rx_t^{IPCA} = \alpha + \beta.rx_t^s + \epsilon_t$ for computing the inflation-indexed bond-stock betas (Figure 6). The patterns of the coefficients, P-values and R^2s are very similar from the nominal bond beta, but here the R^2s are slightly smaller showing that nominal bonds are more correlated with stocks than inflation bonds.

Figure 6: $rx_t^{IPCA} = \alpha + \beta.rx_t^s + \epsilon_t$



Mainly on 2015 to 2020 appears a gap between both betas (Figure 7), that was already visible on the difference between their excess returns (Figure 4). Nominal bonds excess returns are more volatile and has larger bond-stock betas. This comes from deviations of realized inflation (which inflation bonds are adjusted by ex-post) and expected inflation (see Figure 26 in the Appendix). For example, in 2015, the inflation was higher than expected, providing higher returns for the inflation bond. From late 2016 to early 2018, people expected inflation to be higher than it really was, and nominal bonds were priced with higher yields, providing higher returns than inflation-indexed bonds. Furthermore, the risk premium for holding a fixed rate bond when the future inflation is unknown is also relevant. Lowenkron and Garcia (2007) provided earlier evidences from Brazilian bonds showing that higher deviations from realized inflation in relation to expected inflation generated higher expected inflation and higher inflation risk premium components on the nominal term structure. These extra components come from higher uncertainties perceived by investors about the capacity of the Central Bank to control inflation. Our calculations of bond to stocks betas show that this inflation risk components, related to uncertainties in Brazil about the future path of inflation, is correlated with the stock prices. These mechanisms and its risk premia consequences will be better analyzed with the term structure model we estimate on the next section.

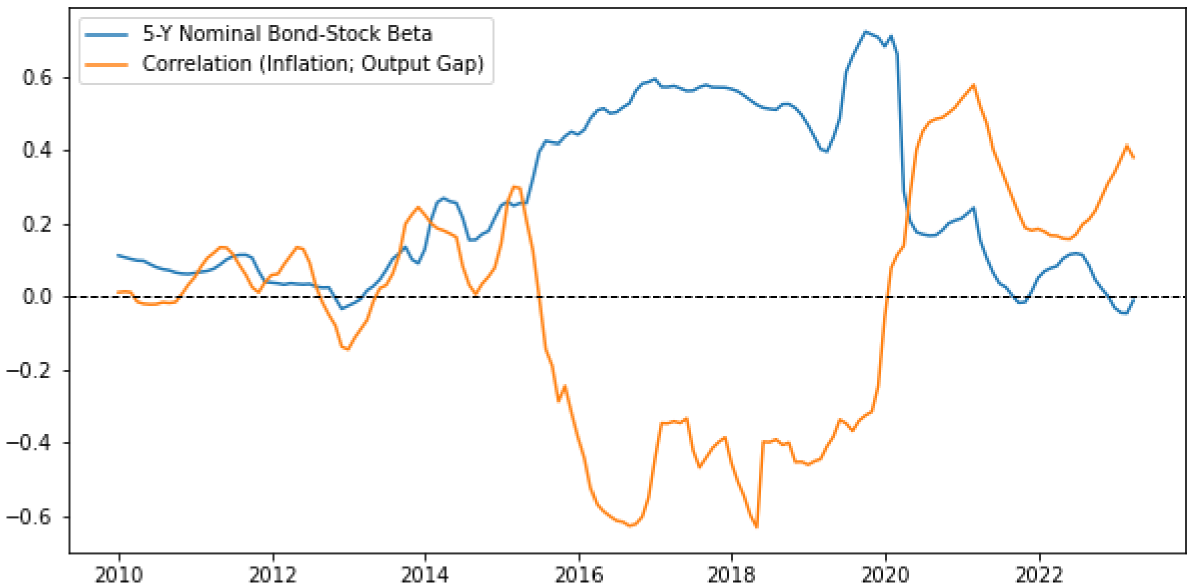
Figure 7: Nominal Bond-Stock beta and Inflation-Indexed Bond-Stock Beta



2.3 Macroeconomic Rationale

Below (Figure 8), we plot the betas with rolling correlations (3 years windows) of inflation with the output gap.[8] In the Appendix (see Figure 25), we compare also using the output growth (annual log difference of the IBC-br), reaching similar results. It is visible that the period of higher R^2 and most positive bond-stock beta coincides with the period of sustained negative correlation between the inflation and output (2015 $\sim$ 2019).

Figure 8: Bond-Stock Betas and Correlation Between Inflation and Output Gap



In 2015-16, the stagflation promoted a positive bond beta as in the 1980's USA. In

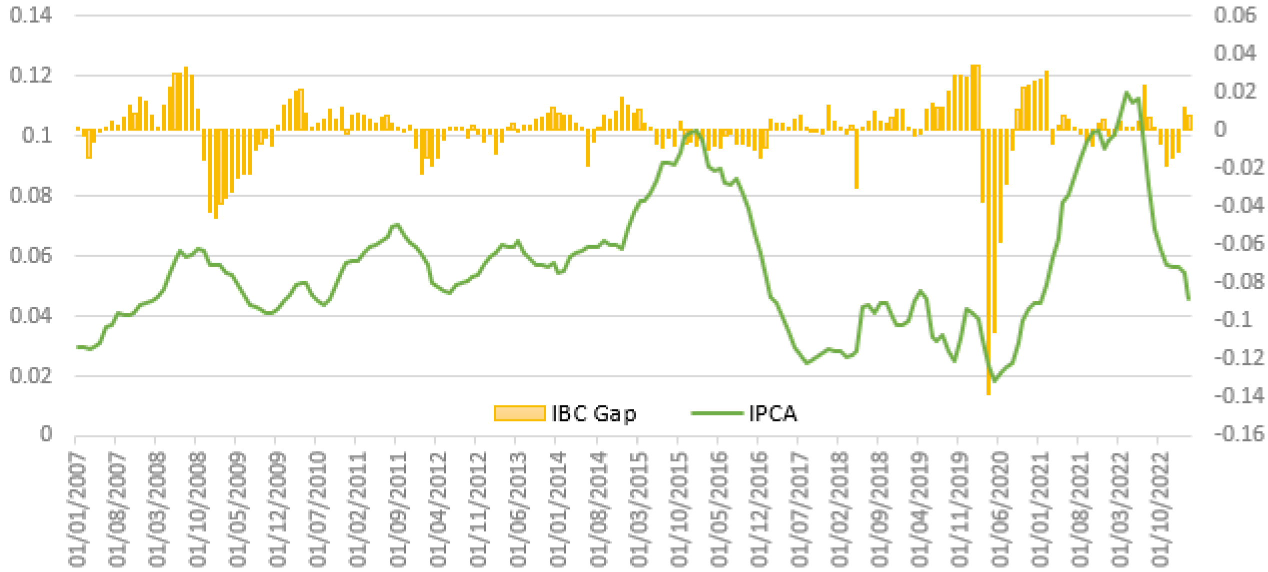
[8]Inflation is the difference of the log of the price index IPCA, provided by Ipeadata. The Output Gap is the deviation from the trend given by the HP filter, we use the log of the IBC-br from the Brazilian Central Bank for proxying the GDP to have monthly data.

2017-2019, the desinflation implemented increased the bond prices, and slightly positive GDP growth lead to increase in stock prices.

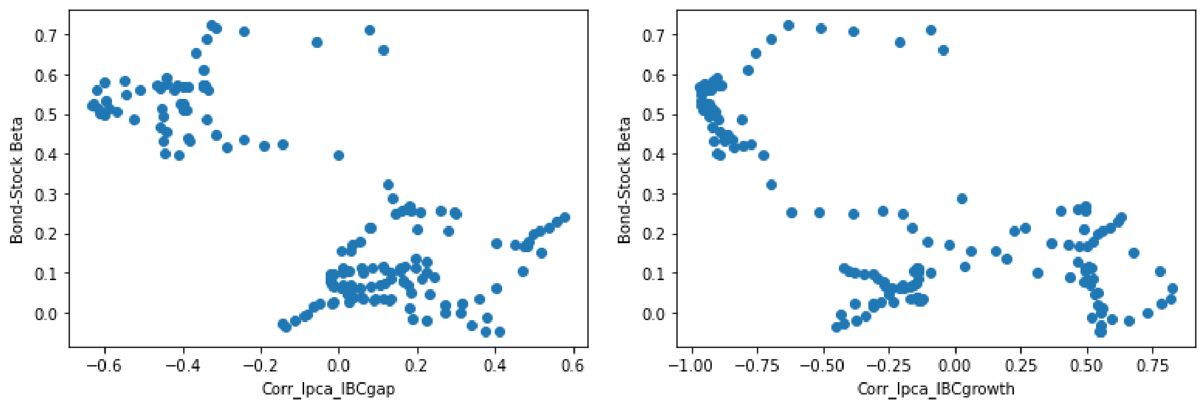
The monetary policy might have contributed as well. Recalling to Pflueger (2023), a quick-acting inflation-focused monetary policy works in favor of a positive beta. Carvalho and Muinhos (2023) estimate the Taylor Rule parameters of the Brazilian Central Bank (BCB). They estimate a high smoothing coefficient of 0.94 indicating an inertial policy rate. For the inflation weight parameter, they estimate it to be high from 2003 to 2010. From 2011 to 2016, it is smaller, indicating a dovish overly stimulative monetary policy. And from 2016 and on it continuously increased, but not as much as before 2011, showing again a more inflation focused monetary policy. The coefficient for the output gap remains quite stable, but higher around 2012-2013 and 2018-2019.

Taking this into analysis, the inflation-focused monetary policy from after 2016 might have contributed, in a first moment, for both bond and stock prices going down, reinforcing a positive bond beta. Then, with inflation under control and lower interest rates, both stock and bond prices turned up again.

Figure 9: Inflation and Output Gap



But one question that arises is why, in Brazil, procyclical inflation (positive inflation-output correlation) did not generate negative bond betas? The scatter plots below help to show that negative inflation-output correlation generated positive bond-stock beta, but positive inflation-output correlation did not generated negative betas. Around 2013-2014 the correlation was positive, the monetary policy was dovish but the bond-stock beta did not turned negative. More explicitly in the pandemic period of 2020-2022, the correlation became considerably positive and again the beta did not flipped sign. We argue that, differently from the USA, there is a behavior linked to country risk from the Brazilian economy, as an emerging market, that generates a more positive beta, a positive 'beta bias', in relation to a developed economy.



2.4 Excess Returns Regressions

Then, we analyze the associations of different variables with the 5-year bonds excess returns, and the stocks excess returns, separately. First, we present just regressions of one asset excess return into one explanatory variable at each time. Obviously, here we are forcing no causal interpretation. Ahead on section 2.6, we implement jointly correlations analysis with the assets excess returns and the explanatory variables all together. The S&P 500 is from Robert Shiller´s website[9]. The EMBI+Br was collected from Ipeadata, the Brazil 5-year CDS, the CRB Commodity Index and the U.S. Dollar Index (DXY) are from Thomson Reuters[10]. The foreign exchange rate and the foreign exchange orders movement for the financial sector are from the BCB. Current Account, Financial Account and Portfolio Investment from the Balance of Payments are from the BCB, and the Foreign Investment Flows is from CVM. The Global Factor (Miranda-Agrippino and Rey (2020)), from Miranda-Agrippino's website, is the first principal component of a data-set until 2019 of equity, commodity and corporate bond prices of all the world.

[9]https://shillerdata.com/

[10]The CDS series starts in Dec/2007.

Table 2: Bond Returns Regressions (Standardized)

Regression of 5-Year Nominal Bond Excess Return on:	$\hat{\beta}$	P-Value	R^2
Annual Log Diff IPCA	-0.5873	0.000	0.345
Annual Log Diff CDS	-0.5789	0.001	0.335
Annual Log Diff EMBI	-0.4477	0.007	0.200
Portfolio Investment	0.2418	0.131	0.058
FX Orders	-0.2247	0.120	0.050
Annual Log Diff FX	-0.1967	0.277	0.039
Financial Account	0.1535	0.363	0.024
Output Gap	-0.1134	0.257	0.013
Annual Log Diff Dollar Index	-0.0999	0.396	0.010
Annual Log Diff S&P500	-0.0713	0.549	0.005
Output Growth	-0.0231	0.871	0.001
Global Factor Agrippino-Rey	0.0245	0.877	0.001

Table 3: Stock Returns Regressions (Standardized)

Regression of Stocks Excess Return on:	$\hat{\beta}$	P-Value	R^2
Annual Log Diff CDS	-0.8443	0.000	0.713
Annual Log Diff EMBI	-0.7504	0.000	0.563
Annual Log Diff FX	-0.6389	0.000	0.408
Annual Log Diff S&P500	0.6193	0.000	0.383
Annual Log Diff IPCA	-0.5873	0.000	0.345
Annual Log Diff Dollar Index	-0.5100	0.002	0.260
Global Factor Agrippino-Rey	0.4568	0.010	0.209
Annual Log Diff IBC	0.4542	0.002	0.206
IBC Gap	0.3399	0.050	0.116
Financial Account	0.3387	0.023	0.115
FX Orders	0.0621	0.766	0.004
Portfolio Investment	-0.0002	0.999	0.000

Inflation was the variable with highest R^2 for explaining nominal bonds returns. The only other two variables that achieved a significant relation with bonds were the country risk variables CDS and EMBI.

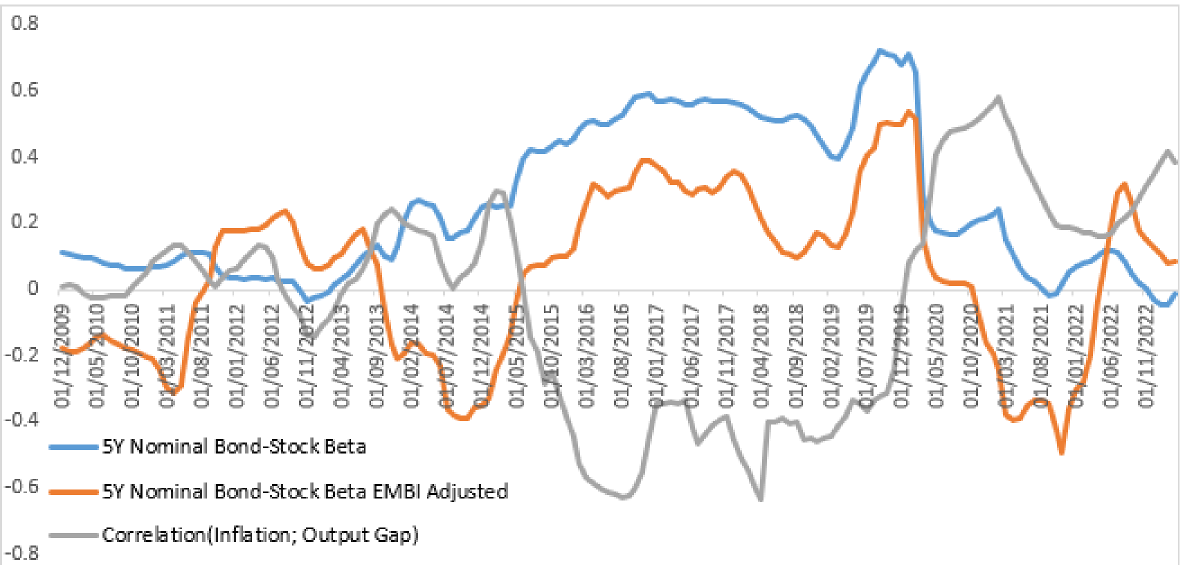
The output was relevant for explaining stocks returns, but was not the most relevant. Again the CDS and EMBI arised with higher explanatory power. They also showed up relevant for bond returns in the same direction, with negative coefficient. So an increase in the value of country risk was associated with lower bond and stock prices, forcing a positive bond-stock beta.

Volatile movements on country risk, then, makes the bond-stock beta more positive. A more positive bond-stock beta generates a higher term premium for holding a bond that performs bad on recessions. The country risk premium, then, contributes to higher bonds yields, not only by itself, but also through the 'positive beta effect'.

Furthermore, some international variables appeared correlated with stocks returns. As an emerging economy subject to the Global Financial Cycles, a strengthening of the Dollar (Dollar Index) is associated with lower Brazilian stock prices, and a rise on risky asset prices around the world (Miranda-Agrippino and Rey's Global Factor) spills positively to Brazilian assets too. An improvement in the Financial Account was also significant for stocks, but not for bonds, and other international flows variables do not seem to be relevant. Intriguingly, international prices movements were reflected on prices on Brazil, even the foreign exchange rate signaled these movements (R^2 = 4% for bonds and 41% for stocks), but they did not reflected as much on quantities at first sight.

There was procyclical inflation around 2013-2014 and 2020-22, but the bond-stock beta did not turned negative, and, on this period, the correlation of both bonds and stocks returns with country risk remained very negative (see Figures 27 and 28 in the Appendix). Taking this into consideration, we run again the rolling regressions of bonds returns into stocks returns but now controlling for the country risk variables[11], and plot it, in Figure 10, together with the standard bond-stock beta and the output-inflation correlation:

Figure 10: Nominal Bond-Stock Beta controlling for EMBI+Br



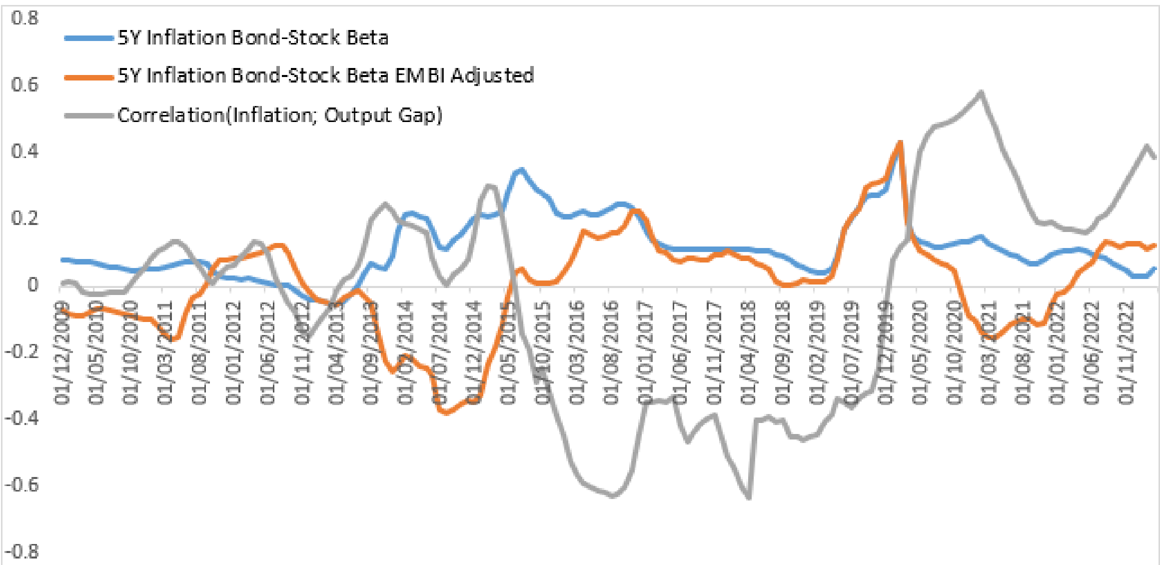
[11]The plots here are controlling for the EMBI+Br. Since the CDS starts just in Dec/2007, we plot controlling for it in the Appendix, Figure 29.

Cleaning the country risk effect, we reach more similar results found for the USA. The positive inflation-output correlation of 2013-2014, controlling for the EMBI or CDS, generated a negative bond-stock beta. That was a moment of presidential election and the country risk was very volatile making bonds and stocks prices go up and down together. Nevertheless, the economy was booming along with procyclical inflation. Cleaning from the country risk effect, the inflation would be driving bond prices down and the output growth was leading the stocks prices up, reaching the negative bond beta.

The pandemic was also a period of volatility for country risk. Controlling for this, we had in the beginning a bust, with inflation and output falling, then inflation and output growing again. This positive correlation generates negative bond beta.

For the real (inflation-indexed) bonds (Figure 11), the patterns are very similar. Interestingly, around 2017 to early 2019, the real bond-stock beta was positive but very small. By controlling for the EMBI (Figure 11) it gets even smaller, and by controlling for the CDS (Figure 29) it even becomes slightly negative. That was a period of desinflation. The monetary policy became more inflation focused, keeping higher real rates along with the decline of inflation, driving real bond prices down. It also was the moment of recuperation from the 2016 crisis, that output and stocks turned to rise again.

Figure 11: Inflation Bond-Stock Beta controlling for EMBI+Br



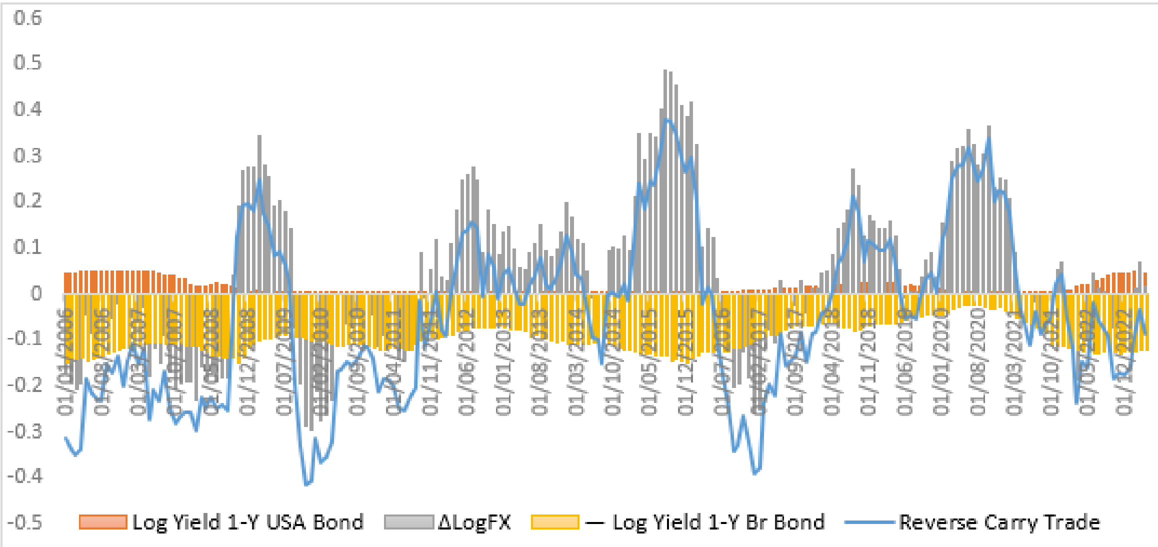
We also plotted on the Appendix using quarterly returns on quarterly frequency (Figure 30), and using annual returns on quarterly frequency (Figure 31), but the patters are clearer and more informative using annual returns on monthly frequency as in the main text above.

2.5 Portfolio Choice Implications

This paper shows that bonds and stocks returns have been positively correlated in Brazil, so bonds are risky in the sense that perform bad on high marginal utility recession moments. As argued, an important component for this relation is the behavior of Brazilian country risk. A question that arises is what alternative could serve as an effective hedge, considering the failure of bonds for this purpose?

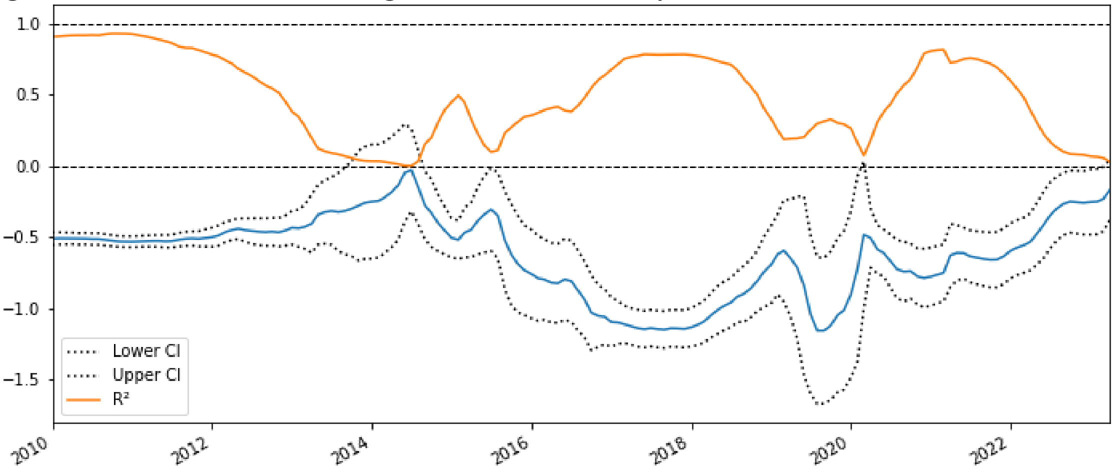
Exploring the country risk implications for bonds and stocks, the reverse carry trade works well hedging equity risk. The central reason for this is that, as in Figure 12, most of the variation of the reverse carry trade returns comes from movements on the exchange rate.

Figure 12: Reverse Carry Trade Components



The exchange rate, stock prices and country risk are all very correlated. When risk increases, driving stock prices down, the exchange rate depreciates, increasing the reverse carry trade returns. So country risk movements take stocks and reverse carry trade returns to opposite direction, making it a good hedge. By plotting the rolling regressions of the reverse carry trade into stocks excess returns, the beta is always negative, supporting the hedge property.

Figure 13: 3Y Window Rolling OLS Reverse Carry Trade into Stocks Excess Returns

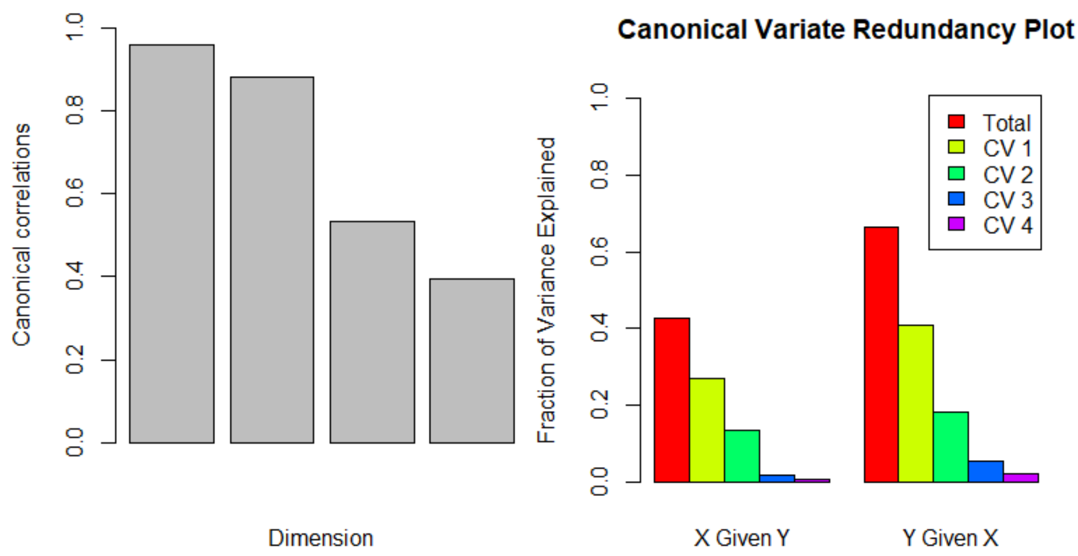


2.6 Canonical Correlation Analysis (CCA)

Now, for better analyzing the correlations of all these variables together in a single framework, we implemented a Canonical Correlation Analysis. We create 2 groups of standardized variables. "Returns" group (Y): Nominal Bond Excess Returns, Inflation Bonds Excess Returns, Stocks Excess Returns and Reverse Carry Trade. "Factors" group (X): Output, Inflation, Financial Account, Portfolio Investment, FX Orders from the financial sector, and the annual log difference of the EMBI, CDS and the Dollar Index.[12]

The CCA is an interesting method for analyzing correlations with and within 2 groups of variables. The procedure is to compute the correlation $\Sigma_{YY}^{-1}\Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}$ matrix of the variables of both groups, and their eigenvectors and eigenvalues. Similarly to a Principal Components Analysis, the eigenvalues are ordered and its respective eigenvector is used for building a new matrix of linear combinations of the original variables for each group. So, the algorithm gets the two groups of variables (2 matrices with the variables and its observations), and create a new matrix for each group. Each column of the new matrix is build with linear combinations of the variables of its group. The columns (called canonical variables) in the new matrix are built orthogonally (eigenvectors). Each column of the first group matrix is constructed to match the correspondent column of the second group matrix, with weights that maximize the correlation between the pair of corresponding columns of each matrix (this comes by ordering the eigenvalues).

[12]We do not include on this analysis the exchange rate because of the obvious correlation with the carry trade, and the Miranda-Agrippino and Rey Global Factor because it ends in 2019.



The first 2 pairs of canonical variables have a (canonical) correlation of 0.96 and 0.88 each, and for the last 2 pairs 0.53 and 0.40. The "factors" group (X) explains 67% of the variance of the "returns" group (Y). The first 2 Canonical Variables (CVs) embrace most of this variability. So I focus on the first two pairs that presented linear combinations of the variables with highest correlation. Roughly speaking, I look here for synthetic vectors that contains the information of the returns that are most related to the information of the macro factors.

Below on Table 4, I compute the correlations between the variables and the canonical variables. Analyzing the first pair of canonical variables, we have that on the returns group, the bonds (0.40, 0.40) and stocks (0.84) returns have positive loadings, with higher weight for stocks, and highly negative loading for the reverse carry trade (-0.87). On the first canonical variable from the "factors" group, the heaviest loadings are negative from the risk variables EMBI (-0.93) and CDS (-0.94). Inflation (-0.11) has also a negative but much smaller loading on this dimension. The output growth and gap (0.58 and 0.32) and the financial account (0.32) have positive considerable loadings. So on this dimension, the stocks and the reverse carry trade are the most affected variables, and the country risk variables are their strongest drivers. Higher country risk decreases bonds and stock prices. But also the movements in the output and financial account are considerable. Financial account or output increases with bonds and stocks prices. The Dollar Index has also a negative correlation (-0.45) with the assets returns, showing that moments of the strengthening of the Dollar are associated with fall in stocks and bonds prices (increase in the Brazilian bonds rates).

On the second dimension of canonical variables, by the returns group, all variables have the same loadings negative sign, with more weights for the nominal and inflation bonds (-0.77 and -0.45). On the factors group, the most relevant variable is now the inflation (0.84), arising as the most relevant variable for explaining the movements on bonds.

So on this dimension inflation is highly negative correlated with bonds. Furthermore, portfolio investment (-0.47) and FX orders (0.63) now arise with more relevance on this dimension. The signs also have economic interpretation. Portfolio investment in Brazil and the bonds returns have negative sign, so they are positively correlated. FX orders and bonds returns have opposite signs, so lower demand for foreign exchange is associated with higher bond prices; analogously, higher demand for Brazilian *reais* is associated with higher bond prices. So on this second dimension that gave more weight for bonds and inflation changes, foreign flows received also considerable weights.

Inflation is the main driver of bond returns. Country risk variables are the most important for stocks returns, but also is a relevant channel for bonds returns. The literature points the output as the relevant variable for stocks representing the real economy, but in Brazil this variable is relevant but not as much as risk perception. Furthermore, international flows variables as financial account, FX orders and portfolio investment appear to comove with stocks and bonds by this Canonical Correlation Analysis. Then, in Brazil, the correlation of inflation and output is a determinant of bond and stock co-movements. But new variables, as country risk present strong effect on the stocks and bonds in the same direction.

Table 4: CCA

Weighting of each factor for the factors Canonical Variables (CVs)				
	Factors CV1	Factors CV2	Factors CV3	Factors CV4
IBC Gap	0.3179375	0.09279283	-0.44841086	-0.088535087
Output Growth	0.5801595	0.27230173	-0.29236118	0.170145522
IPCA	-0.1062532	0.84398694	-0.03953313	-0.206814060
Annual Diff Log CDS	-0.9380998	0.32443463	0.00867503	0.030163526
Dollar Index	-0.4536806	0.03573905	0.27493925	-0.136101727
Financial Account	0.3183480	-0.12645543	-0.20915276	0.009349445
Portfolio Investment	-0.1991482	-0.47096701	-0.06913548	-0.384528911
Annual Diff Log EMBI	-0.9288534	0.12057968	-0.07563586	-0.241248853
FX Orders	0.2949593	0.63421327	0.28970659	0.255360811

Weighting of each return for the returns CVs				
	Returns CV1	Returns CV2	Returns CV3	Returns CV4
Excess Return Nominal Bond	0.3986532	-0.7722427	0.4845653	-0.09956555
Excess Return Inflation Bond	0.4036196	-0.4514611	0.5432085	-0.58154853
Excess Return Stocks	0.8367642	-0.3054112	-0.4410691	-0.10958020
Reverse Carry Trade	-0.8693078	-0.1803647	-0.1879379	-0.42006183

3 Term Structure Model

On section 2 we provided empirical evidence of positive correlation between bond returns and stocks returns. We also argued that these asset prices are correlated with country risk movements and whether inflation is procyclical or countercyclical. To account for bond risk premia in Brazil given by this positive correlation, we use the multifactor term structure model from Campbell, Sunderam, and Viceira (2017). The main question this model was built to answer is how a changing covariance of bonds and stocks affects the bond prices. The model relies on a constant variance for the Stochastic Discount Factor (SDF) in order to isolate the effects of changing bond-stock covariance and aims to capture the low-frequency movements of this covariance.[13]

We must point that, in 2019, the US stock market capitalization to GDP was 158%[14] and the US treasury bond market to GDP was 80%[15]. In Brazil, the stock market capitalization in 2019 was 63%[16] of the GDP and the government bond market was 58%[17]. The stock market in the US is larger and more developed than in Brazil. So we must enter the caveat that the correlation of bonds with the stock market in the US may more clearly inform the risk premium on bonds due to the covariance with the state of the economy than in Brazil. However, taking Brazil as an emerging economy, we consider its stock market as sizeable enough for representing the state of the Brazilian economy and, as argued, the country risk effect boosts this correlation, making the risk premium for bonds covarying with stocks even more positive.

3.1 Model Specifications

The dynamics of the model is given by the following processes:

[13] We present here the specifications from the model in section 3.1 and explain the main dynamics in sections 3.3 and 3.4, for further details see Campbell, Sunderam, and Viceira (2017).

[14] Source: World Bank

[15] Market Value of Marketable Treasury Debt/GDP. Source: Federal Reserve Bank of Dallas

[16] Source: World Bank

[17] Federal Public Debt held by the public/GDP. Source: National Treasury of Brazil

$$-m_{t+1} = x_t + \frac{\sigma_m^2}{2} + \varepsilon_{m,t+1} \tag{1}$$

$$x_{t+1} = \mu_x(1-\phi_x) + \phi_x x_t + \psi_t \varepsilon_{x,t+1} + \varepsilon_{X,t+1} \tag{2}$$

$$\psi_{t+1} = \mu_\psi(1-\phi_\psi) + \phi_\psi \psi_t + \varepsilon_{\psi,t+1} \tag{3}$$

$$\pi_{t+1} = \lambda_t + \xi_t + \frac{\sigma_\pi^2}{2}\psi_t^2 + \psi_t \varepsilon_{\pi,t+1} \tag{4}$$

$$\lambda_{t+1} = \lambda_t + \varepsilon_{\Lambda,t+1} \tag{5}$$

$$\xi_{t+1} = \phi_\xi \xi_t + \psi_t \varepsilon_{\xi,t+1} \tag{6}$$

m_t is the log of the real stochastic discount factor. The short-term log real interest rate x_t follows an AR(1) process and is hit by both homoscedastic and heteroscedastic shocks. The heteroscedasticity of the real rate comes from ψ_t. λ_t and ξ_t are the permanent and transitory components of expected inflation, respectively. Both of them follow AR(1) processes, and the volatility of the expected transitory inflation is governed by ψ_t also. Besides the expected inflation given by the sum of these two components, the realized log inflation π_{t+1} is subject to shocks $\varepsilon_{\pi,t+1}$ weighted by ψ_t. And ψ_t is a common factor state variable that accounts for the volatility of the real interest rate and the inflation, and also their covariance with the SDF, representing then the risk premia. The model runs in quarterly frequency.

All the shocks $\varepsilon_{m,t+1}$, $\varepsilon_{x,t+1}$, $\varepsilon_{X,t+1}$, $\varepsilon_{\psi,t+1}$, $\varepsilon_{\pi,t+1}$, $\varepsilon_{\Lambda,t+1}$ and $\varepsilon_{\xi,t+1}$ are zero-mean normally distributed with a constant variance-covariance matrix. So the volatility of shocks to the real interest rate and to the inflation will change according to the state variable ψ_t multiplying them. Also, the only unconstrained covariances parameters are the covariances with the shock to the SDF (σ_{xm}, σ_{Xm}, $\sigma_{\Delta m}$, $\sigma_{\xi m}$, $\sigma_{\psi m}$, $\sigma_{\pi m}$), and $\sigma_{\xi\pi}$, all the other covariances between shocks are set to zero. This is for the model account the risk premia of the variables covarying with the SDF.

Real and nominal bond prices are given respectively by:

$$p_{n,t} = A_n + B_n.x_t + B_{\psi,n}.\psi_t + C_{\psi,n}.\psi_t^2 \tag{7}$$

$$p_{n,t}^{\$} = A_n^{\$} + B_n^{\$}.x_t + B_{\lambda,n}^{\$}.\lambda_t + B_{\xi,n}^{\$}.\xi_t + B_{\psi,n}^{\$}.\psi_t + C_{\psi,n}^{\$}.\psi_t^2 \tag{8}$$

One can solve recursively for the parameters of the real bond prices using the standard pricing equation:

$$P_{n,t} = E_t\left[\exp\left\{p_{n-1,t+1} + m_{t+1}\right\}\right] = E_t\left[\exp\left\{\begin{array}{c} A_{n-1} + B_{x,n-1}x_{t+1} \\ +B_{\psi,n-1}\psi_{t+1} - x_t - \frac{1}{2}\sigma_m^2 - \varepsilon_{m,t+1} \end{array}\right\}\right] \tag{9}$$

Also for the nominal bond prices:

$$
\begin{aligned}
P^{\$}_{n,t} =& E_t \left[\exp \left\{ p^{\$}_{n-1,t+1} + m_{t+1} - \pi_{t+1} \right\} \right] \\
=& E_t \left[\exp \left\{ \begin{array}{c} A^{\$}_{n-1} + B^{\$}_{x,n-1} x_{t+1} + B^{\$}_{\lambda,n-1} \lambda_{t+1} + B^{\$}_{\xi,n-1} \xi_{t+1} + B^{\$}_{\psi,n-1} \psi_{t+1} \\ + C^{\$}_{\psi,n-1} \psi^2_{t+1} - x_t - \frac{1}{2} \sigma^2_m - \varepsilon_{m,t+1} - \lambda_t - \xi_t - \frac{1}{2} \psi^2_t \sigma^2_\pi - \psi_t \varepsilon_{\pi,t+1} \end{array} \right\} \right]
\end{aligned} \tag{10}
$$

And by knowing that the 1-period real and nominal bond prices are respectively given by:

$$P_{1,t} = E_t \left[\exp \left\{ m_{t+1} \right\} \right] = \exp \left\{ -x_t \right\} \tag{11}$$

$$P^{\$}_{1,t} = E_t \left[\exp \left\{ m_{t+1} - \pi_{t+1} \right\} \right] = \exp \left\{ -x_t - \lambda_t - \xi_t + z_t \psi_t \sigma_{m\psi} \right\} \tag{12}$$

The details of the derivation can be found at the online appendix of Campbell, Sunderam, and Viceira (2017). And the shocks to stocks returns are linear combination of shocks to the real interest rate and shocks to the log SDF:

$$r_{e,t+1} - E_t r_{e,t+1} = \beta_{ex} \varepsilon_{x,t+1} + \beta_{eX} \varepsilon_{X,t+1} + \beta_{em} \varepsilon_{m,t+1} + \varepsilon_{e,t+1} \tag{13}$$

This term structure model is estimated with the unscented Kalman filter of Julier and Uhlmann (1997).

3.2 Data for Model Estimation

The model uses 12 measurement equations for the Kalman filter algorithm. We estimate the model with the closest Brazilian data to the US data used in the Campbell, Sunderam, and Viceira (2017) paper. For nominal and real bonds, they use the zero-coupon nominal bonds and Treasury inflation protected securities (TIPS) from Gürkaynak, Sack and Wright (2007,2010) database. We use the ID x fixed rate swap (*swap DI x pré*) provided by B3 – Brazil Stock Exchange and Over-the-Counter Market - through the *rb3* R package for the nominal bonds yields on constant maturities of 3 months, 1 year, 3 years and 10 years starting in September 2005 (the first four measurement equations). For the 10-year constant maturity real bonds, we use the yields of NTN-Bs provided by the *Tesouro Nacional* (the National Treasury of Brazil) starting in January 2005 (the fifth measurement equation). They provide daily data of these real bonds yields traded in the bond market but, since the Brazilian Treasury does not very frequently issue 10-year inflation-indexed bonds, the maturity is not always exactly of 10 years and varies from 7 to 11 years. There is no liquid market for swaps of inflation-indexed bonds as for the nominal bonds, and ANBIMA only publishes the IDkA indices but does not publish the yields, so this is the closest measure of 10-year real bonds yields available, plotted in Figure 32 in the Appendix.

For the observed price index they use the CPI and we use the IPCA provided by Ipeadata. They also use data on the median forecast of GDP deflator inflation one quarter ahead from the Survey of Professional Forecasters for inflation expectations. We use the data on the median forecast of IPCA one month ahead from the Focus Survey and sum for having the three months ahead expectations, starting in September 2003 provided by the Brazilian Central Bank through the *rbcb* R package (sixth and seventh measurement equations).

For equity returns they use CRSP value-weighted index comprising the stocks traded in the NYSE, AMEX, and NASDAQ, we use the Ibovespa provided by Thomson Reuters (eighth measurement).

The estimation period starts in the last quarter of 2005 and goes until the third quarter of 2023 (end of September 2023). Since we have daily data for bonds and stocks prices, the model gets the last yield value of each quarter, and quarterly covariances and variances are calculated from all the days of each quarter. The variances of the real and nominal bonds returns and their covariances with stocks returns are the last four measurement equations.

3.3 Parameter Estimates

The estimates are in table 3.3. The persistence parameters estimates are slightly higher than for the US case, and are precisely estimated with standard errors of 0.000.

The volatility of shocks to the SDF are high and precisely estimated indicating considerable movements on this variable. Looking for the real interest rate shocks, the volatility of the homoscedastic shock σ_X is very low and non-significant statistically, and the volatility of the shock σ_x linked to movements in the risk premium ψ_t is high and precisely estimated with a low standard error, indicating this is shock is more relevant for fitting the data. The standard deviation for the parameter of the volatility of the permanent component of inflation σ_Λ is undefined since it is a unit-root process. Similarly to the US estimation, the volatility of ψ_t is small indicating its characteristic of capturing low-frequency movements in risk premia over time.

The loadings βs of stocks returns on the shocks to the SDF and to the real interest rate are significantly positive, implying a positive equity premium.

Now we turn to analyze the correlations of the different shocks with the shock to the SDF. These correlations are crucial in the model for understanding how the state variable ψ_t captures the risk premia. The correlation ρ_{xm} of the shocks to the SDF with the shocks to the real interest rate weighted by the ψ_t variable has a statistically significant estimate of -0.19. For the US, this variable was estimated with a considerably higher standard deviation, so for the Brazilian case the movements it is representing are more relevant.

With this negative correlation estimated, recall the processes 1 and 2 for m_{t+1} and x_{t+1}, and suppose a negative shock for $\varepsilon_{m,t+1}$ and positive shock for $\varepsilon_{x,t+1}$. The negative shock will increase m_{t+1}, meaning the economy is diving into a bad moment of recession with stock prices falling. But the effect of the positive shock into the real interest rate will depend on the sign of ψ_t[18]. If ψ_t is positive, real rates are increasing in this bad moment and real bond prices are falling, indicating that real bonds are risky for having its price comoving negatively with the SDF and imply positive risk premium for holding them. This risk premium is accounted by ψ_t. Conversely, if in a recession the real rates fall, ψ_t will be negative to fit this comovement, and real bonds will input a negative risk premium for working as hedge assets. So ψ_t is identified over time by the volatility of the real rate and its covariance with the SDF. ρ_{Xm} is estimated to be small with a very high standard deviation. This correlation is non-significant for the US as well, and Campbell, Sunderam, and Viceira (2017) says that: "This implies that bond risk premia are not just linear in ψ_t but almost proportional to it". Also similarly to the US, $\rho_{\Lambda m}$ is small and non-significant.

$\rho_{\xi m}$ is significantly estimated to be -0.703. Following the same reasoning, if the expected transitory inflation is countercyclical, ψ_t is positive and is accounting for the risk premium in the nominal term structure for having higher expected inflation, higher nominal bond yields, and thus lower nominal bond prices in recession moments.

Estimates for the US for $\rho_{\xi\pi}$ and $\rho_{\pi m}$ are insignificant, but our estimation for Brazil shows that these parameters are precisely estimated with standard errors of 0.000, indicating relevance for explaining the bond risk premia. This is in line with earlier Brazilian evidences from Lowenkron and Garcia (2007). They show that short run inflation surprises (realized - expected inflation) affected positively inflation expectations, because of indexation and lack of credibility of the Central Bank. A significantly positive estimate for $\rho_{\xi\pi}$ captures exactly this. They argue that: "If the cause of the effect of short run inflation suprise on 12 month inflation expectation is solely indexation, there is no reason for an increase in the uncertainty when the econnomy is hit by a positive inflation shock: we know that prices will be re-adjusted in the future with certainty. However, if there is lack of credibility on monetary policy, there will be an increase in the uncertainty in inflation itself. This will be captured by the inflation risk premium." We provide here updated evidences that this continues to be relevant for inflation risk premium on bonds, and will also be accounted in ψ_t through the significantly negative estimate for $\rho_{\pi m}$.

Finally, the estimate for $\rho_{\psi m}$ is smaller than for the US, but still significantly positive.

[18] ψ_t can be positive or negative. Campbell, Sunderam, and Viceira (2017) estimation for the US shows that ψ_t was positive during the 1980's and 90's, and turned negative in the 2000's, fitting the flip of the sign of the bond to stock beta in the US.

Parameter	Estimate	Std Err
μ_x x 10^3	17.812	0.981
μ_ψ x 10^3	15.644	1.120
ϕ_x	0.983	0.000
ϕ_ξ	0.894	0.000
ϕ_ψ	0.923	0.000
σ_m x 10^2	26.251	0.003
σ_x x 10^1	1.275	0.006
σ_X x 10^4	0.004	0.059
σ_Λ x 10^4	20.145	0.988
σ_ξ x 10^1	2.605	0.000
σ_π	1.000	
σ_ψ x 10^3	0.670	0.105
β_{em} x 10^1	1.536	0.003
β_{ex} x 10^2	2.455	0.009
β_{eX}	0.113	0.000
ρ_{xm}	-0.190	0.009
ρ_{Xm} x 10^3	-0.839	13.212
$\rho_{\Lambda m}$ x 10^4	1.603	3.391
$\rho_{\xi m}$	-0.703	0.001
$\rho_{\xi\pi}$	0.014	0.000
$\rho_{\pi m}$ x 10^2	-3.460	0.000
$\rho_{\psi m}$	0.024	0.004

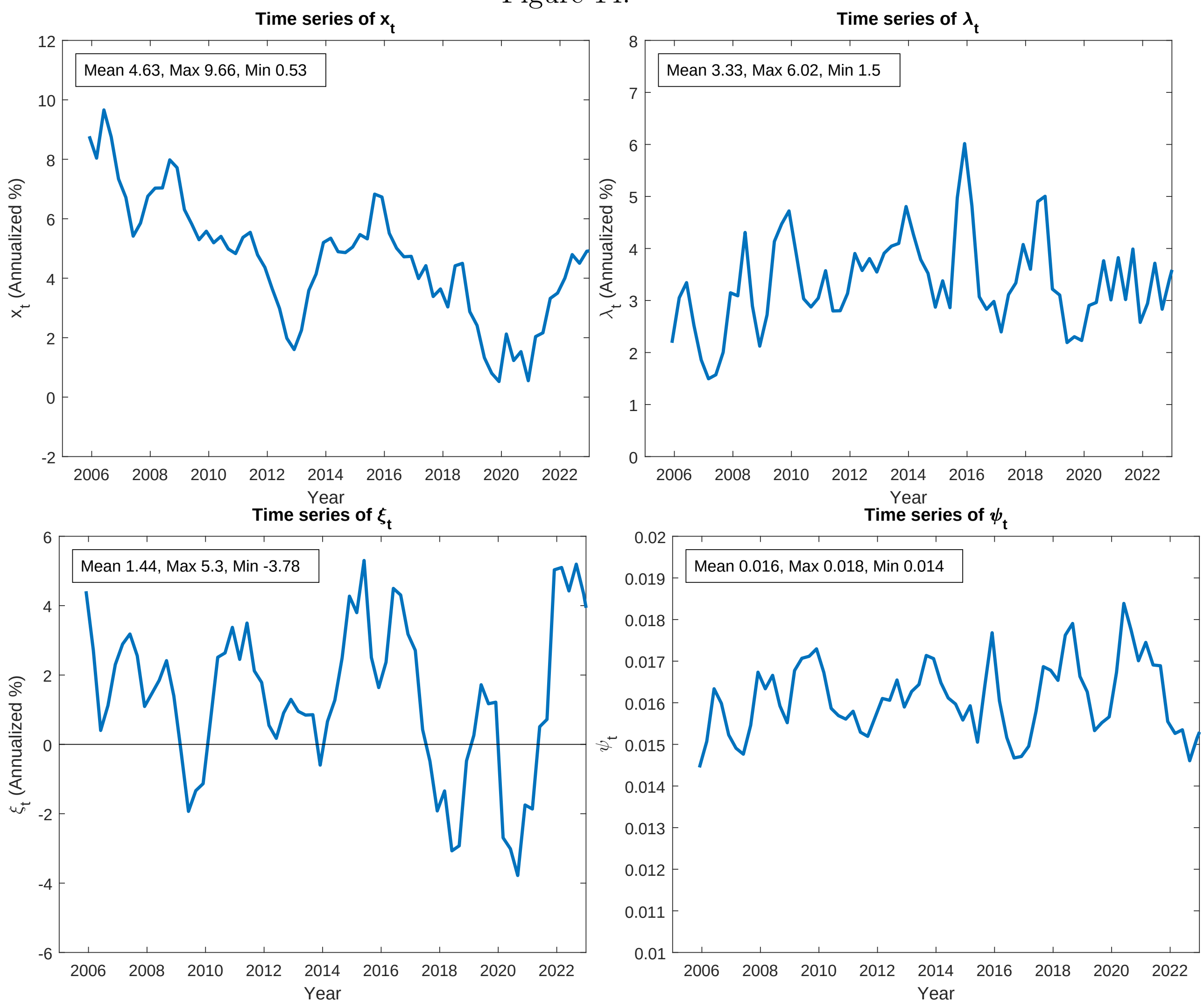
Table 5: Campbell, Sunderam, and Viceira (2017) Model Parameter Estimates for Brazil. Note: This table reports parameter estimates and standard errors, estimated using maximum likelihood and an unscented Kalman filter. The first block reports the means of the real rate x_t and the state variable ψ_t which governs the time variation in both the volatility of inflation and the real rate and their covariance with the SDF, m_t; the second block reports persistence parameters for these two state variables and ξ_t, the transitory component of expected inflation; the third block reports the volatilities of shocks; the fourth block reports the loadings of equities on shocks; and the fifth block reports the correlations between the shocks. σ_π is not estimated but normalized to 1.

3.4 Model Results

Below (Figure 14) we plot the time series of the state variables. The real interest rate x_t was most of the time countercyclical, working in favor of higher risk premium for bonds. It presented for example spikes during the global financial crisis and the 2015-16 Brazilian crisis, and big falls during 2012-13 (along with GDP growth) and in 2020 (along with

GDP fall). The permanent component of expected inflation λ_t kept around 3.3% with no clear consistent trends of rise or fall. The transitory component of expected inflation ξ_t was high during the global financial crisis, and the periods of 2010-11, 2014-16 and 2021-23, indicating that agents expected inflation to be temporarily high during them. Around 2009, 2018 and 2020 agents expected the inflation to be temporarily too low, generating lower yields for short-term bonds. ψ_t, which captures the risk premia from bonds volatility and covariance with stocks, showed up positive during the entire sample period ranging from 0.014 to 0.018.[19]

Figure 14:



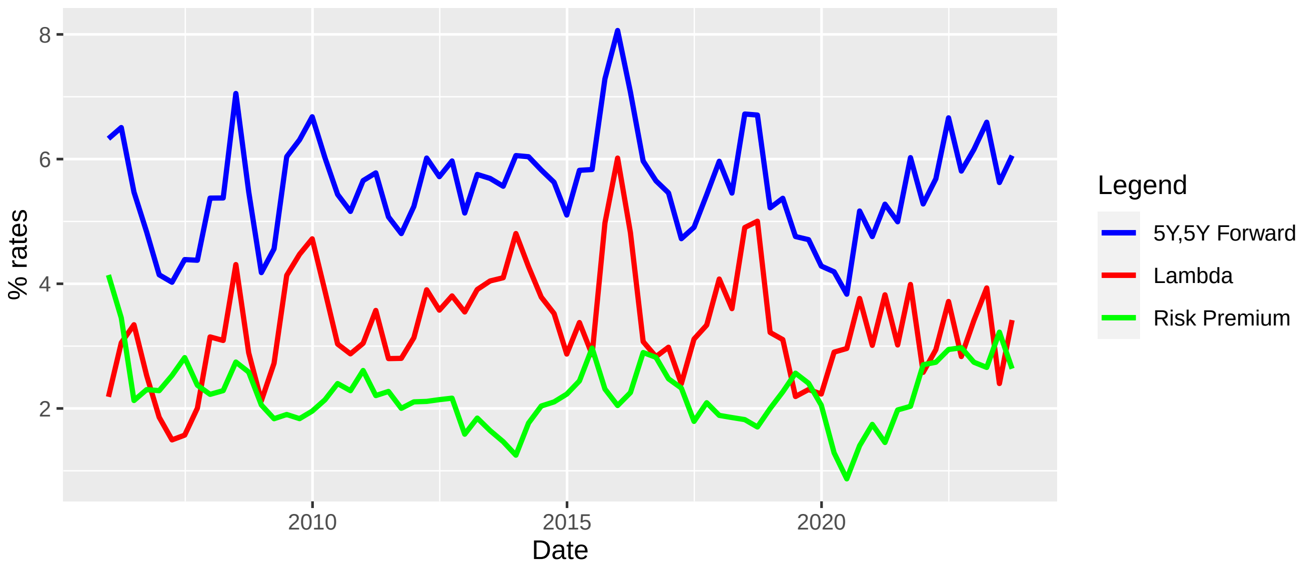
In order to analyze the capacity of the model to decompose the Brazilian inflation expectations, we compare the permanent component of the expected inflation with the 5-Year, 5-Year forward inflation expectation rate[20], see Figure 15. The 5-Year, 5-Year forward is a measure of the expected inflation starting from 5 years ahead from today,

[19]For comparison, ψ_t in the US ranges from 0.020 in early 1980's to -0.013 in the late 2000's.

[20]We calculate it with the same data for the nominal and real interest rates used for the model, further details on the methodology can be found at https://fred.stlouisfed.org/series/T5YIFR.

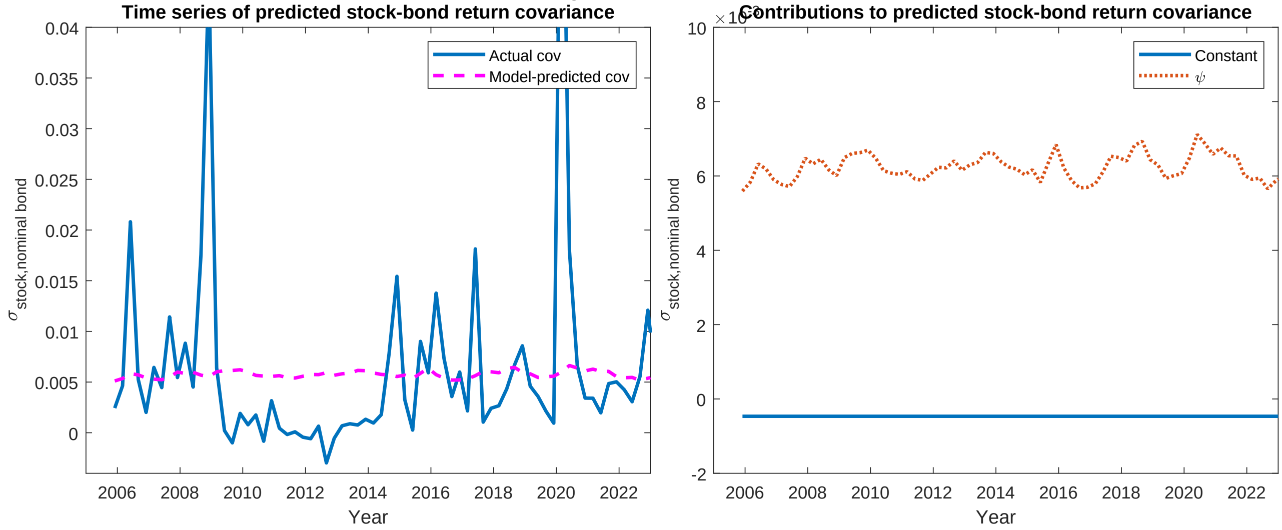
for the next 5 following years. It is calculated by the ratio from the 10 year and the 5 year breakeven inflation. It is capturing the expected inflation from the market from 5 to 10 years ahead, so this is capturing a long-term non-transitory inflation expectation. This is the reason why it works as a proxy for the permanent component of the expected inflation. The Figure 15 below shows that indeed the λ_t from the model was able to capture the most of the variations of this variable. Since the 5-year, 5-year forward is given by the breakeven inflation from the long-term bonds, it also incorporates the risk premium component of the long-term bond in addition to the inflation expectation. We also plot in green the risk premium given by the difference between both. For clarification, this risk premium given by the difference from the 5-year, 5-year forward and the Lambda is different from the risk premium given by the ψ_t from the model. The risk premium on the Figure 15 is the required for holding a 10-year nominal bond over its last 5 years. The ψ_t from the model accounts just for the share of this risk premium given by holding a bond whose components have larger or smaller volatility and that comoves negatively or positively with the SDF. For this, a model is necessary.

Figure 15: 5-Year, 5-Year Forward and the λ_t



Below (Figure 16) we plot the quarterly realized covariance of stocks and bonds returns with daily data for each quarter, together with the prediction of the model. Similarly to the US estimation from Campbell, Sunderam, and Viceira (2017), the model captures, through ψ_t, the low frequency trend of this covariance to be positive, missing high frequency movements.

Figure 16:



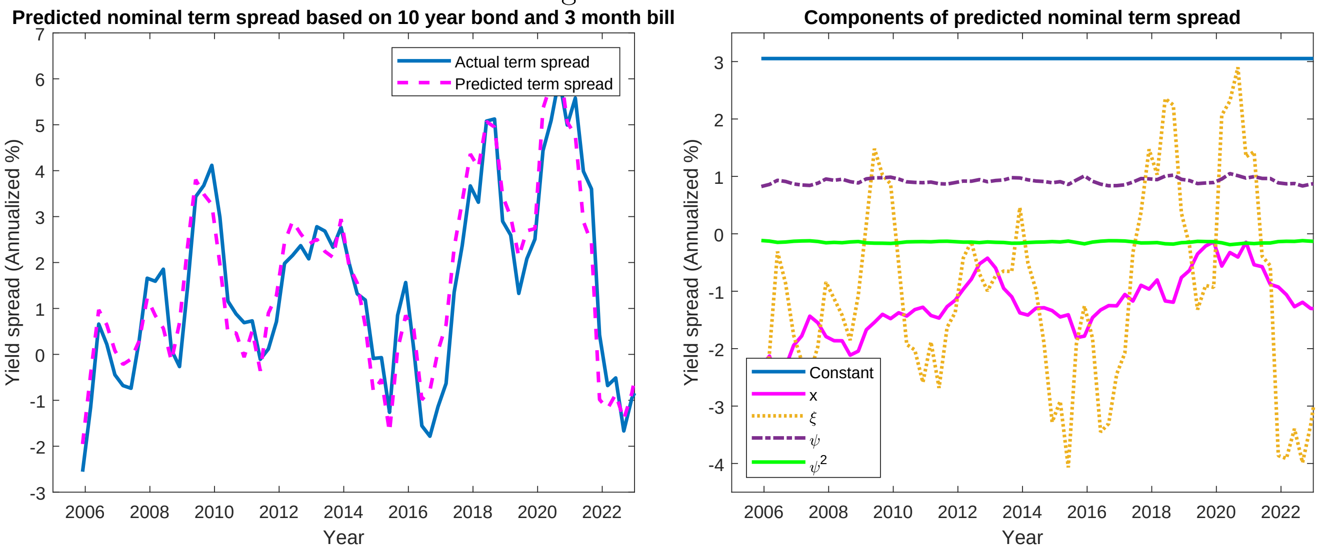
Below (Figure 17) we plot the yield spread between the 10-year nominal bond and the 3-month bill. On the left, the figure shows that the model fitted well the term spread. On the right, the figure shows the model components for this spread. Similarly to the US, changes in the real rate and in the expected inflation were relevant for changes in the term spread. It is visible, for example, around 2015-16 and 2022-23 when inflation was expected to be temporarily high, increasing short-term yields in relation to long-term yields, driving down the term spread. And around 2018 and 2020, when inflation was expected to be temporarily low making the opposite happen.

A big difference in relation to the US is the role of ψ_t for this spread, keeping around 0 there, just with a tiny rise in the early 1980's and a tiny fall in the late 2000's. As Campbell, Sunderam, and Viceira (2017) point out: "The state variable ψ_t and its square have only very small effects on the spread. Even though ψ_t determines the risk premium, the variation in the risk premium is neither large nor persistent enough to be a dominant influence on the yield spread in our model." Our estimation for Brazil shows that the risk premium component of the yield spread summarized by ψ_t corresponded to around 1 positive percentage point during the sample period.

By decomposing the components of the long-term bond with the model, and estimating a common factor that accounts for the volatility of these components and their covariance with the SDF, we accounted for a considerably positive risk premium. Investors require higher yields for risky long-term assets that are volatile and comove positively with the economy. We accounted that 1 percentage point of the term premium comes from this risky dynamics of the Brazilian bonds.

Movements on Brazilian country risk, affecting the volatilities of the interest rates, inflation expectations, and how they covary with the SDF considerably affected the choice of investors to hold long-term bonds. We conclude that this riskier dynamics of the bond components generated larger and more persistent variation in the risk premium.

Figure 17:



4 Term Structure Model with Country Risk

We propose here a modification of Campbell, Sunderam, and Viceira (2017)'s model including a country risk component. We have argued on section 2 that movements on country risk are relevant for the volatility and comovements between bonds and stocks prices. We provided empirical evidence that country risk, bonds and stocks prices are significantly correlated. The asset pricing theory states that this 'risky' comovement with the state of the economy implies a positive risk premium for bonds. So we quantify how much the components of long-term bonds contributed for this on section 3 with a term structure model. Now on this section, we implement a modification trying to explicit the country risk component on the model.

4.1 Modification Proposed

We introduce a new state variable δ_t to be the country risk in the model, following an AR(1) process:

$$\delta_{t+1} = \mu_\delta(1 - \phi_\delta) + \phi_\delta \delta_t + \psi_t \varepsilon_{\delta,t+1} + \varepsilon_{\Delta,t+1}. \tag{14}$$

We also use a new measurement equation for the unscented Kalman filter algorithm relating the δ_t with the EMBI+Br. We introduce the δ_t on the price equations of real and nominal bonds, respectively, as:

$$P_{n,t} = \exp\left\{A_n + B_{x,n}x_t + B_{\delta,n}\delta_t + B_{\psi,n}\psi_t + C_{\psi,n}\psi_t^2\right\}. \tag{15}$$

$$P^{\$}_{n,t} = \exp\left\{A^{\$}_n + B^{\$}_{x,n}x_t + B^{\$}_{\delta,n}\delta_t + B^{\$}_{\lambda,n}\lambda_t + B^{\$}_{\xi,n}\xi_t + B_{\psi,n}\psi_t + C^{\$}_{\psi,n}\psi^2_t\right\}. \tag{16}$$

In order to be able to solve recursively for the new coefficients $B_{\delta,n}$ and $B^{\$}_{\delta,n}$, to keep the variables with an economic interpretation, and to fit the information given by the EMBI+Br with the new measurement equation, we model the δ_t as a component of the short-term real interest rate of the economy. Considering Brazil as an emerging economy, the Brazilian short-term rates have an extra risk premium component in relation to the US treasury bonds in order to avoid capital outflows and large currency depreciations. This interest rate differential is obviously related to the perception of Brazilian country risk. The EMBI+Br captures this difference between interest rates across both countries. With the price equations 15 and 16, this country risk component also propagates for the long-term rates. Then we have that the SDF becomes:

$$-m_{t+1} = x_t + \delta_t + \frac{\sigma^2_m}{2} + \varepsilon_{m,t+1} \tag{17}$$

The short-term bond prices:

$$P_{1,t} = E_t\left[\exp\left\{m_{t+1}\right\}\right] = \exp\left\{-x_t - \delta_t\right\} \tag{18}$$

$$P^{\$}_{1,t} = E_t\left[\exp\left\{m_{t+1} - \pi_{t+1}\right\}\right] = \exp\left\{-x_t - \delta_t - \lambda_t - \xi_t + z_t\psi_t\sigma_{m\psi}\right\} \tag{19}$$

Solving recursively with the standard pricing equation for real and nominal bonds:

$$P_{n,t} = E_t\left[\exp\left\{p_{n-1,t+1} + m_{t+1}\right\}\right] = E_t\left[\exp\left\{\begin{array}{c} A_{n-1} + B_{x,n-1}x_{t+1} + B_{\delta,n-1}\delta_{t+1} \\ +B_{\psi,n-1}\psi_{t+1} - x_t - \delta_t - \frac{1}{2}\sigma^2_m - \varepsilon_{m,t+1} \end{array}\right\}\right] \tag{20}$$

$$\begin{aligned} P^{\$}_{n,t} =& E_t\left[\exp\left\{p^{\$}_{n-1,t+1} + m_{t+1} - \pi_{t+1}\right\}\right] \\ =& E_t\left[\exp\left\{\begin{array}{c} A^{\$}_{n-1} + B^{\$}_{x,n-1}x_{t+1} + B^{\$}_{\delta,n-1}\delta_{t+1} + B^{\$}_{\lambda,n-1}\lambda_{t+1} + B^{\$}_{\xi,n-1}\xi_{t+1} + B^{\$}_{\psi,n-1}\psi_{t+1} \\ +C^{\$}_{\psi,n-1}\psi^2_{t+1} - x_t - \delta_t - \frac{1}{2}\sigma^2_m - \varepsilon_{m,t+1} - \lambda_t - \xi_t - \frac{1}{2}\psi^2_t\sigma^2_\pi - \psi_t\varepsilon_{\pi,t+1} \end{array}\right\}\right] \end{aligned} \tag{21}$$

We provide details on the model derivation in the Appendix B.

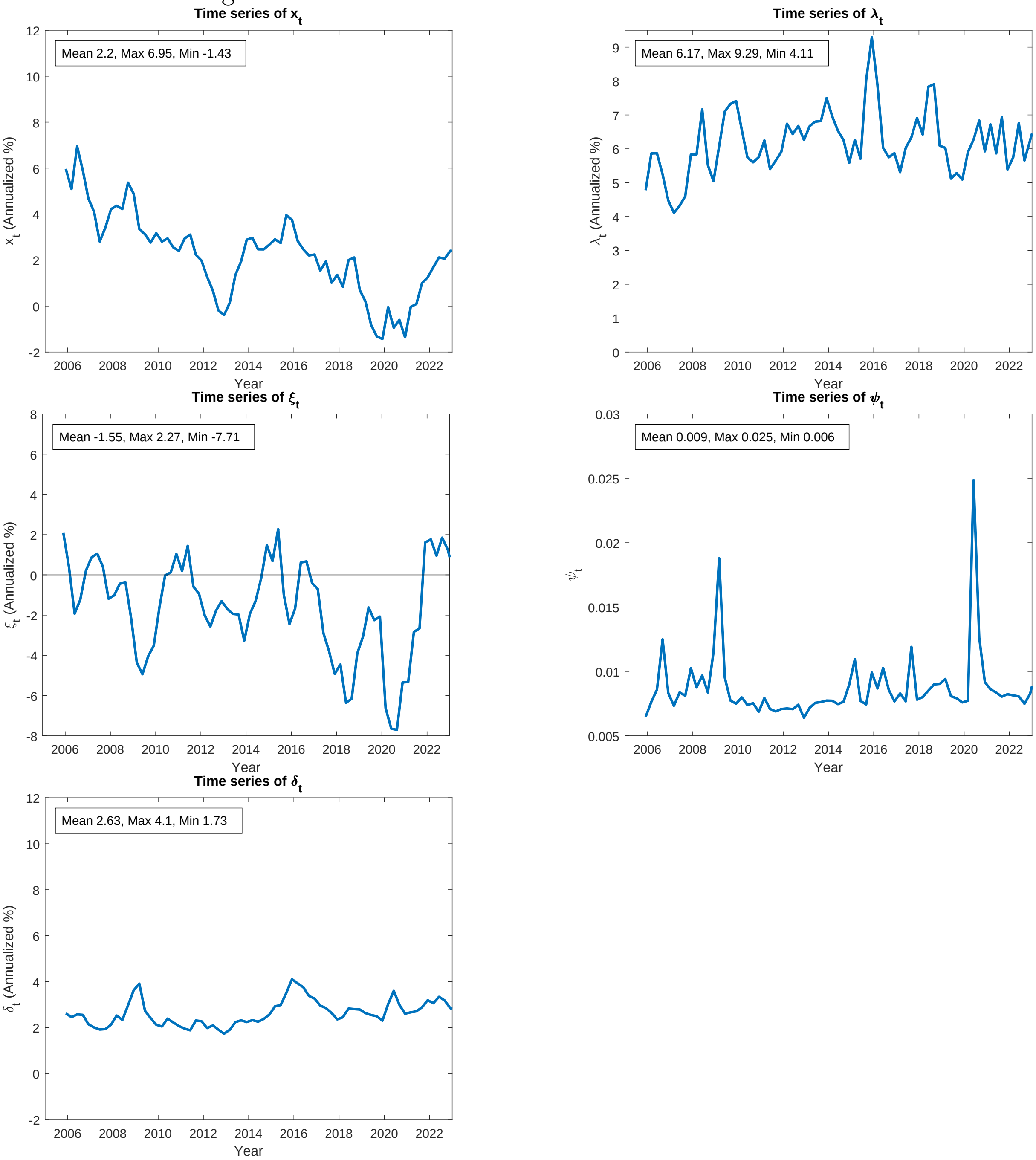
Below on table 6 we report the estimates of the augmented model parameters. In general, the standard errors increased considerably and most of the estimates became statistically not significant, so the model's precision diminished, resulting in a poorer fit for the data. The signs of ρ_{xm} and $\rho_{\xi m}$, which are very informative for tracking the bond risk premia, kept negative as before. Despite not reaching statistical significance, the correlation between shocks to country risk and the SDF was estimated as negative. This suggests that the country risk component exhibited a positive comovement with the SDF, contributing to a positive risk premium in the term structure. The estimates for $\rho_{\xi\pi}$ and $\rho_{\pi m}$, which were highly informative about the inflationary risk premium in the original version of the model, now became not significant as well.

Modified model		
Parameter	Estimate	Std Err
μ_x x 10^3	20.288	24.635
μ_ψ x 10^3	2.688	6.766
μ_δ x 10^3	6.505	1.740
ϕ_x	0.989	0.017
ϕ_ξ	0.891	0.032
ϕ_ψ	0.785	0.099
ϕ_δ	0.775	0.079
σ_m x 10^2	3.001	0.001
σ_x x 10^1	0.469	0.529
σ_X x 10^4	17.677	5.531
σ_Λ x 10^4	12.506	3.873
σ_ξ x 10^1	7.030	7.221
σ_π	1.000	
σ_ψ x 10^3	5.118	8.851
σ_δ x 10^1	0.738	0.990
σ_Δ x 10^4	9.423	3.203
β_{em} x 10^1	1.696	0.543
β_{ex} x 10^2	-1.211	78.477
β_{eX}	1.971	0.114
ρ_{xm}	-0.465	0.325
ρ_{Xm} x 10^3	0.039	0.197
$\rho_{\Lambda m}$ x 10^4	-6.990	3.673
$\rho_{\xi m}$	-0.402	0.207
$\rho_{\xi\pi}$	-0.074	1.352
$\rho_{\pi m}$ x 10^2	0.699	56.278
$\rho_{\psi m}$	0.685	0.622
$\rho_{\delta m}$	-0.013	0.041
$\rho_{\Delta m}$ x 10^3	0.040	0.013

Table 6: Campbell, Sunderam, and Viceira (2017)'s Model Augmented with Country Risk Parameter Estimates for Brazil. Note: This table reports parameter estimates and standard errors, estimated using maximum likelihood and an unscented Kalman filter. The first block reports the means of the real rate x_t, the country risk δ_t and the state variable ψ_t which governs the time variation in both the volatility of inflation and the real rate and their covariance with the SDF, m_t; the second block reports persistence parameters for these three state variables and ξ_t, the transitory component of expected inflation; the third block reports the volatilities of shocks; the fourth block reports the loadings of equities on shocks; and the fifth block reports the correlations between the shocks. σ_π is not estimated but normalized to 1.

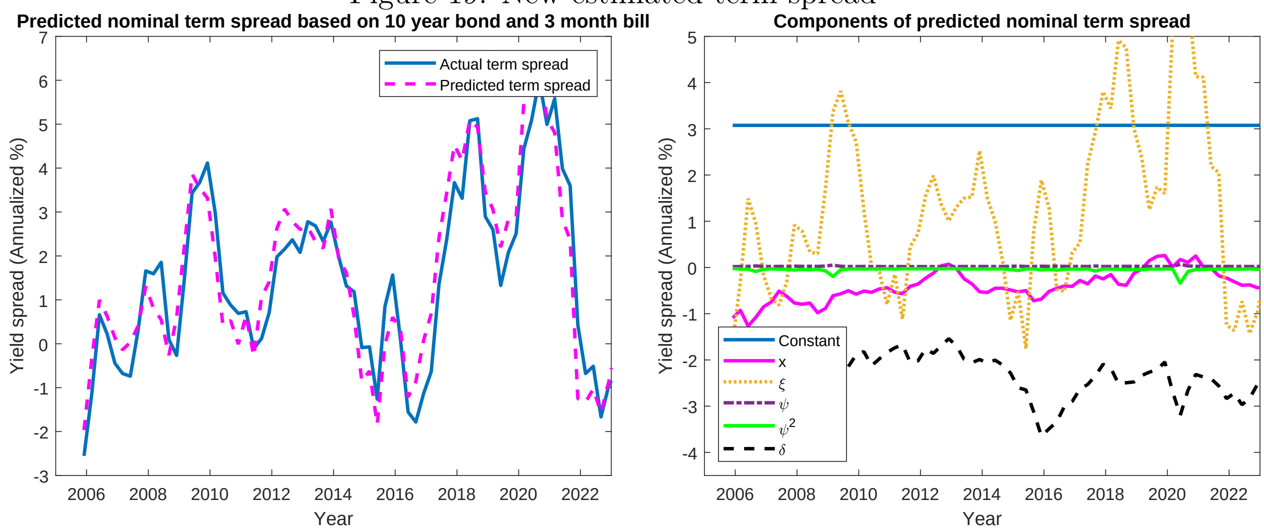
Below on Figure 18 we plot the new time series of the state variables from the modified model. The x_t and both components of inflation expectations exhibited similar patterns but with upward and downward shifts. As expected, the x_t shifted downward due to the introduction of a new component representing the country risk, which the original model's real interest rate was incorporating. The permanent component of inflation expectation λ_t shifted upward to slightly above the 5-Year, 5-Year Forward. And the transitory component ξ_t shifted downward to be negative almost all the time period. The introduction of the new component of country risk δ_t seems to have unrealistically distorted both components of inflation expectations at their levels. The ψ_t kept positive all the time, but smaller and with higher variability.

Figure 18: Time series of new estimated state variables



Below we plot on Figure 19 the decomposition of the yield spread in the new state variables. With the shift of ξ_t, the transitory component of expected inflation was the most relevant variable for generating a positive yield spread. The modified model suggests that, in most of the time period, the market agents expected that the inflation was temporarily too low, but would increase in the long term, implying higher yield spreads. Since the x_t became smaller, it contributed in a smaller degree for the yield spread. The country risk state variable contributed negatively for the yield spread. We see that in moments of higher Brazilian country risk perception, as in 2016, the short rates increased much more proportionally to the long-term rates, diminishing the yield spread. Furthermore, the contribution of the ψ_t state variable dropped to around 0. This might have occurred because of the higher standard deviations of the volatilities of shocks weighted by ψ_t and their covariances with the shocks to the SDF, making it difficult to filter ψ_t by the Kalman filter.

Figure 19: New estimated term spread



By introducing a new unobserved component to decompose the bond rates, the components from the original model shifted and changed their contribution for the interest rates. By introducing the country risk component, the x_t and the ψ_t decreased, suggesting that they were encompassing movements in the country risk in the original model specification. The state variables from the original model already accounted for volatilities influenced by changes in country risk perception. However, explicitly introducing the country risk component in the new model may have led to distortions, causing the new model to fit the data less effectively than the original model.

5 Conclusion

On this paper, we compare the determinants of the bond to stock beta over time and its risk premia implications in Brazil, in relation to the US. As endorsed by the US literature, the cyclicality of inflation, and the monetary policy influence on it, are relevant for making bonds work as hedge or risky assets.

However, the Brazilian position of an emerging economy more subject to internal and external imbalances made the Brazilian bonds even riskier by comoving with the stock market. Country risk variables as the CDS and the EMBI are highly correlated with both stocks and bonds returns. By implementing a Canonical Correlation Analysis for accounting for correlations within several variables together, variables representing the Global Financial Cycles and international flows appeared to comove with both bonds and stocks in the same direction, as well. So, volatility related to risk perception made the bond to stock beta more positive.

By estimating a multifactor term structure, we are able to track the covariance of each bond component with the SDF and account the implications of the positive beta for risk premia. In general, countercyclical real interest rate and expected inflation accounted for positive risk premia. Also, Brazilian bonds incorporate considerable inflation risk premium due to inflation surprises. The volatility of these components and how the covary with the state of the economy have direct connections with country risk movements, generating a risk premium on the yield spread. The higher yields, generated by the volatility of the components of long-term bonds and their risky covariance with the state of the economy, contributed with larger financing cost for the Brazilian government.

We see that attempts to write models where movements in country risk or in the sovereign default probability generate bonds with higher term premia, due to a risky pattern of comovements with the state of the economy, as promising for future research and important for taking into considerations while studying bond markets of emerging economies.

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A Figures

Figure 20: Bonds Returns with Swap and IDkA

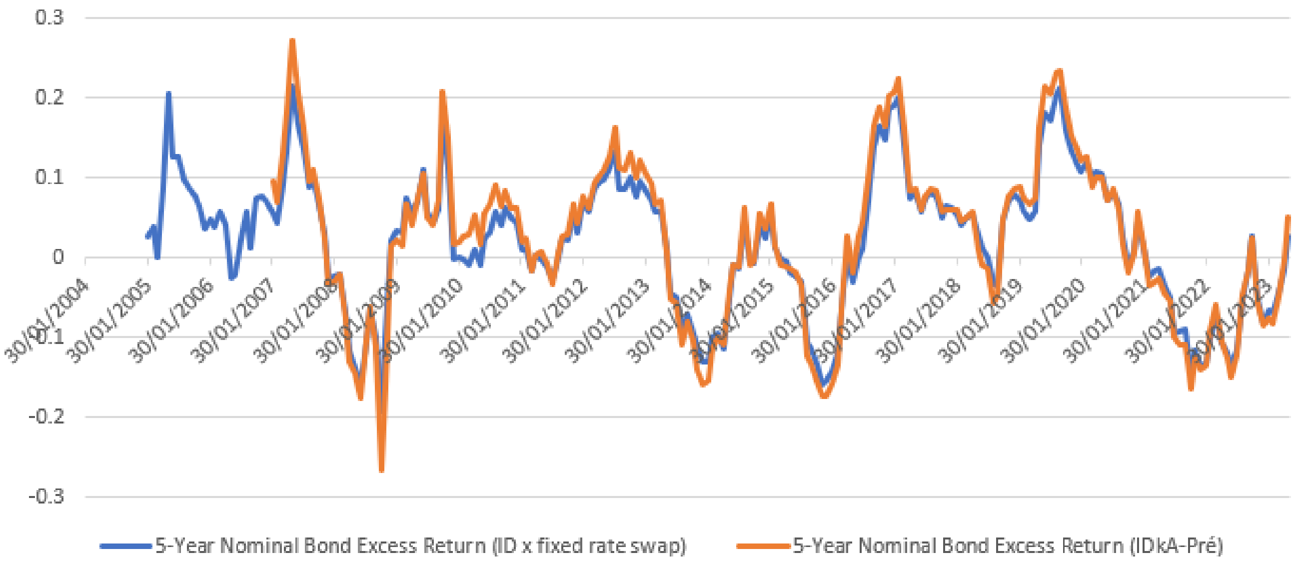


Figure 21: CDI accumulated (left), Ibov and 5Y Nominal Bond Return (right)

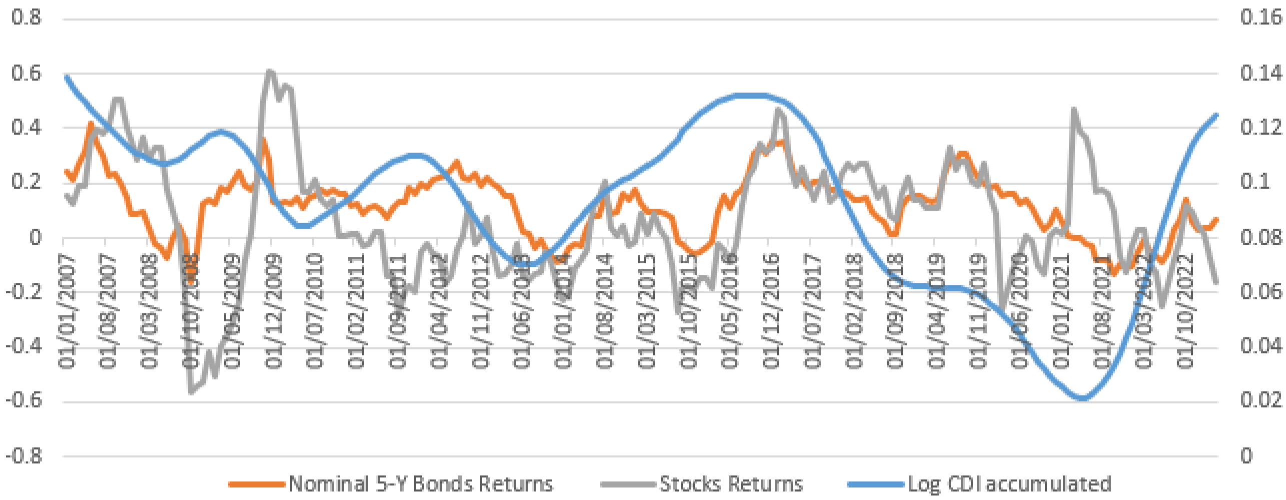


Figure 22: Excess returns over CDI and 1-year Bill

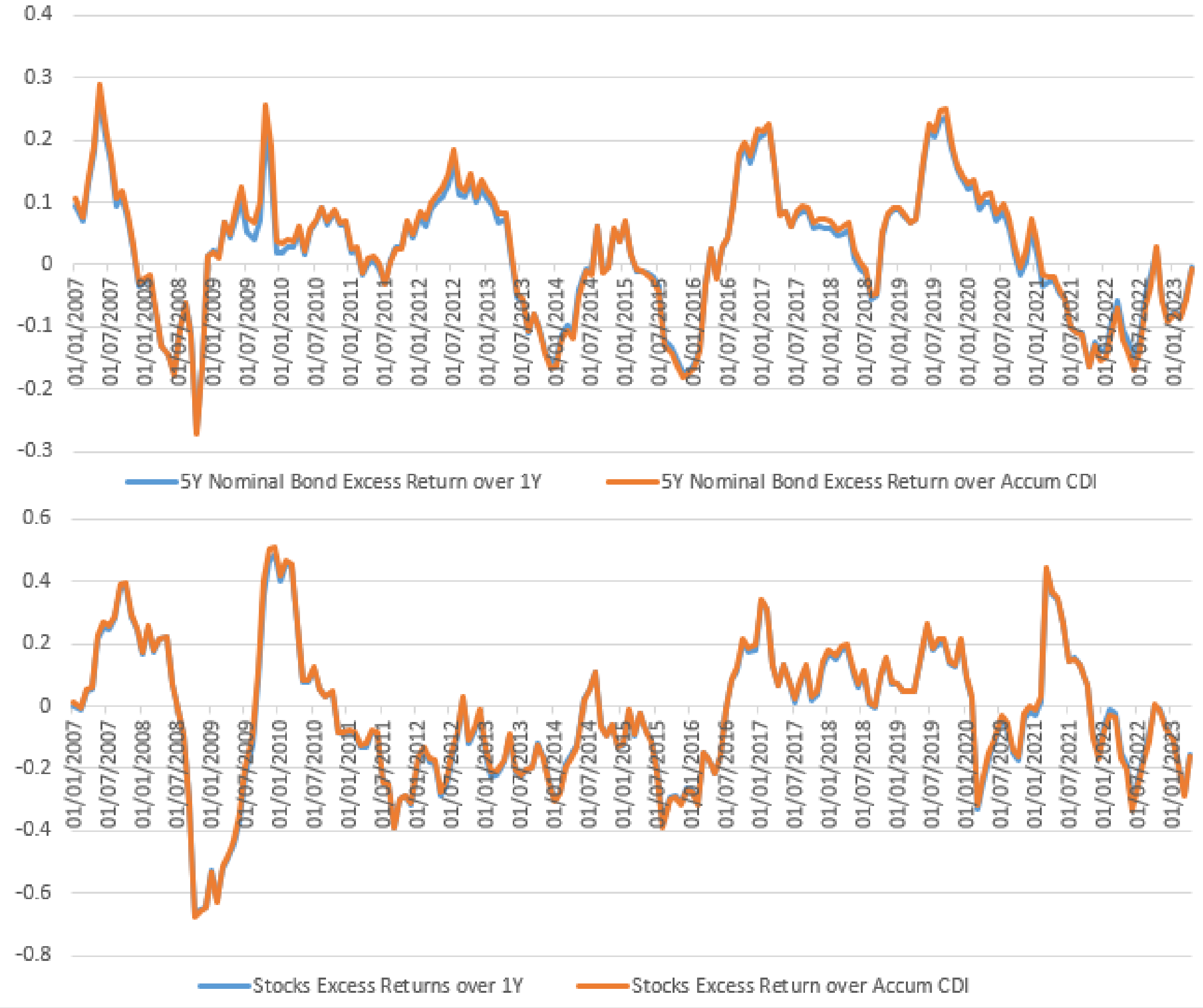


Figure 23: 10Y and 5Y Nominal Bond Excess Returns

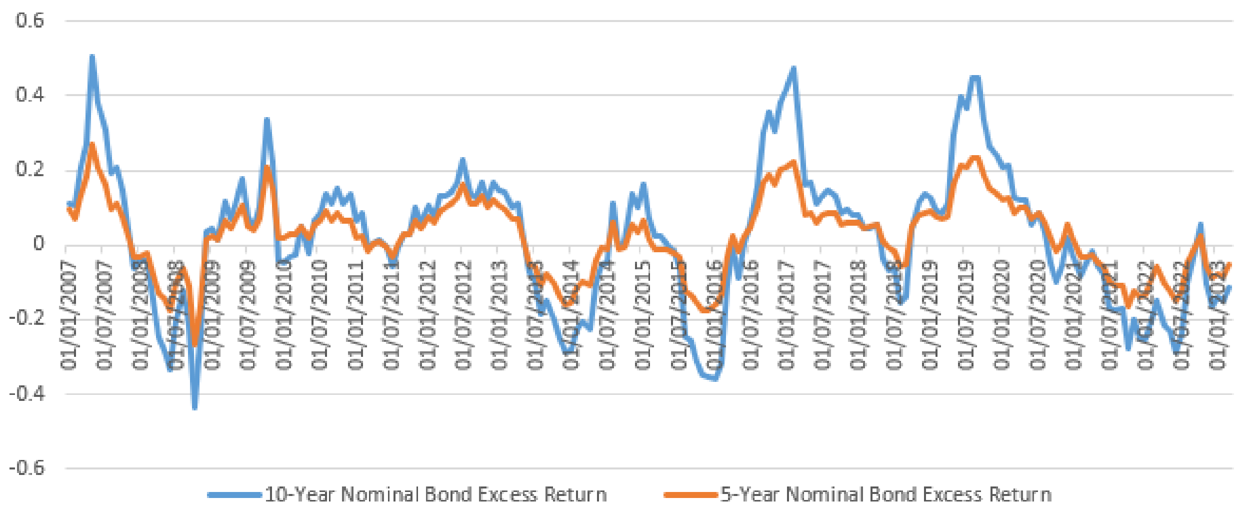


Figure 24: βs from $rx_t^{Pre5} = \alpha + \beta.rx_t^s + \epsilon_t$, using DI x fixed rate swap

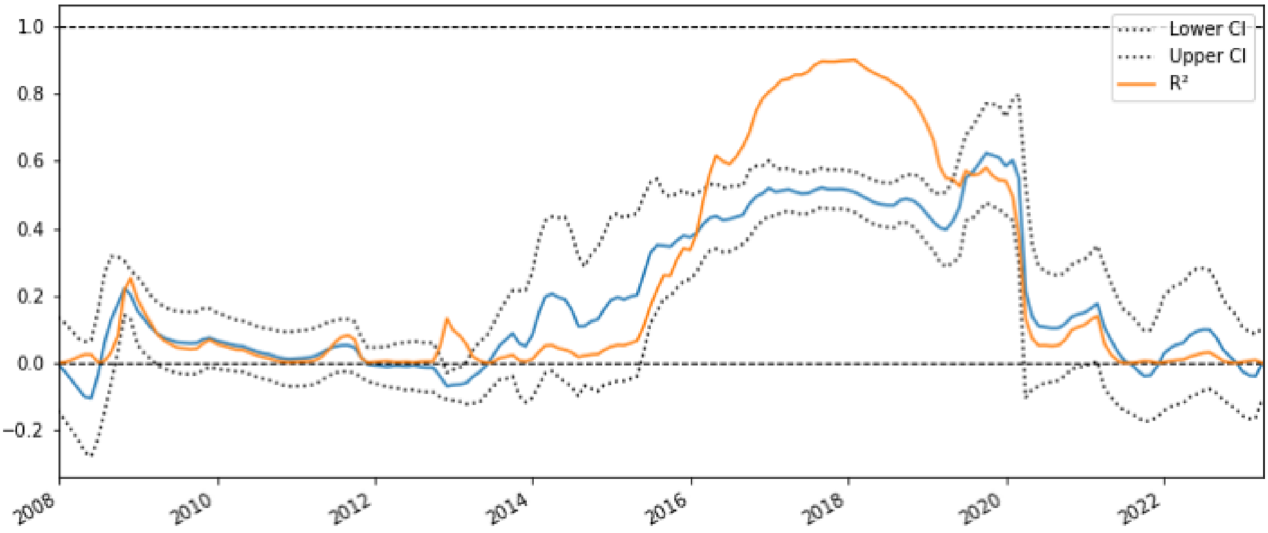


Figure 25: Correlation of Inflation and Annual Log Difference of IBC-Br

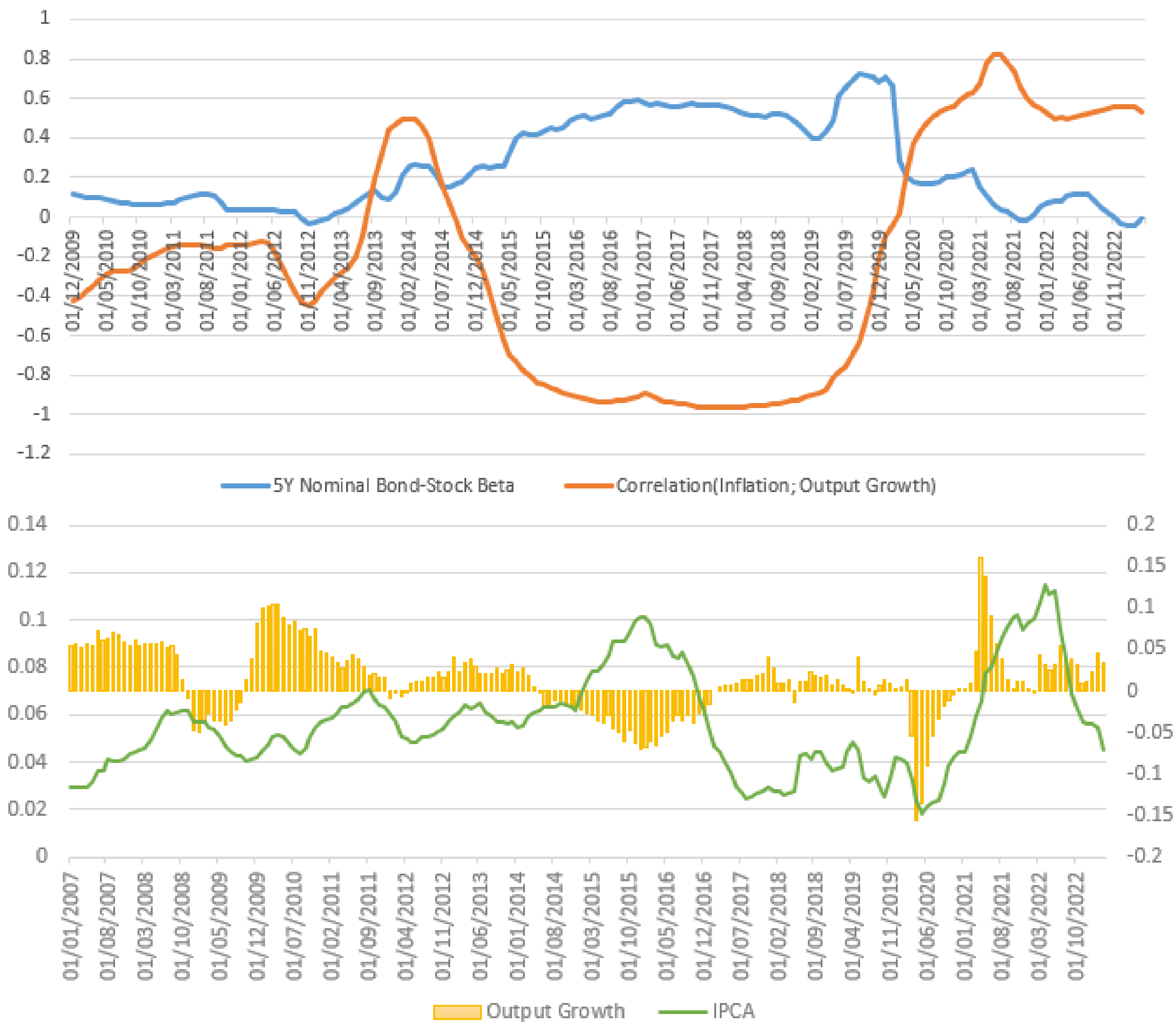


Figure 26: Inflation Ex-Ante and Ex-Post

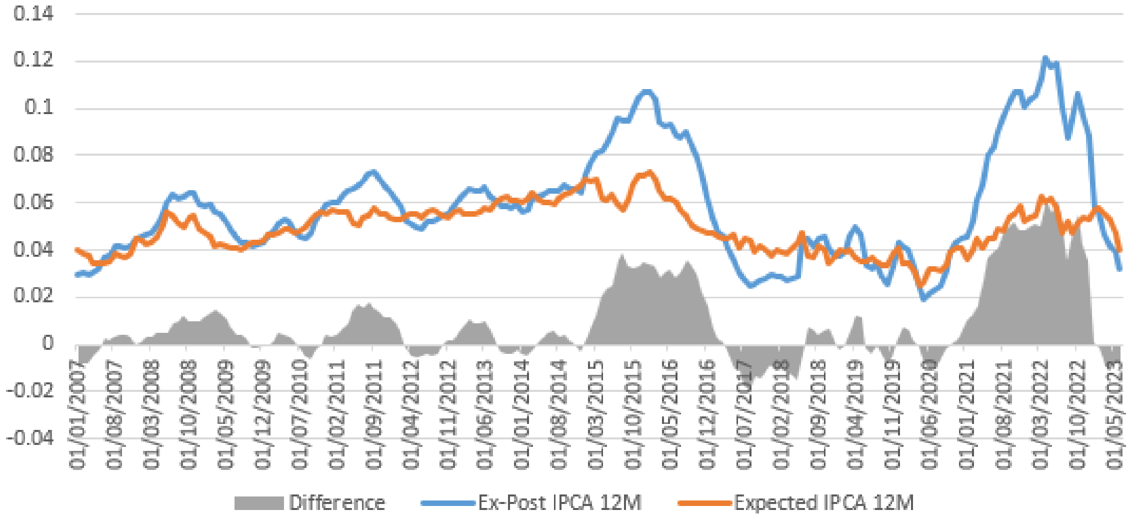


Figure 27: Excess Returns and CDS

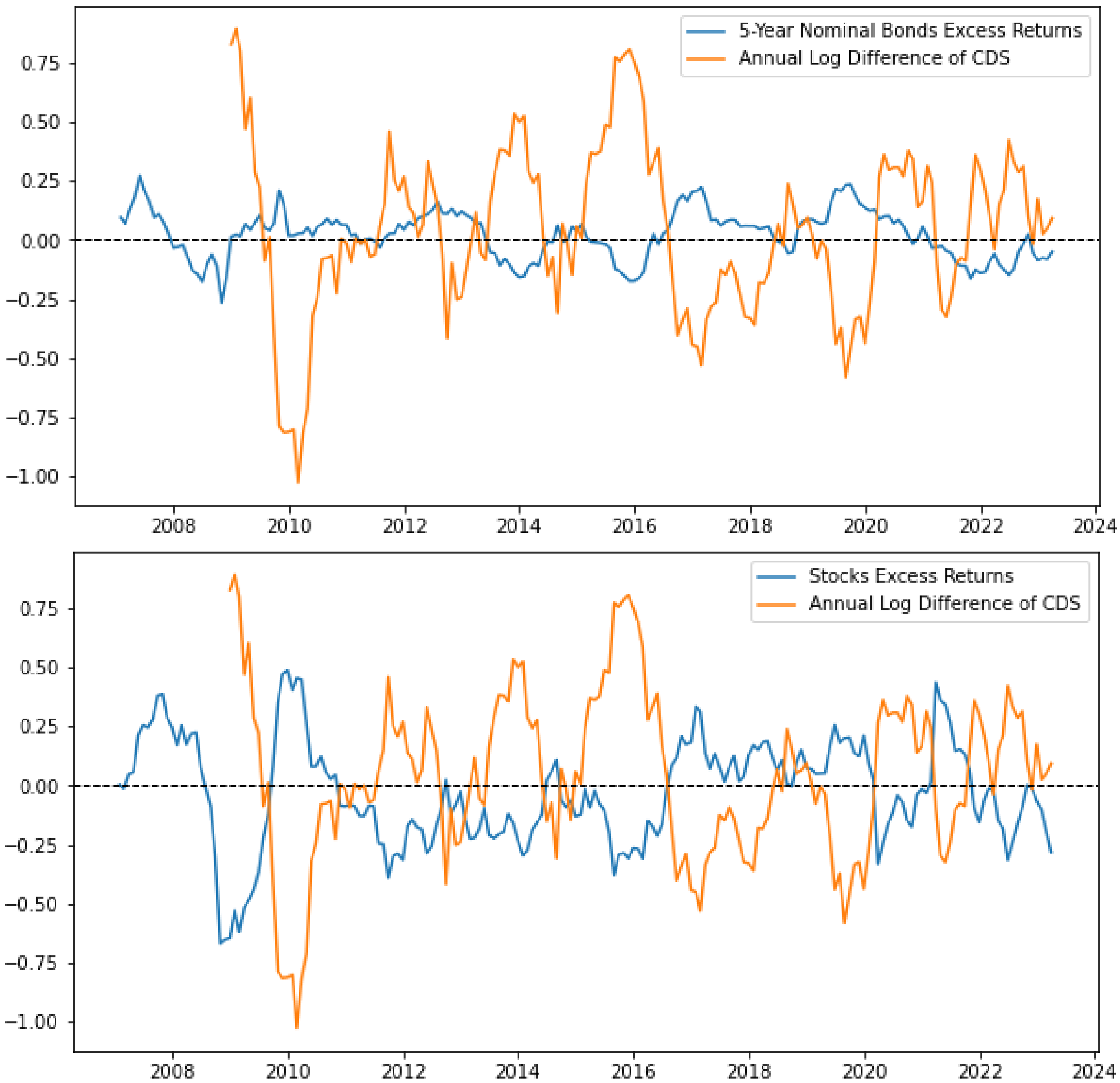


Figure 28: Excess Returns and EMBI+Br

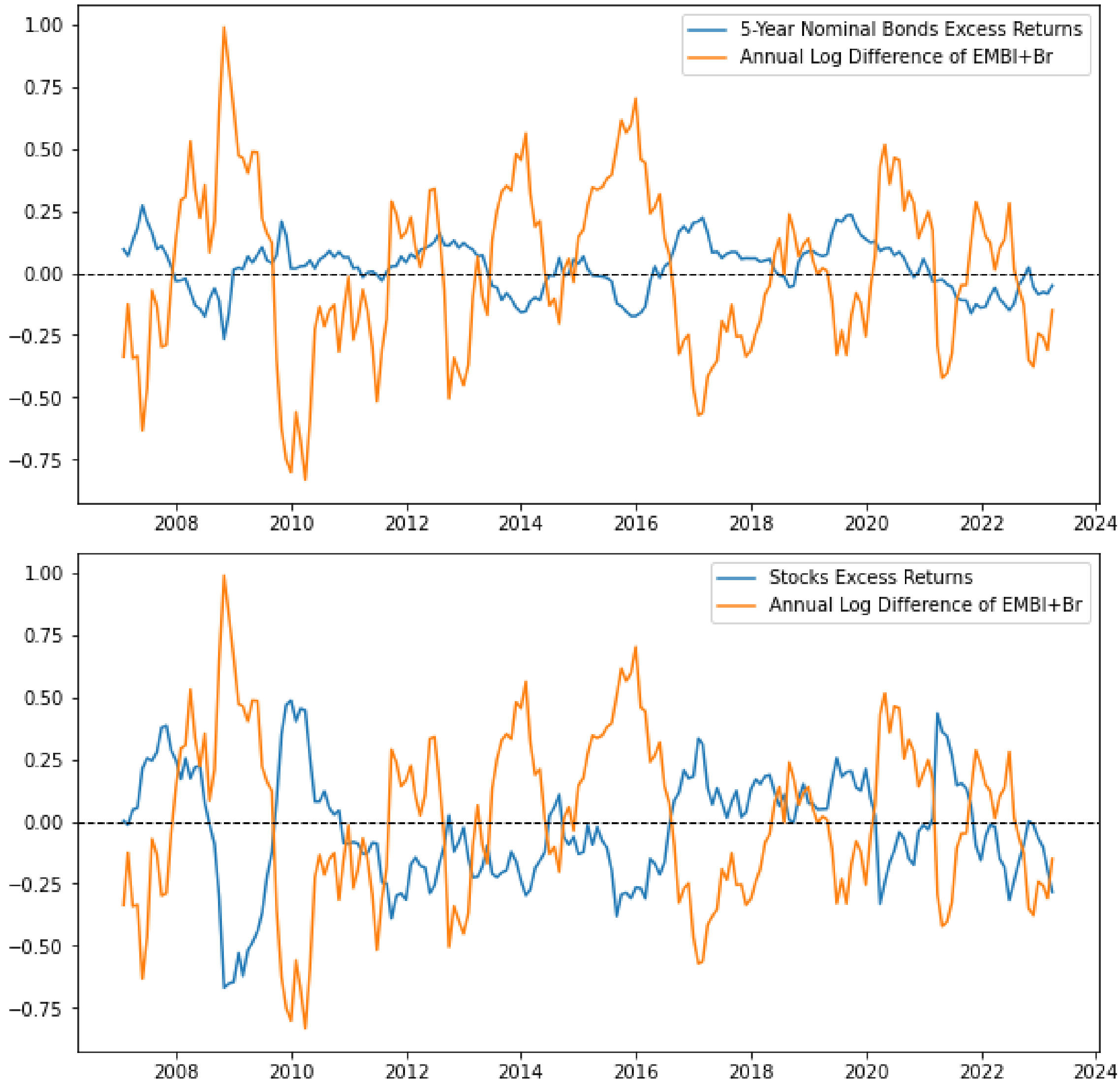


Figure 29: Bond-Stock Betas controlling for CDS

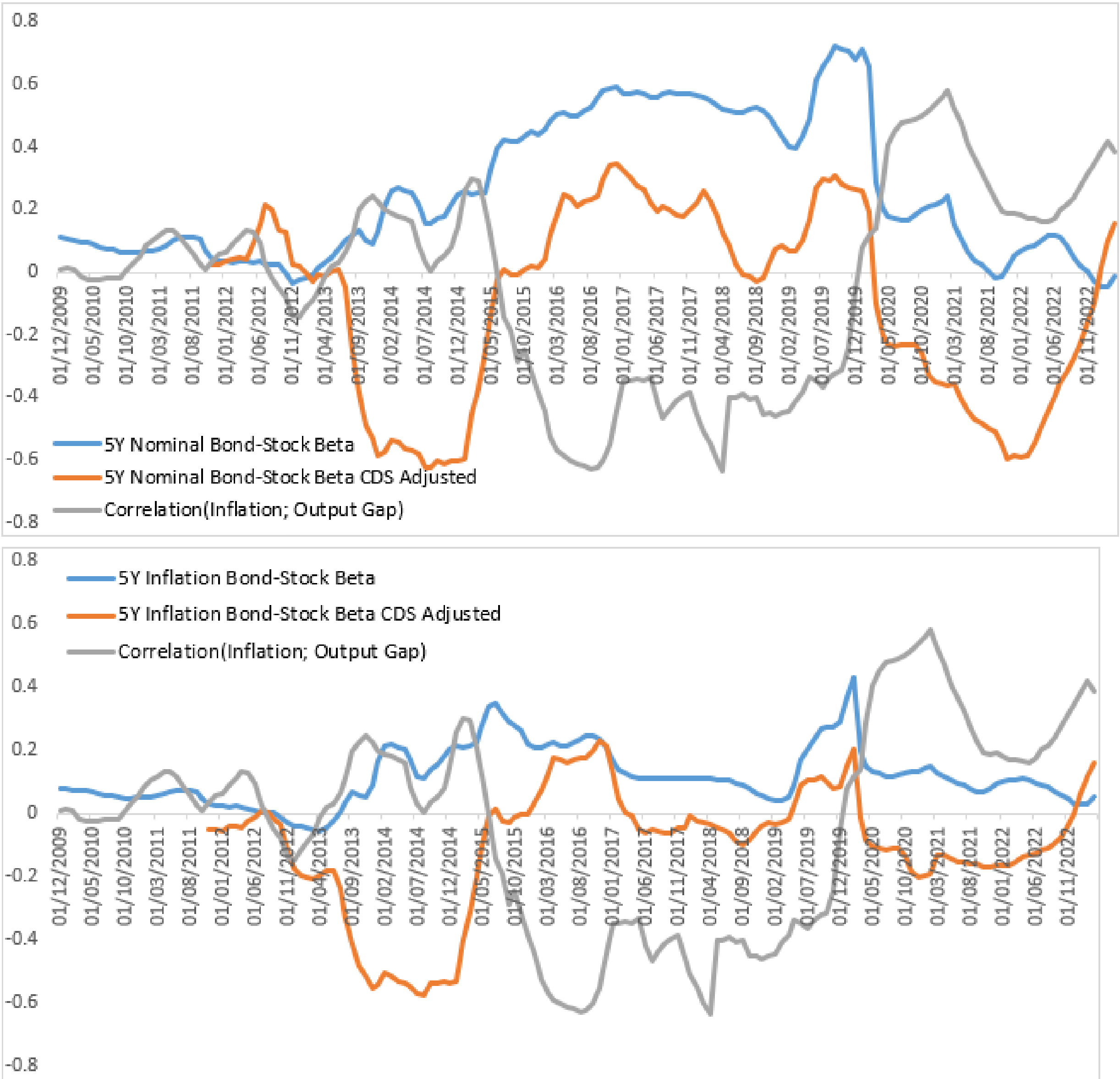
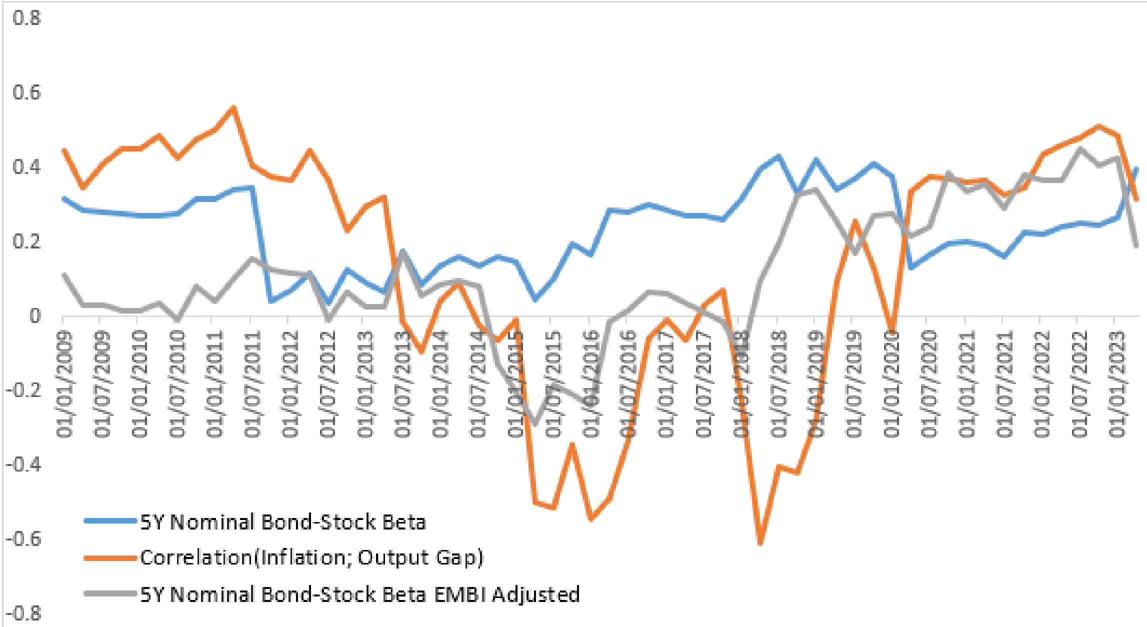


Figure 30: Bond-Stock Betas with quarterly returns on quarterly frequency



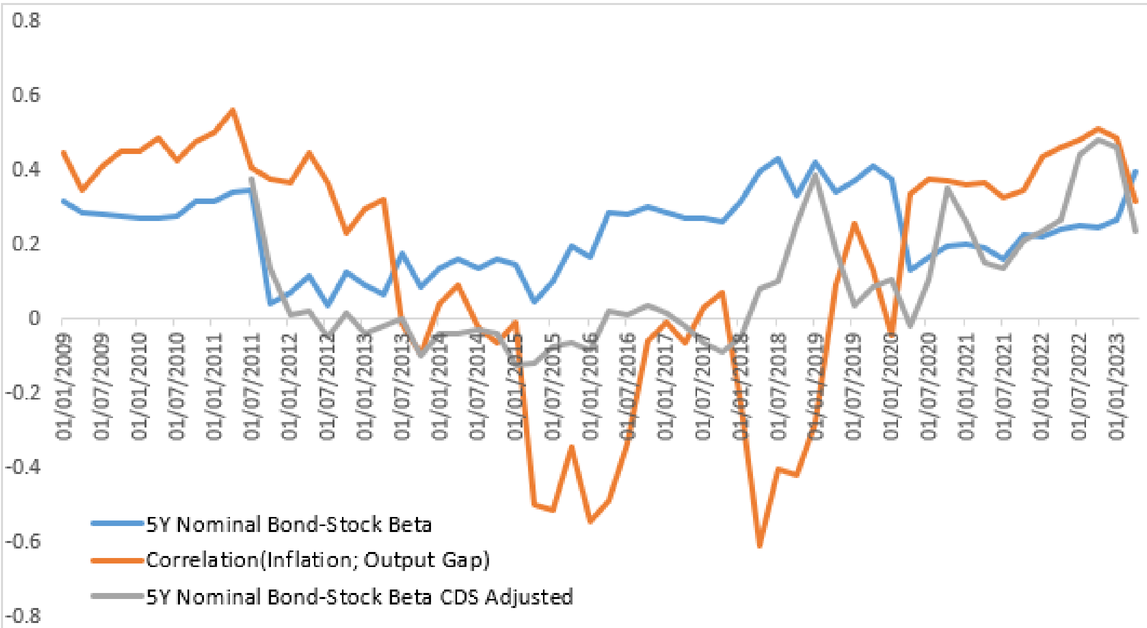
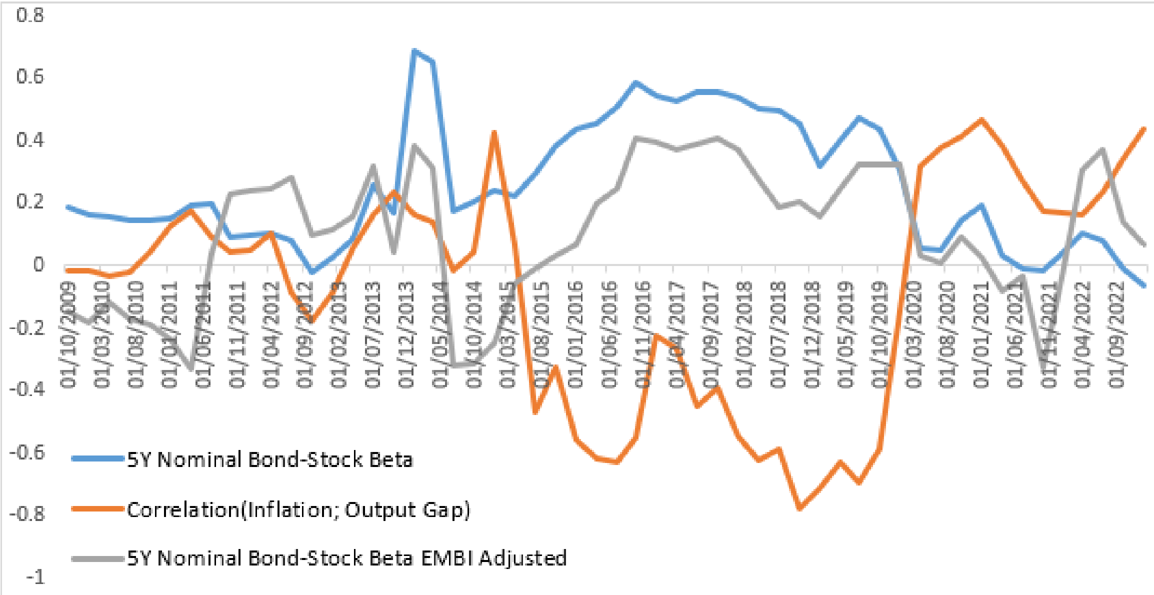


Figure 31: Bond-Stock Betas with annual returns on quarterly frequency



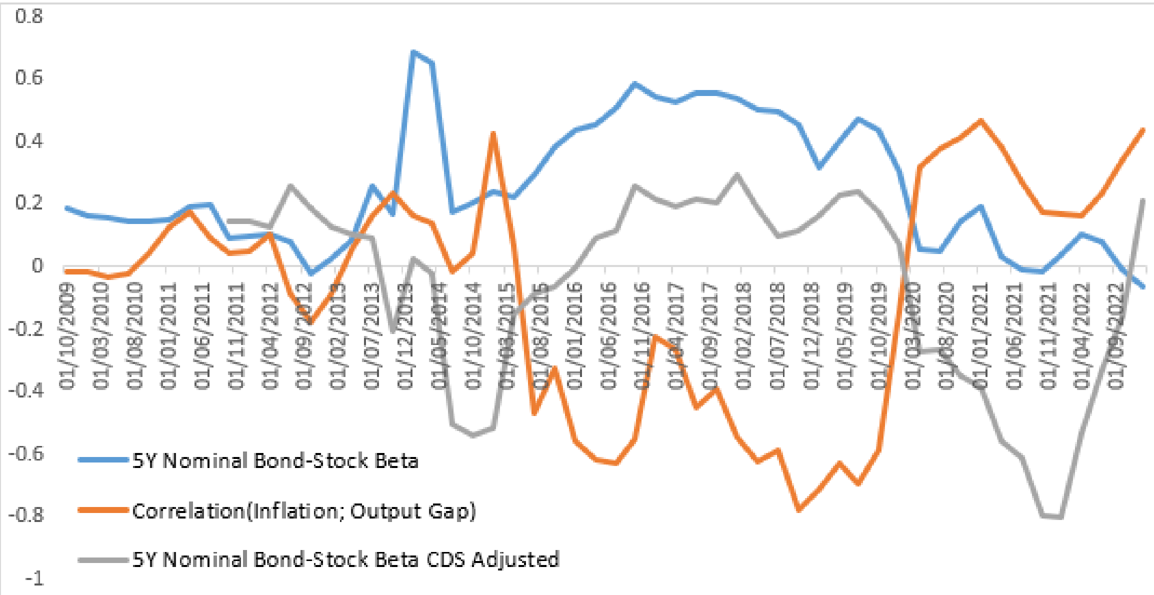
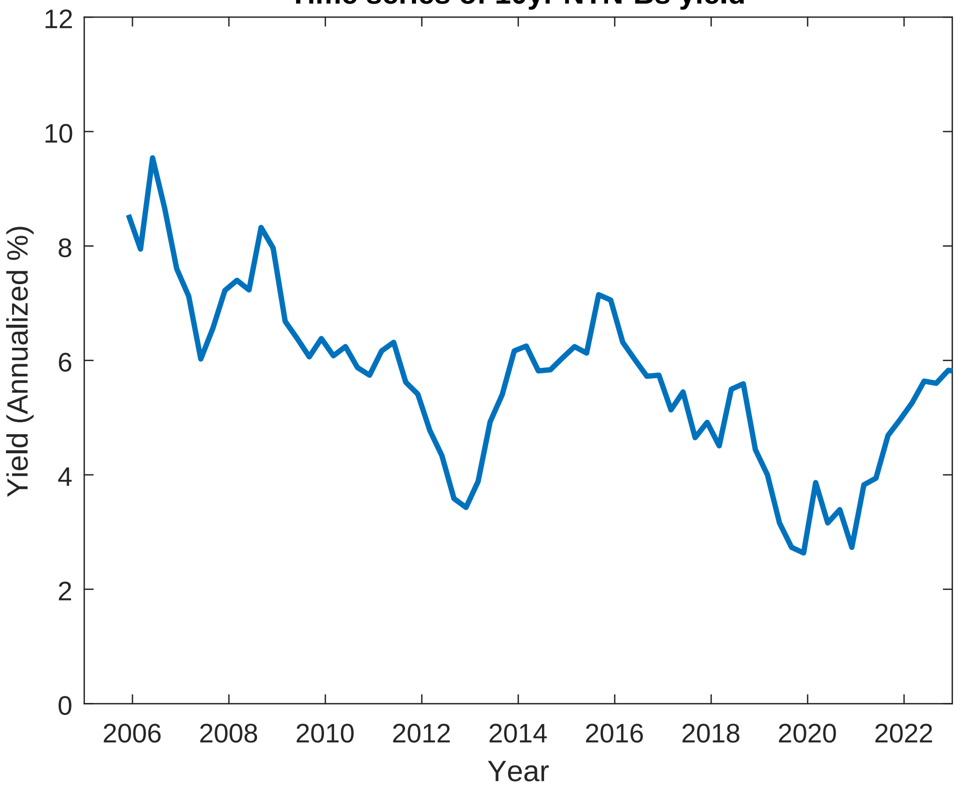


Figure 32:
Time series of 10yr NTN-Bs yield



B Derivation of the Model with Country Risk

This section of the appendix reports the solution for a more general version of the model where the volatility of the stochastic discount factor (SDF) is allowed to vary over time. The volatility of the SDF is controlled by the state variable z_t, which is modeled as following an AR(1) process. The solution to the simplified model presented in the main text of the paper obtain when we set $z_t = 1$ and constant. This more general version of the model was used just in the first working paper version of Campbell, Sunderam and Viceira (2017). The full derivation of the model in the Appendix from campbell2017inflation is made with this variable z_t, so we solve here the modified model with this variable z_t just for comparability purposes. For clarification, the parts of the model written in blue disappear by fixing $z_t = 1$. The modifications made by introducing the country-risk variable δ_t are in red.

The dynamics of the model is given by the following processes:

$$-m_{t+1} = x_t + \delta_t + \frac{\sigma_m^2}{2} z_t^2 + z_t \varepsilon_{m,t+1} \tag{22}$$

$$x_{t+1} = \mu_x(1 - \phi_x) + \phi_x x_t + \psi_t \varepsilon_{x,t+1} + \varepsilon_{X,t+1} \tag{23}$$

$$\psi_{t+1} = \mu_\psi(1 - \phi_\psi) + \phi_\psi \psi_t + \varepsilon_{\psi,t+1} \tag{24}$$

$$\pi_{t+1} = \lambda_t + \xi_t + \frac{\sigma_\pi^2}{2} \psi_t^2 + \psi_t \varepsilon_{\pi,t+1} \tag{25}$$

$$\lambda_{t+1} = \lambda_t + \psi_t \varepsilon_{\lambda,t+1} + \varepsilon_{\Lambda,t+1} \tag{26}$$

$$\xi_{t+1} = \phi_\xi \xi_t + \psi_t \varepsilon_{\xi,t+1} \tag{27}$$

$$z_{t+1} = \mu_z(1-\phi_z) + \phi_z z_t + \varepsilon_{z,t+1} \tag{28}$$

$$\delta_{t+1} = \mu_\delta(1-\phi_\delta) + \phi_\delta \delta_t + \psi_t \varepsilon_{\delta,t+1} + \varepsilon_{\Delta,t+1} \tag{29}$$

B.1 Pricing Equations

Real Term Structure The price of a single-period zero-coupon real bond satisfies

$$\begin{aligned} P_{1,t} &= E_t\left[M_{t+1} \times 1\right] = E_t\left[\exp\left\{m_{t+1}\right\}\right] \\ &= \exp\left\{-x_t - \delta_t - \frac{1}{2} z_t^2 \sigma_m^2 + \frac{1}{2} z_t^2 \sigma_m^2\right\} = \exp\left\{-x_t - \delta_t\right\} \end{aligned} \tag{30}$$

So, the single-period zero-coupon real bond yield equals:

$$y_{1,t} = -p_{1,t} = x_t + \delta_t$$

We conjecture that the price function is exponential affine in x_t and z_t with the form

$$P_{n,t} = \exp\left\{A_n + B_{x,n} x_t + B_{\delta,n}\delta_t + B_{z,n} z_t + B_{\psi,n}\psi_t + C_{z,n} z_t^2 + C_{\psi,n}\psi_t^2 + C_{z\psi,n} z_t \psi_t\right\}.$$

The standard pricing equation implies

$$\begin{aligned} P_{n,t} = &\; E_t\left[\exp\left\{p_{n-1,t+1} + m_{t+1}\right\}\right] = E_t\left[\exp\left\{\begin{array}{c} A_{n-1} + B_{x,n-1} x_{t+1} + B_{\delta,n-1}\delta_{t+1} + B_{z,n-1} z_{t+1} \\ + B_{\psi,n-1}\psi_{t+1} + C_{z,n-1} z_{t+1}^2 + C_{\psi,n-1}\psi_{t+1}^2 \\ + C_{z\psi,n-1} z_{t+1}\psi_{t+1} - x_t - \delta_t - \frac{1}{2} z_t^2 \sigma_m^2 - z_t \varepsilon_{m,t+1} \end{array}\right\}\right] \\ = &\; \exp\left\{\begin{array}{c} A_{n-1} + B_{x,n-1}\left(\left(1-\phi_x\right)\mu_x + \phi_x x_t\right) + B_{\delta,n-1}\left(\left(1-\phi_\delta\right)\mu_\delta + \phi_\delta \delta_t\right) \\ + B_{z,n-1}\left(\left(1-\phi_z\right)\mu_z + \phi_z z_t\right) + B_{\psi,n-1}\left(\left(1-\phi_\psi\right)\mu_\psi + \phi_\psi \psi_t\right) \\ + C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)^2 + C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)^2 \\ + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) - x_t - \delta_t - \frac{1}{2} z_t^2 \sigma_m^2 \end{array}\right\} \\ &\; \times E_t\left[\exp\left\{\mathbf{d}_1' \boldsymbol{\omega}_{t+1} + \boldsymbol{\omega}_{t+1}' \mathbf{D}_2 \boldsymbol{\omega}_{t+1}\right\}\right] \end{aligned}$$

where $\boldsymbol{\omega}'_{t+1} = (\varepsilon_{X,t+1}, \varepsilon_{m,t+1}, \varepsilon_{x,t+1}, \varepsilon_{z,t+1}, \varepsilon_{\psi,t+1}, \varepsilon_{\delta,t+1}, \varepsilon_{\Delta,t+1}) \sim N(0, \boldsymbol{\Sigma}_\omega)$,

$$\mathbf{d}_1 = \begin{pmatrix} B_{x,n-1} \\ -z_t \\ B_{x,n-1}\psi_t \\ B_{z,n-1} + 2C_{z,n-1}\left(\mu_z(1-\phi_z) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi \psi_t\right) \\ B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi \psi_t\right) + C_{z\psi,n-1}\left(\mu_z(1-\phi_z) + \phi_z z_t\right) \\ B_{\delta,n-1}\psi_t \\ B_{\Delta,n-1} \end{pmatrix}$$

$$\mathbf{D}_2 = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & & & & \\ & & & & & \\ & & C_{z,n-1} & \frac{1}{2}C_{z\psi,n-1} & 0 & 0 \\ 0 & \cdots & \frac{1}{2}C_{z\psi,n-1} & C_{\psi,n-1} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

Following Campbell, Chan, and Viceira (2003), we complete the square to calculate

$$\begin{aligned} E_t\left[\exp\left\{\mathbf{d}'_1\boldsymbol{\omega}_{t+1} + \boldsymbol{\omega}'_{t+1}\mathbf{D}_2\boldsymbol{\omega}_{t+1}\right\}\right] &= \frac{\left|\boldsymbol{\Sigma}_\omega\right|^{-1/2}}{\left|\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right|^{1/2}} \exp\left\{\frac{1}{2}\mathbf{d}_1\left(\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right)^{-1}\mathbf{d}'_1\right\} \\ &= \exp\left\{-\frac{1}{2}\log\left|\boldsymbol{\Sigma}_\omega\right| + \frac{1}{2}\log\left|\mathbf{G}\right| + \frac{1}{2}\mathbf{d}_1\mathbf{G}\mathbf{d}'_1\right\} \end{aligned}$$

where $\mathbf{G} = \left(\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right)^{-1}$. Let g_{ij} be the ij-th element of $\mathbf{G}$. Then expanding and collecting terms gives

$$p_{n,t} = \left[\begin{array}{c} A_{n-1} + B_{x,n-1}\left((1-\phi_x)\,\mu_x + \phi_x x_t\right) + B_{\delta,n-1}\left((1-\phi_\delta)\,\mu_\delta + \phi_\delta \delta_t\right) \\ +B_{z,n-1}\left((1-\phi_z)\,\mu_z + \phi_z z_t\right) + B_{\psi,n-1}\left((1-\phi_\psi)\,\mu_\psi + \phi_\psi \psi_t\right) \\ +C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)^2 + C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)^2 \\ +C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) \\ -x_t - \delta_t - \frac{1}{2} z_t^2 \sigma_m^2 - \frac{1}{2}\log\left|\mathbf{\Sigma}_\omega\right| + \frac{1}{2}\log\left|\mathbf{G}\right| + \frac{1}{2} g_{11} B_{x,n-1}^2 + \frac{1}{2} g_{22} z_t^2 + \frac{1}{2} g_{33} B_{x,n-1}^2 \psi_t^2 \\ +\frac{1}{2} g_{44}\left(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right)^2 \\ +\frac{1}{2} g_{55}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)^2 \\ +\frac{1}{2} g_{66} B_{\delta,n-1}^2 \psi_t^2 + \frac{1}{2} g_{77} B_{\delta,n-1}^2 - g_{12} B_{x,n-1} z_t + g_{13} B_{x,n-1}^2 \psi_t \\ +g_{14} B_{x,n-1}(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)) \\ +g_{15} B_{x,n-1}(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)) \\ +g_{16} B_{x,n-1} B_{\delta,n-1} \psi_t + g_{17} B_{x,n-1} B_{\delta,n-1} - g_{23} B_{x,n-1} z_t \psi_t \\ -g_{24} z_t\left(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right) \\ -g_{25} z_t\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right) \\ -g_{26} z_t B_{\delta,n-1} \psi_t - g_{27} z_t B_{\delta,n-1} \\ +g_{34} B_{x,n-1} \psi_t(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)) \\ +g_{35} B_{x,n-1} \psi_t\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right) \\ +g_{36} B_{\delta,n-1} B_{x,n-1} \psi_t^2 + g_{37} B_{\delta,n-1} B_{x,n-1} \psi_t \\ +g_{45}(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)) \\ \times\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right) \\ +g_{46} B_{\delta,n-1} \psi_t(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)) \\ +g_{47} B_{\delta,n-1}(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)) \\ +g_{56} B_{\delta,n-1} \psi_t\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right) \\ +g_{57} B_{\delta,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right) \\ +g_{67} B_{\delta,n-1}^2 \psi_t \end{array} \right]$$

Thus, equating coefficients yields

$$A_n = \begin{bmatrix} A_{n-1} + B_{x,n-1}(1-\phi_x)\mu_x + B_{\delta,n-1}(1-\phi_\delta)\mu_\delta + B_{z,n-1}(1-\phi_z)\mu_z \\ +B_{\psi,n-1}(1-\phi_\psi)\mu_\psi + C_{z,n-1}\mu_z^2(1-\phi_z)^2 \\ +C_{\psi,n-1}\mu_\psi^2(1-\phi_\psi)^2 + C_{z\psi,n-1}\mu_z(1-\phi_z)\mu_\psi(1-\phi_\psi) \\ -\frac{1}{2}\log|\mathbf{\Sigma}_\omega| + \frac{1}{2}\log|\mathbf{G}| + \frac{1}{2}g_{11}B_{x,n-1}^2 \\ +\frac{1}{2}g_{44}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi(1-\phi_\psi)\right)^2 \\ +\frac{1}{2}g_{55}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right)^2 \\ +\frac{1}{2}g_{77}B_{\delta,n-1}^2 \\ +g_{14}B_{x,n-1}(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi(1-\phi_\psi)) \\ +g_{15}B_{x,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right) \\ +g_{17}B_{x,n-1}B_{\delta,n-1} \\ +g_{45}(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi(1-\phi_\psi)) \\ \times\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right) \\ +g_{47}B_{\delta,n-1}(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi(1-\phi_\psi)) \\ +g_{57}B_{\delta,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right) \end{bmatrix}$$

$$\mathrm{B}_{x,n} = B_{x,n-1}\phi_x - 1$$

$$\mathrm{B}_{\delta,n} = B_{\delta,n-1}\phi_\delta - 1$$

$$\mathrm{B}_{z,n} = \begin{bmatrix} B_{z,n-1}\phi_z + 2C_{z,n-1}\mu_z(1-\phi_z)\phi_z + C_{z\psi,n-1}\mu_\psi(1-\phi_\psi)\phi_z \\ +2g_{44}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi(1-\phi_\psi)\right)C_{z,n-1}\phi_z \\ +g_{55}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right)C_{z\psi,n-1}\phi_z \\ -g_{12}B_{x,n-1} + 2g_{14}B_{x,n-1}C_{z,n-1}\phi_z + g_{15}B_{x,n-1}C_{z\psi,n-1}\phi_z \\ -g_{24}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi(1-\phi_\psi)\right) \\ -g_{25}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right) \\ -g_{27}B_{\delta,n-1} \\ +g_{45}\begin{bmatrix} 2C_{z,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right) \\ +C_{z\psi,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi(1-\phi_\psi)\right) \end{bmatrix}\phi_z \\ +2g_{47}B_{\delta,n-1}C_{z,n-1}\phi_z \\ +g_{57}B_{\delta,n-1}C_{z\psi,n-1}\phi_z \end{bmatrix}$$

$$
\mathrm{B}_{\psi,n} = \left[\begin{array}{c} B_{\psi,n-1}\phi_\psi + 2C_{\psi,n-1}\mu_\psi (1-\phi_\psi)\phi_\psi + C_{z\psi,n-1}\mu_z (1-\phi_z)\phi_\psi + g_{13}B_{x,n-1}^2 \\ +g_{14}B_{x,n-1}C_{z\psi,n-1}\phi_\psi + 2g_{15}B_{x,n-1}C_{\psi,n-1}\phi_\psi + +g_{16}B_{x,n-1}B_{\delta,n-1} \\ +g_{44}(B_{z,n-1} + 2C_{z,n-1}\mu_z (1-\phi_z) + C_{z\psi,n-1}\mu_\psi (1-\phi_\psi)) C_{z\psi,n-1}\phi_\psi \\ +2g_{55}B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi (1-\phi_\psi) + C_{z\psi,n-1}\mu_z(1-\phi_z))C_{\psi,n-1}\phi_\psi \\ +g_{34}B_{x,n-1}(B_{z,n-1} + 2C_{z,n-1}\mu_z (1-\phi_z) + C_{z\psi,n-1}\mu_\psi (1-\phi_\psi) \\ +g_{35}B_{x,n-1}(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi (1-\phi_\psi) + C_{z\psi,n-1}\mu_z (1-\phi_z)) \\ +g_{37}B_{\delta,n-1}B_{x,n-1} \\ +g_{45}\left[\begin{array}{c} 2C_{\psi,n-1}(B_{z,n-1} + 2C_{z,n-1}\mu_z (1-\phi_z) + C_{z\psi,n-1}\mu_\psi (1-\phi_\psi)) \\ +C_{z\psi,n-1}(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi (1-\phi_\psi) + C_{z\psi,n-1}\mu_z (1-\phi_z)) \end{array} \right]\phi_\psi \\ +g_{46}B_{\delta,n-1}(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi(1-\phi_\psi)) \\ +g_{47}B_{\delta,n-1}C_{z\psi,n-1}\phi_\psi) \\ +g_{56}B_{\delta,n-1}(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}\mu_z(1-\phi_z)) \\ +2g_{57}B_{\delta,n-1}C_{\psi,n-1}\phi_\psi + g_{67}B_{\delta,n-1}^2 \end{array} \right]
$$

$$
C_{z,n} = \left[\begin{array}{c} C_{z,n-1}\phi_z^2 - \frac{1}{2}\sigma_m^2 + \frac{1}{2}g_{22} + 2g_{44}C_{z,n-1}^2\phi_z^2 + \frac{1}{2}g_{55}C_{z\psi,n-1}^2\phi_z^2 \\ -2g_{24}C_{z,n-1}\phi_z - g_{25}C_{z\psi,n-1}\phi_z + 2g_{45}C_{z,n-1}C_{z\psi,n-1}\phi_z^2 \end{array} \right]
$$

$$
C_{\psi,n} = \left[\begin{array}{c} C_{\psi,n-1}\phi_\psi^2 + \frac{1}{2}g_{33}B_{x,n-1}^2 + \frac{1}{2}g_{44}C_{z\psi,n-1}^2\phi_\psi^2 \\ +2g_{55}C_{\psi,n-1}^2\phi_\psi^2 + \frac{1}{2}g_{66}B_{\delta,n-1}^2 + g_{34}B_{x,n-1}C_{z\psi,n-1}\phi_\psi \\ +2g_{35}B_{x,n-1}C_{\psi,n-1}\phi_\psi + g_{36}B_{\delta,n-1}B_{x,n-1} + 2g_{45}C_{\psi,n-1}C_{z\psi,n-1}\phi_\psi^2 \\ +g_{46}B_{\delta,n-1}C_{z\psi,n-1}\phi_\psi + 2g_{56}B_{\delta,n-1}C_{\psi,n-1}\phi_\psi \end{array} \right]
$$

$$
C_{z\psi,n} = \left[\begin{array}{c} C_{z\psi,n-1}\phi_z\phi_\psi + 2g_{44}C_{z,n-1}C_{z\psi,n-1}\phi_z\phi_\psi + 2g_{55}C_{\psi,n-1}C_{z\psi,n-1}\phi_z\phi_\psi \\ -g_{23}B_{x,n-1} - g_{24}C_{z\psi,n-1}\phi_\psi - 2g_{25}C_{\psi,n-1}\phi_\psi - g_{26}B_{\delta,n-1} \\ +2g_{34}B_{x,n-1}C_{z,n-1}\phi_z + g_{35}B_{x,n-1}C_{z\psi,n-1}\phi_z + g_{45}C_{z\psi,n-1}^2\phi_\psi\phi_z \\ +2g_{46}B_{\delta,n-1}C_{z,n-1}\phi_z + g_{56}B_{\delta,n-1}C_{z\psi,n-1}\phi_z \end{array} \right]
$$

Nominal Term Structure The price of a single-period zero-coupon nominal bond satisfies

$$
\begin{aligned} P_{1,t}^{\$} &= E_t[M_{t+1} \times 1 \div \Pi_{t+1}] \equiv E_t[\exp\{m_{t+1} - \pi_{t+1}\}] \\ &= \exp\{-x_t - \delta_t - \lambda_t - \xi_t + z_t\psi_t\sigma_{m\psi}\} \end{aligned} \tag{31}
$$

since $z_t\varepsilon_{m,t+1}$ and $\psi_t\varepsilon_{\pi,t+1}$ are jointly conditional normal, so the single-period zero-coupon nominal bond yield equals:

$$
y_{1,t}^{\$} = -p_{1,t}^{\$} = x_t + \delta_t + \lambda_t + \xi_t - z_t\psi_t\sigma_{m\psi}
$$

We conjecture that the price function is exponential linear quadratic in the state variables with the form:

$$
P_{n,t}^{\$} = \exp\left\{A_n^{\$} + B_{x,n}^{\$}x_t + B_{\delta,n}^{\$}\delta_t + B_{z,n}^{\$}z_t + B_{\lambda,n}^{\$}\lambda_t + B_{\xi,n}^{\$}\xi_t + B_{\psi,n}\psi_t + C_{z,n}^{\$}z_t^2 + C_{\psi,n}^{\$}\psi_t^2 + C_{z\psi,n}^{\$}z_t\psi_t\right\}.
$$

The standard pricing equation then implies

$$P^{\$}_{n,t} = E_t\left[\exp\left\{p^{\$}_{n-1,t+1} + m_{t+1} - \pi_{t+1}\right\}\right]$$

$$= E_t\left[\exp\left\{\begin{array}{c} A^{\$}_{n-1} + B^{\$}_{x,n-1}x_{t+1} + B^{\$}_{\delta,n-1}\delta_{t+1} + B^{\$}_{z,n-1}z_{t+1} + B^{\$}_{\lambda,n-1}\lambda_{t+1} + B^{\$}_{\xi,n-1}\xi_{t+1} \\ +B^{\$}_{\psi,n-1}\psi_{t+1} + C^{\$}_{z,n-1}z^2_{t+1} + C^{\$}_{\psi,n-1}\psi^2_{t+1} + C^{\$}_{z\psi,n-1}z_{t+1}\psi_{t+1} \\ -x_t - \delta_t - \frac{1}{2}z_t^2\sigma_m^2 - z_t\varepsilon_{m,t+1} - \lambda_t - \xi_t - \frac{1}{2}\psi_t^2\sigma_\pi^2 - \psi_t\varepsilon_{\pi,t+1} \end{array}\right\}\right]$$

$$= \exp\left\{\begin{array}{c} A^{\$}_{n-1} + B^{\$}_{x,n-1}\left(\mu_x\left(1-\phi_x\right) + \phi_x x_t\right) + B^{\$}_{\delta,n-1}\left(\mu_\delta\left(1-\phi_\delta\right) + \phi_\delta\delta_t\right) \\ +B^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + B^{\$}_{\lambda,n-1}\left(\mu_\lambda + \lambda_t\right) + B^{\$}_{\xi,n-1}\phi_\xi\xi_t \\ +B^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)^2 \\ +C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)^2 \\ +C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) \\ -x_t - \delta_t - \frac{1}{2}z_t^2\sigma_m^2 - \lambda_t - \xi_t - \frac{1}{2}\psi_t^2\sigma_\pi^2 \end{array}\right\}$$

$$\times E_t\left[\exp\left\{\mathbf{d}^{\$\prime}_1\boldsymbol{\omega}^{\$}_{t+1} + \omega^{\$\prime}_{t+1}\mathbf{D}^{\$}_2\boldsymbol{\omega}^{\$}_{t+1}\right\}\right]$$

where $\boldsymbol{\omega}^{\$\prime}_{t+1} = \left(\varepsilon_{X,t+1}, \varepsilon_{\Lambda,t+1}, \varepsilon_{\lambda,t+1}, \varepsilon_{m,t+1}, \varepsilon_{\pi,t+1}, \varepsilon_{x,t+1}, \varepsilon_{\xi,t+1}, \varepsilon_{z,t+1}, \varepsilon_{\psi,t+1}, \varepsilon_{\delta,t+1}, \varepsilon_{\Delta,t+1}\right) \sim N\left(0, \boldsymbol{\Sigma}^{\$}_{\omega}\right)$,

$$\mathbf{d}^{\$}_1 = \begin{pmatrix} B^{\$}_{x,n-1} \\ B^{\$}_{\lambda,n-1} \\ B^{\$}_{\lambda,n-1}\psi_t \\ -z_t \\ -\psi_t \\ B^{\$}_{x,n-1}\psi_t \\ B^{\$}_{\xi,n-1}\psi_t \\ B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) \\ B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) \\ B^{\$}_{\delta,n-1}\psi_t \\ B^{\$}_{\delta,n-1} \end{pmatrix}$$

$$\mathbf{D}^{\$}_2 = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & & & & \\ & & & & & \\ & & C^{\$}_{z,n-1} & \frac{1}{2}C^{\$}_{z\psi,n-1} & 0 & 0 \\ 0 & \cdots & \frac{1}{2}C^{\$}_{z\psi,n-1} & C^{\$}_{\psi,n-1} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

Following Campbell, Chan, and Viceira (2003), we complete the square to calculate

$$E_t\left[\exp\left\{\mathbf{d}^{\$\prime}_1\boldsymbol{\omega}^{\$}_{t+1} + \boldsymbol{\omega}^{\$\prime}_{t+1}\mathbf{D}^{\$}_2\boldsymbol{\omega}^{\$}_{t+1}\right\}\right] = \exp\left\{-\frac{1}{2}\log\left|\boldsymbol{\Sigma}^{\$}_{\omega}\right| + \frac{1}{2}\log\left|\mathbf{G}^{\$}\right| + \frac{1}{2}\mathbf{d}^{\$}_1\mathbf{G}^{\$}\mathbf{d}^{\$\prime}_1\right\}$$

where $\mathbf{G}^{\$} = \left(\boldsymbol{\Sigma}^{\$-1}_{\omega} - 2\mathbf{D}^{\$}{}_2\right)^{-1}$. Let $g^{\$}_{ij}$ be the ij-th element of $\mathbf{G}^{\$}$. Then expanding and collecting terms gives

$$
p^{\$}_{n,t} = \left[\begin{array}{c}
A^{\$}_{n-1} + B^{\$}_{x,n-1}\left(\mu_x\left(1-\phi_x\right)+\phi_x x_t\right) + B^{\$}_{\delta,n-1}\left(\mu_\delta\left(1-\phi_\delta\right)+\phi_\delta\delta_t\right) \\
+B^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + B^{\$}_{\lambda,n-1}\lambda_t + B^{\$}_{\xi,n-1}\phi_\xi\xi_t \\
+B^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)^2 \\
+C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)^2 + C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) \\
-x_t-\delta_t - \frac{1}{2}z_t^2\sigma_m^2 - \lambda_t - \xi_t - \frac{1}{2}\psi_t^2\sigma_\pi^2 \\
-\frac{1}{2}\log\left|\Sigma^{\$}_\omega\right| + \frac{1}{2}\log\left|\mathbf{G}^{\$}\right| + \frac{1}{2}g^{\$}_{11}B^{\$2}_{x,n-1} + \frac{1}{2}g^{\$}_{22}B^{\$2}_{\lambda,n-1} + \frac{1}{2}g^{\$}_{33}B^{\$2}_{\lambda,n-1}\psi_t^2 + \frac{1}{2}g^{\$}_{44}z_t^2 \\
+\frac{1}{2}g^{\$}_{55}\psi_t^2 + \frac{1}{2}g^{\$}_{66}B^{\$2}_{x,n-1}\psi_t^2 + \frac{1}{2}g^{\$}_{77}B^{\$2}_{\xi,n-1}\psi_t^2 \\
+\frac{1}{2}g^{\$}_{88}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)^2 \\
+\frac{1}{2}g^{\$}_{99}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)^2 \\
+\frac{1}{2}g^{\$}_{10,10}B^{\$2}_{\delta,n-1}\psi_t^2 + \frac{1}{2}g^{\$}_{11,11}B^{\$2}_{\delta,n-1} + g_{12}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1} + g_{13}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}\psi_t \\
-g_{14}B^{\$}_{x,n-1}z_t - g_{15}B^{\$}_{x,n-1}\psi_t + g_{16}B^{\$2}_{x,n-1}\psi_t + g_{17}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}\psi_t \\
+g^{\$}_{18}B^{\$}_{x,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right) \\
+g^{\$}_{19}B^{\$}_{x,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right) \\
+g^{\$}_{1,10}B^{\$}_{x,n-1}B^{\$}_{\delta,n-1}\psi_t + g^{\$}_{1,11}B^{\$}_{x,n-1}B^{\$}_{\delta,n-1} + g^{\$}_{23}B^{\$2}_{\lambda,n-1}\psi_t - g^{\$}_{24}B^{\$}_{\lambda,n-1}z_t \\
-g^{\$}_{25}B^{\$}_{\lambda,n-1}\psi_t + g^{\$}_{26}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1}\psi_t + g^{\$}_{27}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\psi_t \\
+g^{\$}_{28}B^{\$}_{\lambda,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right) \\
+g^{\$}_{29}B^{\$}_{\lambda,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right) \\
+g^{\$}_{2,10}B^{\$}_{\lambda,n-1}B^{\$}_{\delta,n-1}\psi_t + g^{\$}_{2,11}B^{\$}_{\lambda,n-1}B^{\$}_{\delta,n-1} - g^{\$}_{34}B^{\$}_{\lambda,n-1}z_t\psi_t \\
-g^{\$}_{35}B^{\$}_{\lambda,n-1}\psi_t^2 + g^{\$}_{36}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1}\psi_t^2 + g^{\$}_{37}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\psi_t^2 \\
+g^{\$}_{38}B^{\$}_{\lambda,n-1}\psi_t\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right) \\
+g^{\$}_{39}B^{\$}_{\lambda,n-1}\psi_t\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right) \\
+g^{\$}_{3,10}B^{\$}_{\lambda,n-1}B^{\$}_{\delta,n-1}\psi_t^2 + g^{\$}_{3,11}B^{\$}_{\lambda,n-1}B^{\$}_{\delta,n-1}\psi_t + g^{\$}_{45}z_t\psi_t - g^{\$}_{46}B^{\$}_{x,n-1}z_t\psi_t \\
-g^{\$}_{47}B^{\$}_{\xi,n-1}z_t\psi_t \\
-g^{\$}_{48}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)z_t \\
-g^{\$}_{49}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)z_t \\
-g^{\$}_{4,10}B^{\$}_{\delta,n-1}z_t\psi_t - g^{\$}_{4,11}B^{\$}_{\delta,n-1}z_t - g^{\$}_{56}B^{\$}_{x,n-1}\psi_t^2 - g^{\$}_{57}B^{\$}_{\xi,n-1}\psi_t^2 \\
-g^{\$}_{58}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\psi_t \\
-g^{\$}_{59}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\psi_t \\
-g^{\$}_{5,10}B^{\$}_{\delta,n-1}\psi_t^2 - g^{\$}_{5,11}B^{\$}_{\delta,n-1}\psi_t + g^{\$}_{67}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}\psi_t^2 \\
+g^{\$}_{68}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)B^{\$}_{x,n-1}\psi_t \\
+g^{\$}_{69}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)B^{\$}_{x,n-1}\psi_t \\
\vdots
\end{array} \right]
$$

$$\left[\begin{array}{c} +g^{\$}_{6,10}B^{\$}_{x,n-1}B^{\$}_{\delta,n-1}\psi_t^2+g^{\$}_{6,11}B^{\$}_{x,n-1}B^{\$}_{\delta,n-1}\psi_t \\ +g^{\$}_{78}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)B^{\$}_{\xi,n-1}\psi_t \\ +g^{\$}_{79}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)B^{\$}_{\xi,n-1}\psi_t \\ +g^{\$}_{7,10}B^{\$}_{\xi,n-1}B^{\$}_{\delta,n-1}\psi_t^2+g^{\$}_{7,11}B^{\$}_{\xi,n-1}B^{\$}_{\delta,n-1}\psi_t \\ +g^{\$}_{89}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right) \\ \times\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right) \\ +g^{\$}_{8,10}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)B^{\$}_{\delta,n-1}\psi_t \\ +g^{\$}_{8,11}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)B^{\$}_{\delta,n-1} \\ +g^{\$}_{9,10}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)B^{\$}_{\delta,n-1}\psi_t \\ +g^{\$}_{9,11}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)B^{\$}_{\delta,n-1} \\ +g^{\$}_{10,11}B^{\$2}_{\delta,n-1}\psi_t \end{array}\right]$$

Thus, equating coefficients yields

$$A^{\$}_n=\left[\begin{array}{c} A^{\$}_{n-1}+B^{\$}_{x,n-1}\mu_x(1-\phi_x)+B^{\$}_{\delta,n-1}\mu_\delta\left(1-\phi_\delta\right)+B^{\$}_{z,n-1}\mu_z(1-\phi_z) \\ +B^{\$}_{\psi,n-1}\mu_\psi(1-\phi_\psi)+C^{\$}_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2 \\ +C^{\$}_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) \\ -\frac{1}{2}\log\left|\mathbf{\Sigma}^{\$}_\omega\right|+\frac{1}{2}\log\left|\mathbf{G}^{\$}\right|+\frac{1}{2}g^{\$}_{11}B^{\$2}_{x,n-1}+\frac{1}{2}g^{\$}_{22}B^{\$2}_{\lambda,n-1}+g_{12}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1} \\ +g_{18}B^{\$}_{x,n-1}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)+C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\ +g_{19}B^{\$}_{x,n-1}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\ +g^{\$}_{1,11}B^{\$}_{x,n-1}B^{\$}_{\delta,n-1}+\frac{1}{2}g^{\$}_{88}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)+C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2 \\ +\frac{1}{2}g^{\$}_{99}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2+\frac{1}{2}g^{\$}_{11,11}B^{\$2}_{\delta,n-1} \\ +g^{\$}_{28}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)+C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\lambda,n-1} \\ +g^{\$}_{29}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\lambda,n-1} \\ +g^{\$}_{2,11}B^{\$}_{\lambda,n-1}B^{\$}_{\delta,n-1}+g^{\$}_{89}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)+C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\ \times\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\ +g^{\$}_{8,11}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)+C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\delta,n-1} \\ +g^{\$}_{9,11}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\delta,n-1} \end{array}\right]$$

$$B_{x,n}^{\$} = B_{x,n-1}^{\$}\phi_x - 1$$

$$B_{\delta,n}^{\$} = B_{\delta,n-1}^{\$}\phi_\delta - 1$$

$$B_{\lambda,n}^{\$} = B_{\lambda,n-1}^{\$} - 1$$

$$B_{\xi,n}^{\$} = B_{\xi,n-1}^{\$}\phi_\xi - 1$$

$$B_{z,n}^{\$} = \left[\begin{array}{c}
\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)\right)\phi_z \\
-g_{14}B_{x,n-1}^{\$} + 2g_{18}B_{x,n-1}^{\$}C_{z,n-1}^{\$}\phi_z + g_{19}B_{x,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_z - g_{24}^{\$}B_{\lambda,n-1}^{\$} \\
+2g_{88}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z,n-1}^{\$}\phi_z \\
+g_{99}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}^{\$}\mu_z\left(1-\phi_z\right)\right)C_{z\psi,n-1}^{\$}\phi_z \\
+2g_{28}^{\$}B_{\lambda,n-1}^{\$}C_{z,n-1}^{\$}\phi_z + g_{29}^{\$}B_{\lambda,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_z \\
-g_{48}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)\right) \\
-g_{49}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}^{\$}\mu_z\left(1-\phi_z\right)\right) \\
-g_{4,11}^{\$}B_{\delta,n-1}^{\$} \\
+g_{89}^{\$}\left[\begin{array}{c}
2C_{z,n-1}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}^{\$}\mu_z\left(1-\phi_z\right)\right) \\
+C_{z\psi,n-1}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)\right)
\end{array}\right]\phi_z \\
+2g_{8,11}^{\$}C_{z,n-1}^{\$}\phi_z B_{\delta,n-1}^{\$} \\
+g_{9,11}^{\$}C_{z\psi,n-1}^{\$}\phi_z B_{\delta,n-1}^{\$}
\end{array}\right]$$

$$B^{\$}_{\psi,n} = \left[\begin{array}{c} \left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right) + C^{\$}_{z\psi,n-1}\mu_{z}\left(1-\phi_{z}\right)\right)\phi_{\psi} \\ +g_{13}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1} - g_{15}B^{\$}_{x,n-1} + g_{16}B^{\$2}_{x,n-1} + g_{17}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1} \\ +g_{18}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi} + 2g_{19}B^{\$}_{x,n-1}C^{\$}_{\psi,n-1}\phi_{\psi} + g^{\$}_{1,10}B^{\$}_{x,n-1}B^{\$}_{\delta,n-1} \\ +g^{\$}_{23}B^{\$2}_{\lambda,n-1} - g^{\$}_{25}B^{\$}_{\lambda,n-1} + g^{\$}_{26}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1} + g^{\$}_{27}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1} \\ +g^{\$}_{28}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi} + 2g^{\$}_{29}B^{\$}_{\lambda,n-1}C^{\$}_{\psi,n-1}\phi_{\psi} + g^{\$}_{2,10}B^{\$}_{\lambda,n-1}B^{\$}_{\delta,n-1} \\ +g^{\$}_{38}B^{\$}_{\lambda,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_{z}\left(1-\phi_{z}\right) + C^{\$}_{z\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right)\right) \\ +g^{\$}_{39}B^{\$}_{\lambda,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right) + C^{\$}_{z\psi,n-1}\mu_{z}\left(1-\phi_{z}\right)\right) + g^{\$}_{3,11}B^{\$}_{\lambda,n-1}B^{\$}_{\delta,n-1} \\ +g^{\$}_{88}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_{z}\left(1-\phi_{z}\right) + C^{\$}_{z\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right)\right)C^{\$}_{z\psi,n-1}\phi_{\psi} \\ +2g^{\$}_{99}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right) + C^{\$}_{z\psi,n-1}\mu_{z}\left(1-\phi_{z}\right)\right)C^{\$}_{\psi,n-1}\phi_{\psi} \\ -g^{\$}_{58}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_{z}\left(1-\phi_{z}\right) + C^{\$}_{z\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right)\right) \\ -g^{\$}_{59}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right) + C^{\$}_{z\psi,n-1}\mu_{z}\left(1-\phi_{z}\right)\right) - g^{\$}_{5,11}B^{\$}_{\delta,n-1} \\ +g^{\$}_{68}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_{z}\left(1-\phi_{z}\right) + C^{\$}_{z\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right)\right)B^{\$}_{x,n-1} \\ +g^{\$}_{69}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right) + C^{\$}_{z\psi,n-1}\mu_{z}\left(1-\phi_{z}\right)\right)B^{\$}_{x,n-1} \\ +g^{\$}_{6,11}B^{\$}_{x,n-1}B^{\$}_{\delta,n-1} + g^{\$}_{78}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_{z}\left(1-\phi_{z}\right) + C^{\$}_{z\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right)\right)B^{\$}_{\xi,n-1} \\ +g^{\$}_{79}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right) + C^{\$}_{z\psi,n-1}\mu_{z}\left(1-\phi_{z}\right)\right)B^{\$}_{\xi,n-1} + g^{\$}_{7,11}B^{\$}_{\xi,n-1}B^{\$}_{\delta,n-1} \\ +g^{\$}_{89}\left[\begin{array}{c} 2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_{z}\left(1-\phi_{z}\right) + C^{\$}_{z\psi,n-1}\mu_{\psi}(1-\phi_{\psi})\right)C^{\$}_{\psi,n-1} \\ +\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right) + C^{\$}_{z\psi,n-1}\mu_{z}\left(1-\phi_{z}\right)\right)C^{\$}_{z\psi,n-1} \end{array} \right]\phi_{\psi} \\ +g^{\$}_{8,10}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_{z}\left(1-\phi_{z}\right) + C^{\$}_{z\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right)\right)B^{\$}_{\delta,n-1} \\ +g^{\$}_{8,11}C^{\$}_{z\psi,n-1}\phi_{\psi}B^{\$}_{\delta,n-1} \\ +g^{\$}_{9,10}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_{\psi}\left(1-\phi_{\psi}\right) + C^{\$}_{z\psi,n-1}\mu_{z}\left(1-\phi_{z}\right)\right)B^{\$}_{\delta,n-1} \\ +2g^{\$}_{9,11}C^{\$}_{\psi,n-1}\phi_{\psi}B^{\$}_{\delta,n-1} + g^{\$}_{10,11}B^{\$2}_{\delta,n-1} \end{array} \right]$$

$$C^{\$}_{z,n} = \left[\begin{array}{c} C^{\$}_{z,n-1}\phi_{z}^{2} - \frac{1}{2}\sigma_{m}^{2} + \frac{1}{2}g^{\$}_{44} + 2g^{\$}_{88}C^{\$2}_{z,n-1}\phi_{z}^{2} + \frac{1}{2}g^{\$}_{99}C^{\$2}_{z\psi,n-1}\phi_{z}^{2} \\ -2g^{\$}_{48}C^{\$}_{z,n-1}\phi_{z} - g^{\$}_{49}C^{\$}_{z\psi,n-1}\phi_{z} + 2g^{\$}_{89}C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\phi_{z}^{2} \end{array} \right]$$

$$C^{\$}_{\psi,n} = \left[\begin{array}{c} \frac{1}{2}g^{\$}_{33}B^{\$2}_{\lambda,n-1} + C^{\$}_{\psi,n-1}\phi_{\psi}^{2} - \frac{1}{2}\sigma_{\pi}^{2} + \frac{1}{2}g^{\$}_{55} + \frac{1}{2}g^{\$}_{77}B^{\$2}_{\xi,n-1} + \frac{1}{2}g^{\$}_{88}C^{\$2}_{z\psi,n-1}\phi^{\$}_{\psi} + \frac{1}{2}g^{\$}_{10,10}B^{\$2}_{\delta,n-1} \\ +\frac{1}{2}g^{\$}_{66}B^{\$2}_{x,n-1} + g^{\$}_{36}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1} + 2g^{\$}_{39}B^{\$}_{\lambda,n-1}C^{\$}_{\psi,n-1}\phi_{\psi} + g^{\$}_{38}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi} \\ +g^{\$}_{37}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1} - g^{\$}_{35}B^{\$}_{\lambda,n-1} + g^{\$}_{3,10}B^{\$}_{\lambda,n-1}B^{\$}_{\delta,n-1} \\ +2g^{\$}_{99}C^{\$2}_{\psi,n-1}\phi_{\psi}^{2} - g^{\$}_{56}B^{\$}_{x,n-1} - g^{\$}_{57}B^{\$}_{\xi,n-1} - g^{\$}_{58}C^{\$}_{z\psi,n-1}\phi_{\psi} \\ -2g^{\$}_{59}C^{\$}_{\psi,n-1}\phi_{\psi} - g^{\$}_{5,10}B^{\$}_{\delta,n-1} + g^{\$}_{67}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1} + g^{\$}_{68}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi} \\ +2g^{\$}_{69}B^{\$}_{x,n-1}C^{\$}_{\psi,n-1}\phi^{\$}_{\psi} + g^{\$}_{6,10}B^{\$}_{x,n-1}B^{\$}_{\delta,n-1} + g^{\$}_{78}B^{\$}_{\xi,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi} \\ +2g^{\$}_{79}B^{\$}_{\xi,n-1}C^{\$}_{\psi,n-1}\phi_{\psi} + g^{\$}_{7,10}B^{\$}_{\xi,n-1}B^{\$}_{\delta,n-1} + 2g^{\$}_{89}C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi}^{2} \\ +g^{\$}_{8,10}C^{\$}_{z\psi,n-1}\phi_{\psi}B^{\$}_{\delta,n-1} + 2g^{\$}_{9,10}C^{\$}_{\psi,n-1}\phi_{\psi}B^{\$}_{\delta,n-1} \end{array} \right]$$

$$C_{z\psi,n}^{\$}=\left[\begin{array}{c} g_{39}^{\$}B_{\lambda,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_z+2g_{38}^{\$}B_{\lambda,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_z-g_{34}^{\$}B_{\lambda,n-1}^{\$}+C_{z\psi,n-1}^{\$}\phi_z\phi_\psi+g_{45}^{\$} \\ -g_{46}^{\$}B_{x,n-1}^{\$}-g_{47}^{\$}B_{\xi,n-1}^{\$}-g_{4,10}^{\$}B_{\delta,n-1}^{\$}+2g_{88}^{\$}C_{z,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_z\phi_\psi \\ +2g_{99}^{\$}C_{\psi,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_z\phi_\psi-g_{48}^{\$}C_{z\psi,n-1}^{\$}\phi_\psi-2g_{49}^{\$}C_{\psi,n-1}^{\$}\phi_\psi \\ -2g_{58}^{\$}C_{z,n-1}^{\$}\phi_z-g_{59}^{\$}C_{z\psi,n-1}^{\$}\phi_z \\ +2g_{68}^{\$}B_{x,n-1}^{\$}C_{z,n-1}^{\$}\phi_z+g_{69}^{\$}B_{x,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_z+2g_{78}^{\$}B_{\xi,n-1}^{\$}C_{z,n-1}^{\$}\phi_z \\ +g_{79}^{\$}B_{\xi,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_z+g_{89}^{\$}\left(4C_{z,n-1}^{\$}C_{\psi,n-1}^{\$}+C_{z\psi,n-1}^{\$2}\right)\phi_z\phi_\psi \\ +2g_{8,10}^{\$}C_{z,n-1}^{\$}\phi_z B_{\delta,n-1}^{\$}+g_{9,10}^{\$}C_{z\psi,n-1}^{\$}\phi_z B_{\delta,n-1}^{\$} \end{array}\right]$$

where $B_{x,1}^{\$}=-1$, $B_{\delta,1}^{\$}=-1$, $B_{\lambda,1}^{\$}=-1$, $B_{\xi,1}^{\$}=-1$, $C_{z\psi,1}^{\$}=\sigma_{m\pi}$ and all other coefficients are zero at $n=1$.

Stock Returns We model the unexpected stock return as

$$r_{e,t+1}-E_t r_{e,t+1}=\beta_{ex}\varepsilon_{x,t+1}+\beta_{eX}\varepsilon_{X,t+1}+\beta_{em}\varepsilon_{m,t+1},$$

then we have:

$1=E_t\left[\exp\left(r_{e,t+1}+m_{t+1}\right)\right]$

$$=\exp\left(E_t r_{e,t+1}-x_t-\delta_t-\frac{1}{2}z_t^2\sigma_m^2\right)\exp\left(\begin{array}{c} \frac{1}{2}\beta_{ex}^2\sigma_x^2+\frac{1}{2}\beta_{eX}^2\sigma_X^2+\frac{1}{2}\beta_{em}^2\sigma_m^2+\frac{1}{2}z_t^2\sigma_m^2 \\ +\beta_{ex}\beta_{em}\sigma_{xm}-\beta_{ex}z_t\sigma_{xm}+\beta_{eX}\beta_{em}\sigma_{X,m} \\ -\beta_{eX}z_t\sigma_{Xm}-\beta_{em}z_t\sigma_m^2 \end{array}\right),$$

so that

$$r_{e,t+1}=\left[\begin{array}{c} -\frac{1}{2}\beta_{ex}^2\sigma_x^2-\frac{1}{2}\beta_{eX}^2\sigma_X^2-\frac{1}{2}\beta_{em}^2\sigma_m^2-\beta_{ex}\beta_{em}\sigma_{xm}-\beta_{eX}\beta_{em}\sigma_{X,m}+x_t+\delta_t \\ +\left(\beta_{ex}\sigma_{xm}+\beta_{eX}\sigma_{Xm}+\beta_{em}\sigma_m^2\right)z_t+\beta_{ex}\varepsilon_{x,t+1}+\beta_{eX}\varepsilon_{X,t+1}+\beta_{em}\varepsilon_{m,t+1} \end{array}\right]$$

Stock-Real Bond Return Covariance We have

$$r_{n,t+1}-E_t r_{n,t+1}=\left[\begin{array}{c} B_{x,n-1}\psi_t\varepsilon_{x,t+1}+B_{x,n-1}\varepsilon_{X,t+1}+B_{\delta,n-1}\psi_t\varepsilon_{\delta,t+1}+B_{\delta,n-1}\varepsilon_{\Delta,t+1} \\ +C_{z,n-1}\varepsilon_{z,t+1}^2+C_{\psi,n-1}\varepsilon_{\psi,t+1}^2+C_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ +\left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\varepsilon_{z,t+1} \\ +\left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array}\right]$$

Since the ε 's are conditionally jointly normal and mean zero we have $\mathrm{Cov}_t\left(\varepsilon_{a,t+1},\varepsilon_{b,t+1}^2\right)=0$ and $\mathrm{Cov}_t\left(\varepsilon_{a,t+1},\varepsilon_{b,t+1}\varepsilon_{c,t+1}\right)=0$ for all a,b,c. We impose that the only non-zero covariance of $\varepsilon_{X,t+1}$ is σ_{Xm}, and of $\varepsilon_{\Lambda,t+1}$ is $\sigma_{\Lambda m}$. Then, we have

$$\mathrm{Cov}_t\left(r_{e,t+1}, r_{n,t+1}\right) = \beta_{ex}\begin{pmatrix} \left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{x.z} \\ + \left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{x,\psi} \end{pmatrix}$$

$$+ \beta_{eX}B_{x,n-1}\sigma_X^2$$

$$+ \beta_{em}\begin{pmatrix} B_{x,n-1}\sigma_{Xm} + B_{\delta,n-1}\sigma_{\Delta m} \\ + \left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{z,m} \\ + \left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{\psi,m} \end{pmatrix}$$

$$+ \begin{bmatrix} \beta_{ex}\left(2C_{z,n-1}\sigma_{xz}\phi_z + C_{z\psi,n-1}\sigma_{x\psi}\phi_z\right) \\ +\beta_{em}\left(2C_{z,n-1}\sigma_{zm}\phi_z + C_{z\psi,n-1}\sigma_{\psi m}\phi_z\right) \end{bmatrix} z_t$$

$$+ \begin{bmatrix} \beta_{ex}\left(B_{x,n-1}\sigma_x^2 + B_{\delta,n-1}\sigma_{x\delta} + C_{z\psi,n-1}\sigma_{xz}\phi_\psi + 2C_{\psi,n-1}\sigma_{x\psi}\phi_\psi\right) \\ +\beta_{em}\left(B_{x,n-1}\sigma_{xm} + B_{\delta,n-1}\sigma_{m\delta} + C_{z\psi,n-1}\sigma_{zm}\phi_\psi + 2C_{\psi,n-1}\sigma_{\psi m}\phi_\psi\right) \end{bmatrix}\psi_t$$

Stock-Nominal Bond Return Covariance

We model the unexpected stock return as

$$r_{e,t+1} - E_t r_{e,t+1} = \beta_{ex}\varepsilon_{x,t+1} + \beta_{eX}\varepsilon_{X,t+1} + \beta_{em}\varepsilon_{m,t+1}$$

We have

$$r^\$_{n,t+1} - E_t r^\$_{n,t+1} = \begin{bmatrix} B^\$_{x,n-1}\psi_t\varepsilon_{x,t+1} + B^\$_{x,n-1}\varepsilon_{X,t+1} + B^\$_{\delta,n-1}\psi_t\varepsilon_{\delta,t+1} + B^\$_{\delta,n-1}\varepsilon_{\Delta,t+1} \\ +B^\$_{\lambda,n-1}\psi_t\varepsilon_{\lambda,t+1} + B^\$_{\lambda,n-1}\varepsilon_{\Lambda,t+1} + B^\$_{\xi,n-1}\psi_t\varepsilon_{\xi,t+1} \\ +C^\$_{z,n-1}\varepsilon^2_{z,t+1} + C^\$_{\psi,n-1}\varepsilon^2_{\psi,t+1} + C^\$_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ + \left(B^\$_{z,n-1} + 2C^\$_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C^\$_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right)\varepsilon_{z,t+1} \\ + \left(B^\$_{\psi,n-1} + 2C^\$_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C^\$_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{bmatrix}$$

$$\mathrm{Cov}_t\left(r_{e,t+1}, r^\$_{n,t+1}\right) = \beta_{ex}\begin{pmatrix} \left(B^\$_{z,n-1} + 2C^\$_{z,n-1}\mu_z\left(1-\phi_z\right) + C^\$_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{x.z} \\ + \left(B^\$_{\psi,n-1} + 2C^\$_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^\$_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{x,\psi} \end{pmatrix}$$

$$+ \beta_{eX}B^\$_{x,n-1}\sigma_X^2$$

$$+ \beta_{em}\begin{pmatrix} B^\$_{x,n-1}\sigma_{Xm} + B^\$_{\lambda,n-1}\sigma_{\Lambda m} + B^\$_{\delta,n-1}\sigma_{\Delta m} \\ + \left(B^\$_{z,n-1} + 2C^\$_{z,n-1}\mu_z\left(1-\phi_z\right) + C^\$_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{z,m} \\ + \left(B^\$_{\psi,n-1} + 2C^\$_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^\$_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{\psi,m} \end{pmatrix}$$

$$+ \begin{bmatrix} \beta_{ex}\left(2C^\$_{z,n-1}\sigma_{xz}\phi_z + C^\$_{z\psi,n-1}\sigma_{x\psi}\phi_z\right) \\ +\beta_{em}\left(2C^\$_{z,n-1}\sigma_{zm}\phi_z + C^\$_{z\psi,n-1}\sigma_{\psi m}\phi_z\right) \end{bmatrix} z_t$$

$$+ \begin{bmatrix} \beta_{ex}\begin{pmatrix} B^\$_{x,n-1}\sigma_x^2 + B^\$_{\delta,n-1}\sigma_{x\delta} + B^\$_{\lambda,n-1}\sigma_{x,\lambda} + B^\$_{\xi,n-1}\sigma_{x,\xi} \\ +C^\$_{z\psi,n-1}\sigma_{xz}\phi_\psi + 2C^\$_{\psi,n-1}\sigma_{x\psi}\phi_\psi \end{pmatrix} \\ +\beta_{em}\begin{pmatrix} B^\$_{x,n-1}\sigma_{xm} + B^\$_{\delta,n-1}\sigma_{m\delta} + B^\$_{\lambda,n-1}\sigma_{m,\lambda} + B^\$_{\xi,n-1}\sigma_{m,\xi} \\ +C^\$_{z\psi,n-1}\sigma_{zm}\phi_\psi + 2C^\$_{\psi,n-1}\sigma_{\psi m}\phi_\psi \end{pmatrix} \end{bmatrix}\psi_t$$

Volatility of Real Bond Returns We have

$$r_{n,t+1}-E_t r_{n,t+1} = \left[\begin{array}{c} B_{x,n-1}\psi_t\varepsilon_{x,t+1} + B_{x,n-1}\varepsilon_{X,t+1} + B_{\delta,n-1}\psi_t\varepsilon_{\delta,t+1} + B_{\delta,n-1}\varepsilon_{\Delta,t+1} \\ +C_{z,n-1}\varepsilon^2_{z,t+1} + C_{\psi,n-1}\varepsilon^2_{\psi,t+1} + C_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ +\left(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\varepsilon_{z,t+1} \\ +\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array}\right]$$

$$\begin{aligned}
\text{Var}_t\left(r_{n,t+1}\right) = & \left[\begin{array}{c} B^2_{x,n-1}\sigma^2_X + B^2_{\delta,n-1}\sigma^2_\Delta + 2C^2_{z,n-1}\sigma^4_z + 2C^2_{\psi,n-1}\sigma^4_\psi + C^2_{z\psi,n-1}\left(\sigma^2_z\sigma^2_\psi + \sigma^2_{z\psi}\right) \\ +\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2\sigma^2_z \\ +\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2\sigma^2_\psi \\ +2\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\ \times\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{z,\psi} \end{array}\right] \\
& + \left[\begin{array}{c} 4\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z,n-1}\phi_z\sigma^2_z \\ +2\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{z\psi,n-1}\phi_z\sigma^2_\psi \\ +2\left[\begin{array}{c} 2C_{z,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\ +C_{z\psi,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \end{array}\right]\phi_z\sigma_{z,\psi} \end{array}\right] z_t \\
& + \left[\begin{array}{c} 2\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z\psi,n-1}\phi_\psi\sigma^2_z \\ +4\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{\psi,n-1}\phi_\psi\sigma^2_\psi \\ +2\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B_{x,n-1}\sigma_{xz} \\ +2\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B_{x,n-1}\sigma_{x\psi} \\ +2\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B_{\delta,n-1}\sigma_{\delta z} \\ +2\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B_{\delta,n-1}\sigma_{\delta\psi} \\ +2\left[\begin{array}{c} 2C_{\psi,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\ +C_{z\psi,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \end{array}\right]\phi_\psi\sigma_{z,\psi} \end{array}\right]\psi_t \\
& + \left[4C^2_{z,n-1}\phi^2_z\sigma^2_z + C^2_{z\psi,n-1}\phi^2_z\sigma^2_\psi + 4C_{z,n-1}C_{z\psi,n-1}\phi^2_z\sigma_{z,\psi}\right]z^2_t \\
& + \left[\begin{array}{c} B^2_{x,n-1}\sigma^2_x + B^2_{\delta,n-1}\sigma^2_\delta + C^2_{z\psi,n-1}\phi^2_\psi\sigma^2_z + 4C^2_{\psi,n-1}\phi^2_\psi\sigma^2_\psi + 2C_{z\psi,n-1}\phi_\psi B_{x,n-1}\sigma_{xz} \\ +4C_{\psi,n-1}\phi_\psi B_{x,n-1}\sigma_{x\psi} + 4C_{\psi,n-1}C_{z\psi,n-1}\phi^2_\psi\sigma_{z,\psi} + 2B_{x,n-1}B_{\delta,n-1}\sigma_{x\delta} \end{array}\right]\psi^2_t \\
& + \left[\begin{array}{c} 4C_{z,n-1}C_{z\psi,n-1}\phi_z\phi_\psi\sigma^2_z + 4C_{\psi,n-1}\phi_\psi C_{z\psi,n-1}\phi_z\phi_\psi\sigma^2_\psi + 4C_{z,n-1}\phi_z B_{x,n-1}\sigma_{xz} \\ +2C_{z\psi,n-1}\phi_z B_{x,n-1}\sigma_{x\psi} + 2\left(4C_{z,n-1}C_{\psi,n-1} + C^2_{z\psi,n-1}\right)\sigma_{z\psi}\phi_\psi\phi_z \\ +4C_{z,n-1}\phi_z B_{\delta,n-1}\sigma_{\delta z} + 2C_{z\psi,n-1}\phi_z B_{\delta,n-1}\sigma_{\delta\psi} \end{array}\right]z_t\psi_t
\end{aligned}$$

Volatility of Nominal Bond Returns

We have

$$r^\$_{n,t+1}-E_t r^\$_{n,t+1} = \left[\begin{array}{c} B^\$_{x,n-1}\psi_t\varepsilon_{x,t+1} + B^\$_{x,n-1}\varepsilon_{X,t+1} + B^\$_{\delta,n-1}\psi_t\varepsilon_{\delta,t+1} + B^\$_{\delta,n-1}\varepsilon_{\Delta,t+1} \\ +B^\$_{\lambda,n-1}\psi_t\varepsilon_{\lambda,t+1} + B^\$_{\lambda,n-1}\varepsilon_{\Lambda,t+1} + B^\$_{\xi,n-1}\psi_t\varepsilon_{\xi,t+1} \\ +C^\$_{z,n-1}\varepsilon^2_{z,t+1} + C^\$_{\psi,n-1}\varepsilon^2_{\psi,t+1} + C^\$_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ +\left(B^\$_{z,n-1} + 2C^\$_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C^\$_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\varepsilon_{z,t+1} \\ +\left(B^\$_{\psi,n-1} + 2C^\$_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C^\$_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array}\right]$$

$$
\mathrm{Var}_t\left(r^{\$}_{n,t+1}\right) = \left[\begin{array}{c}
B^{\$2}_{x,n-1}\sigma^2_X + B^{\$2}_{\delta,n-1}\sigma^2_\Delta + B^{\$2}_{\lambda,n-1}\sigma^2_\Lambda + 2C^{\$2}_{z,n-1}\sigma^4_z + 2C^{\$2}_{\psi,n-1}\sigma^4_\psi \\
+C^{\$2}_{z\psi,n-1}\left(\sigma^2_z\sigma^2_\psi + \sigma^2_{z\psi}\right) \\
+\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2\sigma^2_z \\
+\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2\sigma^2_\psi \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\lambda,n-1}\sigma_{\psi,\Lambda} \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\delta,n-1}\sigma_{\psi,\Delta} \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
\times\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{z,\psi}
\end{array}\right]
$$

$$
+\left[\begin{array}{c}
4\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z,n-1}\sigma^2_z\phi_z \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{z\psi,n-1}\sigma^2_\psi\phi_z \\
++2C^{\$}_{z\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{\psi,\Lambda}\phi_z + 2C^{\$}_{z\psi,n-1}B^{\$}_{\delta,n-1}\sigma_{\psi,\Delta}\phi_z \\
+2\left[\begin{array}{c}
2C^{\$}_{z,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+C^{\$}_{z\psi,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)
\end{array}\right]\sigma_{z,\psi}\phi_z
\end{array}\right]z_t
$$

$$
+\left[\begin{array}{c}
2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{x,n-1}\sigma_{xz} \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{x,n-1}\sigma_{x\psi} \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\delta,n-1}\sigma_{\delta z} \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\delta,n-1}\sigma_{\delta\psi} \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z\psi,n-1}\sigma^2_z\phi_\psi \\
+4\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{\psi,n-1}\sigma^2_\psi\phi_\psi \\
+2B^{\$2}_{\lambda,n-1}\sigma_{\lambda,\Lambda} + 2B^{\$2}_{\delta,n-1}\sigma_{\delta,\Delta} \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\lambda,n-1}\sigma_{z,\lambda} \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\lambda,n-1}\sigma_{\psi,\lambda} \\
+2B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\sigma_{\Lambda,\xi} + 2B^{\$}_{\delta,n-1}B^{\$}_{\xi,n-1}\sigma_{\Delta,\xi} + 4C^{\$}_{\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{\psi,\Lambda}\phi_\psi + 4C^{\$}_{\psi,n-1}B^{\$}_{\delta,n-1}\sigma_{\psi,\Delta}\phi_\psi \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\xi,n-1}\sigma_{\xi,z} \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\xi,n-1}\sigma_{\psi,\xi} \\
+2\left[\begin{array}{c}
2C^{\$}_{\psi,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
+C^{\$}_{z\psi,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)
\end{array}\right]\sigma_{z,\psi}\phi_\psi
\end{array}\right]\psi_t
$$

$$
+\left[\; 4C^{\$2}_{z,n-1}\phi^2_z\sigma^2_z + C^{\$2}_{z\psi,n-1}\phi^2_z\sigma^2_\psi + 4C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\sigma_{z,\psi}\phi^2_z \;\right]z^2_t
$$

$$
+\left[\begin{array}{c}
B^{\$2}_{x,n-1}\sigma_x^2 + B^{\$2}_{\delta,n-1}\sigma_\delta^2 + B^{\$2}_{\lambda,n-1}\sigma_\lambda^2 + B^{\$2}_{\xi,n-1}\sigma_\xi^2 + 2B^{\$}_{\xi,n-1}B^{\$}_{\delta,n-1}\sigma_{\xi,\delta} \\
+2B^{\$}_{x,n-1}B^{\$}_{\delta,n-1}\sigma_{x,\delta} + 2B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}\sigma_{x,\lambda} + 2B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}\sigma_{x,\xi} \\
+2B^{\$}_{\lambda,n-1}B^{\$}_{\delta,n-1}\sigma_{\lambda,\delta} + 2C^{\$}_{z\psi,n-1}B^{\$}_{x,n-1}\sigma_{xz}\phi_\psi + 4C^{\$}_{\psi,n-1}B^{\$}_{x,n-1}\sigma_{x\psi}\phi_\psi \\
+2C^{\$}_{z\psi,n-1}B^{\$}_{\delta,n-1}\sigma_{\delta z}\phi_\psi + 4C^{\$}_{\psi,n-1}B^{\$}_{\delta,n-1}\sigma_{\delta\psi}\phi_\psi \\
+C^{\$2}_{z\psi,n-1}\phi_\psi^2\sigma_z^2 + 4C^{\$2}_{\psi,n-1}\phi_\psi^2\sigma_\psi^2 + 2B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\sigma_{\xi\lambda} + 2C^{\$}_{z\psi,n-1}B^{\$}_{\lambda,n-1}\sigma^{\$}_{z,\lambda}\phi_\psi \\
+4C^{\$}_{\psi,n-1}B^{\$}_{\lambda,n-1}\sigma^{\$}_{\psi,\lambda}\phi_\psi + 2C^{\$}_{z\psi,n-1}B^{\$}_{\xi,n-1}\sigma_{\xi,z}\phi_\psi + 4C^{\$}_{\psi,n-1}B^{\$}_{\xi,n-1}\sigma_{\psi,\xi}\phi_\psi \\
+4C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\sigma^{\$}_{z,\psi}\phi_\psi^2
\end{array}\right]\psi_t^2
$$

$$
+\left[\begin{array}{c}
4C^{\$}_{z,n-1}B^{\$}_{x,n-1}\sigma_{xz}\phi_z + 2C^{\$}_{z\psi,n-1}B^{\$}_{x,n-1}\sigma_{x\psi}\phi_z + 4C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\sigma_z^2\phi_z\phi_\psi \\
+4C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\sigma_\psi^2\phi_z\phi_\psi + 4C^{\$}_{z,n-1}B^{\$}_{\lambda,n-1}\sigma_{z,\lambda}\phi_z + 2C^{\$}_{z\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{\psi,\lambda}\phi_z \\
+4C^{\$}_{z,n-1}B^{\$}_{\xi,n-1}\sigma_{\xi,z}\phi_z + 2C^{\$}_{z\psi,n-1}B^{\$}_{\xi,n-1}\sigma_{\psi,\xi}\phi_z \\
+4C^{\$}_{z,n-1}B^{\$}_{\delta,n-1}\sigma_{\delta,z}\phi_z + 2C^{\$}_{z\psi,n-1}B^{\$}_{\delta,n-1}\sigma_{\psi,\delta}\phi_z \\
+2\left(4C^{\$}_{z,n-1}C^{\$}_{\psi,n-1} + C^{\$2}_{z\psi,n-1}\right)\sigma_{z\psi}\phi_\psi\phi_z
\end{array}\right]z_t\psi_t
$$