Frictionless Inflation*

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Abstract

The existence of menu costs of price adjustment is one of the leading explanations for the delayed response of prices to changes in economic conditions. This paper investigates the relationship between menu costs of price adjustment at the micro level and aggregate inflation dynamics. For that purpose, it introduces a measure of frictionless inflation that estimates counterfactual inflation in the absence of menu costs. The measure is based on a novel smoother for a state-space representation that describes pricing dynamics at the micro level as implied by a random menu cost model. Combining the smoother with rich micro price data underlying the UK CPI, I produce a measure of frictionless inflation for the UK at a monthly frequency from 1997 to 2018. The analysis of that measure yields three main findings: (i) menu costs matter for aggregate inflation dynamics, but their importance decreased over time; (ii) the response of frictionless inflation to a monetary policy shock is at odds with the monetary transmission mechanism from the basic new Keynesian model and (iii) frictionless inflation contains useful information to forecast CPI inflation.

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1 Introduction

The idea that prices adjust sluggishly to changes in economic conditions lies at the heart of the new Keynesian framework that has emerged as the workhorse for the analysis of monetary policy and its implications for inflation, economic fluctuations and welfare (Woodford, 2003; Galí, 2008). One explanation for the delayed response of prices to shocks is the existence of menu costs; that is, the existence of costs that have to be incurred whenever prices are changed, independently of the size of the change.[1] In the presence of menu costs, price changes will only occur when the resulting increase in profits is sufficiently large to outweigh the associated costs (Mankiw, 1985). In addition to their intuitive appeal, so-called menu cost models are also consistent with some of the stylised facts about price setting observed in micro price data (Klenow and Malin, 2010; Nakamura and Steinsson, 2008, 2013).

This paper evaluates the importance of menu costs at the micro level for aggregate inflation dynamics. In order to do that, the main challenge is to construct a measure of the aggregate inflation that would have been observed in the counterfactual scenario where all the price-setters in the economy were subject to the same economic conditions but could change their prices without incurring in menu costs. Under the assumption that menu costs are the only friction in firms' pricing decisions, this measure is equivalent to the inflation that would result from price setters charging a sequence of prices that maximise their profits period by period. I label that measure *frictionless inflation*.[2,3]

The first contribution of this paper is to provide a methodology that allows the estimation of frictionless inflation at the quote-line level using only information contained in observed prices. The fundamental insight underlying this methodology is to think about individual inflation as an imperfect measurement of frictionless inflation that is observed in the data because of the existence of menu costs. More formally, this methodology is based on a nonlinear and non-Gaussian state-space representation of pricing dynamics at the microeconomic level that is microfounded by a random menu cost model (Stokey, 2009; Alvarez, Bihan and Lippi, 2016). In that representation, the problem of estimating frictionless inflation is tantamount to obtaining the smoothed estimates of an unobserved state variable. Under a set of assumptions about

[1]The classic example of menu costs of price adjustment is the problem of a restaurant owner that has to print new menus whenever the price of an item is changed. In a broader sense, menu costs can be thought of as resulting from costs of information, decision and implementation of a pricing strategy.

[2]I am not the first to use this terminology. The price that maximise firm's profits in the absence of price adjustment frictions have received different labels in the literature, such as, *frictionless profit-maximising prices* (Alvarez, Bihan and Lippi, 2016), *frictionless optimum* (Midrigan, 2011, figure 3) or *static desired price* (Nakamura and Steinsson, 2010, figure 4). In a related paper, Bonomo, Correa and Medeiros (2013) using a different methodology also produce a measure of *frictionless optimal price inflation* for Brazil. Further discussion in the literature review.

[3]In a broad sense, the notion of "frictionless" is necessarily dependent on which frictions are present in the model chosen to characterise price setting decisions. This paper studies price setting decisions as implied by a random menu cost model in which the only mechanism preventing price adjustment is the presence of menu costs. Therefore, the notion of frictionless here adopted is with respect to *one* particular friction, menu costs, and not to other frictions that have been proposed in the literature such as sticky-information (Mankiw and Reis, 2002) or information capacity constraints (Woodford, 2009).

processes governing the evolution of the unobserved state variables, I show that, despite the "kinks" in the system induced by the presence of menu costs, the smoothing problem can be solved in closed form. The methodology introduced here is transferrable to other contexts where fixed costs of adjustment have been used to justify the empirical "lumpiness" of microeconomic dynamics and the optimal policies are characterised by an inaction region and a single reset point. Examples include cash balances, investment and employment.[4]

The second contribution is to provide a measure of frictionless inflation for the UK at a monthly frequency from 1997 to 2018. To construct this measure, I combine the above methodology with micro-price data underlying the UK Consumer Price Index (CPI). The final dataset contains over 23 million quote-lines and over 2.2 million unique-price trajectories and, most importantly, contains price quotes of products representative of all broadly defined consumption categories except for education. Those prices are collected in thousands of outlets across the UK and cover between 55% to 65% of the published CPI by weight.

In terms of findings, first and foremost, I find that menu costs at the microeconomic level matter for aggregate inflation dynamics. I find, however, that the extent to which menu costs matter for aggregate inflation dynamics decreased over time. This conclusion is based on three key observations. First, at any point in time the comparison of frictionless inflation with its regular price inflation counterpart shows that the presence of menu costs can account for a difference of up to 2.29 percentage points on year over year inflation. Second, the time-series of frictionless inflation and regular price inflation co-move positively but not perfectly so (correlation of 0.83 over the whole sample). Third, consistent with a decrease in the importance of menu costs in more recent periods, the differences between the two time-series decreases, whereas their correlation increases in the second half of the sample.

The constructed time-series of frictionless inflation is then used to test two predictions from the basic new Keynesian model in Galí (2008). First, the model predicts that the difference between regular price inflation and frictionless inflation (the inflation wedge) should be negatively correlated with changes in the output gap. Second, in response to a monetary policy shock, frictionless inflation should decrease more than regular price inflation upon impact of the shock and less in subsequent periods. In the data, I find that the co-movement between the inflation wedge and changes in the output gap is in line with the new Keynesian model predictions, but the responses to monetary policy shocks are not. In particular, and in contrast with the monetary policy transmission mechanism in the basic New Keynesian model, I find that frictionless inflation does not react significantly to monetary policy shock at any horizon up to 3 years.

Finally, I investigate to what extent the constructed measure of frictionless inflation contains information that could be used to improve the forecasts of published CPI inflation. In particular,

[4] A non-exhaustive list includes: Miller and Orr (1966), Frenkel and Jovanovic (1980), Alvarez and Lippi (2009) on cash balances; Dixit and Pindyck (1994), Caballero, Engel and Haltiwanger (1995) on investment; Caballero, Engel and Haltiwanger (1997), Elsby and Michaels (2019) on employment.

I revisit the forecasting exercise of Blinder and Reis (2005) and compare forecasts from a linear forecasting regression that uses only the information on past inflation with forecasts from a regression that also combines information on past inflation and past frictionless inflation. At different forecasting horizons, I find evidence that frictionless inflation can contain useful information to forecast published inflation both in sample and out-of-sample.

Relation to the literature This paper relates to four strands of literature. First, it relates to a literature quantifying the size of price adjustment costs and their importance for individual pricing decisions (Levy, Bergen, Dutta and Venable, 1997; Blinder, Canetti, Lebow and Rudd, 1998; Dutta, Bergen, Levy and Venable, 1999; Zbaracki, Ritson, Levy, Dutta and Bergen, 2004; Anderson, Jaimovich and Simester, 2015). It contributes to this literature in two fundamental ways. First, in contrast with previous papers that have focused on quantifying the costs of price adjustments, this paper focuses not on the size of menu costs *per se* but on their influence on aggregate inflation dynamics. In spirit, this approach is similar to Gorodnicheko and Weber (2016) who focus on the implications of menu costs for the responses of the stock market returns of different firms to a monetary policy surprise. The second contribution is in terms of scope, since my measure of frictionless inflation is based on prices of hundreds of different products that are representative of the typical basket of consumer goods and were collected in thousands of outlets across the UK. This contrasts with the existing literature that relies on very detailed micro data that covers a limited set of products or store chains.

Second, it connects with other papers that have estimated closely related measures of inflation, most notably, the *reset price inflation* from Bils, Klenow and Malin (2012) and the *frictionless optimal price inflation* from Bonomo, Correa and Medeiros (2013). In a context where price-setters follow Ss pricing rules, as it will be assumed in section 2, all these three measures are theoretically equivalent.[5] Having said that, this paper differs and complements Bils, Klenow and Malin (2012) and Bonomo, Correa and Medeiros (2013) in three dimensions. First, the methodology used to construct my measure of frictionless inflation is diametrically different from the existing ones. Second, I use micro price data underlying the UK CPI, whilst Bils, Klenow and Malin (2012) use US CPI micro data and Bonomo, Correa and Medeiros (2013) use Brazilian CPI micro data. Third, the measures of inflation in Bils, Klenow and Malin (2012) and Bonomo, Correa and Medeiros (2013) are used to test implications from different models of price rigidities whereas in my case the estimated measure of frictionless inflation is the key counterfactual of interest to evaluate the impact of menu costs on aggregate inflation dynamics.[6]

[5]My measure of frictionless inflation is conceptually equivalent to the frictionless optimal price inflation from Bonomo, Correa and Medeiros (2013), that is, the inflation that would be observed in the counterfactual scenario where all price-setters chose the prices that maximise their static profits period by period. That measure is conceptually different from the reset price inflation. In the words of Bils, Klenow and Malin (2012, p. 2803), "[...] new prices need not be viewed as frictionless spot prices. If future spot prices are expected to differ from the current spot price, then a newly set price may be influenced by future expected spot prices. Thus, reset price inflation can deviate from spot price inflation.". However, if price-setters follow Ss pricing rules, then frictionless prices and reset prices will differ by a *constant* and, hence, frictionless inflation and reset price inflation coincide. This point is also made in Bonomo, Correa and Medeiros (2013, pp. 19-20).

[6]The key focus of the methodology introduced in Bonomo, Correa and Medeiros (2013) is the estimation of

Third, the proposed state-space representation of pricing dynamics at the individual level is microfounded by a random menu cost model (Dotsey, King and Wolman, 1999; Caballero and Engel, 2007; Stokey, 2009; Nakamura and Steinsson, 2010; Costain and Nakov, 2011a,b; Alvarez, Bihan and Lippi, 2016) which is estimated allowing for rich parameter heterogeneity in order to account for the observed heterogeneity in pricing moments across goods (Carvalho, 2006; Nakamura and Steinsson, 2010; Gautier and Bihan, 2018). This paper contributes to this class of models by developing a methodology that allows the estimation of frictionless inflation based only on the information contained in the observed price spells. As I shall discuss in section 2, this methodology can be applied to other contexts that are characterised by the presence of fixed adjustment costs and "lumpy" microeconomic adjustments, such as, the demand for cash (Miller and Orr, 1966; Frenkel and Jovanovic, 1980), investment (Caballero and Engel, 1999; Baley and Blanco, 2019) and labor adjustment (Caballero, Engel and Haltiwanger, 1997; Elsby and Michaels, 2019).

Finally, this paper also relates to a vast literature on filtering and smoothing in nonlinear and non-Gaussian state-space representations. The intersection of this literature with macroeconomics is mostly confined to the estimation of nonlinear DSGE models (Herbst and Schorfheide, 2015), but it has applications to a wide array of problems across different fields such as GPS tracking, brain imaging, audio signal processing and autonomous navigation.[7] In linear and Gaussian state-space representations filtering and smoothing problems can be solved in closed form (Kalman, 1960; Rauch, Tung and Striebel, 1965). However, for nonlinear non-Gaussian state-space representations filtering and smoothing problems in general do not admit closed form solutions and require the use of some form of numerical approximation.[8] Building from the forward filtering-backward smoothing recursion techniques presented in Kitagawa (1987, 1994, 1996), this paper contributes to the existing literature by deriving a closed-form solution for the filtering and smoothing problems in a particular state-space representation that is consistent with the optimal pricing dynamics implied a random menu cost model. The reduction in computational burden from having closed form solutions allows me to solve the smoothing problem for each of the millions of individual quote-lines underlying the construction of the UK CPI.

Structure of the paper The remainder of the paper is structured as follows. Section 2 introduces a state-space representation of pricing dynamics at the microeconomic level. Section 3 starts from that representation and presents a methodology to estimate frictionless inflation

strategic complementarities and the measure of frictionless optimal price inflation and it's subsequent analysis are produced as a by-product of their main exercise.

[7]Some applications of filtering and smoothing problems and underlying references are presented in Särkkä (2013, chapter 1) and in Chen (2003, section VIII).

[8]The literature on different methods is large and it includes: extended Kalman filters and smoothers, the unscented Kalman filters and smoothers, grid based approximation methods, sequential Monte Carlo methods or particle filters and smoothers. All these methods are presented in Särkkä (2013). A good survey of particle filters in particular is Doucet and Johansen (2011). Some of these methods in the literature are not suitable for my application as they require differentiability of the measurement equation (e.g. filters and smoothers of the extended Kalman family). Others can require a prohibitive computational time to implement at the scale necessary to re-construct the whole CPI (e.g. particle filters).

from observed quote-lines. Section 4 illustrates the properties of the proposed methodology through a Monte Carlo experiment. Section 5 describes the micro price data used to construct the measure of frictionless inflation for the UK. Section 6 discusses the parameter estimates, the model fit and the dynamics of frictionless inflation for the UK from 1997 to 2018. Section 7 uses the constructed measure of frictionless inflation to test two predictions from the basic new Keynesian model. Section 8 investigates whether frictionless inflation contains useful information to forecast actual published inflation. Section 9 concludes.

2 The microeconomic dynamic of prices with menu costs

This section derives a state-space representation of pricing dynamics at the microeconomic level. That representation will be foundational for the estimation of frictionless inflation and it must satisfy three requirements. First, it has to be consistent with the theoretically implied optimal pricing policy of a price-setter that faces fixed adjustment costs. Second, as a data generating process for a large number of quote-lines it has to be flexible enough to match the main facts about prices observed in the micro price data. Third, it has to provide a connection between the evolution of the observed price inflation and the unobserved frictionless inflation that is the ultimate object of interest.

2.1 Microfoundation: a random menu cost model

The microfoundation for the state-space representation of pricing here proposed is a random menu cost model. The main feature that distinguishes a random menu cost model from a "canonical" menu cost model (Barro, 1972; Sheshinski and Weiss, 1977, 1983; Golosov and Lucas, 2007) is that the size of the menu cost is allowed to stochastically change over time. The class of random menu cost models is here preferred over its canonical counterpart because the later cannot generate the large number of small price changes that is observed in micro price data (Midrigan, 2011). By adding the possibility that the menu costs change over time, the class of random menu cost models, also known as second-generation state-dependent models, has more flexibility to match the shape of the distribution of price changes observed in the data and, in particular, the large number of small price changes (Klenow and Kryvstov, 2008).[9]

[9] Adding the possibility that menu costs change stochastically over time is not the only way of matching the shape of the distribution of price changes observed in the micro price data. Building on the empirical evidence from Lach and Tsiddon (2007), Midrigan (2011) introduced economies of scope in price setting in a general equilibrium framework by considering a setting where multi-product firms pay a single fixed cost to re-price all of their goods. He shows that not only his model can better fit the features of the micro price data than Golosov and Lucas (2007) but also has different implications for the real effects of monetary policy. Alvarez *et al.* (2016) combine multi-product firms with a random menu cost and argue that that combining these two features is important to match the shape of the distribution of price changes observed in their data, in particular, the positive excess kurtosis. In this paper, the random menu cost model for a single product is chosen instead of a multi-product for two fundamental reasons. First and foremost, the micro price data underlying the CPI that is used here contains typically the quote-line of one particular item sold in a particular outlet and, although the outlet itself can be viewed as a multi-product firm, the data collected is typically for one single item and not for all the products sold in the outlet. Second, the presence of multiple products would necessarily increase the dimensionality of the state-space representation making it intractable for the propose of non-linear smoothing and filtering.

Since Dotsey, King and Wolman (1999), different variants of the random menu cost model have been used in the literature.[10] Because of its tractability and transferability, I choose environment presented in Stokey (2009) and Alvarez, Bihan and Lippi (2016).

Environment Consider a firm that sells a single good and chooses its prices to maximize the expected discounted value of its profit flows. Time is continuous and discounted at a constant rate r. To change prices the firms must pay an adjustment cost. In a period of length dt that cost is equal to a positive constant c with probability $1 - \lambda dt$ or zero with probability λdt. Let p_t denote the log of the nominal price and $p_t^\star$ denote the log of the *frictionless price*, that is, the log of the price that would maximize the firm's instantaneous flow of profits. It is assumed that $p_t^\star$ follows a brownian motion with drift, that is, $dp_t^\star = \mu dt + \sigma dW_t$ where dW_t is the increment to a standard Wiener process. Define the price gap as $\tilde{x}_t \equiv p_t - p_t^\star$. It is assumed that the profit flow of the firm can be written as $\Pi(\tilde{x}_t; \psi)$, where ψ is a vector of structural parameters that characterise the shape of the profit function. Finally, it is assumed that $\Pi(\cdot; \psi)$ is continuous, strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, +\infty)$.

In this environment, the optimal pricing policy is fully characterized by an inaction region (b, B) and a return point $S \in (b, B)$. The optimal pricing behavior is given by two cases. First, upon arrival of a free adjustment opportunity, change nominal prices such that the price gap equals the return value. Second, if prices cannot be changed costlessly, pay the adjustment cost to reset the price gap to the return value only if the current price gap is outside the inaction region, otherwise, do not change prices.[11]

2.2 A state-space representation of price setting

Given the optimal pricing behavior implied the random menu cost model environment, a three equation state-space representation of the pricing dynamics is proposed. The first equation is the measurement equation that describes the evolution of cumulated price inflation (observable) as a function of two unobservable states, namely, cumulated frictionless price inflation and the arrival of costless adjustment opportunities. The remaining two equations are transition equations that describe the evolution of the unobserved states.[12] Before discussing each of these equations, it is necessary to define some variables and notation.

[10] Some examples include Caballero and Engel (2007), the smoothly state-dependent pricing model of Costain and Nakov (2011a,b) and the CalvoPlus model in Nakamura and Steinsson (2010).

[11] For a formal proof, refer to chapter 7 in Stokey (2009). In particular, propositions 7.1 to 7.3 and the extension to include random opportunities of costless adjustment in section 7.5.

[12] The reason why the state-space representation is cast in terms of cumulated inflation and cumulated frictionless inflation and not in terms of prices and frictionless prices is the lack of parameter identification in the representation in levels. In a separate note, I formally study parameter identification through moments of the distribution of price changes in the context of the random menu cost model in Alvarez, Bihan and Lippi (2016) that underlies the state-space representation of pricing here proposed. On the one hand, I show that the return point S is not locally identified through any subset of commonly used moments of the distribution of price changes. On the other hand, the differences $(S - B)$ and $(S - b)$ and the remaining parameters (μ, σ and λ) are locally identified as long as the frequency of price changes is included in the set of moments used for estimation. Expressing the dynamics of pricing through the cumulated inflation and its frictionless counterpart allows me to estimate only parameters that are locally identified in that representation.

Definitions and notation In the data, each firm is equivalent to a quote-line which is uniquely identified by a triplet of subscripts (i, j, t) where i refers to a particular outlet, j a particular product and t a month. The optimal inaction region for a given quote-line is given by $(b_{i,j},\, B_{i,j})$ and the optimal return point by $S_{i,j}$. The cumulated inflation at the quote-line level is defined as $Z_{i,j,t} \equiv p_{i,j,t} - p_{i,j,0}$ where $p_{i,j,0}$ is the (log of) the initial price for that quote-line. Analogously, the cumulated frictionless inflation is defined as $Z^{\star}_{i,j,t} \equiv p^{\star}_{i,j,t} - p^{\star}_{i,j,0}$ where $p^{\star}_{i,j,0}$ is the (log of) the initial frictionless price for that quote-line. Notice that by construction $Z_{i,j,0} = Z^{\star}_{i,j,0} = 0$. Define the re-centred price gap as $x_{i,j,t} \equiv p_{i,j,t} - p^{\star}_{i,j,t} - S_{i,j}$ and the re-centred inaction region as $(\underline{x}_{i,j}, \bar{x}_{i,j})$ where $\underline{x}_{i,j} \equiv b_{i,j} - S_{i,j}$ and $\bar{x}_{i,j} \equiv B_{i,j} - S_{i,j}$. Henceforth, price gap implicitly refers to the re-centred price gap and inaction region refers to the re-centred inaction region. Note that using the definitions above one can write $x_{i,j,t} = Z_{i,j,t} - Z^{\star}_{i,j,t} + x_{i,j,0}$. Finally, let $L_{i,j,t}$ be an indicator variable equal to one if prices can be changed for free.

The measurement equation The dynamics of cumulated inflation is given by:

$$Z_{i,j,t} = \underbrace{Z_{i,j,t-1}}_{\substack{\text{Cumulated}\\ \text{inflation}\\ \text{if price is}\\ \text{unchanged}}} d_{i,j,t} + \underbrace{(Z^{\star}_{i,j,t} - x_{i,j,0})}_{\substack{\text{Value of cumulated}\\ \text{inflation that closes the}\\ \text{price gap}}} (1 - d_{i,j,t}) \tag{1}$$

where $d_{i,j,t}$ is an indicator variable equal to one when not changing the prices is optimal. Whether inaction is optimal depends on the cumulated frictionless inflation and the arrival of costless adjustment opportunities,

$$d_{i,j,t} = \underbrace{\mathbb{1}\{Z_{i,j,t-1} - Z^{\star}_{i,j,t} + x_{i,j,0} \in (\underline{x}_{i,j}, \bar{x}_{i,j})\}(1 - L_{i,j,t})}_{\substack{\text{=1 if price cannot be changed for free}\\ \text{and price gap at previous period price}\\ \text{is inside the inaction region}}} + \underbrace{\mathbb{1}\{Z^{\star}_{i,j,t} = Z_{i,j,t-1} + x_{i,j,0}\}L_{i,j,t}}_{\substack{\text{=1 if price can be changed for free}\\ \text{but the price gap is already closed}\\ \text{at previous period price}}} \tag{2}$$

Notice that (1) and (2) describe the dynamics of cumulated inflation in a form that follows from the theoretically optimal pricing policy in the random menu cost environment and, at the same time, provide a (non-linear) connection between the observed cumulated inflation and its frictionless counterpart.

Transition equation for frictionless inflation I assume that cumulated frictionless inflation evolves according to the following transition equation,

$$Z^{\star}_{i,j,t} = \mu_{i,j} + Z^{\star}_{i,j,t-1} + \varepsilon_{i,j,t} \tag{3}$$

where $\varepsilon_{i,j,t} \sim \mathcal{N}(0, \sigma^2_{\varepsilon,i,j})$ with $\varepsilon_{i,j,t}$ independent and identically distributed across i, j and t. There are three main motivations for the choice of (3). First, the random walk with drift is the closest discrete-time counterpart of the assumption that frictionless prices evolve according to brownian motion with drift as in the random menu cost environment used as a microfoundation. Second, beyond the realm of pricing models, the representation in (3) brings the state-space representation here proposed closer to a wider range of stochastic impulse control problems in which in the absence of control the relevant state variable is assumed to follow a brownian motion with drift (see section 2.4). Third, as we shall later see, the normality assumption will allow the derivation of closed form expressions for the filtering and smoothing probability densities despite the kinks present in the measurement equation.

In the pricing literature, some papers have assumed normality of idiosyncratic shocks affecting the firm's frictionless price (e.g Golosov and Lucas, 2007; Nakamura and Steinsson, 2010) but also other distributions have been considered. For instance, Gertler and Leahy (2008) and Midrigan (2011) use a Poisson distribution for productivity and good's quality shocks, respectively. More recently, Karadi and Reiff (2019) use a mixture of normal distributions for good's quality shocks. Given the unobservable nature of the shocks that affect firm's frictionless prices, the choice of what distribution to use is largely driven by the ability of each specification of matching features of the micro price data. In section 6.2, it will be shown that using (3) and allowing for parameter heterogeneity at the product level produces a good fit of the UK micro price data, specially, of the overall distribution of price changes.

It is important to notice two limitations of (3). First, there are no strategic complementarities in the sense that there is no dependency of frictionless prices on the prices observed in the other quote-lines. In a general equilibrium setup this could be justified in setups like Golosov and Lucas (2007) and Alvarez, Bihan and Lippi (2016). A second limitation of (3) is that there are no common shocks driving the fluctuations of frictionless prices. Later, I will partially address this drawback by assuming that trend in the frictionless prices is common across quote-lines of a given product.[13]

Transition equation for costless adjustment opportunities I assume that the arrival of costless adjustment opportunities is given by,

$$L_{i,j,t} = \mathbb{1}\{\nu_{i,j,t} \leqslant \lambda_{i,j}\} \tag{4}$$

where $\nu_{i,j,t} \sim \text{Uniform}\,(0,1)$ and independent and identically distributed across i, j and t. It is common in the class of random menu cost models to assume that the menu cost draws are independent and identically distributed across firms and time. Some specifications assume those draws are done from a specific distribution with weakly positive support (e.g. Dotsey,

[13]In a specification like (3) one way of introducing common shocks would be to have a factor structure in the disturbances with a common component across quote-lines of a product, for example, to include a disturbance of the form $\omega_{j,t} + \varepsilon_{i,j,t}$ where $\omega_{j,t} \sim \mathcal{N}(0, \sigma_j^2)$ and i.i.d across j and t. This specification would pose two additional challenges. The first one is identification, as it would be difficult to separate the variances of the two shocks from moments of the distribution of price changes. Second, and perhaps most importantly, this would make the filtering and smoothing problems more difficult to solve since the density of frictionless inflation would have to be conditioned not only on the observed inflation for that quote-line but also on the observed price-spells for *all* the other quote-lines of the same product. I am currently working on this extension.

King and Wolman, 1999). Here, in line with the environment used in the microfoundation for the state-space representation, it assumed that the menu cost is drawn from a two-point distribution with values zero or some positive constant.[14] Equations (1) to (4) form a nonlinear state-space representation of the pricing dynamics at the quote-line level that will be used to estimate frictionless inflation.

2.3 Semi-structural representation

The state-space representation in (1) to (4) is semi-structural. On the one hand, it is structural since the form of the measurement equation is theoretically grounded on the optimal behavior of a firm in a random menu cost environment. On the other hand, it is reduced form because the six parameters that enter that representation are functions of deep structural parameters that are left unspecified, such as those that describe the firm's production function, the demand that the firm faces, the size of the menu costs or any structural trends in factors that affect the firm's marginal cost. Also, the idiosyncratic shocks entering (3) are reduced form ones as they represent a combination of a variety of structural shocks that affect the firm's frictionless prices (e.g. technology shocks, demand shocks, quality shocks, among others).

I adopt this semi-structural approach instead of a fully structural one for three fundamental reasons. First and most importantly, for the main purpose of obtaining estimates of frictionless inflation it is sufficient to focus on the parameters that enter the state-space representation (1) to (4). Second, with this approach less restrictions on specific unknown functional forms are needed which is an advantage both from the estimation perspective, since imposing incorrect restrictions could lead to inconsistent estimates of the parameters of interest, and from the perspective of transferability of the current methodology to other contexts in which (S, s) type of behaviors is optimal. Third, the use of a reduced form representation reduces the computational costs of simulation-based estimation which facilitates parameter estimation allowing for heterogeneity across products which, in turn, allows the match empirical features of the distribution of price changes at more disaggregated levels.[15]

2.4 Generalizations and special cases

In terms of the pricing literature, the random menu cost environment used as a microfoundation of our state-representation nests two important models of price adjustment as special cases. The canonical menu cost model used in Golosov and Lucas (2007) is obtained when the are no possibilities of adjusting for free ($\lambda = 0$). The Calvo (1983) model is obtained if changing prices is infinitely costly ($c \to \infty$), in that case, the arrival of a free adjustment opportunity

[14] This is also similar to the specification of the CalvoPlus model in Nakamura and Steinsson (2010) with the exception that there the menu cost can take two positive values (low versus high menu costs).

[15] The reduction in computational burden comes from the fact that for a given combination of the parameters of interest, the state-space representation can be used to directly simulate the distribution of price changes without the need of solving the underlying economic model first. Given the non-convexities due to the presence of fixed costs of adjustment, solving the model typically requires the use of global methods (e.g. value function interaction) which can be costly in terms of computational time.

which can be equivalently interpreted as the arrival of the "Calvo fairy".[16] This environment can also be adapted to some other contexts where fixed adjustment costs have been used to rationalize the "lumpiness" of adjustments observed in the micro data and, notably, to any context where the optimal policy is derived or assumed to take the form of an inaction region and a single return point. Examples include the model of demand for money by firms of Miller and Orr (1966), the model of demand for cash by Frenkel and Jovanovic (1980), the model of investment in Caballero and Engel (1999) and the models of labor adjustments in Caballero, Engel and Haltiwanger (1997) and Elsby and Michaels (2019).[17]

3 Constructing a measure of frictionless inflation

Given the state-space representation of pricing dynamics introduced in the previous section, this section presents a methodology to estimate cumulated frictionless inflation at the quote-line level. The key insight that underlies the methodology is that cumulated frictionless inflation is a hidden state variable and to estimate it given all the data available is tantamount to solving a specific smoothing problem. This section starts by formally defining that problem and subsequently presents its solution in two blocks: one to deal with parameter estimation and other to solve for the smoothing density taking parameters as given.

3.1 Frictionless inflation estimation: a smoothing problem

Consider the state-space representation of price dynamics from the previous section:

$$Z_{i,j,t} \;=\; Z_{i,j,t-1}\, d_{i,j,t} \;+\; (Z^{\star}_{i,j,t} - x_{i,j,0})\,(1 - d_{i,j,t}) \tag{5}$$

$$Z^{\star}_{i,j,t} \;=\; \mu_{i,j} \;+\; Z^{\star}_{i,j,t-1} \;+\; \varepsilon_{i,j,t} \tag{6}$$

$$L_{i,j,t} \;=\; \mathbb{1}\{\nu_{i,j,t} \leqslant \lambda_{i,j}\} \tag{7}$$

where $d_{i,j,t}$ is given by (2), $\varepsilon_{i,j,t} \sim \mathcal{N}(0, \sigma^2_{\varepsilon,i,j})$ and $\nu_{i,j,t} \sim \text{Uniform}\,(0,1)$ both independent across i, j and t and $Z_{i,j,0} = Z^{\star}_{i,j,0} = 0$. The state-space representation above is taken as the data generating process for each quote-line in the micro price data. Let $Z^{T_{i,j}}_{i,j} = \{Z_{i,j,t}\}^{T_{i,j}}_{t=0}$ denote all the observed data for a given quote-line and $\Theta_{i,j}$ denote vector containing all the parameters in the data generating process for that quote-line. The aim is to obtain a *smoothed estimate* of

[16] In terms of the reduced form parameters in our state-space representation this could be obtained by imposing $\underline{x}_{i,j} \to -\infty$ and $\bar{x}_{i,j} \to \infty$.

[17] It is important to notice that the state-space representation here proposed and all the associated filtering and smoothing methods are based on a policy of the form (b, S, B) which arises in an environment where the fixed costs are the *only* form of adjustment costs. For prices this is typically the case, but for other contexts like investment, inventories and durable consumption the adjustment cost can involve a fixed *and* a proportional cost. In that case, the optimal policy is characterized by four parameters (b, q, Q, B) and the optimal policy is given by inaction if the relevant state is in (b, B), adjust such that the state jumps to q (Q) if the state is smaller (bigger) than b (B) (Stokey, 2009, chapter 8). Extensions of the methodology here proposed to encompass also those cases are left for future research.

the hidden state $Z^{\star}_{i,j,t}$ or, more concretely, to obtain an estimate of $\mathbb{E}\left[Z^{\star}_{i,j,t} \,|\, Z^{T_{i,j}}_{i,j}; \Theta_{i,j}\right]$.[18]

3.2 Parameter estimation: A two-stage procedure

To obtain the smoothed estimates of frictionless inflation it is necessary to estimate the parameters that enter the state-space representation (5) to (7). For each quote-line these parameters are: the initial price gap ($x_{i,j,0}$), the lower and upper bounds of the inaction region ($\underline{x}_{i,j}$, $\bar{x}_{i,j}$), the drift of cumulated frictionless inflation process ($\mu_{i,j}$), the volatility of idiosyncratic shocks to frictionless inflation ($\sigma_{\varepsilon,i,j}$) and the rate of arrival of free adjustment opportunities ($\lambda_{i,j}$).

Following a large literature that has used micro price data to estimate or calibrate parameters of pricing models, the estimation procedure proposed is based on the moment conditions from the distribution of price changes. Due to the limited number of price changes observed per-quote line, it would be impossible to allow the parameters to vary at the quote-line level as in the most general formulation (5) to (7). Instead, with the exception of the initial price gap, all the parameters will be assumed to vary at the product level.[19] More precisely, for a given product in the data the vector of parameters of interest is $\Theta_j = \{\theta_j, \{x_{i,j,0}\}_{i=1}^{\mathrm{N}_j}\}$ where $\theta_j = \{\underline{x}_j, \bar{x}_j, \mu_j, \sigma_{\varepsilon,j}, \lambda_j\}$ and N_j is the number of quote-lines for a given product. With parameter heterogeneity and the absence of any other common components across products, the estimation of parameters of interest can done for each product separately.

Given that the state-space representation (5) to (7) does not admit a closed form solution for the distribution of price changes as a function of the parameters of interest, this paper uses simulated method of moments (SMM) (Gouriéroux and Monfort, 1996; Adda and Cooper, 2003; Davidson and MacKinnon, 2004) to estimate the vector of parameters Θ_j at the product level. More precisely, let $\mathcal{Z}_j = \{\{Z(\varepsilon_{i,j,t}, \nu_{i,j,t}, \theta^{o}_j, x^{o}_{i,j,0}\}_{t=0}^{T_{i,j}}\}_{i=1}^{N_j}$ be a panel of observed data with all the quote-lines of a given product where θ^{o}_j and $x^{o}_{i,j,0}$ denote the true (unknown) parameter values. Let $\mathcal{Z}^s_j(\Theta_j) = \{\{Z(\varepsilon^s_{i,j,t}, \nu^s_{i,j,t}, \theta_j, x_{i,j,0}\}_{t=0}^{T_{i,j}}\}_{i=1}^{N_j}$ be one panel of simulated data obtained fixing a value of Θ_j, drawing sequences of shocks $\{\{\varepsilon^s_{i,j,t}, \nu^s_{i,j,t}\}_{t=0}^{T_{i,j}}\}_{i=1}^{N_j}$ from their respective distributions and using the state-representation to generate the data. Each panel of simulated data is indexed by $s = 1, ..., S$ and has the same structure as the panel of observed data in terms of the number of quote-lines and their initial and end dates.[20] Finally, denote by $m(\cdot)$ a

[18] Note that the smoothing problem is conditional on the all the observations for that quote-line. This is the case due to the absence of any linkages across the quote-lines, so that conditioning on all the available measurements (i.e. all the quote-lines) is equivalent to conditioning only on the measurements for that specific quote-line. In the most general form were linkages across quote-lines are allowed, the smoothing problem is to obtain an estimate of $\mathbb{E}\left[Z^{\star}_{i,j,t} \,|\, \mathcal{Z}; \Theta\right]$ where $\mathcal{Z}$ denotes the panel with all the quote-lines observed and Θ is the vector of parameters that describe the data generating process for all the quote-lines.

[19] A product in the data is the same as an *item* in the ONS terminology. An item is identified by a unique six digit code. This level of disaggregation is still less disaggregated than bar code level data as it does not specify a specific product variety (including a specific brand) but it is the most disaggregated information that is given to price collectors. Examples include: "Large loaf white sliced 800 grams" (item id 210101) or "Ultra-low sulphur diesel 10L" (item 610304). More details on the data in section 5.

[20] The general principle in simulated based estimation is that of treating real and simulated data as similarly as possible. In a recent paper, Berger, Caballero and Engel (2018) show that using a number of agents in simulations that is larger than the number of agents in actual data can lead to underestimate the shock persistence in a

$p \times 1$ vector containing moment conditions that are functions of a panel of data. The estimator for the SMM is defined as,

$$\widehat{\Theta}_j = \arg\min_{\Theta_j} \left\| \Omega^{\frac{1}{2}} \left(m(\mathcal{Z}_j) - \frac{1}{S} \sum_{s=1}^{S} m\left(\mathcal{Z}_j^s(\Theta_j)\right) \right) \right\|^2 \tag{8}$$

where $\|\cdot\|$ denotes the Euclidean norm and Ω is a symmetric and positive definite weight matrix.

Since the vector Θ_j for each product contains five common parameters plus one initial re-centred price gap for each quote-line, avoiding under-identification in (8) would require a prohibitively large number of moment conditions even for relatively small number of quote-lines per product. To overcome this problem, this paper uses a two-stage procedure. The first stage estimates the vector of common parameters by using a set of moment conditions that do *not* depend on the values of the initial price gaps. In the second step, the vector of common parameters is fixed and a set of moment conditions that depend on both the common parameters and the initial price gap are used to estimate the initial price gaps for each quote-line.

3.2.1 First stage: common parameters

The choice of moment conditions for the first stage is based on the following result,

Proposition 1 *Consider the state-space representation given by* (5), (6) *and* (7). *For a given quote-line, let* $\tau_{i,j}^1$ *denote the time at which the first price change is observed. For any* $t > \tau_{i,j}^1$, $\Delta p_{i,j,t}$ *does not depend on* $x_{i,j,0}$.

Proof. See appendix A.1 □

This result explores the fact that whenever a price change occurs it must be such that it closes the re-centred price gap and what happens from then onwards does not depend on the previous values of the price gap. In other words, price changes that occur after the re-centred price gap has been closed once will not depend on the value of the re-centred price gap when the quote-line started. This is illustrated in figure 1.

For a given product, let Δp_j be a vector containing all the price changes and $\Delta p_{j\,|\,t>\tau^1}$ be a vector containing all the price changes that occur after the first price change for each quote-line. Denote by Δp_j^s and $\Delta p_{j\,|\,t>\tau^1}^s$ their respective counterparts in simulated data.[21] Proposition 1 implies that whereas Δp_j and Δp_j^s are functions of both common parameters and initial price gaps, $\Delta p_{j\,|\,t>\tau^1}$ and $\Delta p_{j\,|\,t>\tau^1}^s$ depend only on the vector of common parameters (θ_j).

The vector of common parameters is estimated from,

Calvo model (see table 1, p. 12). More details on how simulated data is generated can be found in appendix B.

[21] Implicitly, any quote-line that does not have any price change will be excluded from both either the actual data or the simulated data.

$$\widehat{\theta}_j = \arg\min_{\theta_j} \left\| g(\Delta p_{j|t>\tau^1}) - \frac{1}{S}\sum_{s=1}^{S} g\left(\Delta p^s_{j|t>\tau^1}(\theta_j)\right) \right\|^2 \tag{9}$$

where $g(\cdot)$ denotes a vector valued function containing the following moments: the frequency of price changes; the first and the ninety-ninth percentiles of the distribution of non-zero price changes and all the multiple of five percentiles of the distribution of non-zero price changes.[22] The percentiles of the distribution of non-zero price changes along with the frequency are used for parameter identification, whereas other moments of the distribution of price changes that have been commonly used in the literature (e.g. frequency of price increases and decreases, variance of price changes, skewness and kurtosis) are left to evaluate the model fit (see section 6.2).

3.2.2 Second stage: initial price gaps

To estimate the initial price gap for each quote-line, the common parameters estimated in the first stage are taken as given and the moments conditions have to depend on the initial price gap. More concretely, for each quote-line the initial price gap is estimated from,

$$\hat{x}_{i,j,0} = \arg\min_{x_{i,j,0}} \left\| h(\Delta p_{i,j}) - \frac{1}{S}\sum_{s=1}^{S} h\left(\Delta p^s_{i,j}(x_{i,j,0}, \hat{\theta}_j)\right) \right\|^2 \tag{10}$$

where $\Delta p_{i,j}$ is a vector containing all the price changes for a given quote-line of a product (and $\Delta p^s_{i,j}$ is its counterpart in simulated data), $\hat{\theta}_j$ is the vector of common parameters estimated from (9) and $h(\cdot)$ is a function containing two moment conditions, namely, the number of periods elapsed until the first price change and the value of the first price change.[23] For the quote-lines for which no price change is observed, the initial price gap is estimated by taking the mean of the estimated initial price gaps for the other quote-lines for that specific product.

Parameter identification For the true parameter value Θ^o_j to be globally identified requires that it is the unique minimizer of the population counterparts of (9) and (10). For that to be the case, the objective functions cannot be flat at the true parameter value. For the initial price gap in (10), the inclusion of the value of the first price change in the moment conditions is important. The time elapsed until the first price change is typically concave in the initial price gap with a maximum achieved in an interior point of the inaction region but the value of

[22] The identity is taken as the weight matrix in objective function to estimate the common parameters and the number of replications is set to 50. The SMM estimator is consistent for a given fixed S and for any weight matrix (Gouriéroux and Monfort, 1996, Proposition 2.3), so neither of these choices affects estimator consistency. Other choices of the weight matrix and a large number of replications could yield more precise estimates.

[23] In this case the difference in the data moments and the moments in simulated data in (10) is expressed as percentage deviation from the moment observed in the data. Because of the difference in scales (time elapsed until the first price change is an integer whereas the value of the first price change is typically of the order 10^{-1}), failing to do so with the euclidean norm in the objective function would imply a disproportional weight being given to the time elapsed until the first price change. This and other computational details can be found in appendix B.

the first price change is usually strictly decreasing in the price gap which is helpful to separate cases where there are two initial price gaps that yield the same expected time until the first price adjustment.[24] For some values of the true parameter value, identification from (9) can be challenging specially for the two boundaries of the inaction region. To understand why, consider for instance a case where true parameter values are such that no price changes are triggered by crossing the boundaries of the inaction region (e.g. cases closer to a pure Calvo model) the objective function will be flat at the true parameter value as increasing the boundaries of the inaction region will not affect any price changes and, hence, will not affect any of the moments of the distribution of price changes and, consequently, the objective function. In other words, global identification in (9) requires that at the true parameter value both of boundaries are binding for at least some of the price changes. In terms of the choice of the moment conditions, the inclusion of extreme percentiles of the distribution of the non-zero price changes is important specially in cases where very few price changes occur at the boundaries.[25]

3.3 The smoothed density function

The last ingredient missing to obtain the smoothed estimates of cumulated frictionless inflation is the smoothed density. This section presents closed form solutions for the smoothed density and formally defines the smoothed estimates.[26] Before presenting the results, the relevant notation is introduced.

Notation For notational simplicity, given that all the results are presented at the quote-line level, the subscripts i and j are omitted. For any two real numbers $a < b$, denote by $\mathbb{Z}_{[a,b]}$ the set of all integers in $[a, b]$. Upper case letters are used to denote random variables whereas their lower case counterparts denote realizations of those random variables. For a continuous random variable X, the notation $f_{X|Y}(x|y)$ is used to denote its probability density function evaluated at a specific value x conditional on the random variable Y taking the value y. If X is discrete then $f_{X|Y}(x|y)$ is used to denote its probability mass function. The function $\delta(\cdot)$ is used to denote the Dirac delta function. The functions $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal probability density function and cumulative distribution function, respectively. The subscript t is used to refer to a specific point in time whereas the superscript t is used to refer to the sequence of all random variables or realizations up to that moment in time.[27] Without loss of generality, all the results in this section are presented for an arbitrary quote-line z^T

[24] For the typical relationship between the time elapsed until the first price change and the initial price refer to figure 5.3 in Stokey (2009). The fact that the value of the first price change is strictly decreasing in the initial gap can be intuitively understood. For a quote-line that starts with a negative price gap it is likely that the first price change is triggered either by crossing the lower bound of the inaction region or by receiving an opportunity of adjusting prices for free while the price gap is still negative, in either case, that would generate a positive value for the first price price change. For quote-lines that start with a positive price gap the converse applies and it is more likely that the first price change will be negative.

[25] This point will be illustrated further for one of the data generating processes considered in the Monte Carlo experiment in section 4.

[26] Solving for the smoothed density requires solving for the filtered density first. Expressions for the filtered density are presented in lemmas 2 and 3 in appendix A.2.

[27] For example, z_t refers to a realization of the random variable Z_t at time t whereas z^t refers to all realizations until t, that is, $z^t = \{z_{\tau^0}, \dots, z_t\}$.

that starts at time τ^0 and finishes at time T. Let $K \geqslant 0$ denote the number of non-zero price changes observed for that quote-line and τ^k denote the period at which the k-th non-zero price change occurred. Given the structure of (5) any such quote-line can be fully characterized by the sequences $\{\tau^1, \ldots, \tau^K\}$ and $\{z_{\tau^1}, \ldots, z_{\tau^K}\}$.[28]

For expositional purposes, the expressions for the smoothed density are divided in three cases. First, for the initial period and any periods where a non-zero price change is observed. Second, for periods of inaction for which some non-zero price change is observed afterwards. Last, for periods of inaction for which no non-zero price change is observed before the end of the quote. In all the results, the vector of parameters Θ is taken as given.

Proposition 2 *Consider a quote-line z^T generated from the state-space representation* (5) *to* (7)*. Suppose $t = \tau^k$ for some $k \in \mathbb{Z}_{[0,K]}$, then it holds that,*

$$f_{Z_t^\star \,|\, Z^T;\Theta}\left(z^\star | z^T; \Theta\right) \;=\; \delta\left(z^\star - c^k\right) \tag{11}$$

where $c^k = 0$ for $k = 0$ or $c^k = z_{\tau^k} + x_0$ for $k \in \mathbb{Z}_{[1,K]}$.

Proof. See appendix A.2 □

The result above states that the smoothed density of cumulated frictionless inflation is degenerate at the first time period or whenever a non-zero price change is observed. For the initial time period τ^0 that is trivial since by definition $Z^\star_{\tau^0} = 0$. For the time periods where a non-zero price change is observed this follows immediately, since from (5) whenever a non-zero price change occurs it must be such that at the new value of cumulated inflation the price gap is closed which is the case if and only if $z_t - Z_t^\star + x_0 = 0$.

Now we turn to the time periods of inaction but for which a non-zero price change is observed before the end of the quote-line,

Proposition 3 *Consider a quote-line z^T generated from the state-space representation* (5) *to* (7)*. Suppose $t \in (\tau^k, \tau^{k+1})$ for some $k \in \mathbb{Z}_{[0,K-1]}$. Define $\Delta^k \equiv \tau^{k+1} - \tau^k$, $b \equiv t - \tau^k$, $\underline{Z}^k \equiv \mathbb{1}\{k = 0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \bar{x}$, $\bar{Z}^k \equiv \mathbb{1}\{k = 0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \underline{x}$, $\mathcal{I}^k \equiv (\underline{Z}^k, \bar{Z}^k)$ and c^k is defined as in proposition 2. Then, ignoring terms that are zero almost everywhere, it holds that,*

$$f_{Z_t^\star \,|\, Z^T;\Theta}\left(z^\star | z^T; \Theta\right) \;\propto\; \frac{1}{\breve{\sigma}_b^k}\phi\left(\frac{z^\star - \breve{\mu}_b^k}{\breve{\sigma}_b^k}\right)\; \beta_b^k(z^\star)\; \chi_b^k(z^\star)\; \mathbb{1}\left\{z^\star \in \mathcal{I}^k\right\} \tag{12}$$

where the functions $\beta_b^k(\cdot)$ and $\chi_b^k(\cdot)$ are defined recursively as,

[28] In particular, given those two sequences it follows that: $z_t = 0$ for $t \in [\tau^0, \tau^1)$; $z_t = z_{\tau^k}$ for $t \in [\tau^k, \tau^{k+1})$ and $k < K$; $z_t = z_{\tau^K}$ for $t \in [\tau^K, T]$.

$$\beta_b^k(x) = \begin{cases} 1, \; \textit{if } b = 1 \\ \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\tilde{\sigma}_{b-1}} \phi\left(\frac{y - \tilde{\mu}_{b-1}^k(x)}{\tilde{\sigma}_{b-1}}\right) \beta_{b-1}^k(y)\, dy, \; \textit{if } b > 1 \end{cases} \tag{13}$$

$$\chi_b^k(x) = \begin{cases} 1, \; \textit{if } b = \Delta^k - 1 \\ \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\ddot{\sigma}_{b+1}^k} \phi\left(\frac{y - \ddot{\mu}_{b+1}^k(x)}{\ddot{\sigma}_{b+1}^k}\right) \chi_{b+1}^k(y)\, dy, \; \textit{if } b < \Delta^k - 1 \end{cases} \tag{14}$$

and the means and standard deviations of the distributions are given by,

$$\breve{\mu}_b^k = \frac{bc^{k+1} + (\Delta^k - b)c^k}{\Delta^k} \quad \textit{and} \quad \breve{\sigma}_b^k = \sqrt{\frac{b(\Delta^k - b)}{\Delta^k}}\, \sigma_\varepsilon \tag{15}$$

$$\tilde{\mu}_b^k(x) = \frac{c^k + bx}{b+1} \quad \textit{and} \quad \tilde{\sigma}_b = \sqrt{\frac{b}{b+1}}\, \sigma_\varepsilon \tag{16}$$

$$\ddot{\mu}_b^k(x) = \frac{c^{k+1} + (\Delta^k - b)x}{\Delta^k - b + 1} \quad \textit{and} \quad \ddot{\sigma}_b^k = \sqrt{\frac{\Delta^k - b}{\Delta^k - b + 1}}\, \sigma_\varepsilon \tag{17}$$

Proof. See appendix A.2.5 □

Intuition for this result can be better conveyed through figure 2. First, given that the density refers to a period of inaction, the last term in (12) arises naturally since it results in a zero probability mass being assigned for any value of cumulated frictionless inflation term that would yield a price gap outside the inaction region. Second, the time periods τ^1 and τ^2 can be considered as reference periods relative to a given time period as these are the last and next periods for which the values of cumulated frictionless inflation are known (equal to the reference values c^1 and c^2, respectively). The mean of the normal distribution in the first term of (12) is the linear interpolation of those two reference values, whereas the standard deviation is proportional to the standard deviation of the idiosyncratic shocks driving frictionless inflation but maximised in the middle of these two reference periods.

Finally, for a given time period the density of cumulated frictionless inflation evaluated at a certain value will also depend on the fact that in between the two reference periods it must remain inside the inaction region. Consider the example in figure 2 and suppose that the trend in frictionless inflation is positive. Intuitively, what is the likelihood that at time t frictionless cumulated inflation was equal to $z^\star$? For that to be the case it must be that in the periods between t and τ^2 there was a sequence of consecutive negative idiosyncratic shocks that offset the positive trend such that that cumulated frictionless inflation was equal to c^2 at time τ^2 and did not cross the upper bound of the inaction region. Whilst possible, such a sequence negative shocks is rather unlikely to happen specially given that the shocks are normally distributed with zero mean.

The recursions $\beta_b^k(z^\star)$ and $\chi_b^k(z^\star)$ capture precisely that intuition. In particular, the backward recursion $\beta_b^k(z^\star)$ captures the probability that in period t the cumulated frictionless inflation was equal to some value $z^\star$ given that b periods before it was equal to c^k and it cannot have left the inaction region in any period between τ^k and t. On the other hand, the forward recursion $\chi_b^k(z^\star)$ captures the probability that in period t the cumulated frictionless inflation was equal to some value $z^\star$ given that $\tau^{k+1}-t$ periods afterwards it was to equal c^{k+1} and it cannot have left the inaction region at any point in between t and τ^{k+1}. By entering multiplicatively in (12), these two recursions deform the normal density from the first term to take into account for the boundaries of the inaction region. If the boundaries were not present, as it would be the case for a pure Calvo model, the smoothed density would be given only by the first term in (12) and its expected value would simply be given by the linear interpolation of the two reference values (see corollary 1).

The last proposition characterises the smoothed density for the periods of inaction for which no non-zero price change is observed before the end of the quote-line,

Proposition 4 *Consider a quote-line z^T generated from the state-space representation* (5) *to* (7)*. Suppose $t \in (\tau^K, T]$. Define $b \equiv t-\tau^K$, $\Delta^K \equiv T-\tau^K$ and the remaining objects defined as in propositions 2 and 3. Then, ignoring terms that are zero almost everywhere, it holds that,*

$$f_{Z_t^\star \mid Z^T;\Theta}\left(z^\star | z^T;\Theta\right) \;\propto\; \frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^K}{\sigma_b}\right)\; \beta_b^K(z^\star)\; \iota_b^K(z^\star)\; \mathbb{1}\left\{z^\star \in \mathcal{I}^K\right\} \tag{18}$$

where the function $\beta_b^K(\cdot)$ is defined in (13)*, $\iota_b^K(\cdot)$ is defined recursively as,*

$$\iota_b^K\left(x\right) = \begin{cases} 1, \text{ for } b=\Delta^K \\ \int_{\underline{Z}^K}^{\bar{Z}^K} \frac{1}{\sigma_\varepsilon}\,\phi\left(\frac{y-(\mu+x)}{\sigma_\varepsilon}\right)\, \iota_{b+1}^K\left(y\right)dy, \text{ for } b<\Delta^K \end{cases} \tag{19}$$

and the mean and standard deviation are given by,

$$\mu_b^k = b\mu + c^k \;\;\text{ and }\;\; \sigma_b = \sqrt{b}\,\sigma_\varepsilon \tag{20}$$

Proof. See appendix A.2.6 □

The intuition for the above result is similar to that of proposition 3 with the fundamental difference that in this case there is no period after t for which cumulated frictionless inflation is known. This changes the expression for the smoothed density relative to (12) in three ways. First, the mean of the normal density in the first term is no longer the linear interpolation between two known values, but instead it is given by a linear extrapolation starting from the last known value (c^K) and the rate of growth is given by the drift in the cumulated frictionless inflation process. Second, the standard deviation of the distribution in the first term is still proportional to that of the idiosyncratic shocks but now it is increasing in the distance from the last reference period. Third, the forward recursion $\iota_b^K(z^\star)$ in this case simply captures the probability that at time t cumulated frictionless inflation was equal to $z^\star$ given that in the next $T-t$ periods it cannot have left the inaction region. This case is illustrated in figure 3.

3.4 Smoothed estimates

Combining all the results presented in this section, the smoothed estimates of cumulated frictionless inflation can be formally defined,

Definition *Given a quote-line z^T, the smoothed estimates of $Z_t^\star$, denoted by $\widehat{Z}_t^\star$, are given by,*

$$\widehat{Z}_t^\star = \mathbb{E}\left[Z_t^\star \,|\, z^T; \widehat{\Theta}\right] = \int_{-\infty}^{\infty} z^\star\, f_{Z_t^\star\,|\,Z^T;\Theta}(z^\star|z^T;\widehat{\Theta})\, dz^\star \tag{21}$$

where $f_{Z_t^\star\,|\,Z^T;\Theta}(z^\star|z^T;\widehat{\Theta})$ is obtained from (11), (12) *and* (18) *and $\widehat{\Theta}$ are estimates of the parameter vector for that quote-line obtained from the two-stage procedure in* (9) *and* (10).

The measure of frictionless inflation proposed in this paper is based on computing (21) for each quote-line and period in the micro price data. Finally, for the case of a pure Calvo model the expression in (21) can be simplified further. This is formalised in the following result,

Corollary 1 *Consider a quote-line z^T and let $\widehat{\Theta}^C$ denote a vector of estimated parameters obtained from* (9) *and* (10) *by imposing the restrictions $\underline{x} = -\infty$ and $\bar{x} = +\infty$. Then, the smoothed estimates of $Z_t^\star$ are given by,*

$$\widehat{Z}_t^\star = \mathbb{E}\left[Z_t^\star \,|\, z^T; \widehat{\Theta}^C\right] = \begin{cases} c^k, \text{ if } t = \tau^k \text{ for some } k \in \mathbb{Z}_{[0,K]} \\ \breve{\mu}_b^k, \text{ if } t \in (\tau^k, \tau^{k+1}) \text{ for some } k \in \mathbb{Z}_{[0,K-1]} \\ \mu_b^K, \text{ if } t > \tau^K \end{cases} \tag{22}$$

where c^k is as defined in proposition 2, $\breve{\mu}_b^k$ is defined in (15) *and μ_b^k is defined in* (20).

Proof. See appendix A.2.7 □

The above result follows from the fact that the pure Calvo model is obtained by removing the boundaries of the inaction region. In that case, the last three terms in the smoothed densities (12) and (18) are always equal to one and the smoothed estimates are given by the mean of the first term in those densities.

4 Frictionless inflation: Monte Carlo evidence

Before turning to actual micro price data, this section presents a Monte Carlo experiment to illustrate the properties of the methods introduced in the previous section.

4.1 Monte Carlo setup

Three different data generating processes (DGPs) are considered (see table 1). In all the DGPs the upper and lower bounds of the inaction region are set to 10% and -10%, the drift in cumulated inflation process is set to 0.2% per month and the standard deviation of the idiosyncratic shocks to frictionless inflation is set to 5% per month. The key parameter that varies across DGPs is the rate of arrival of costless adjustment opportunities (λ) and it controls how close the DGP is from two well known benchmarks: the pure menu cost model and the Calvo model. That parameter alone has strong implications for the shape of the distribution of non-zero price changes as illustrated in figure 4. For example, in the first DGP with a low value of λ only 6% of all the price changes occur due to the arrival of free adjustment opportunities whereas in the third DGP that figure is around 87%, which visually translates into a larger mass of small price changes for the third DGP. Price gaps are chosen as equally spaced points inside the inaction region.[29] Finally, for each of the DGPs balanced panels of three sample sizes are considered: 100 quote-lines over 5 years ($N = 100$ and $T = 60$), 100 quote-lines over 20 years ($N = 100$ and $T = 240$) and 300 quote-lines over 5 years ($N = 300$ and $T = 60$).

4.2 Monte Carlo results

For each combination of true parameter values and sample size, 1000 samples of artificial panel data are generated according to the state-space representation in (5), (6) and (7). In each of these samples the two-stage procedure is used to estimate the common parameters and initial price gaps.

4.2.1 First stage: common parameters

Figure 5 contains the kernel densities of estimated common parameters across Monte Carlo replications. First and most importantly, the density becomes more concentrated around the true parameter values with the increase in total sample size - as expected from a consistent estimator. Second, in line with with the choice of first-stage moments motivated by proposition 1, all the results in figure 5 were obtained even though the simulated data in (9) was generated assuming a zero initial price gap for every quote-line whereas the Monte Carlo samples were generated using linearly spaced initial price gaps. Third, for larger values of the arrival of costless adjustment opportunities the boundaries of the inaction region are estimated less precisely, which can be rationalised from the fact that the increase in the mass of non-zero price changes

[29] Other rules could be used here, but it is fundamental that the values of those conditions are kept fixed across Monte Carlo replications. This rule ensures that as it only depends on the boundaries of the inaction region and the cross-sectional sample size which are both kept fixed across replications.

around those boundaries is smaller the larger is that parameter. Most importantly, in spite of that decrease in precision, the boundary parameters were estimated without signs of lack of parameter identification. This reinforces the idea that the inclusion of extreme percentiles of the distribution of non-zero price changes in (9) is helpful for parameter identification, specially in cases where relatively few price changes are triggered by crossing the boundaries of the inaction region.[30]

4.2.2 Second stage: initial price gaps

For each combination of common parameters and sample size, figure 6 presents the histograms for the difference between the estimated initial price gaps ($\hat{x}_{i,j,0}$) and their true values ($x^{o}_{i,j,0}$) across quote-lines and Monte Carlo replications. Notice first that all the histograms are centred at zero which is indicative that the two moments used for the estimation do contain some useful information to estimate the initial price gap. Nonetheless, and in contrast with the estimates of the common parameters in figure 5, the estimates of the initial price gaps do not become more concentrated around their true parameter values with the increase in total sample size. The underlying reason for that is that regardless of the total number of time periods that a quote-line is observed there can only be *one* first price change for each individual. In other words, the estimates of the initial price gap in (10) are based on statistics that involve one single realisation of a random variable and laws of large numbers will not apply.[31]

5 Micro Price Data

To produce estimates of frictionless inflation this paper uses publicly available data on locally collected price quotes underlying the construction of the UK Consumer Price Index (CPI). In order to produce the CPI, the Office for National Statistics (ONS) collects on a monthly basis prices of different goods and services that are selected to be representative of general expenditure across the whole of the UK. There are two price collection methods: central or local. Central collection is used for goods and services for which the price is the same for all UK residents or the regional price variation can be collected with no field work, for example, via internet, telephone or e-mail enquiries. Since some of these price quotes could reveal the identity of the price setter, the ONS excludes those from the publicly available data. For the remaining goods and services, which account for about 60% of the aggregate CPI by weight, price collectors on behalf on the ONS visit every month thousands of shops in over 140 locations spread over the United Kingdom to record over 100,000 prices on hand-held computers. The

[30]Some versions of this experiment that did not include extreme percentiles of the distribution of non-zero price changes yielded a non-negligible fraction of extreme estimates for the boundary parameters, suggesting the lack of parameter identification.

[31]To illustrate this point, consider the problem of estimating the mean of a random variable $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ by taking the first observation from a sample $\{x_1, x_2, \ldots, x_N\}$ of iid observations, in other words, consider $\hat{\mu}_x = x_1$ as an estimator for μ_x. Despite being unbiased (in this particular case), the estimator $\hat{\mu}_x$ will not converge in probability to μ_x as there are no laws of large numbers that apply for an estimator based on a single observation.

baseline sample comprises locally collected price quotes from 1996m1 to 2018m1.[32]

5.1 Sample Selection

Some price quotes available in the baseline sample are excluded from the final sample used for the construction of frictionless inflation measures. First, price quotes that did not enter the actual CPI calculation because they did not pass the ONS internal validation procedures are excluded.[33] Second, some price quotes referring to research items that did not enter the actual CPI calculation were also excluded. Third, any price quotes that were not uniquely identified by a concatenation of month, shop code, region and item identifier were also excluded from the final sample.[34] Fourth, since the estimation of common parameters is conducted at the item level and it is based on the moments of the distribution of regular price changes excluding the first price changes in any quote-line, any item that has less than 100 such price changes is excluded from the sample.

5.2 Product substitutions, sales and quote-line gaps

The random menu cost model that motivates state-space representation of pricing in (5) to (7) does not include sales or product substitutions and, hence, that state-space representation should be interpreted as a data generating process for *regular prices*. In order to have a good match between the underlying theory and the data, it is necessary to remove product substitutions and temporary sales from the data in order to obtain quote-lines of regular prices before estimating frictionless inflation.

Product substitutions For each outlet in which prices are collected, the price collectors start by choosing, among all products matching the specification of each item to be priced, one product that is representative of what people buy in the area. Once a product is chosen, the price collector returns to the outlet every month to collect prices of that same product. However, in some cases the price collector might be forced to change the product being priced either because it becomes unavailable or because the producer changed its physical characteristics such as weight or size. Whenever those forced substitutions occur the price collector should flag the respective price quote accordingly. To deal with product substitutions, whenever a substitution flag is present a break in the original price trajectory is generated.

[32] Bunn and Ellis (2012) were the first to have access and document stylised facts about consumer prices in the United Kingdom. Due to its public availability and disaggregation level, there is a growing number of papers using the micro price data provided by the ONS. A non-exhaustive list includes: Chu *et al.* (2018), Petrella, Santoro and Simonsen (2018), Carvalho and Kryvstov (2018), Blanco and Cravino (2019), Kryvstov and Vincent (2019) and Hobijn, Nechio and Shapiro (2019). Since this is not the first paper to use this dataset, this section focuses on the most important aspects of the data to produce a measure of frictionless inflation.

[33] The internal validation procedures used to ensure the prices have been accurately recorded and the indicator codes have also have been used sensibly and correctly are described in ONS (2014, chapter 6).

[34] In the ONS internal systems price quotes are uniquely identified by a concatenation of month, shop code, location and item identifier. For confidentiality reasons, the ONS does not publish the location variable, hence, the use of the region variable as a proxy. Nonetheless, it happens in some instances that two different outlets are the same region and have the same shop code and, in those cases, the two price trajectories could not be separated.

Product sales The prices quotes collected by the price collectors are the product shelf-prices and this means the prices would reflect any temporary sales at the time of collection. If the collected price is a sale price, the price collectors should flag it with the respective sales indicator. Those sales flags are used to obtain quote lines of regular prices. First, collected prices that are not on sale are immediately considered as the regular prices for the period. Second, if the collected price is on sale the last regular price observed is used as the regular price for that period. If the last regular price change is not available, the first regular price observed *after* the current observation is used instead. Third, if no regular prices are observed before or after the sales observation it is because the respective quote-line is composed exclusively of sales prices and in those cases the quote-line is excluded from the sample.

Quote-line gaps After cleaning the data following the steps above it is possible that the resulting quote line contains gaps, for example, because some price quote in the middle of the quote line was not validated internally by the ONS. Since to produce estimates of the frictionless inflation requires a non singleton quote-line of contiguous observations of regular prices, whenever the resulting quote-line contains gaps it is split into separate quote-lines of contiguous observations. Any resulting singleton quote lines are excluded from the sample.

An example To illustrate how to go from the primary data provided by the ONS to the quote lines of regular prices used to produces estimates of frictionless inflation, fictitious price quotes of an item collected in a particular outlet are considered in figure 7. Although price quotes are available for a 4 year period, some of them did not pass the ONS internal checks and, because of that, did not contribute to the CPI in the respective periods. Those price quotes are excluded from the final sample. Moreover, in one of the periods the price collector was forced to substitute the product for which prices were being recorded for another brand and/or variety that also matches the specification of the item to be priced. Since from that period onwards it is effectively a different product that is being priced a quote break is created. Finally, during the whole period the price collector also indicated two episodes of temporary sales and in those periods regular prices are imputed for sales prices following the procedure above described. More precisely, for the first sales spell the last regular price observed is imputed in place of the sales prices whereas for the second sales spell the next regular price is imputed since the last regular price is not observed. The outcome of this process are four quote lines of contiguous regular price observations which include a total of six regular price changes and three regular price changes when excluding the first. Estimates for the frictionless cumulated inflation are obtained for each quote line separately.

Final Sample The final sample used to produce frictionless inflation measures spans the period from 1996m1 to 2018m1 and comprises over 23 million price quote observations, over 2.2 million quote-lines and 979 unique items.

5.3 From price quotes to price indices

To produce estimates of frictionless inflation at the aggregate level, the estimates of cumulated frictionless inflation obtained from computing (21) for each quote-line in the data are aggregated to produce higher level indices. The aggregation procedure adapts the methodology used to construct the actual CPI as described in ONS (2014).

From price quotes to elementary aggregates The lowest level of aggregation at which price-quotes are aggregated into indices is at the *stratum* level. Each item in the data can be stratified in four different ways: by region, by shop type, by region and shop type or not stratified.[35] For each stratum, the individual price quotes are aggregated to produce an *elementary aggregate index*. The method primarily used to produce elementary aggregates in the CPI is the geometric mean (also known as the Jevons formula), more precisely, the elementary aggregate for stratum s of item j in month t with base period t^b is given by,

$$\mathcal{I}^{EA}_{s,j,t|t^b} = \left(\prod_{i=1}^{N_{s,j}} \frac{P_{i,j,t}}{P_{i,j,t^b}} \right)^{\frac{1}{N_{s,j}}} \tag{23}$$

where $P_{i,j,t}$ and P_{i,j,t^b} are the prices of the same product collected in a particular outlet in period t and in the base period t^b and $N_{s,j}$ is the number of price quotes in stratum s of item j after using the central shop weights as a replication factors. Using the previous month as the base period and the definition of cumulated individual inflation, the above expression can be re-written as,

$$\mathcal{I}^{EA}_{s,j,t|t-1} = \exp \left\{ \frac{1}{N_{s,j}} \sum_{i=1}^{N_{s,j}} \Delta Z_{i,j,t} \right\} \tag{24}$$

The regular price elementary aggregates are obtained from (24) with $\Delta Z^{\text{reg}}_{i,j,t}$. Similarly, the frictionless elementary aggregates are obtained from (24) with $\Delta \widehat{Z}^{\star}_{i,j,t}$.[36]

From elementary aggregates to higher level indices Indices for higher levels are weighted averages of the elementary aggregate indices. First, combining elementary aggregates with stratum weights yields item level indices. Second, item level indices and the respective item weights are combined to produces Classification of Individual Consumption by Purpose (COICOP) in-

[35] There are 12 regions in total (London, Southeast, Southwest, East Anglia, East Midlands, Yorks and Humber, Northwest, North, Wales, Scotland and Northern Ireland) and two shop types (multiple if the shop has 10 or more outlets or independent if it has less than 10 outlets).

[36] A key difference between the elementary aggregates produced following (24) and the elementary aggregates underlying the published CPI is that the later uses the previous month of January as the base period. Because of the breaks in quote-lines created to deal with product substitutions and quote-line gaps, using the previous January as the base period would require to either: (i) drop any observations that do not have information for $\Delta Z^{\text{reg}}_{i,j,t}$ in the previous month of January which are almost a half of the total sample or (ii) impute values for $\Delta Z^{\text{reg}}_{i,j,t^b}$ and $\Delta Z^{\star}_{i,j,t^b}$ at the base period. I choose to change the base period and chain-link the elementary aggregate indices every month over either of these two options.

dices at the class, group and division levels and the aggregate CPI.

6 A measure of frictionless inflation

This section presents the reduced form parameter estimates obtained for each item in the micro price data, analyses the model fit and presents a measure of year over year frictionless inflation for the UK from 1997 to 2018.

6.1 Reduced form parameter estimates

There is substantial heterogeneity in the estimated reduced form parameters at the item level (figure 8 and table 2). This section investigates whether this heterogeneity in estimated reduced form parameters can be rationalised in light of their theoretically implied relationships and key moments of the distribution of price changes.

Inaction region asymmetry A dimension of interest is the extent to which price setters pricing policies are characterised by symmetric inaction region boundaries. In terms of estimated parameters, a simple way of measuring the asymmetry of the inaction region is by taking the sum of the two boundaries, that is, $\mathcal{A}_j = \hat{\bar{x}}_j + \hat{\underline{x}}_j$. In the data, that measure ranges from -171% to 206% and 58% of the items display negative asymmetry.[37] A common explanation for asymmetric pricing policies is the presence of non-zero trend inflation, since in a pure menu cost model with positive trend inflation the optimal policy is characterised by price increases that are larger than price decreases (Ball and Mankiw, 1994). This explanation is qualitatively in line with the heterogeneity observed in the cross section of items since the correlation between $\mathcal{A}_j$ and the estimated drift of the frictionless inflation process of -0.41 (figure 9, top-left panel). Nonetheless, that explanation alone is quantitatively insufficient to rationalise all the heterogeneity in the asymmetry measure across different items. Other potential explanations for asymmetric adjustment policies include differences in size of the menu cost to increase or decrease prices and asymmetries in the profit loss function.[38]

Trends in frictionless inflation The estimated frictionless inflation trend ($\hat{\mu}_j$) across items is typically in the range of -1.4% to 1.2% per month. The median estimated frictionless inflation trend across products is 0.2% per month which is consistent with the average monthly CPI inflation for the period analysed of 0.17%. In the data, the correlation across items between estimated frictionless inflation trend and the average size of price changes is 0.63 (figure 9, top right-panel). Moreover, approximately one-fourth of the items have negative estimated trends. These negative estimated trends can be explained by the fact that, despite the positive

[37] The measure of asymmetry is computed only for items that have at least one price changed triggered by crossing the upper bound of the inaction region and one price change triggered by crossing the lower bound of the inaction region (total of 424 items).

[38] It is relatively common to assume a zero trend inflation and profit flow function that is quadratic in the price gap. These two assumptions give rise to an optimal symmetric Ss policy that can be solved in closed form (Dixit, 1991). Despite their analytical convenience, it is noticeable that these assumptions would be at odds with the pricing behaviour observed for most items in the UK micro price data.

aggregate inflation, some individual items have become less expensive over time.[39]

Standard deviation of idiosyncratic shocks Differences in dispersion of price changes are mostly accounted by differences in the volatility of idiosyncratic shocks. More precisely, the correlation between the standard deviation of price changes and the estimated standard deviation of idiosyncratic shocks ($\hat{\sigma}_{\varepsilon,j}$) across items is 0.86 (figure 9, bottom-left panel). The median value of the estimated standard deviation of the idiosyncratic shocks for the UK micro price data is 7.2% which is in line with the figures reported in Gautier and Bihan (2018). Estimating a random menu cost model across 227 products underlying the French CPI, Gautier and Bihan (2018) find a median across products for the unconditional productivity standard deviation to be either 5% (in a model without strategic complementarities) or 9% (in a model with strategic complementarities).

Arrival of costless adjustment opportunities The heterogeneity in frequency of price changes across items in the micro price data is mostly accounted by the estimated probability of a costless adjustment opportunity ($\hat{\lambda}_j$). Across all the items in the sample they correlate at 0.91 (figure 9, bottom-right panel). This high correlation resounds the fact that, in the UK data, most of the price changes are estimated to be due to the arrival of costless adjustment opportunities.

State-dependent versus time-dependent pricing An interesting feature of the random menu cost model is that it allows for the occurrence of both time-dependent and state-dependent price changes. A way of measuring which of these two extremes can better account for the pricing dynamics in the micro data is to compute the ratio between the number of costless price changes over the total number of price changes.[40] For a given item, a value of this measure closer to a hundred percent indicates that most of the price changes are triggered by the arrival of costless adjustment opportunities and, hence, the pricing dynamics is closer to that implied by a pure Calvo model. Computing this ratio using all the price changes in my final sample yields a figure of 88%. For individual items this measure ranges from 7.5% to 100% with an average value of 87%. Across different COICOP divisions, this ratio ranges from 66% for restaurants and hotels to 94% in food and non-alcoholic beverages (figure 10). The large number of time-dependent price changes is broadly in line with previous findings in the literature. For instance, Nakamura and Steinsson (2010, footnote 25) using data underlying the US CPI find that roughly 75% of price changes occur in the low menu cost state. Gautier and Bihan (2018, tables 5 and 6) find that costless adjustment opportunities account for 80% of the price changes in micro price data underlying the French CPI. Moreover, Blanco (2017) also using UK CPI micro price data finds that 93% of all the price changes are due to either

[39]Most examples are from items classified as: recreation and culture (e.g personal CD player, CD radio cassette, computer diskettes etc); furniture, household equipment and maintenance (e.g automatic washing machine, vacuum cleaners etc) and clothing and footwear (e.g. boy's casual short sleeve shirt).

[40]More formally, for each item in the data this statistic is computed as $\mathcal{C}_j = \sum_i \sum_t \mathbb{1}\{\Delta p_{i,j,t} \in (-\hat{\underline{x}}_j, 0) \cup (0, -\hat{\underline{x}}_j)\} / \sum_i \sum_t \mathbb{1}\{\Delta p_{i,j,t} \neq 0\}$.

free adjustment opportunities or fat-tailed idiosyncratic shocks.[41]

6.2 Model fit

Figures 11 to 16 investigate the extent to which the proposed state-space representation of pricing dynamics with parameter heterogeneity at the item level can account for the patterns of price changes observed in the micro price data. In terms of the moments directly targeted in the first-stage of parameter estimation, the proposed representation of pricing dynamics can in general match the percentiles of the distribution of non-zero price changes and the model fit tends to be better at the tails of the distribution (figures 11 and 12). In terms of the non-targeted moments, the worse fit of the middle percentiles of the distribution of non-zero price changes does not affect the model's ability to match some of the moments (e.g. mean or variance of price changes) but it does imply that for a non-negligible proportion of items the model cannot account for the values of robust skewness and robust kurtosis observed in the micro price data (figure 13). The inability of the single product version of the random menu cost model to match the kurtosis observed in the micro price data has been pointed before by Alvarez, Bihan and Lippi (2016). It is interesting to notice that for the micro price data underlying the UK CPI that is also the case for some individual items but the mismatch is much smaller when considering the distribution for the price changes across all items (figure 14). Finally, across all individual items considered the actual distribution of regular price changes can take a wide variety of shapes and inevitably the proposed state-space representation will fit some items better than others. Figures 15 and 16 illustrate the variety of shapes that the distribution of price changes can take across items and the model fit for the items with worst and best model fits, respectively.

6.3 A measure of frictionless inflation for the United Kingdom: 1997 - 2018

The measure of frictionless inflation for the UK against its regular price counterpart is presented in figure 17. In a nutshell, the conclusion to be drawn from that figure is that menu costs matter for aggregate inflation dynamics, their importance is decreasing over time period analysed.

Menu costs matter for aggregate inflation dynamics This comes from the fact that the two lines do not perfectly overlap over the period analysed. The correlation between the two-series from 1997 to 2018 is equal to 0.83 and the difference between regular price inflation and its frictionless counterpart can range from -0.79 and 2.29 percentage points. On average over

[41] More precisely, the ratio of free to total price adjustments is only 48% but adding the price changes after fat-tailed idiosyncratic shocks the figure increases to 93% (p.17). The combination of the two cases is a measure of how close the slope of the Phillips curve in the model in Blanco (2017) is from a Calvo model and that measure is more comparable to the ratio of free price changes to total price changes that I consider. Intuitively, in the state-space representation of the random menu cost model in (5) to (7), the boundaries of the inaction region are allowed to vary freely to account for the extremes in the distribution whereas the arrival of the costless adjustment opportunities accounts for the "middle" of the distribution. In Blanco (2017) also the middle of the distribution is accounted by the arrival of costless adjustment opportunities but the tails are accounted by fat-tailed idiosyncratic shocks. Despite these methodological differences, it is interesting that we find similar figures in terms of how close the model is from a pure Calvo model.

the whole period analysed, frictionless inflation is 0.55 percentage points lower than regular price inflation and the difference is mostly driven by observations prior to 2004. Moreover, from figure 17 it is also the case that the time-series for frictionless inflation is smoother that its regular price counterpart. More precisely, the historical standard deviation of the time-series for frictionless inflation is 20% lower than the standard deviation for the time-series of regular price inflation (0.91 against 1.13), also, the time-series of frictionless inflation exhibits higher persistence as it is more autocorrelated than regular price inflation at any horizon up to three years.

Their importance is decreasing over time From figure 17 it is also clear that the time-series of frictionless inflation is in general closer to its regular price counterpart after mid-2000s. In particular, considering only the periods prior to 2004 the correlation between the two time-series is 0.52 against 0.94 in the post-2004 period. Moreover, the average difference between regular price inflation and frictionless inflation decreases from 1.14 pp prior to 2004 to 0.27 pp post 2004. Interpreted through the lens of the random menu cost model that underlies the construction of the frictionless inflation measure, this change in the correlation of the two-time series can be caused by a narrowing of the boundaries of the inaction region or by an increase in the rate of costless adjustment opportunities or by a combination of the two.[42] There are two events that are be consistent with a narrowing of the inaction boundaries, namely, the increase in product competition from import penetration from China around the early 2000s and the change in inflation target in December 2003 from 2.5% of the Retail Price Index excluding mortgage payments (RPIX) to 2% of the CPI. The increase in product competition from imports would make deviations from the frictionless prices more costly in terms of profit flows and, hence, a smaller deviation would be enough to trigger the payment of the menu cost narrowing the bands of the inaction region. The decrease in the inflation target could also rationalise a narrowing of the boundaries of the inaction region if interpreted as decreasing the uncertainty regarding aggregate inflation rate. It is more difficult to rationalise an increase in the rate of arrival of costless adjustment opportunities in terms of specific events, but such increase is consistent with an increase in the frequency of price changes that is observed for part of the period between 2005 and 2012 (Petrella *et al.*, 2018, figure 1).

7 Frictionless inflation and the basic New Keynesian model

In this section, the constructed measure of frictionless inflation is used to empirically test two novel predictions from the basic new Keynesian model in Galí (2008, chapter 3). The first prediction concerns the difference between inflation and its frictionless counterpart over the business cycle. The second prediction concerns the difference in their responses to a monetary policy shock.

[42]On the one hand, the narrowing of the inaction region decreases the range of values that $\Delta p^{\star}$ can take. On the other hand, an increase in the rate of arrival of costless adjustment opportunities would decrease the number of periods where $\Delta Z_{i,j,t} = 0$ and $\Delta Z^{\star}_{i,j,t} \neq 0$. In the limit, if $\lambda_{i,j} = 1$ for every product, then $\Delta Z_{i,j,t} = \Delta \widehat{Z}^{\star}_{i,j,t}, \forall t > \tau^{1}_{i,j}$.

7.1 The inflation wedge and the output gap

The three equation New Keynesian model as presented in Galí (2008) implies a relationship between the difference of inflation and its frictionless counterpart, the inflation wedge, and the changes in the output gap.[43] In particular, letting $\tilde{y}_t$ denote the output gap, it can be shown that,

$$\pi_t^\star = \frac{\kappa}{\lambda}\,\Delta\tilde{y}_t + \pi_t \tag{25}$$

where λ and κ are convolutions of structural parameters that take only positive values and, most importantly, κ is the slope of the New Keynesian Phillips curve.[44] This relationship implies that the inflation wedge and the changes in the output gap should be *negatively* correlated. In particular,

$$\frac{\text{Cov}(\pi_t - \pi_t^\star, \Delta\tilde{y}_t)}{\text{Var}(\Delta\tilde{y}_t)} = -\frac{\kappa}{\lambda} < 0 \tag{26}$$

Empirical test Using the constructed measure of frictionless inflation, the prediction in (26) is empirically assessed by a bivariate regression of the form,

$$\pi_t - \pi_t^\star = \alpha + \beta\Delta\tilde{y}_t + w_t \tag{27}$$

where the hypothesis of interest is whether the slope of this regression is negative and statistically significant. The dependent variable is computed by taking the difference between year over year regular price inflation and its frictionless counterpart (the two time-series in figure 17). For the output gap six different measures are considered. First to measure output at the monthly frequency it is used either the monthly GDP index or industrial production index both from the ONS. Second, to measure the gap the deviations of the log of the indices from three types of trends are consider: an HP trend, a cubic trend or the trend extracted from the Hamilton (2018) filter.

Results The slope estimates in (27) across are presented in table 3. In line with the prediction in (26) all the estimated coefficients are negative. In three out of the six specifications considered the slope is also statistically significant at a 10% significance level. Overall, this suggests the co-movement between the inflation wedge and the changes in the output gap is qualitatively in line with a positively slopped Phillips curve implied by nominal pricing frictions as in the three equation new Keynesian model of Galí (2008).[45]

[43] The output gap being defined as "the log deviation of output from it's flexible price counterpart."

[44] Details and derivations of relationships implied by the basic New Keynesian model can be found in appendix C.

[45] Using the same values for the structural parameters as in Galí (2008, p. 52), the model implied value for $-\kappa/\lambda$ is approximately -3 which is in line with some of the point estimates for β reported in table 3.

7.2 Differential responses to a monetary policy shock

The second prediction considered concerns the responses of inflation and frictionless inflation to a monetary policy shock. In particular, the three equation new Keynesian model under an interest rate rule implies that in response to a monetary tightening frictionless inflation should decrease more on impact and less in the subsequent periods. More formally, let ε_t^m denote a monetary policy shock then in the basic new Keynesian model of Galí (2008) it holds that,

$$\frac{\partial \pi_t^\star}{\partial \varepsilon_t^m} < \frac{\partial \pi_t}{\partial \varepsilon_t^m} < 0 \tag{28}$$

$$\frac{\partial \pi_{t+h}}{\partial \varepsilon_t^m} < \frac{\pi_{t+h}^\star}{\partial \varepsilon_t^m}, \text{ for any } h > 0 \tag{29}$$

Empirical test The predictions in (28) and (29) are formally tested by using local projections (Jordà, 2005) to estimate the impulse responses of regular price inflation and frictionless inflation to an externally identified monetary policy shock. More precisely, for horizons $h = 0, 1, \ldots, 36$, I estimate regressions of the following form:

$$\begin{cases} \pi_{t+h} = \alpha_h + \beta_h \widehat{\Delta i_t} + \gamma_h \mathbf{X}_t + \epsilon_t \\ \pi_{t+h}^\star = \alpha_h^\star + \beta_h^\star \widehat{\Delta i_t} + \gamma_h^\star \mathbf{X}_t^\star + \epsilon_t^\star \end{cases} \tag{30}$$

where $\widehat{\Delta i_t}$ denotes the change in the official bank rate instrumented by the high-frequency measure monetary policy surprises from Cesa-Bianchi, Thwaites and Vicondoa (2019) and $\mathbf{X}_t$ and $\mathbf{X}_t^\star$ are vectors of control variables. In the baseline specification the vectors of controls include four lags of regular price inflation and four lags frictionless inflation. In terms of the coefficients β_h and $\beta_h^\star$, the hypothesis of interest are: (i) $\beta_0^\star < \beta_0 < 0$ and (ii) $\beta_h^\star > \beta_h$ for $h > 0$.

Results The impulse responses for regular price inflation and frictionless inflation to a 1 percent unexpected increase in nominal interest rates are depicted in figure 18. The first hypothesis is not supported given that frictionless inflation reacts positively on impact whereas regular inflation is virtually unchanged and neither of the two responses is statistically significant (see figure 19). The second hypothesis is only weakly supported at horizons greater than 21 months when regular price inflation starts declining. It is important to notice that this is not a test of whether monetary policy has effects inflation. From the response of regular price inflation in figure 18 it is indeed the case that regular price inflation declines in response to a monetary tightening albeit with a long delay, that is, there is a substantial prize puzzle. Instead, this exercise should be interpreted as a test of the monetary transmission mechanism implied by the basic new Keynesian model. In that model, frictionless inflation should react quickly to a monetary policy shock and this is at odds with the responses in figure 18 where frictionless inflation does not significantly react to a monetary policy shock at any horizon considered.

8 Can frictionless inflation help to forecast published inflation?

This section investigates to what extent the proposed measure of frictionless inflation contain useful information to forecast the actual published CPI inflation. The intuition behind this hypothesis is simple. Consider a quote-line for which price remains unchanged for 9 months and in the tenth month it change by 30%. The frictionless counterfactual for that quote-line is another quote-line for which the price changes every period by small positive amounts, say, on average 3% every month. Starting from this disaggregated perspective, the question is whether an aggregate measure based on those small changes can contain useful information to forecast the actual inflation rate based on infrequent "lumpy" price adjustments.

A simple forecasting exercise In order to assess the relevance of frictionless inflation to forecast published inflation, the forecasting exercise of Blinder and Reis (2005) is revisited. In particular, I consider linear regressions of the form:

$$\pi_{t,t+h} = \alpha + \mathbf{X}\beta + \varepsilon_t \tag{31}$$

where $\pi_{t,t+h}$ is the inflation between months t and $t+h$ calculated based on the published headline CPI index, $\mathbf{X}$ is a vector containing published inflation over the previous 12 months ($\pi_{t-12,t}$) or frictionless inflation over the previous 12 months ($\pi^{\star}_{t-12,t}$) or both. Forecasting ability is assessed through three different measures. First, the in-sample forecasting ability is evaluated by estimating the different specifications of the above regression and computing the standard error of the regressions. Second, the out-of-sample forecasting ability is assessed by estimating the above regressions using data until January 2008 and computing the root mean squared errors from forecasting inflation from then until the end of the sample. Third, the version of the above regression with both $\pi_{t-12,t}$ and $\pi^{\star}_{t-12,t}$ and the significance of the coefficients is compared.

Results The results for the forecasting exercise are presented in table 4. First, for any forecasting horizon considered the specification with frictionless inflation only has a better in-sample fit than the specification only with headline inflation. Moreover, the gains of including both frictionless and headline inflation are small at short horizons and inexistent for longer horizons. The message for out-of-sample fit is similar since again the specification only with frictionless inflation has better out-of-sample forecasting ability when compared with the specification with the exception that there are typically gains from including both the frictionless and the headline inflation. Finally, regarding coefficient significance only the coefficient on frictionless inflation is significant at any of the horizons considered and more so for longer horizons. Overall, the evidence from the forecasting exercise here presented is indicative that frictionless inflation can contain useful information to forecast the published inflation.

9 Conclusion

Menu costs are one of the leading explanations for the sluggish response of prices to changes in economic conditions. This paper investigated the relationship between menu costs of price adjustment at the micro level and the aggregate inflation dynamics. I introduce a methodology to estimate counterfactual inflation in the absence of menu costs based on information contained in the observed price quotes. Applying this methodology to rich micro data underlying the UK CPI, I find that menu costs matter for aggregate inflation dynamics, but their importance decreased over time. Moreover, the response of frictionless inflation to a monetary policy shock is at odds with the monetary transmission mechanism from the basic new Keynesian model. Finally, frictionless inflation contains useful information to forecast actual CPI inflation.

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Figures and Tables

Figure 1: Illustration of proposition 1

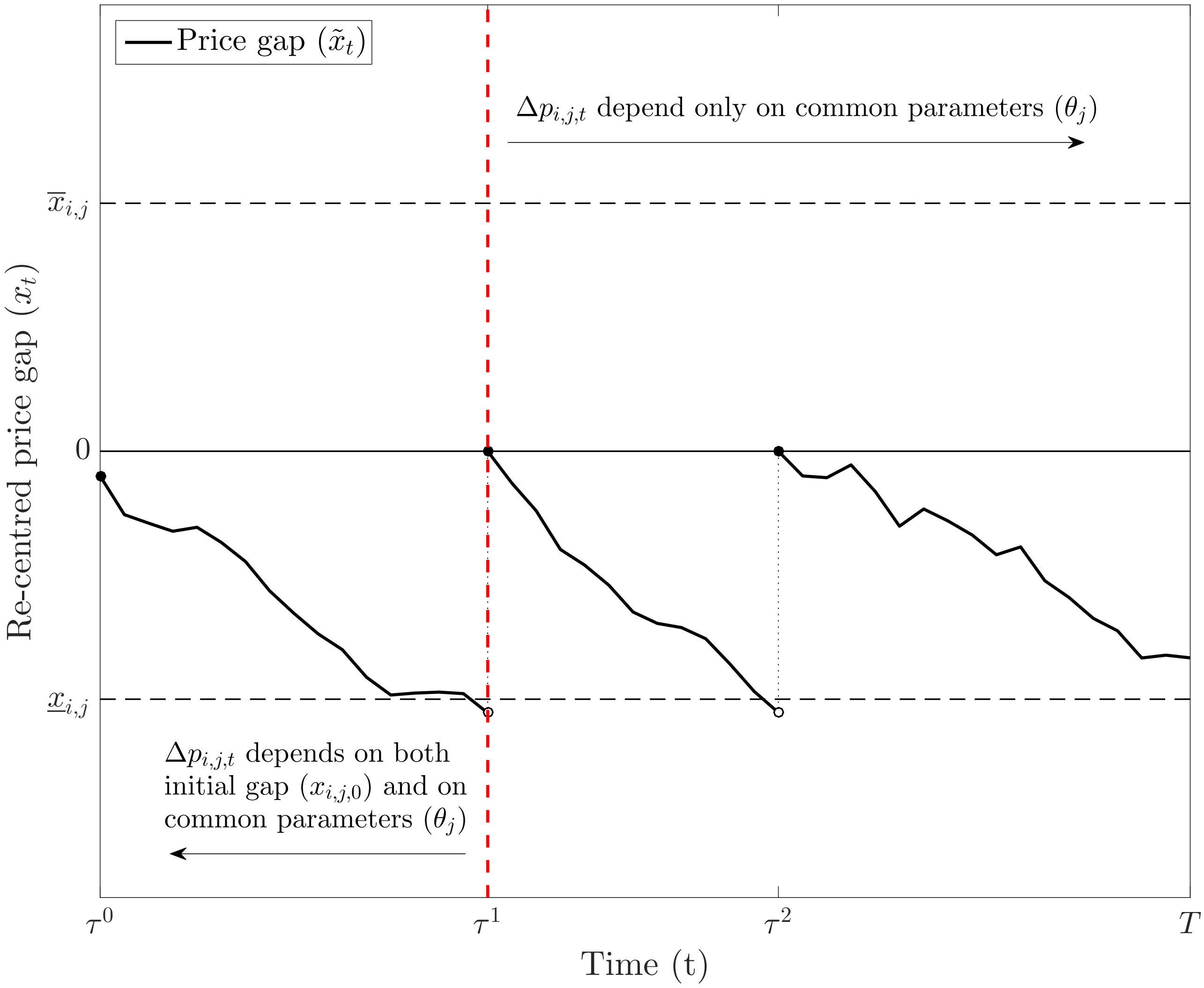
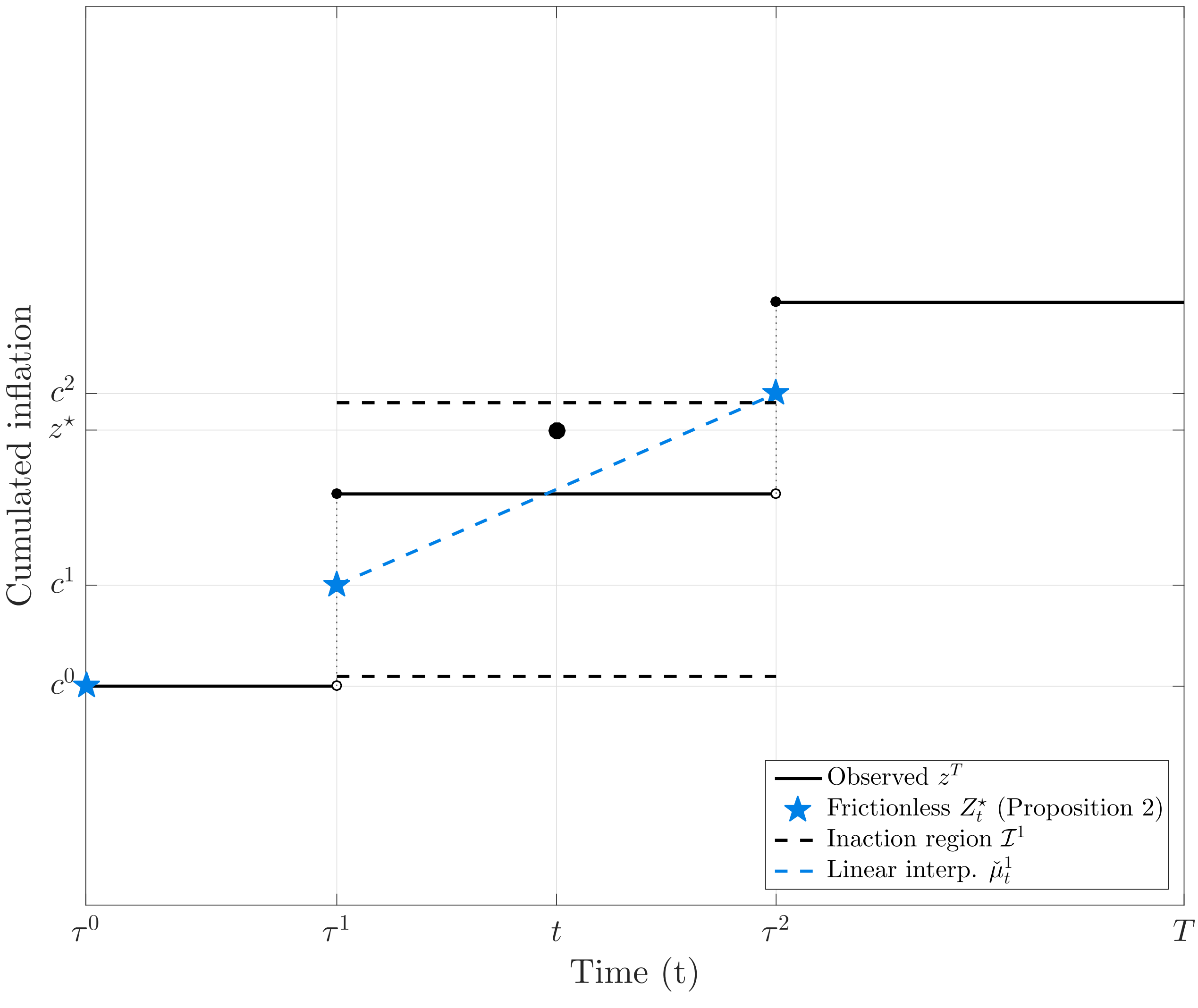
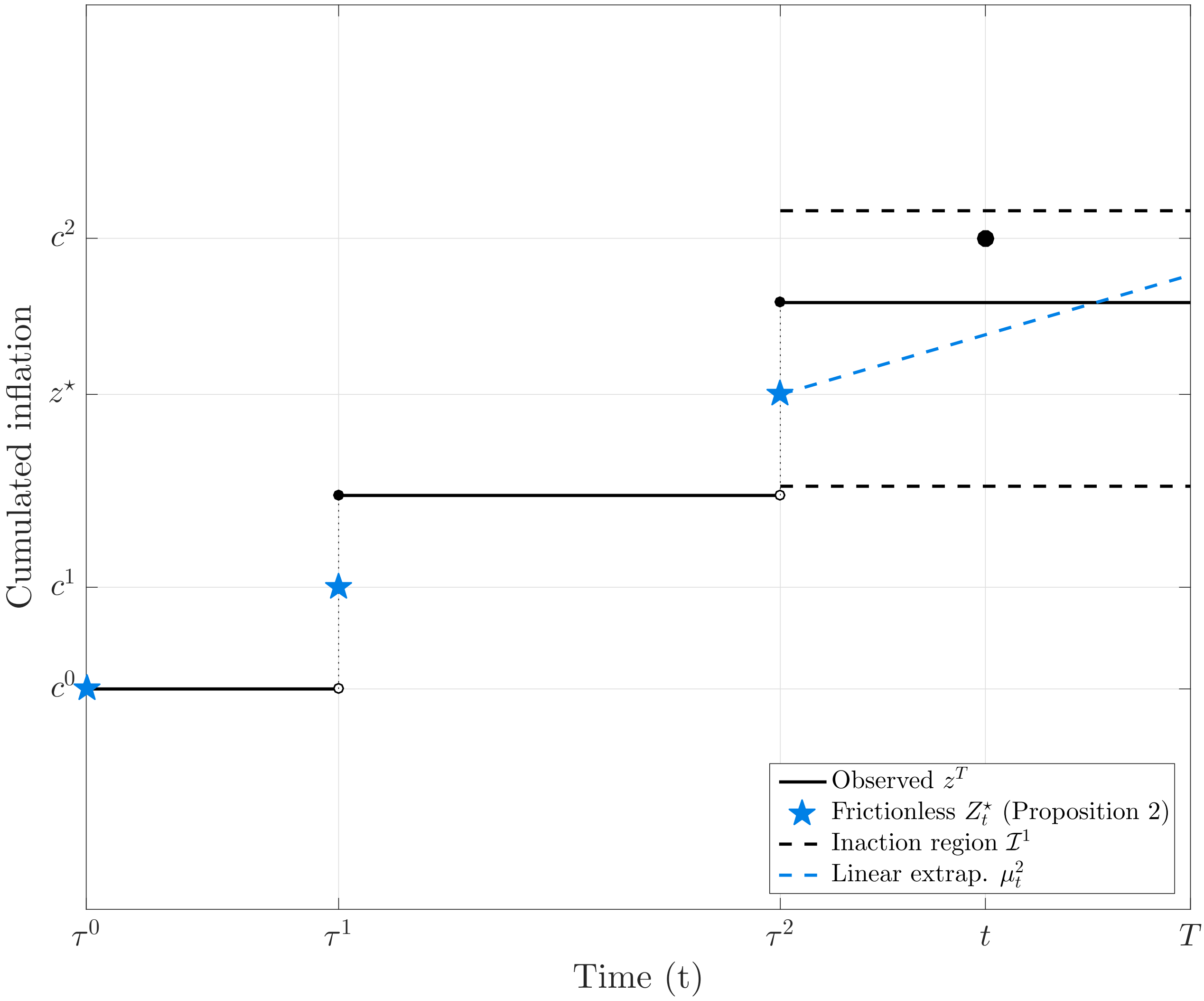


Figure 2: Illustration of proposition 3



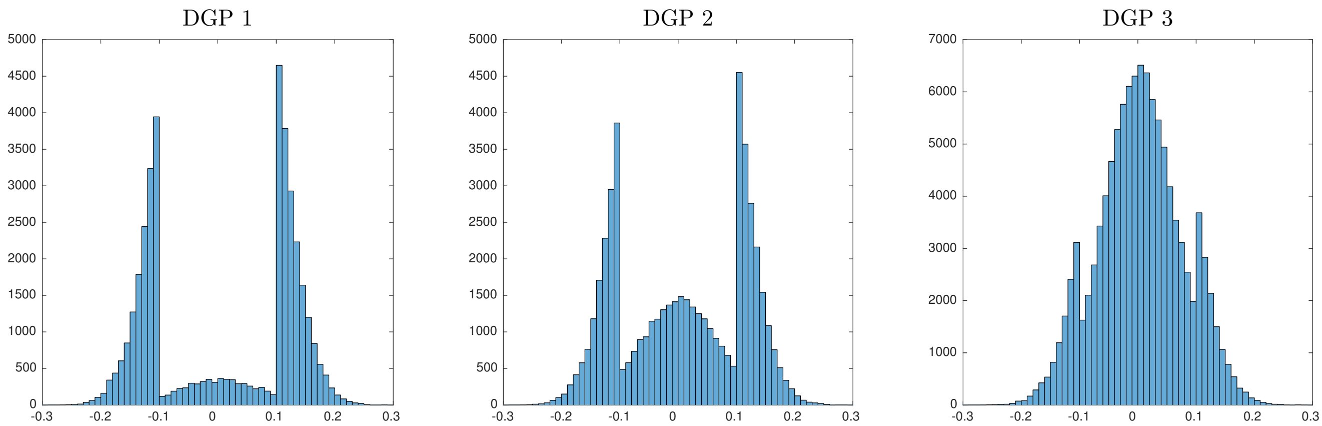
The solid black line illustrates a hypothetical observed quote-line. The dashed black lines show the boundaries of the inaction region in terms of $Z^\star$ for the price spell that starts at τ_1 and ends at τ_2, the dashed blue line connecting the two pentagrams is equal to the linear interpolation ($\check{\mu}_t^1$) and $z^\star$ is an arbitrary value chosen to illustrate the argument.

Figure 3: Illustration of proposition 4



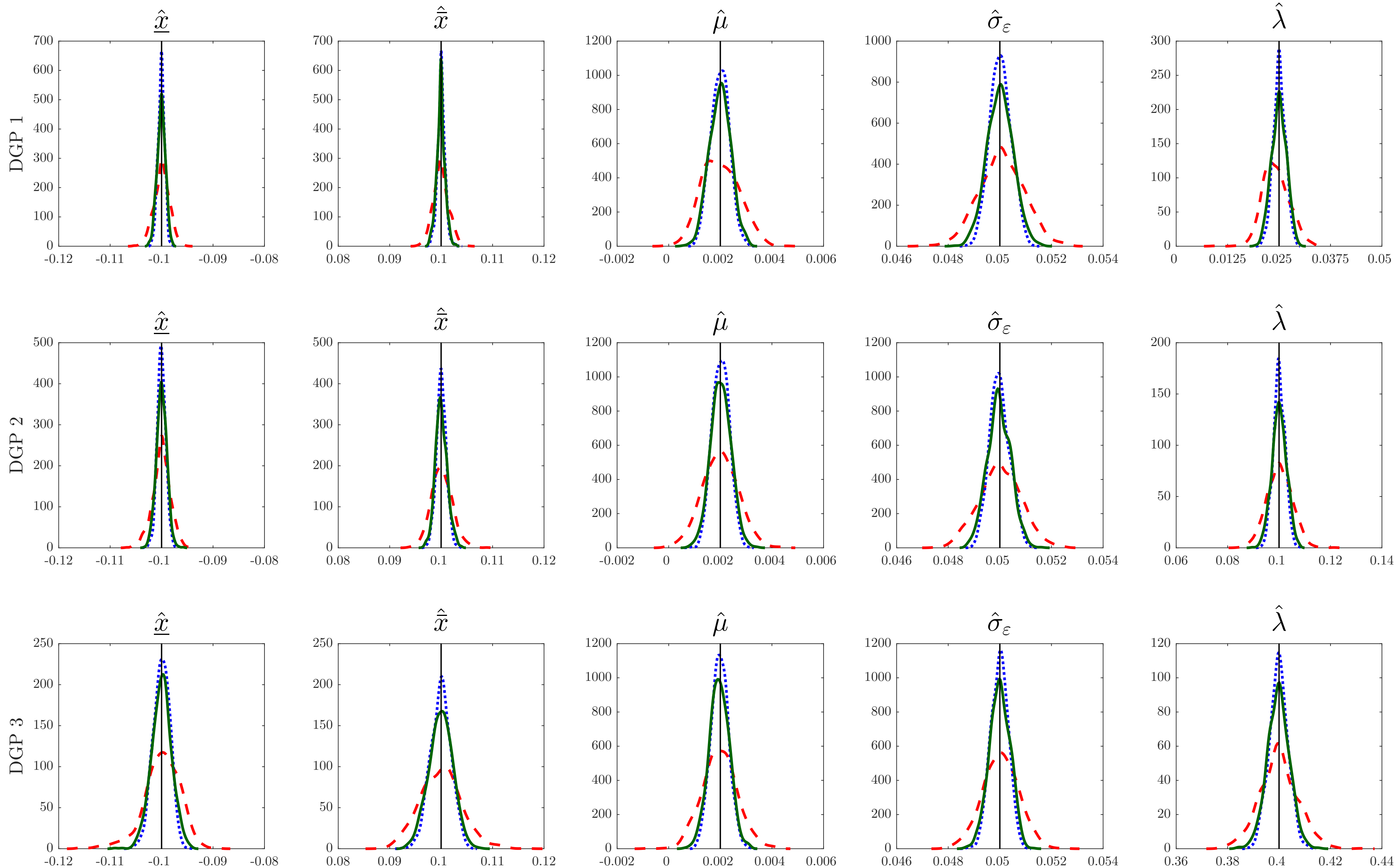
The solid black line illustrates a hypothetical observed quote-line. The dashed black lines show the boundaries of the inaction region in terms of $Z^\star$ for the price spell that starts at τ_1 and ends at τ_2, the dashed blue line connecting the two pentagrams is equal to the linear interpolation ($\check{\mu}_t^1$) and $z^\star$ is an arbitrary value chosen to illustrate the argument.

Figure 4: Price change distribution for DGPs in Monte Carlo experiment



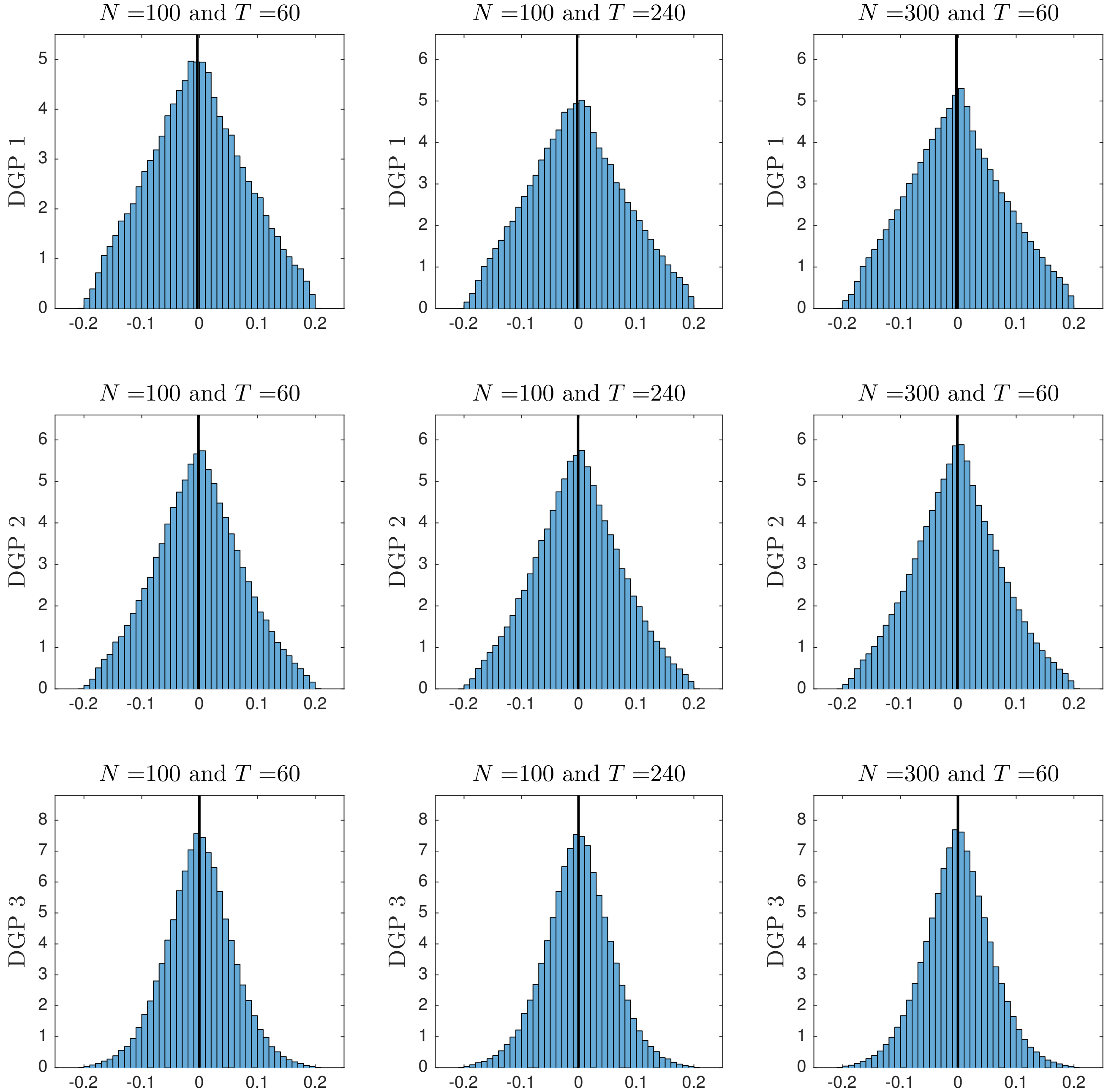
Distributions of price changes are obtained from a sample with a balanced panel with 1000 quote-lines spanning 240 months. The initial conditions are equally spaced points in the inaction region following the same rule as in table 1.

Figure 5: Kernel density of parameter estimates over Monte Carlo replications



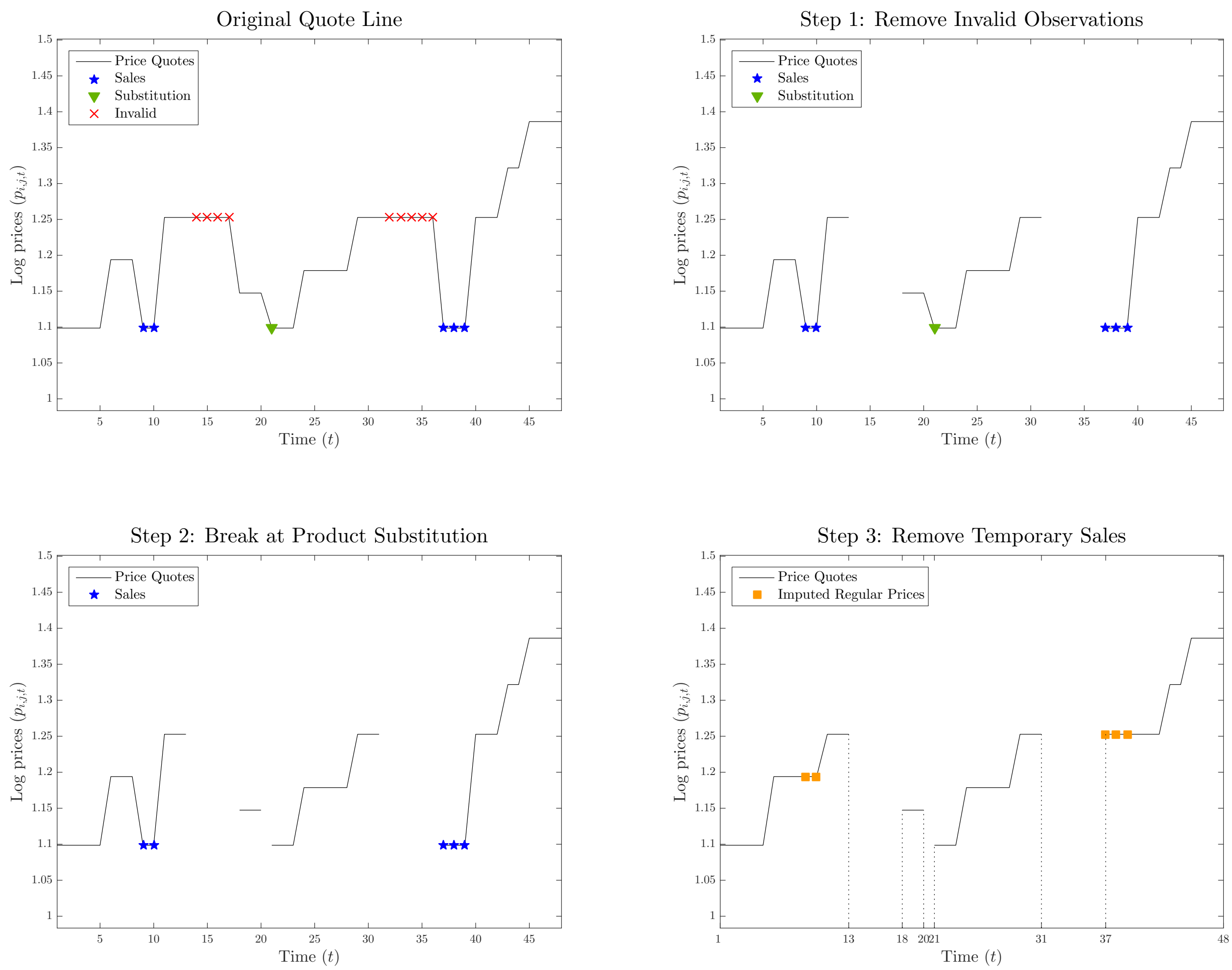
Each line is a normal kernel density estimate of each parameter estimate across 1,000 monte carlo replications. The red dashed line correspond to the case $N = 100$ and $T = 60$, the blue dotted line to the case ($N = 100$ and $T = 240$) and the green solid line to the case to the case ($N = 300$ and $T = 60$). The vertical black solid lines are at the true parameter values. The alternative DGPs are described in table 1.

Figure 6: Histograms of estimates of initial price gaps over Monte Carlo replications



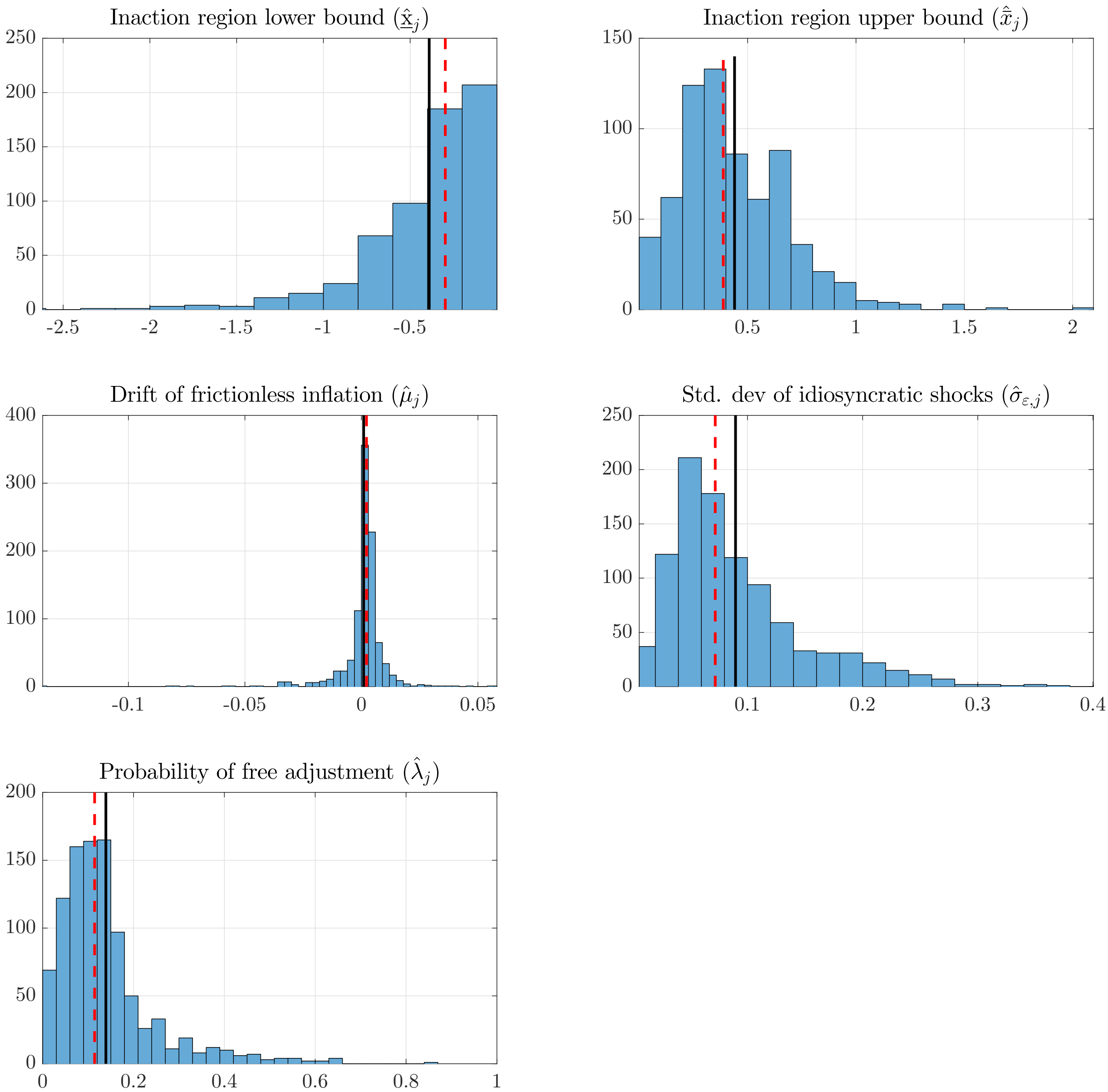
For each individual in each sample an estimate for the initial price gap is obtained $\hat{\hat{x}}_{i,j,t_{i,j}}$. The histograms above are of the difference between the estimated initial gap and the true one. The vertical black solid lines are the means of those differences across individuals and monte carlo replications for a given combination of DGP and sample size. The rule used to generate the initial conditions is described in table 1.

Figure 7: Step by step cleaning of an hypothetical quote-line



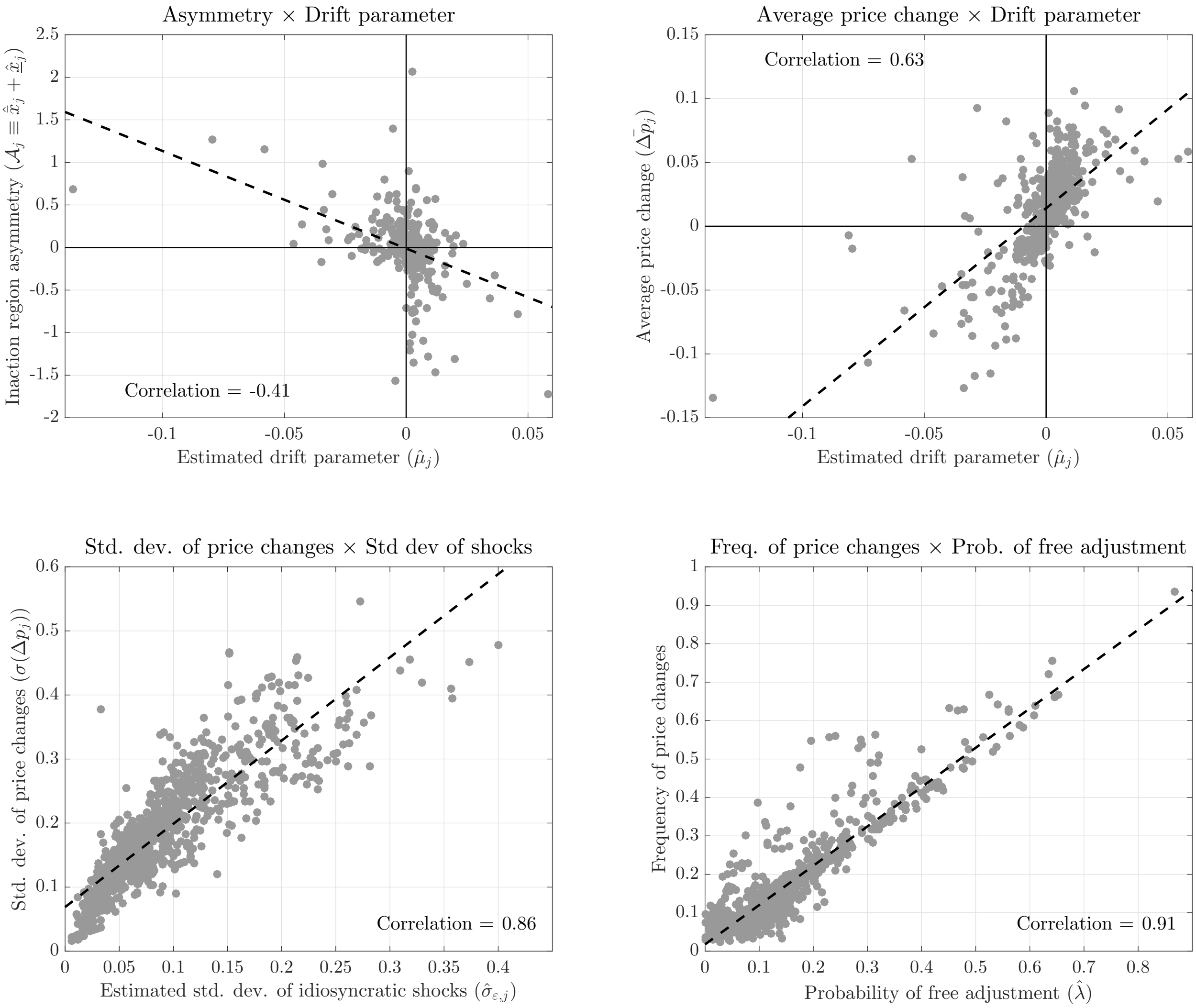
Black solid lines represent raw log price quotes. Blue stars indicate observations flagged as sales. Downward pointing green triangle indicates an observation flagged as a product substitution. Red crosses indicates observations that did not pass the validity checks. Orange squares indicate regular prices imputed for sales prices following the procedure described in the main text.

Figure 8: Distribution of estimated common parameters at the item level



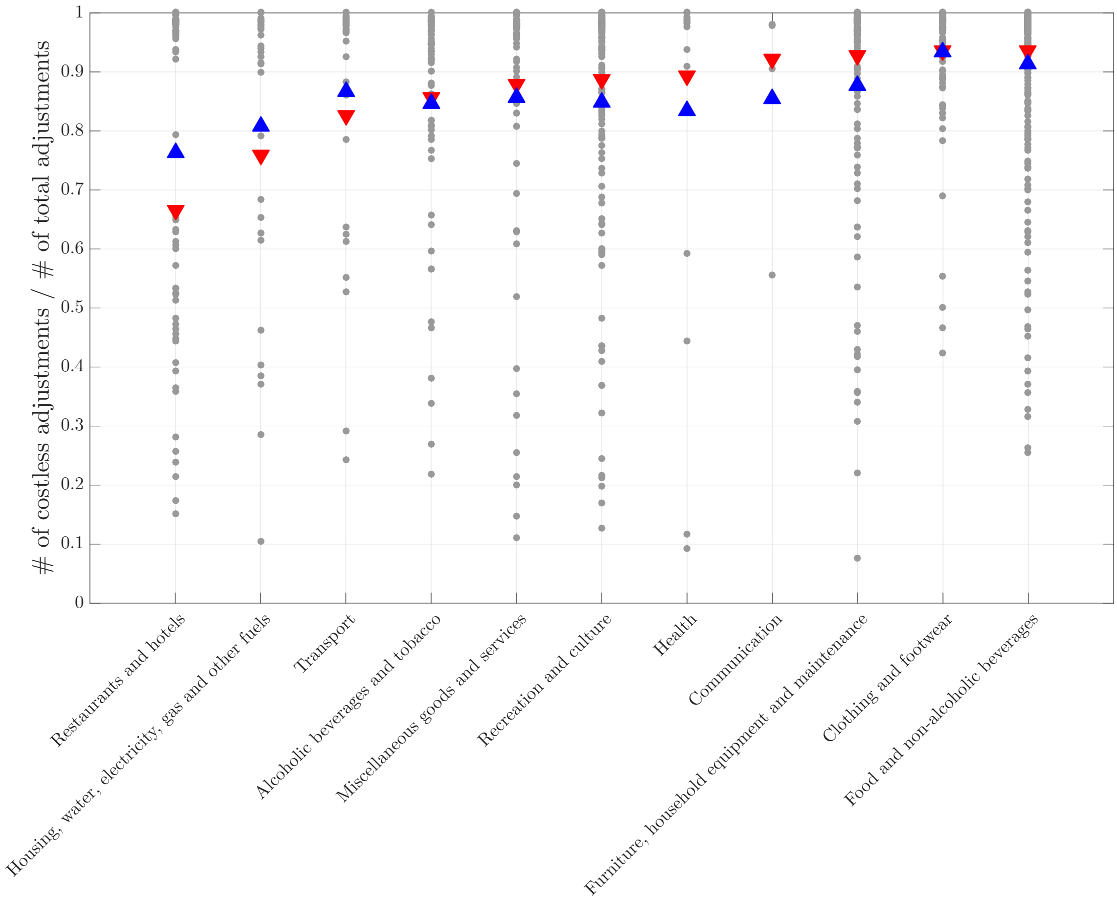
Notes: Histograms of estimated common parameters over different items. Histogram for the inaction region lower bound excludes items for which none of the observed price changes is triggered by crossing the lower bound (358 out of 979 items). Histogram for the inaction region upper bound excludes items for which none of the observed price changes is triggered by crossing the upper bound. Solid black and red dashed lines are, respectively, the mean and the median of estimated parameters calculated from the sample that is used to plot the histogram. Descriptive statistics across all items available in table 2.

Figure 9: Pricing moments and reduced form parameters



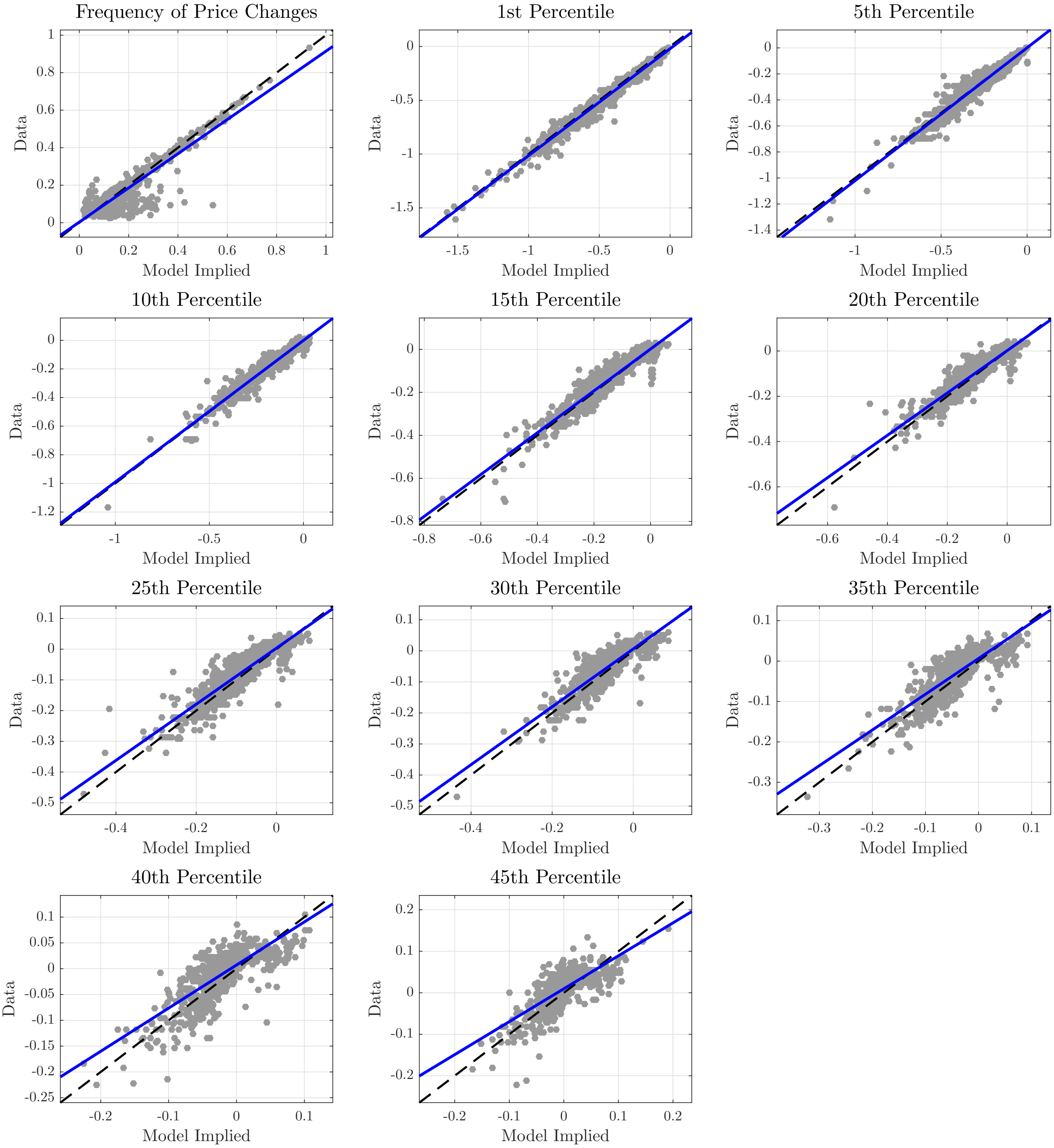
Notes: In all the scatter plots, each grey dot represents a given 6-digit item in the sample. In the first scatter plot only items for which there is at least one price change triggered by crossing the upper and the lower boundaries of the inaction region are considered (total of 424 items). In all the scatter plots, the dashed black line is the best fit line obtained from a bivariate regression of the variable on the y-axis on the variable on the x-axis.

Figure 10: Proportion of costless adjustments by COICOP division



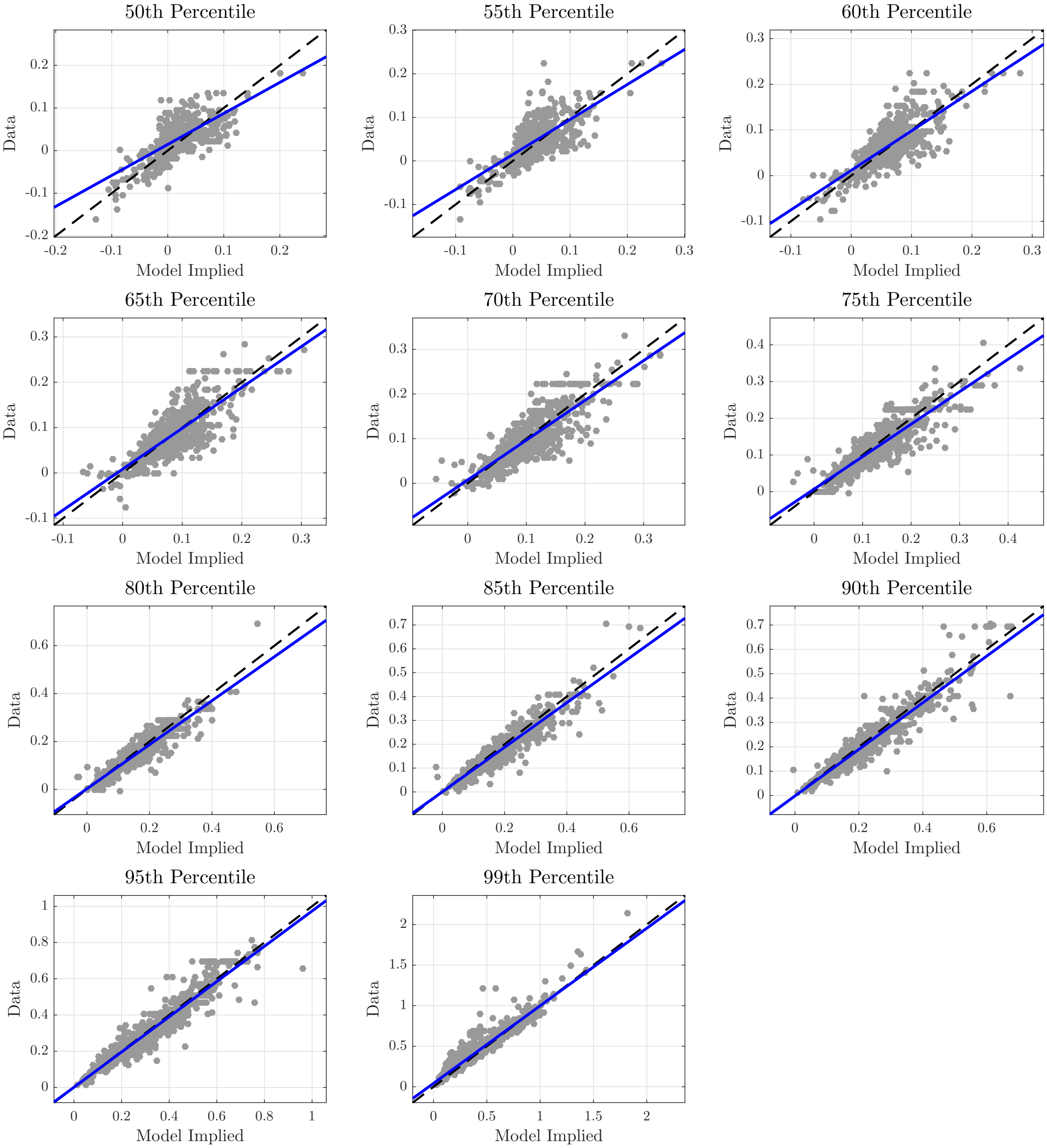
Notes: Each grey dot represents the ratio of costless adjustments over the total number of adjustments for each of the 979 unique items in the sample. The red downward pointing triangle contains this ratio computed using all the price changes for that division. The blue upward pointing triangle contains the (unweighted) average ratio across all items in the division.

Figure 11: Data versus model implied targeted moments (1/2)



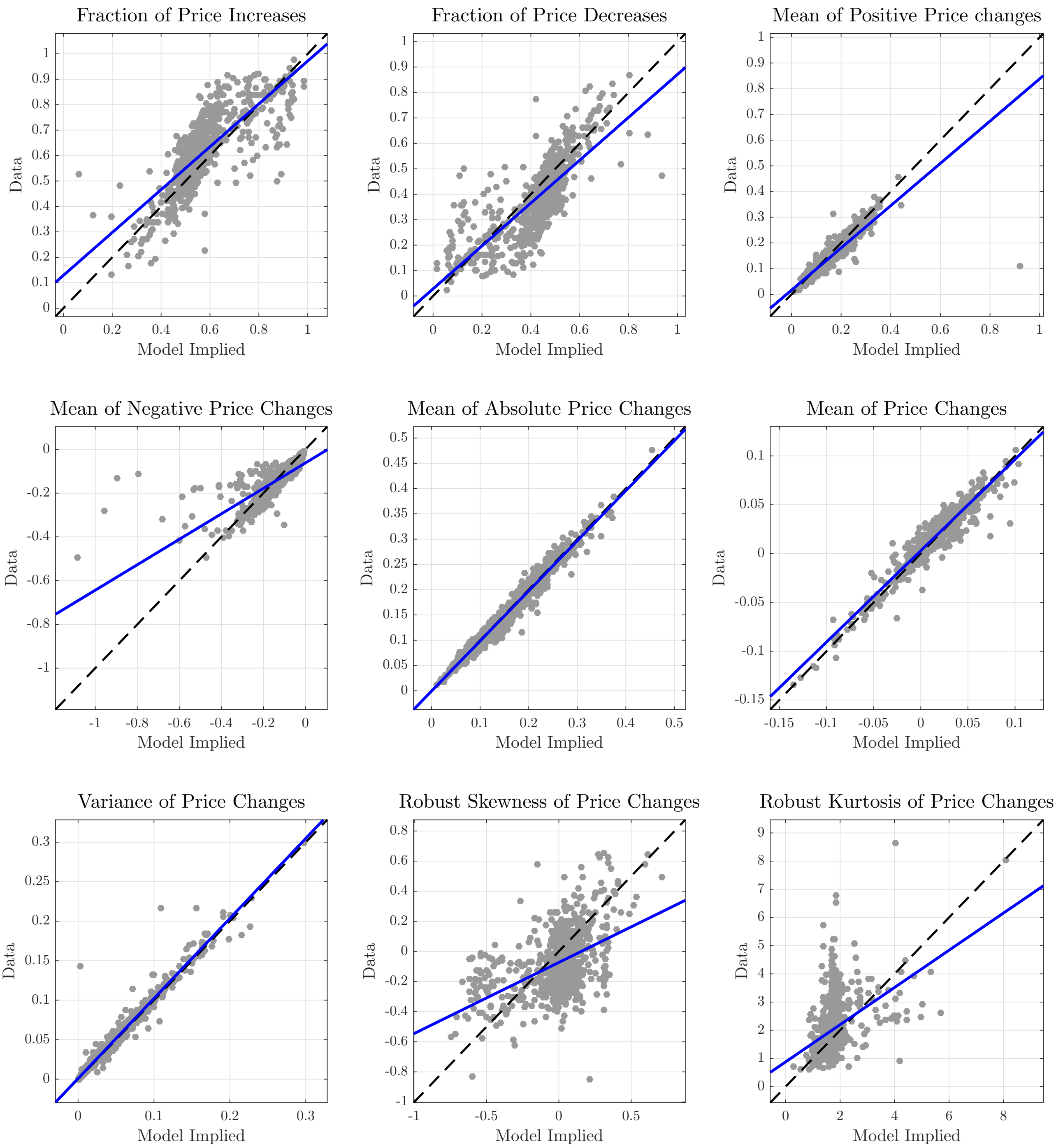
Notes: Each grey dot represents, for a given 6-digit item in the sample, the value of the moment in the data (y-axis) against its model implied counterpart (x-axis). The black dashed line is a 45° line. The solid blue line is the best fit line obtained from a bivariate regression of data moments on their model implied counterparts. Both in the data and in simulated data the moments are computed after excluding the first price change of each quote line. To compute the model implied moments a set of 50 panels is generated using the common parameter estimates from the first stage and the same primitive shocks used for the estimation. The model implied moment is the average of the respective moment across simulated panels. In terms of equation (9), the y-axis in each plot contains $g(\Delta p_{j|t>\tau^1})$ and the x-axis contains the average of $g(\Delta p^s_{j|t>\tau^1}(\widehat{\theta}_j))$ across simulations.

Figure 12: Data versus model implied targeted moments (2/2)



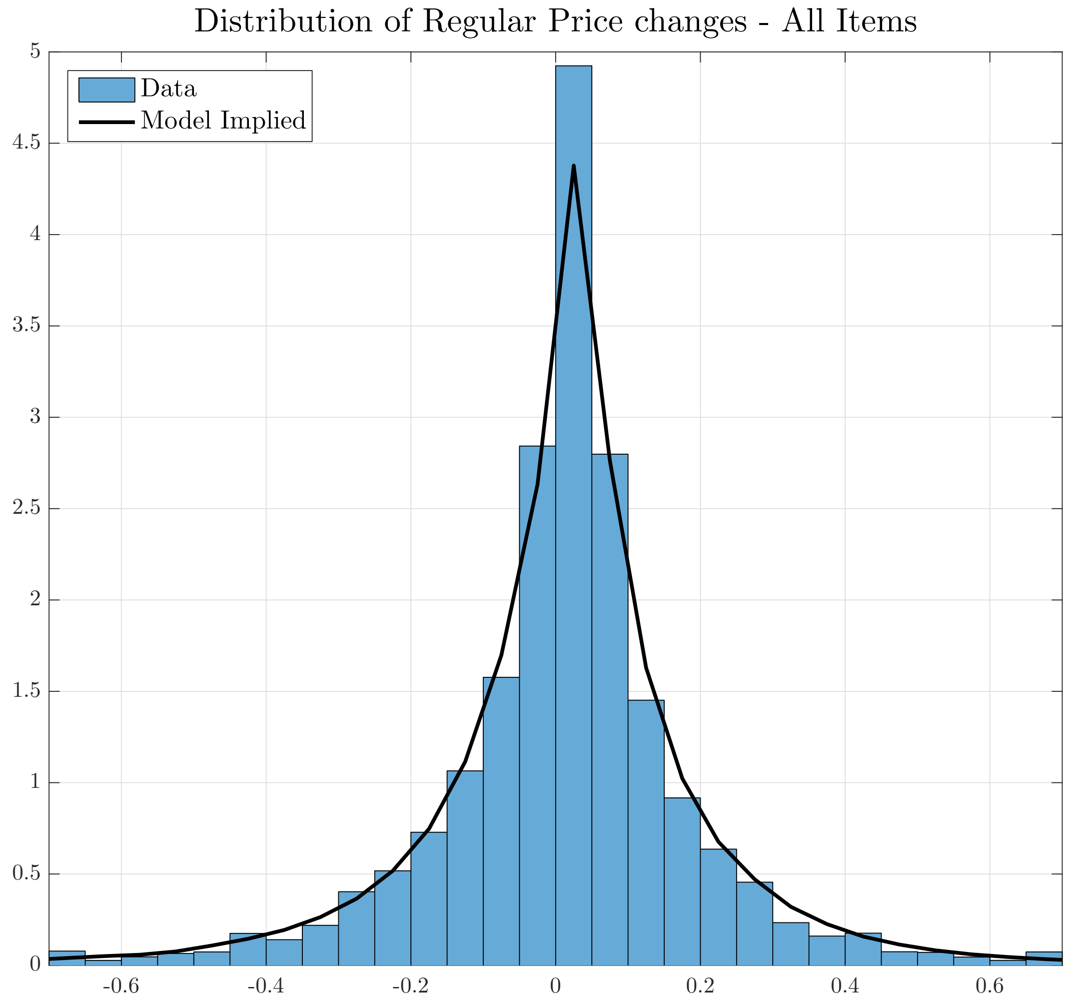
Notes: Each grey dot represents, for a given 6-digit item in the sample, the value of the moment in the data (y-axis) against its model implied counterpart (x-axis). The black dashed line is a 45° line. The solid blue line is the best fit line obtained from a bivariate regression of data moments on their model implied counterparts. Both in the data and in simulated data the moments are computed after excluding the first price change of each quote line. To compute the model implied moments a set of 50 panels is generated using the common parameter estimates from the first stage and the same primitive shocks used for the estimation. The model implied moment is the average of the respective moment across simulated panels. In terms of equation (9), the y-axis in each plot contains $g(\Delta p_{j|t>\tau^1})$ and the x-axis contains the average of $g(\Delta p^s_{j|t>\tau^1}(\widehat{\theta}_j))$ across simulations.

Figure 13: Data versus model implied non targeted moments



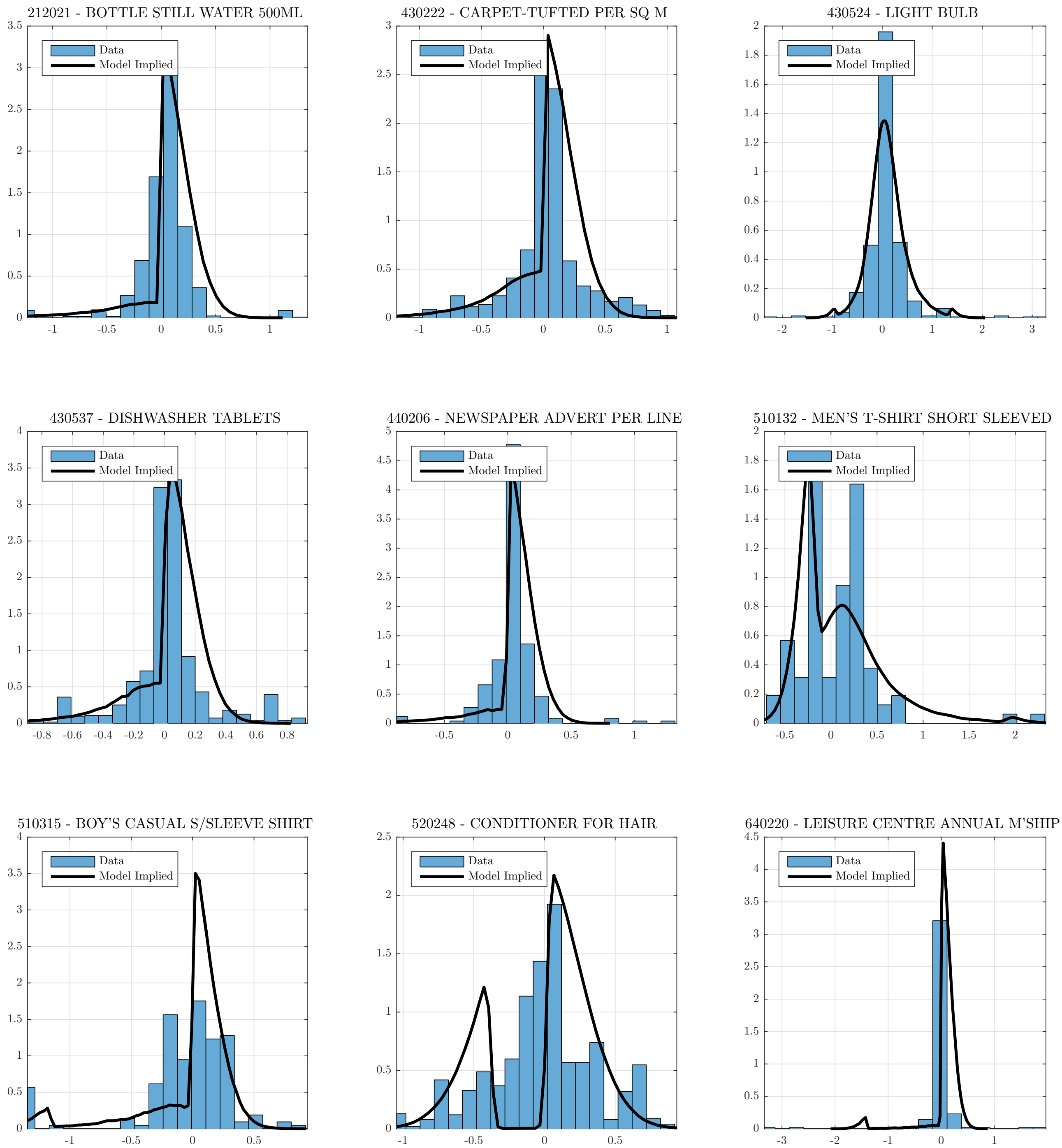
Notes: Each grey dot represents, for a given 6-digit item in the sample, the value of the moment in the data (y-axis) against its model implied counterpart (x-axis). The black dashed line is a 45° line. The solid blue line is the best fit line obtained from a bivariate regression of data moments on their model implied counterparts. Both in the data and in simulated data the moments are computed after excluding the first price change of each quote line. To compute the model implied moments a set of 50 panels is generated using the common parameter estimates from the first stage and the same primitive shocks used for the estimation. The robust skewness and robust kurtosis are computed as in Berger and Vavra (2018, table 1), in particular, Robust-Skew $= (P_{90} + P_{10} - 2P_{50})/(P_{90} - P_{10})$ and Robust-Kurt $= (P_{90} - P_{62.5} + P_{37.5} - P_{10})/(P_{75} - P_{25})$.

Figure 14: Model fit for all items



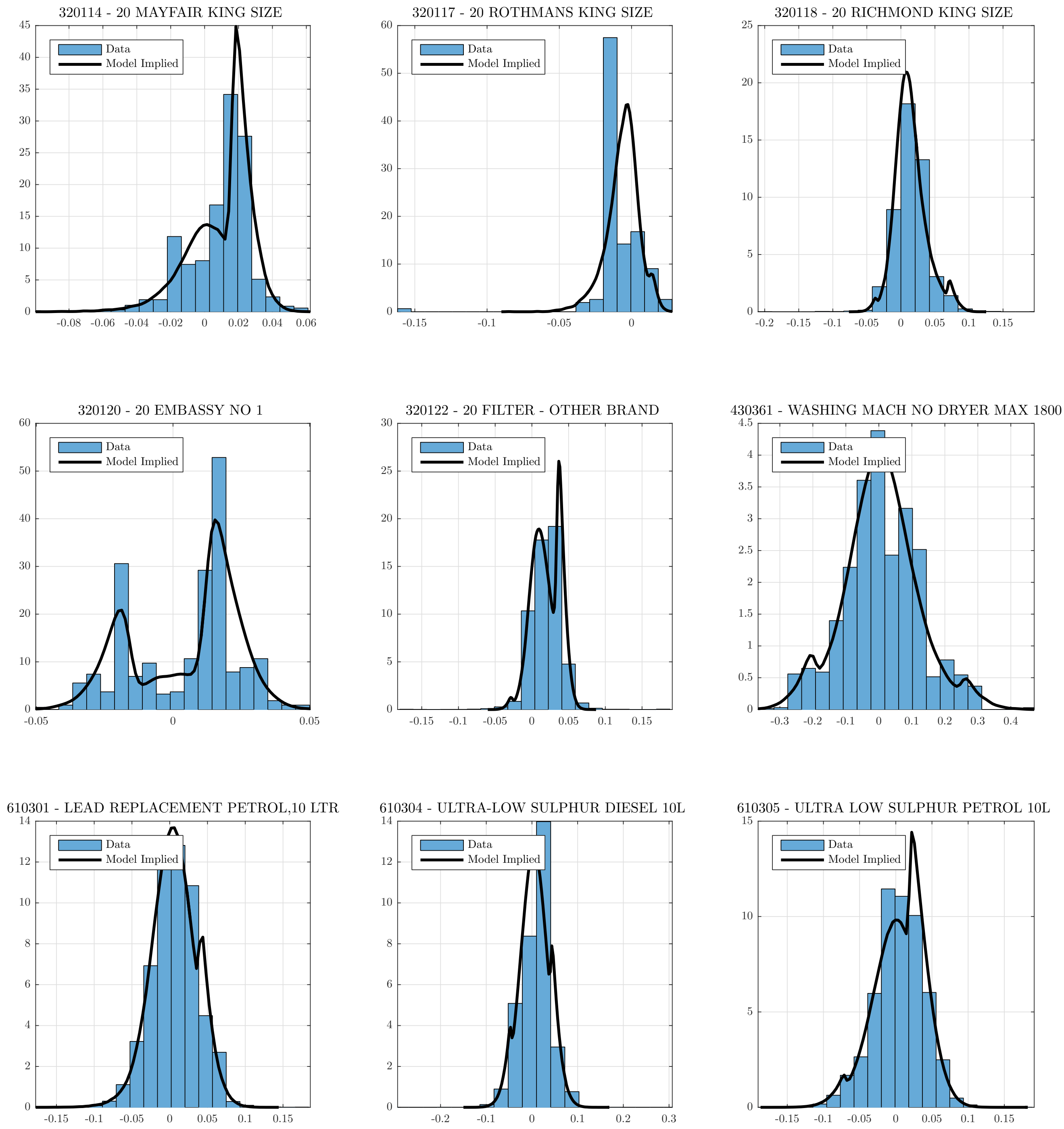
Notes: Blue bars represent the histograms for the distributions of regular price changes observed for the whole dataset. The black solid line is kernel density estimate of the distribution of price changes for all items over 50 panels of simulated data. Simulated data is generated by combining data of separate items and simulating using the the estimated common parameters. When considering the price changes, the first price change in each quote line is excluded. The histogram excludes log price changes that are not in the range $[-0.7, 0.7]$. The excluded observations account for 0.78% of the total number of non-zero price changes.

Figure 15: Model fit for the worst fitting 9 items



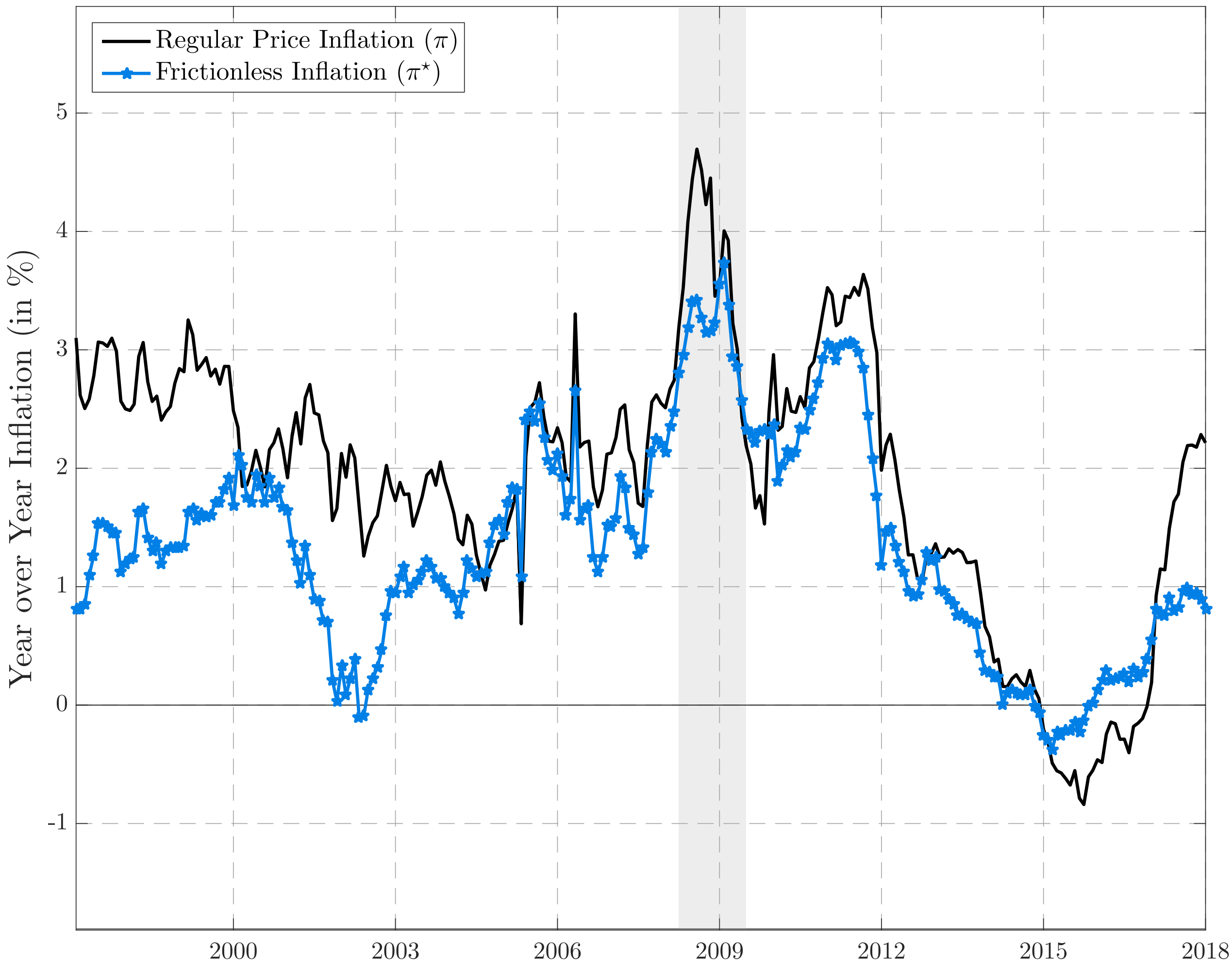
Notes: Blue bars represent the histograms for the distributions of regular price changes observed in the data for a given item. The black solid lines the kernel density estimates of the distribution of price changes over 50 panels of simulated data. Each panel was simulated using the estimated parameter values for the first stage for each item and the same primitive shocks used for estimation. When considering the price changes, the first price change in each quote line is excluded. The items chosen are those for which the first-stage SMM objective function (9) evaluated at the estimated parameters displayed the *largest* values.

Figure 16: Model fit for the best fitting 9 items



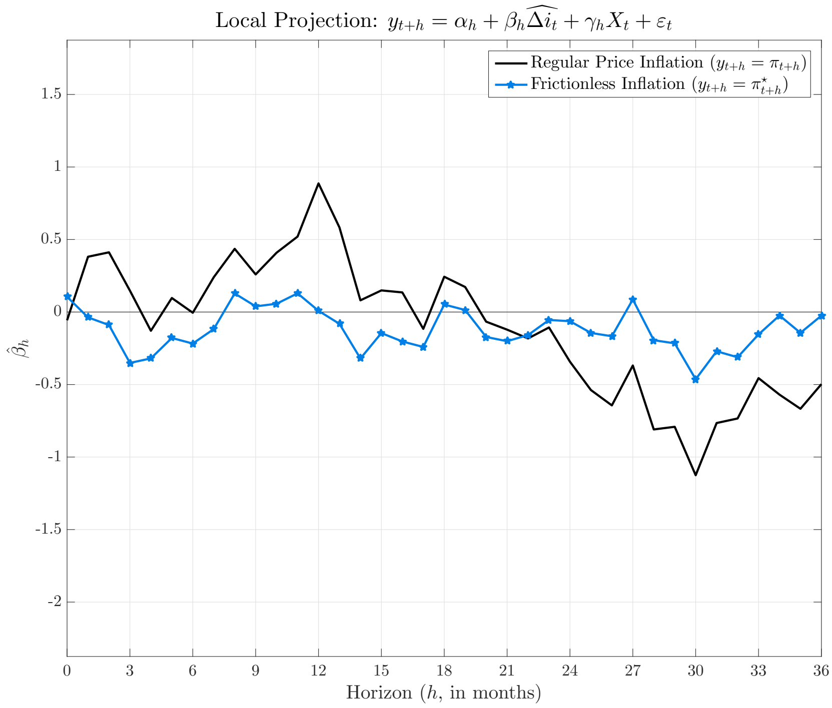
Notes: Blue bars represent the histograms for the distributions of regular price changes observed in the data for a given item. The black solid lines the kernel density estimates of the distribution of price changes over 50 panels of simulated data. Each panel was simulated using the estimated parameter values for the first stage for each item and the same primitive shocks used for estimation. When considering the price changes, the first price change in each quote line is excluded. The items chosen are those for which the first-stage SMM objective function (9) evaluated at the estimated parameters displayed the *smallest* values.

Figure 17: Regular price versus frictionless inflation for the UK (1997m2 - 2018m1)



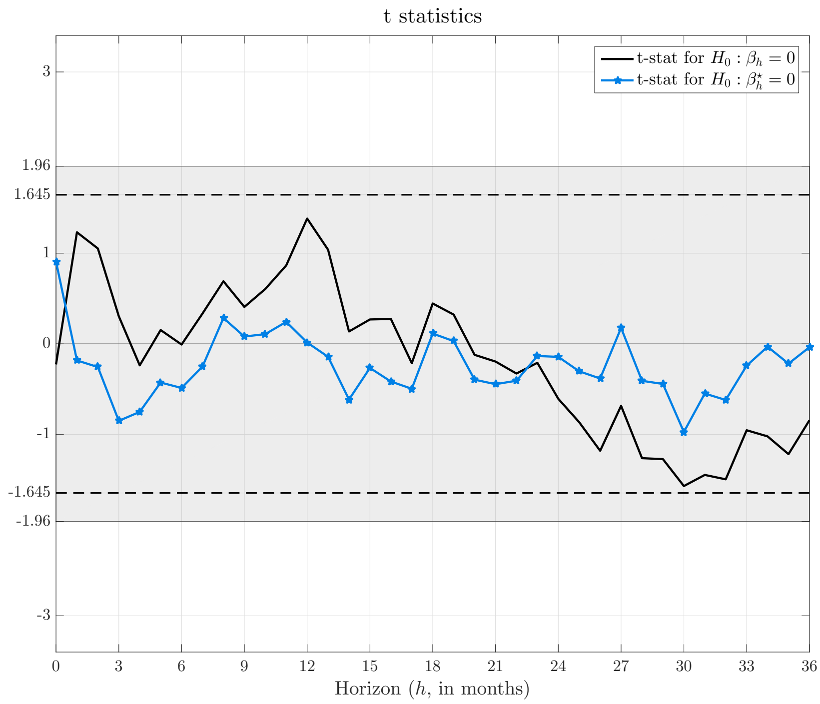
Notes: Regular price and frictionless price indexes are computed at a monthly frequency from weighted averages of the elementary aggregates in (24). Year over year inflation is computed as the percentage variation in the index of a given month against the same month of the previous year. The grey shaded area denotes the great recession.

Figure 18: Impulse responses to a monetary policy shock



Notes: Impulse responses obtained from a local projection of the form $y_{t+h} = \alpha_h + \beta_h \widehat{\Delta i_t} + \gamma_h X_t + \varepsilon_t$ for horizons $h = 0, 1, \ldots, 36$. The change in the nominal Bank of England interest rate is instrumented with the series of high-frequency identified monetary surprises by Cesa-Bianchi, Thwaites and Vicondoa (2019) and the vector of controls X_t includes four lags of regular price inflation and frictionless inflation.

Figure 19: Significance tests of impulse responses to a monetary policy shock



Notes: For a local projection of the form $y_{t+h} = \alpha_h + \beta_h \widehat{\Delta i_t} + \gamma_h X_t + \varepsilon_t$ this plot contains the t-statistics for the null hypothesis $H_0 : \beta_h = 0$ over different horizons. For a given horizon if the line is outside the grey shaded (dashed) area indicates the null can be rejected at the 5% (10%) against a double sided alternative.

Table 1: Data generating processes in the Monte Carlo experiment

DGP	$\underline{x}_j$	$\bar{x}_j$	μ_j	$\sigma_{\varepsilon,j}$	λ_j
1	-0.10	0.10	0.002	0.05	0.025
2	-0.10	0.10	0.002	0.05	0.10
3	-0.10	0.10	0.002	0.05	0.40

Notes: For each of the DGPs above balanced panels of three sizes are considered: 100 quote-lines over 5 years ($N = 100$, $T = 60$); 100 quote-lines over 20 years ($N = 100$, $T = 240$) and 300 quote-lines over 5 years ($N = 300$, $T = 60$). True initial price gaps are set to equally spaced points within the re-centred inaction region.

Table 2: Descriptive statistics for estimated parameters

	$\hat{\underline{x}}_j$	$\hat{\bar{x}}_j$	$\hat{\mu}_j$	$\hat{\sigma}_{\varepsilon,j}$	$\hat{\lambda}_j$
Mean	-2.691	2.271	0.001	0.09	0.139
Median	-0.596	0.568	0.002	0.072	0.114
Std Dev	8.633	6.676	0.011	0.06	0.109
IQR	1.87	1.466	0.004	0.064	0.092
5th Percentile	-9.377	0.129	-0.014	0.023	0.024
10th Percentile	-5.586	0.194	-0.007	0.032	0.039
25th Percentile	-2.104	0.31	0	0.05	0.073
75th Percentile	-0.234	1.776	0.004	0.114	0.165
90th Percentile	-0.056	4.925	0.008	0.179	0.266
95th Percentile	-0.02	8.815	0.012	0.213	0.37
Minimum	-130.84	0	-0.137	0.006	0.001
Maximum	0	130.262	0.058	0.4	0.868
N	979	979	979	979	979

Notes: Descriptive statistics calculated over different items. All the numbers rounded to three decimal places and any number smaller than 5×10^{-4} is displayed as a zero.

Table 3: Inflation wedge over the business cycle

Deviation from	HP trend	Cubic trend	Hamilton trend
Monthly GDP			
	-5.52	-10.62*	-3.06
	(4.56)	(5.93)	(4.18)
Industrial Production			
	-3.11	-4.6*	-3.77*
	(2.44)	(2.45)	(2.21)

Notes: Each cell contains the slope coefficient from a bivariate regression of the form $(\pi_t - \pi_t^{\star}) = \alpha + \beta\Delta\tilde{y}_t + w_t$. In parenthesis are the Newey and West (1987) HAC standard errors with 12 lags. In all the specifications the dependent variable is computed as the difference between year over year regular price inflation and its frictionless counterpart (as depicted in figure 17). The independent variable varies across specifications: output is measured by either the log of a monthly GDP index (top panel) or the log of industrial production index (bottom panel) and the output gap is computed as the log deviation of the given output measure from an HP trend (second column), from a cubic trend (third column) or from a Hamilton (2018) trend (fourth column). * indicates significance at the 10% level, ** indicates significance at the the 5% level and *** indicates significance at the 1% level. All the numbers are rounded to two decimal places.

Table 4: Forecasting Inflation - Frictionless versus Headline

Forecasting horizon	6 Months	12 Months	24 Months	36 Months
In-sample standard error				
Frictionless	0.67	0.95	1.54	2.09
Headline	0.69	0.99	1.78	2.54
Both	0.66	0.94	1.54	2.09
Out-of-sample root mean squared error				
Frictionless	0.81	1.16	1.99	2.40
Headline	0.83	1.30	2.79	4.27
Both	0.77	1.09	1.69	2.00
Multivariate regression coefficients				
Frictionless	0.26*	0.47*	1.32**	2.14***
	(0.14)	(0.25)	(0.52)	(0.67)
Headline	0.08	0.20	0.08	0.08
	(0.12)	(0.19)	(0.45)	(0.69)

Notes: Each cell in the table is derived from a regression of the form $\pi_{t,t+h} = \alpha + \mathbf{X}\beta + \varepsilon_t$ where h denotes a particular forecasting horizon. For each forecasting horizon, $\mathbf{X}$ contains headline published inflation over the previous 12 months ($\pi_{t-12,t}$) or frictionless inflation over the previous 12 months ($\pi^{\star}_{t-12,t}$) or both. The in-sample standard error in the top panel is computed from the whole sample 1997m2 to 2018m1. The out-of-sample root mean squared error in the second panel is computed by estimating the regression using data from 1997m2 to 2008m1 and using the forecasting errors from then to the end of the sample. The third panel reports the regression coefficients of the multivariate regression containing both $\pi_{t-12,t}$ and $\pi^{\star}_{t-12,t}$ over the whole sample 1997m2 to 2018m1. In parenthesis are the respective Newey and West (1987) standard erros with a lag length choice equal to 12. * indicates significance at the 10% level, ** indicates significance at the the 5% level and *** indicates significance at the 1% level. All the numbers are rounded to two decimal places.

Appendix to

Frictionless Inflation

by Miguel Bandeira

A Proofs of propositions and auxiliary results

This appendix contains the proofs of all the propositions in the main text as well as some auxiliary results used in those proofs. Appendix A.1 contains the proofs of results in the estimation block. Appendix A.2 contains the proofs of all the results associated with the smoothing problem, including the solution to the filtering problem.

A.1 Proof of results for the estimation block

Proof of Proposition 1. For any t it follows from (7) that $L_{i,j,t}$ depends only on the parameter $\lambda_{i,j}$ and the realizations of the shock $\nu_{i,j,t}$ which is drawn from a Uniform (0,1), hence, $L_{i,j,t}$ does not depend on $x_{i,j,0}$. Iterating (6) backwards and using $Z^\star_{i,j,0} = 0$ yields,

$$Z^\star_{i,j,t} = t\mu_{i,j} + \sum_{k=1}^{t} \varepsilon_{i,j,k} \tag{P1.1}$$

Therefore, for any t this depends only on the parameters $\mu_{i,j}$ and $\sigma_{\varepsilon,i,j}$. For any $t > \tau^1_{i,j}$, equation (5) implies that $Z_{i,j,t-1} = Z^\star_{i,j,\tau^k_{i,j}} - x_{i,j,0}$ where $\tau^k_{i,j}$ denotes the last time period where a price change occurred. Subtract $Z_{i,j,t-1}$ on both sides of (5) and use $Z_{i,j,t-1} = Z^\star_{i,j,\tau^k_{i,j}} - x_{i,j,0}$ to obtain,

$$\Delta p_{i,j,t} = (Z^\star_{i,j,t} - Z^\star_{i,j,\tau^k_{i,j}})\,(1 - d_{i,j,t}) \tag{P1.2}$$

Finally, given that $Z^\star_{i,j,t}$ is independent of $x_{i,j,0}$ for any t, it remains to be shown that for $t > \tau^1_{i,j}$ also $d_{i,j,t}$ is independent of $x_{i,j,0}$. To see this substitute $Z_{i,j,t-1} = Z^\star_{i,j,\tau^k_{i,j}} - x_{i,j,0}$ in (2) to obtain:

$$d_{i,j,t} = \mathbb{1}\{Z^\star_{i,j,\tau^k_{i,j}} - Z^\star_{i,j,t} \in (\underline{x}_{i,j}, \bar{x}_{i,j})\}(1 - L_{i,j,t}) + \mathbb{1}\{Z^\star_{i,j,t} = Z^\star_{i,j,\tau^k_{i,j}}\}L_{i,j,t} \tag{P1.3}$$

Given that $Z^\star_{i,j,t}$ and $L_{i,j,t}$ do not depend on $x_{i,j,0}$, this completes the proof. □

A.2 Proof of results for filtering and smoothing

This section contains the proofs of all the relevant results used to derive the smoothed probability density function of cumulated frictionless inflation at any point in time. In total this section contains seven results. The first one is an algebraic fact on the product of normal densities (lemma 1). That is followed by two results that are the solution to the filtering problem in state-space representation (5), (6) and (7) (lemmas 2 and 3). This is followed by three results that characterise the smoothed density presented in the main text (propositions 2, 3 and 4) and the proof of smoothed estimates for the pure Calvo model (corollary 1). In addition to the the notation introduced at the beginning of section 3.3, the function $o(\cdot) : \mathbb{R} \to \mathbb{R}$ is used to denote any function that is equal to zero almost everywhere in the real line.

A.2.1 Auxiliary fact on the product of normal distributions

Lemma 1 *Let $x, y, \mu_x, a, c \in \mathbb{R}$, $\sigma_x, \sigma_y \in \mathbb{R}_{>0}$ and $\mu_y = ax + c$. Then,*

$$\frac{1}{\sigma_y}\phi\left(\frac{y-\mu_y}{\sigma_y}\right) \times \frac{1}{\sigma_x}\phi\left(\frac{x-\mu_x}{\sigma_x}\right) = \frac{1}{\tilde{\sigma}_y}\phi\left(\frac{y-\tilde{\mu}_y}{\tilde{\sigma}_y}\right) \times \frac{1}{\tilde{\sigma}_x}\phi\left(\frac{x-\tilde{\mu}_x}{\tilde{\sigma}_x}\right) \tag{L1.1}$$

where,

$$\tilde{\mu}_x = \frac{\sigma_x^2 a(y-c) + \sigma_y^2 \mu_x}{\sigma_y^2 + a^2\sigma_x^2} \tag{L1.2}$$

$$\tilde{\sigma}_x = \frac{\sigma_y \sigma_x}{\sqrt{\sigma_y^2 + a^2\sigma_x^2}} \tag{L1.3}$$

$$\tilde{\mu}_y = a\mu_x + c \tag{L1.4}$$

$$\tilde{\sigma}_y = \sqrt{\sigma_y^2 + a^2\sigma_x^2} \tag{L1.5}$$

Proof of Lemma 1. Using the definition of the standard normal probability density function:

$$\begin{aligned}\frac{1}{\sigma_y}\phi\left(\frac{y-\mu_y}{\sigma_y}\right) \times \frac{1}{\sigma_x}\phi\left(\frac{x-\mu_x}{\sigma_x}\right) &= \frac{1}{2\pi\sigma_y\sigma_x}\exp\left\{-\frac{(y-\mu_y)^2}{2\sigma_y^2} - \frac{(x-\mu_x)^2}{2\sigma_x^2}\right\} \\ &= \frac{1}{2\pi\sigma_y\sigma_x}\exp\left\{-\frac{1}{2\sigma_y^2\sigma_x^2}\underbrace{\left[\sigma_x^2(y-\mu_y)^2 + \sigma_y^2(x-\mu_x)^2\right]}_{(*)}\right\}\end{aligned} \tag{L1.6}$$

Given that $\mu_y = ax + c$, rearrange terms and define $\tilde{\mu}_x \equiv (\sigma_y^2 + a^2\sigma_x^2)^{-1}(\sigma_x^2 a(y-c) + \sigma_y^2\mu_x)$ to obtain,

$$
\begin{aligned}
(*) &= \sigma_x^2(y-c)^2 + (\sigma_x^2 a^2 + \sigma_y^2)\left(x^2 - 2x\tilde{\mu}_x\right) + \sigma_y^2 \mu_x^2 \\
&= \sigma_x^2(y-c)^2 + (\sigma_x^2 a^2 + \sigma_y^2)\left(x^2 - 2x\tilde{\mu}_x + \tilde{\mu}_x^2\right) + \sigma_y^2 \mu_x^2 - (\sigma_x^2 a^2 + \sigma_y^2)\tilde{\mu}_x^2 \\
&= (\sigma_x^2 a^2 + \sigma_y^2)\left(x - \tilde{\mu}_x\right)^2 + \underbrace{\sigma_x^2(y-c)^2 + \sigma_y^2 \mu_x^2 - (\sigma_x^2 a^2 + \sigma_y^2)\tilde{\mu}_x^2}_{(**)} \qquad \text{(L1.7)}
\end{aligned}
$$

Using the definition of $\tilde{\mu}_x$ and rearranging terms yields,

$$
\begin{aligned}
(**) &= \sigma_x^2 \sigma_y^2 (\sigma_x^2 a^2 + \sigma_y^2)^{-1}\left[(y-c)^2 - 2a\mu_x(y-c) + a^2\mu_x^2\right] \\
&= \sigma_x^2 \sigma_y^2 (\sigma_x^2 a^2 + \sigma_y^2)^{-1}(y - (\underbrace{a\mu_x + c}_{\equiv \tilde{\mu}_y}))^2 \qquad \text{(L1.8)}
\end{aligned}
$$

Combine (L1.8) and (L1.7) and plug back in (L1.6),

$$
\frac{1}{\sigma_y}\phi\left(\frac{y-\mu_y}{\sigma_y}\right) \times \frac{1}{\sigma_x}\phi\left(\frac{x-\mu_x}{\sigma_x}\right) = \frac{1}{2\pi\sigma_y\sigma_x}\exp\left\{-\frac{1}{2}\left(\frac{\left(x-\tilde{\mu}_x\right)^2}{(\sigma_x^2 a^2 + \sigma_y^2)^{-1}\sigma_x^2\sigma_y^2} + \frac{\left(y-\tilde{\mu}_y\right)^2}{(\sigma_x^2 a^2 + \sigma_y^2)}\right)\right\}
$$

Finally, define $\tilde{\sigma}_x \equiv (\sigma_y^2 + a^2\sigma_x^2)^{-\frac{1}{2}}(\sigma_y\sigma_x)$ and $\tilde{\sigma}_y \equiv (\sigma_y^2 + a^2\sigma_x^2)^{\frac{1}{2}}$ and rearrange to obtain,

$$
\frac{1}{\sigma_y}\phi\left(\frac{y-\mu_y}{\sigma_y}\right) \times \frac{1}{\sigma_x}\phi\left(\frac{x-\mu_x}{\sigma_x}\right) = \underbrace{\frac{1}{(2\pi)^{\frac{1}{2}}\tilde{\sigma}_y}\exp\left\{-\frac{\left(y-\tilde{\mu}_y\right)^2}{2\tilde{\sigma}_y}\right\}}_{\frac{1}{\tilde{\sigma}_y}\phi\left(\frac{y-\tilde{\mu}_y}{\tilde{\sigma}_y}\right)} \underbrace{\frac{1}{(2\pi)^{\frac{1}{2}}\tilde{\sigma}_x}\exp\left\{-\frac{\left(x-\tilde{\mu}_x\right)^2}{2\tilde{\sigma}_x^2}\right\}}_{\frac{1}{\tilde{\sigma}_x}\phi\left(\frac{x-\tilde{\mu}_x}{\tilde{\sigma}_x}\right)}
$$

This completes the proof. □

A.2.2 The filtered density: main result

Lemma 2 *Consider an arbitrary time period* $t \in \mathbb{Z}_{[\tau^0,T]}$.

If $\exists\, k \in \mathbb{Z}_{[0,K]}$ *such that* $t = \tau^k$,

$$f_{Z_t^\star \,|\, Z^t;\Theta}\left(z^\star | z^t; \theta\right) \;=\; \delta\left(z^\star - c^k\right) \tag{L2.1}$$

where $c^k = 0$ *for* $k = 0$ *or* $c^k = z_{\tau^k} + x_0$ *for* $k \in \mathbb{Z}_{[1,K]}$.

Otherwise, suppose $\nexists\, k \in \mathbb{Z}_{[0,K]}$ *such that* $t = \tau^k$. *In that case, let* τ^k *denote the last period where* (L2.1) *holds, that is, let* k *be such that* $\nexists j \in \mathbb{Z}_{[0,K]}$ *that satisfies* $\tau^k < \tau^j < t$. *For a given combination of* t *and* k, *define:* $b \equiv t - \tau^k$, $\underline{Z}^k \equiv \mathbb{1}\{k = 0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \bar{x}$, $\bar{Z}^k \equiv \mathbb{1}\{k = 0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \underline{x}$ *and* $\mathcal{I}^k \equiv (\underline{Z}^k, \bar{Z}^k)$. *Then, ignoring terms that are zero almost everywhere, it holds that:*

$$f_{Z_t^\star \,|\, Z^t;\Theta}\left(z^\star | z^t; \theta\right) \;\propto\; \frac{1}{\sigma_b}\phi\left(\frac{z^\star - \mu_b^k}{\sigma_b}\right)\; \beta_b^k(z^\star)\; \mathbb{1}\left\{z^\star \in \mathcal{I}^k\right\} \tag{L2.2}$$

where $\beta_b^k(\cdot)$ *is given recursively by,*

$$\beta_b^k\left(x\right) = \begin{cases} 1, \text{ if } b = 1 \\ \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\tilde{\sigma}_{b-1}}\,\phi\left(\frac{y - \tilde{\mu}_{b-1}^k(x)}{\tilde{\sigma}_{b-1}}\right)\, \beta_{b-1}^k\left(y\right) dy, \text{ if } b > 1 \end{cases} \tag{L2.3}$$

and the means and standard deviations of the distributions are given by,

$$\mu_b^k = b\mu + c^k \quad \text{and} \quad \sigma_b = \sqrt{b}\,\sigma_\varepsilon \tag{L2.4}$$

$$\tilde{\mu}_b^k(x) = \frac{c^k + bx}{b+1} \quad \text{and} \quad \tilde{\sigma}_b = \sqrt{\frac{b}{b+1}}\,\sigma_\varepsilon \tag{L2.5}$$

Proof of Lemma 2. Consider each of the cases separately.

Proof of (L2.1): If $t = \tau^0$, the distribution of $Z_t^\star$ must be degenerate at 0 since, by definition, $Z_{\tau^0}^\star = p_t^\star - p_{\tau^0}^\star$. Otherwise, if $t = \tau^k$ for some $k \in \mathbb{Z}_{[1,K]}$ it means that the time period considered corresponds to a period where a non-zero price change is observed. In that case, from (5) it must be the case that $d_t = 0$ and, hence, the distribution of $Z_t^\star$ must be degenerate at $z_t + x_0$. Defining $c^k = 0$ for $k = 0$ and $c^k = z_{\tau^k} + x_0$ for $k \in \mathbb{Z}_{[1,K]}$ and using the Dirac delta to denote a degenerate pdf, (L2.1) follows.

Proof of (L2.2): To establish (L2.2), start from Bayes' rule:

$$f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta) = f_{Z_t^\star|Z_t;Z^{t-1};\Theta}(z^\star|z_t;z^{t-1};\theta)$$

$$= \frac{f_{Z_t|Z_t^\star;Z^{t-1};\Theta}(z_t|z^\star;z^{t-1};\theta) f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta)}{\int f_{Z_t|Z_t^\star;Z^{t-1};\Theta}(z_t|a;z^{t-1};\theta)\, f_{Z_t^\star|Z^{t-1};\Theta}(a|z^{t-1};\theta)\, da} \tag{L2.6}$$

Consider the first term in the numerator of (L2.6) and use the law of total probability:

$$f_{Z_t|Z_t^\star;Z^{t-1};\Theta}(z_t|z^\star;z^{t-1};\theta) = f_{Z_t|Z_t^\star;Z^{t-1};L_t;\Theta}(z_t|z^\star;z^{t-1};0;\theta) \times f_{L_t|Z_t^\star;Z^{t-1};\Theta}(0|z^\star;z^{t-1};\theta)$$

$$+ f_{Z_t|Z_t^\star;Z^{t-1};L_t;\Theta}(z_t|z^\star;z^{t-1};1;\theta) \times f_{L_t|Z_t^\star;Z^{t-1};\Theta}(1|z^\star;z^{t-1};\theta) \tag{L2.7}$$

Since $\nexists k \in \mathbb{Z}_{[0,K]}$ such that $t = \tau^k$, it must be that this is a period of inaction and, hence, $z_t = z_{t-1}$ and $d_t = 1$. Moreover, from $\nexists j \in \mathbb{Z}_{[0,K]}$ such that $\tau^k < \tau^j < t$ it must also hold that $z_{t-1} = z_{\tau^k}$. Using the definition of d_t in (2) and the transition equation for the arrival of costless adjustment opportunities in (7), expression (L2.7) can be written as:

$$f_{Z_t|Z_t^\star;Z^{t-1};\Theta}(z_t|z^\star;z^{t-1};\theta) = \mathbb{1}\{z^\star \in \mathcal{I}^k\} \times (1-\lambda) + \mathbb{1}\{z^\star = z_{\tau^k} + x_0\} \times \lambda \tag{L2.8}$$

where $\mathcal{I}^k \equiv (\underline{Z}^k, \bar{Z}^k)$ and $\underline{Z}^k \equiv \mathbb{1}\{k=0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \bar{x}$ and $\bar{Z}^k \equiv \mathbb{1}\{k=0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \underline{x}$. Substituting (L2.8) in (L2.6) and re-arranging yields:

$$f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta) = \frac{f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta)}{C_t}\mathbb{1}\{z^\star \in \mathcal{I}^k\}$$

$$+ \underbrace{\frac{\lambda}{(1-\lambda)}\frac{f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta)}{C_t}\mathbb{1}\{z^\star = z_{\tau^k} + x_0\}}_{=o(z^\star)} \tag{L2.9}$$

where $C_t \equiv \int_{\underline{Z}^k}^{\bar{Z}^k} f_{Z_t^\star|Z^{t-1};\Theta}(a|z^{t-1};\theta)\, da$ and the second term on the RHS is zero almost everywhere in the real line. Considering now the second term in the numerator of (L2.6) and use the Chapman-Kolmogorov equation to obtain:

$$f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta) = \int f_{Z_t^\star|Z_{t-1}^\star;\Theta}(z^\star|\tilde{z}^\star;\theta) f_{Z_{t-1}^\star|Z^{t-1};\Theta}(\tilde{z}^\star|z^{t-1};\theta)\, d\tilde{z}^\star \tag{L2.10}$$

Using the transition equation for the cumulated frictionless inflation in (6) and the normality of the idiosyncratic shocks (ε_t) yields that,

$$f_{Z_t^\star|Z_{t-1}^\star;\Theta}(z^\star|\tilde{z}^\star;\theta) = \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star - (\mu + \tilde{z}^\star)}{\sigma_\varepsilon}\right) \tag{L2.11}$$

Substituting (L2.10) and (L2.11) in (L2.9),

$$f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta) = C_t^{-1}\int \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star - (\mu + \tilde{z}^\star)}{\sigma_\varepsilon}\right) f_{Z_{t-1}^\star|Z^{t-1};\Theta}(\tilde{z}^\star|z^{t-1};\theta)\, d\tilde{z}^\star\, \mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star) \tag{L2.12}$$

Equation (L2.12) is a *filtering forward recursion*. It expresses the filtered pdf a given time period as a function of the filtered pdf in the previous time period. Filtering forward recursions are common in the nonlinear non-Gaussian filtering literature (see, for instance, Kitagawa (1987, equation 2.3) or Särkkä (2013, theorem 4.1)). The key difference of (L2.12) is that it holds only for *inaction periods* (i.e. $\nexists k \in \mathbb{Z}_{[0,K]}$ such that $t = \tau^k$) for which the last period where $Z^\star$ is known is τ^k (i.e. $\nexists j \in \mathbb{Z}_{[0,K]}$ that satisfies $\tau^k < \tau^j < t$). To complete the proof it remains to be shown that (L2.2) satisfies (L2.12) for any such time period. This is shown by induction.

Base case: Suppose t is an inaction period and that $t = \tau^k + 1$ (so that $b = 1$). In that case, using that at $t = \tau^k$ the filtered pdf is given by (L2.1), then (L2.12) reads,

$$\begin{aligned} f_{Z_{\tau^k+1}^\star|Z^{\tau^k+1};\Theta}(z^\star|z^{\tau^k+1};\theta) &= C_{\tau^k+1}^{-1}\int \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star - (\mu + \tilde{z}^\star)}{\sigma_\varepsilon}\right) \delta(\tilde{z}^\star - c^k)\, d\tilde{z}^\star\, \mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star) \\ &= C_{\tau^k+1}^{-1}\frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star - (\mu + c^k)}{\sigma_\varepsilon}\right)\, \mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star) \\ &= C_{\tau^k+1}^{-1}\frac{1}{\sigma_1}\phi\left(\frac{z^\star - \mu_1^k}{\sigma_1}\right) \beta_1^k(z^\star)\, \mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star) \end{aligned} \tag{L2.13}$$

where: (i) the second equality uses the properties of the Dirac delta function and (ii) the third equality uses the definitions of μ_b^k and σ_b in (L2.4) with $b = 1$ and the fact that $\beta_b^k(x) = 1$ for $b = 1$ from (L2.3). Therefore, (L2.2) satisfies (L2.12) for the base case.

Induction step: Suppose $t-1$ and t are both inaction periods and they are such that $\nexists j \in \mathbb{Z}_{[0,K]}$ that satisfies $\tau^k < \tau^j < t-1$. Define $b \equiv t - \tau^k$ and suppose (L2.2) holds for $t-1$, that is, the filtered pdf at $t-1$ can be written as:

$$f_{Z_{t-1}^\star|Z^{t-1};\Theta}(\tilde{z}^\star|z^{t-1};\theta) = C_{t-1}\frac{1}{\sigma_{b-1}}\phi\left(\frac{\tilde{z}^\star - \mu_{b-1}^k}{\sigma_{b-1}}\right) \beta_{b-1}^k(\tilde{z}^\star)\, \mathbb{1}\{\tilde{z}^\star \in \mathcal{I}^k\} + o(\tilde{z}^\star) \tag{L2.14}$$

where C_{t-1} is a normalisation constant. Substituting (L2.14) in (L2.12) yields,

$$f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta) = \frac{C_{t-1}}{C_t}\int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star-(\mu+\tilde{z}^\star)}{\sigma_\varepsilon}\right)\frac{1}{\sigma_{b-1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b-1}^k}{\sigma_{b-1}}\right)\beta_{b-1}^k(\tilde{z}^\star)d\tilde{z}^\star\mathbb{1}\{z^\star\in\mathcal{I}^k\}+o(z^\star)$$

$$= \frac{C_{t-1}}{C_t}\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\int_{\underline{Z}^k}^{\bar{Z}^k}\frac{1}{\tilde{\sigma}_{b-1}}\phi\left(\frac{\tilde{z}^\star-\tilde{\mu}_{b-1}^k(z^\star)}{\tilde{\sigma}_{b-1}}\right)\beta_{b-1}^k(\tilde{z}^\star)d\tilde{z}^\star\mathbb{1}\{z^\star\in\mathcal{I}^k\}+o(z^\star)$$

$$= \frac{C_{t-1}}{C_t}\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\beta_b^k(z^\star)\mathbb{1}\{z^\star\in\mathcal{I}^k\}+o(z^\star) \tag{L2.15}$$

where: (i) the first equality follows from combining (L2.14) in (L2.12) and rearranging terms; (ii) the second equality uses lemma 1 to combine the two normal pdfs in the integral and the definitions in (L2.4) and (L2.5); (iii) the third equality uses the definition of (L2.3).[46] Therefore, the filtered pdf is again given by (L2.2).

This completes the proof. □

A.2.3 The filtered density: auxiliary result

Lemma 3 *Consider an arbitrary time period $t\in\mathbb{Z}_{[\tau^0,T]}$. If $t=\tau^0$ then,*

$$f_{Z_t^\star\,|\,\Theta}\left(z^\star|\Theta\right) \;=\; \delta\left(z^\star\right) \tag{L3.1}$$

Otherwise, for any $t>\tau^0$ let k be such that $\nexists j\in\mathbb{Z}_{[0,K]}:\tau^k<\tau^j<t$. Define $b\equiv t-\tau^k$, $\underline{Z}^k\equiv\mathbb{1}\{k=0\}x_0+\mathbb{1}\{k\geqslant1\}c^k-\bar{x}$ and $\bar{Z}^k\equiv\mathbb{1}\{k=0\}x_0+\mathbb{1}\{k\geqslant1\}c^k-\underline{x}$. Then,

$$f_{Z_t^\star\,|\,Z^{t-1};\Theta}\left(z^\star|z^{t-1};\Theta\right) \;\propto\; \frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\;\beta_b^k(z^\star) \tag{L3.2}$$

where $\beta_b^k(\cdot)$ is given by (L2.3)*, μ_b^k and σ_b are given by* (L2.4).

Proof of Lemma 3. For the initial time period $t=\tau^0$ expression (L3.1) follows from the fact that the distribution is degenerate at zero since, by definition, $Z_t^\star=p_t^\star-p_{\tau^0}^\star$. For any other time period, use the Chapman-Kolmogorov equation and the pdf for $Z_t^\star$ conditional on $Z_{t-1}^\star$:

$$f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta) = \int\frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star-(\mu+\tilde{z}^\star)}{\sigma_\varepsilon}\right)f_{Z_{t-1}^\star|Z^{t-1};\Theta}(\tilde{z}^\star|z^{t-1};\theta)\,d\tilde{z}^\star \tag{L3.3}$$

It remains to be shown that for any t, the expression for the pdf in (L3.2) satisfies (L3.3). According to the expressions for the filtered pdf in lemma 2 there are two cases to be verified.

[46] In the second equality Lemma 1 is invoked with: $y=z^\star$, $x=\tilde{z}^\star$, $a=1$, $c=\mu$, $\sigma_y=\sigma_\varepsilon$, $\sigma_x=\sigma_{b-1}$ and $\mu_x=\mu_{b-1}^k$.

First, consider the case where $t-1$ is either a period for which a non-zero price change is observed or the initial period, that is, $\exists k \in \mathbb{Z}_{[0,K]}$ such that $t-1=\tau^k$. Combining the filtered density in (L2.1) with (L3.3) yields,

$$
\begin{aligned}
f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta) &= \int \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star-(\mu+\tilde{z}^\star)}{\sigma_\varepsilon}\right)\delta(\tilde{z}^\star-c^k)\,d\tilde{z}^\star \\
&= \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star-(\mu+c^k)}{\sigma_\varepsilon}\right) \\
&= \frac{1}{\sigma_1}\phi\left(\frac{z^\star-\mu_1^k}{\sigma_1}\right)\beta_1^k(z^\star) \qquad \text{(L3.4)}
\end{aligned}
$$

where: (i) the second equality uses the properties of the Dirac delta function; (ii) the third equality used the definitions of μ_b^k and σ_b in (L2.4) for $b=1$ and that $\beta_b^k(x)=1\ \forall x$ if $b=1$ from (L2.3). Therefore, (L3.2) holds.

Second, consider the case where $t-1$ is a period of inaction and the last period for which the value of $Z^\star$ is known is given by τ^k. In that case, the filtered density is given by (L2.2) and (L3.3) reads,

$$
\begin{aligned}
f_{Z_t^\star|Z^{t-1};\Theta}(z^\star|z^{t-1};\theta) &= \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\sigma_\varepsilon}\phi\left(\frac{z^\star-(\mu+\tilde{z}^\star)}{\sigma_\varepsilon}\right)\frac{1}{\sigma_{b-1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b-1}^k}{\sigma_{b-1}}\right)\beta_{b-1}^k(\tilde{z}^\star)\,d\tilde{z}^\star \\
&= C_t\,\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\int_{\underline{Z}^k}^{\bar{Z}^k}\frac{1}{\tilde{\sigma}_{b-1}}\phi\left(\frac{\tilde{z}^\star-\tilde{\mu}_{b-1}^k(z^\star)}{\tilde{\sigma}_{b-1}}\right)\beta_{b-1}^k(\tilde{z}^\star)d\tilde{z}^\star \\
&= C_t\,\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\beta_b^k(z^\star) \qquad \text{(L3.5)}
\end{aligned}
$$

where: (i) follows from substituting (L2.2) in (L3.3) and re-arranging; (ii) the second equality from applying lemma 1 to combine the two normal densities (similarly to the derivation of (L2.15)); (iii) the third equality follows by using the definition of $\beta_b^k(z^\star)$ in (L2.3). This completes the proof. □

A.2.4 Proof of Proposition 2

Proof of proposition 2. For the period $t=\tau^0$, $Z^\star$ is degenerate at zero since, by definition, $Z_t^\star = p_t^\star - p_{\tau^0}^\star$. For any other period where a non-zero price change is observed, $t=\tau^k$ for some $k\in\mathbb{Z}_{[1,K]}$, $Z^\star$ is degenerate at the value that closes the price gap, that is, $Z_t^\star = z_{\tau^k}+x_0$ (see measurement equation (5)). This completes the proof. □

A.2.5 Proof of Proposition 3

Proof of proposition 3. This proof has two blocks. First, a smoothing backward recursion that holds for any $t \in (\tau^k, \tau^{k+1})$ and some $k \in \mathbb{Z}_{[0,K-1]}$ is derived. Second, I verify by induction that (12) satisfies that recursion for any $t \in (\tau^k, \tau^{k+1})$. To derive the smoothing backward recursion I start from,

$$
\begin{aligned}
f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) &= \int f_{Z_t^\star,Z_{t+1}^\star|Z^T;\Theta}(z^\star,\tilde{z}^\star|z^T;\theta)\,d\tilde{z}^\star \\
&= \int f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)\,f_{Z_t^\star|Z_{t+1}^\star;Z^T;\Theta}(z^\star|\tilde{z}^\star;z^T;\theta)\,d\tilde{z}^\star \\
&= \int f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)\,f_{Z_t^\star|Z_{t+1}^\star;Z^t;\Theta}(z^\star|\tilde{z}^\star;z^t;\theta)\,d\tilde{z}^\star \\
&= \int \frac{f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)\,f_{Z_{t+1}^\star|Z_t^\star;\Theta}(\tilde{z}^\star|z^\star;\theta)\,f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta)}{f_{Z_{t+1}^\star|Z^t;\Theta}(\tilde{z}^\star|z^t;\theta)}\,d\tilde{z}^\star \\
&= f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta)\int \frac{f_{Z_{t+1}^\star|Z_t^\star;\Theta}(\tilde{z}^\star|z^\star;\theta)\,f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)}{f_{Z_{t+1}^\star|Z^t;\Theta}(\tilde{z}^\star|z^t;\theta)}\,d\tilde{z}^\star \qquad \text{(P3.1)}
\end{aligned}
$$

This derivation is similar to Kitagawa (1987, equation 2.4) or Särkkä (2013, theorem 8.1). For any $t \in (\tau^k, \tau^{k+1})$, the filtered pdf $f_{Z_t^\star|Z^t;\Theta}(z^\star|z^t;\theta)$ is given by (L2.2) whereas the term in the denominator of the expression inside the integral is given by (L3.2). Let $b = t - \tau^k$, substitute using (L2.2), (L2.11) and (L3.2) and re-arrange to obtain,

$$
\begin{aligned}
f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \frac{C_{t-1}}{C_t^2}\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\beta_b^k(z^\star)\mathbb{1}\{z^\star\in\mathcal{I}^k\}\int \frac{\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right)f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)\beta_{b+1}^k(\tilde{z}^\star)}\,d\tilde{z}^\star \\
+\underbrace{o(z^\star)\frac{1}{C_t}\int \frac{\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right)f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)\beta_{b+1}^k(\tilde{z}^\star)}\,d\tilde{z}^\star}_{=o(z^\star)} \qquad \text{(P3.2)}
\end{aligned}
$$

Let $\tilde{C}_t \equiv C_{t-1}/C_t^2$ and re-arrange terms to obtain,

$$
f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \tilde{C}_t\beta_b^k(z^\star)\mathbb{1}\{z^\star\in\mathcal{I}^k\}\int \frac{\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right)f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)\beta_{b+1}^k(\tilde{z}^\star)}\,d\tilde{z}^\star+o(z^\star) \qquad \text{(P3.3)}
$$

Equation (P3.3) is a *smoothing backward recursion* as it relates the smoothed pdf at time t to the smoothed pdf at time $t+1$. Whilst (P3.1) holds in general for any Markov state-space representation, (P3.3) is specific to the state-space representation in (5) to (7) as it uses the expressions for the filtered density and its auxiliary form derived in lemmas 2 and 3 and that hold for any $t \in (\tau^k, \tau^{k+1})$ for some $k \in \mathbb{Z}_{[0,K-1]}$. It remains to be shown that expression (12) satisfies (P3.3) for any $t \in (\tau^k, \tau^{k+1})$ for some $k \in \mathbb{Z}_{[0,K-1]}$. To do so, the proof proceeds by induction.

Base case: Let $t = \tau^{k+1} - 1$ so that $b = \Delta^k - 1$. In that case, the smoothed pdf at $t+1$ is given by (11). Substituting that in (P3.3),

$$
\begin{aligned}
f_{Z^\star_{\tau^{k+1}-1}|Z^T;\Theta}(z^\star|z^T;\theta) &= \tilde{C}_{\tau^{k+1}-1}\beta^k_{\Delta^k-1}(z^\star)\mathbb{1}\{z^\star \in \mathcal{I}^k\} \\
&\quad \times \int \frac{\frac{1}{\sigma_{\Delta^k-1}}\phi\left(\frac{z^\star-\mu^k_{\Delta^k-1}}{\sigma_{\Delta^k-1}}\right)\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right)\delta(\tilde{z}^\star - c^{k+1})}{\frac{1}{\sigma_{\Delta^k}}\phi\left(\frac{\tilde{z}^\star-\mu^k_{\Delta^k}}{\sigma_{\Delta^k}}\right)\beta^k_{\Delta^k}(\tilde{z}^\star)} d\tilde{z}^\star + o(z^\star) \\
&= \tilde{C}_{\tau^{k+1}-1}\,\beta^k_{\Delta^k-1}(z^\star)\,\mathbb{1}\{z^\star \in \mathcal{I}^k\}\,\frac{\frac{1}{\sigma_{\Delta^k-1}}\phi\left(\frac{z^\star-\mu^k_{\Delta^k-1}}{\sigma_{\Delta^k-1}}\right)\frac{1}{\sigma_\varepsilon}\phi\left(\frac{c^{k+1}-(\mu+z^\star)}{\sigma_\varepsilon}\right)}{\frac{1}{\sigma_{\Delta^k}}\phi\left(\frac{c^{k+1}-\mu^k_{\Delta^k}}{\sigma_{\Delta^k}}\right)\beta^k_{\Delta^k}(c^{k+1})} + o(z^\star) \\
&= \check{C}_{\tau^{k+1}-1}\,\beta^k_{\Delta^k-1}(z^\star)\,\mathbb{1}\{z^\star \in \mathcal{I}^k\}\,\frac{\frac{1}{\check{\sigma}^k_{\Delta^k-1}}\phi\left(\frac{z^\star-\check{\mu}^k_{\Delta^k-1}}{\check{\sigma}^k_{\Delta^k-1}}\right)\phi\left(\frac{c^{k+1}-\mu^k_{\Delta^k}}{\sigma_{\Delta^k}}\right)}{\phi\left(\frac{c^{k+1}-\mu^k_{\Delta^k}}{\sigma_{\Delta^k}}\right)} + o(z^\star) \\
&= \check{C}_{\tau^{k+1}-1}\,\frac{1}{\check{\sigma}^k_{\Delta^k-1}}\phi\left(\frac{z^\star-\check{\mu}^k_{\Delta^k-1}}{\check{\sigma}^k_{\Delta^k-1}}\right)\,\beta^k_{\Delta^k-1}(z^\star)\,\chi^k_{\Delta^k-1}(z^\star)\,\mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star)
\end{aligned}
\tag{P3.4}
$$

where: (i) the second equality follows from the properties of the Dirac delta function; (ii) the third equality uses $\check{C}_{\tau^{k+1}-1} \equiv \tilde{C}_{\tau^{k+1}-1}/\beta^k_{\Delta^k}(c^{k+1})$ and lemma 1 to combine the two normal densities in the numerator as well as the definitions of $\check{\mu}^k_b$ and $\check{\sigma}^k_b$ in (15); (iii) the fourth equality simply rearranges terms and uses that $\chi^k_{\Delta^k-1}(x) = 1, \forall x$ from the recursive definition in (14).[47] This shows that (12) satisfies the smoothing backward recursion (P3.3) for $t = \tau^{k+1} - 1$.

[47]In the third equality lemma 1 is invoked with: $y = c^{k+1}$, $x = z^\star$, $a = 1$, $c = \mu$, $\sigma_y = \sigma_\varepsilon$, $\sigma_x = \sigma_{\Delta^k-1}$ and $\mu_x = \mu^k_{\Delta^k-1}$.

Induction step: Consider two time periods $t, t+1 \in (\tau^k, \tau^{k+1})$ and define $b = t - \tau^k$. Suppose that (12) holds for $t+1$ and substitute that in (P3.3) to obtain,

$$f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \tilde{C}_t \check{C}_{t+1} \beta_b^k(z^\star) \mathbb{1}\{z^\star \in \mathcal{I}^k\}$$

$$\times \underbrace{\int_{\underline{Z}^k}^{\bar{Z}^k} \frac{\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^k}{\sigma_b}\right)\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right)\frac{1}{\check{\sigma}_{b+1}^k}\phi\left(\frac{\tilde{z}^\star-\check{\mu}_{b+1}^k}{\check{\sigma}_{b+1}^k}\right)\beta_{b+1}^k(\tilde{z}^\star)\chi_{b+1}^k(\tilde{z}^\star)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)\beta_{b+1}^k(\tilde{z}^\star)} d\tilde{z}^\star}_{=(*)} + o(z^\star) \tag{P3.5}$$

where $\check{C}_{t+1}$ denotes the normalisation constant of the smoothed pdf at $t+1$. Looking at the integral term only, we have that:

$$\begin{aligned}
(*) &= \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)\frac{1}{\tilde{\sigma}_b}\phi\left(\frac{z^\star-\tilde{\mu}_b^k(\tilde{z}^\star)}{\tilde{\sigma}_b}\right)\frac{1}{\check{\sigma}_{b+1}^k}\phi\left(\frac{\tilde{z}^\star-\check{\mu}_{b+1}^k}{\check{\sigma}_{b+1}^k}\right)\chi_{b+1}^k(\tilde{z}^\star)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^k}{\sigma_{b+1}}\right)} d\tilde{z}^\star \\
&= \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\check{\sigma}_b^k}\phi\left(\frac{z^\star-\check{\mu}_b^k}{\check{\sigma}_b^k}\right)\frac{1}{\ddot{\sigma}_{b+1}^k}\phi\left(\frac{\tilde{z}^\star-\ddot{\mu}_{b+1}^k(z^\star)}{\ddot{\sigma}_{b+1}^k}\right)\chi_{b+1}^k(\tilde{z}^\star)\, d\tilde{z}^\star \\
&= \frac{1}{\check{\sigma}_b^k}\phi\left(\frac{z^\star-\check{\mu}_b^k}{\check{\sigma}_b^k}\right)\int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\ddot{\sigma}_{b+1}^k}\phi\left(\frac{\tilde{z}^\star-\ddot{\mu}_{b+1}^k(z^\star)}{\ddot{\sigma}_{b+1}^k}\right)\chi_{b+1}^k(\tilde{z}^\star)\, d\tilde{z}^\star \\
&= \frac{1}{\check{\sigma}_b^k}\phi\left(\frac{z^\star-\check{\mu}_b^k}{\check{\sigma}_b^k}\right)\chi_b^k(z^\star)
\end{aligned} \tag{P3.6}$$

where: (i) the first equality follows from cancelling out the terms $\beta_{b+1}^k(\tilde{z}^\star)$ and by using lemma 1 to combine the first two normal densities in the numerator along with the definitions of $\tilde{\mu}_b^k(x)$ and $\tilde{\sigma}_b$ in (16); (ii) the second equality follows from cancelling out the first density in the numerator with the one in the denominator and using again lemma 1 to combine the two remaining pdfs along with the definitions of $\ddot{\mu}_b^k$ and $\ddot{\sigma}_b^k$ in (17); (iii) the third equality takes out of the integral terms that do not depend on $\tilde{z}^\star$; (iv) the fourth equality follows from using the definition of $\chi_b^k(\cdot)$ in (14).[48]

[48] In the first equality, lemma 1 is used with: $y = \tilde{z}^\star$, $x = z^\star$, $a = 1$, $c = \mu$, $\sigma_x = \sigma_b$, $\sigma_y = \sigma_\varepsilon$ and $\mu_x = \mu_b^k$. In the second equality lemma 1 is again used but now with: $y = z^\star$, $x = \tilde{z}^\star$, $a = b/(b+1)$, $c = c^k/(b+1)$, $\sigma_y = \tilde{\sigma}_b$, $\sigma_x = \check{\sigma}_{b+1}^k$ and $\mu_x = \check{\mu}_{b+1}^k$.

Finally, define $\check{C}_t = \tilde{C}_t \check{C}_{t+1}$ and plug back (P3.6) into (P3.5) to obtain,

$$f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \check{C}_t \, \frac{1}{\check{\sigma}_b^k}\phi\left(\frac{z^\star - \check{\mu}_b^k}{\check{\sigma}_b^k}\right) \beta_b^k(z^\star)\,\chi_b^k(z^\star)\,\mathbb{1}\{z^\star \in \mathcal{I}^k\} + o(z^\star) \tag{P3.7}$$

Therefore, (12) is again the solution for the smoothing backward recursion in (P3.3).

This completes the proof. □

A.2.6 Proof of Proposition 4

Proof of proposition 4. This proof is similar to that of proposition 3. I will show that (18) is the smoothed density for any $t \in (\tau^K, T]$. For the base case $t = T$, the smoothed pdf must be equal to the filtered pdf. Therefore, I start from the filtered pdf expression in (L2.2) with $k = K$, $t = T$ and $b = \Delta^K$ and show that it is equivalent to (18),

$$\begin{aligned} f_{Z_T^\star|Z^T;\Theta}(z^\star|z^T;\theta) &= C_T \, \frac{1}{\sigma_{\Delta^K}}\phi\left(\frac{z^\star - \mu_{\Delta^K}^K}{\sigma_{\Delta^K}}\right) \beta_{\Delta^K}^K(z^\star)\,\mathbb{1}\{z^\star \in \mathcal{I}^K\} + o(z^\star) \\ &= C_T \, \frac{1}{\sigma_{\Delta^K}}\phi\left(\frac{z^\star - \mu_{\Delta^K}^K}{\sigma_{\Delta^K}}\right) \beta_{\Delta^K}^K(z^\star)\,\iota_{\Delta^K}^K(z^\star)\,\mathbb{1}\{z^\star \in \mathcal{I}^K\} + o(z^\star) \end{aligned} \tag{P4.1}$$

where C_T is the integration constant for the filtered pdf in (L2.2) for $t = T$ and the second equality follows from $\iota_{\Delta^K}^K(x) = 1 \; \forall x$ as defined in (14). Therefore, (18) is satisfied. For $t \in (\tau^K, T)$, it means that the last period where $Z^\star$ is known is τ^K and, in that case, the smoothing backward recursion in (P3.3) reads as:

$$f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \tilde{C}_t \beta_b^K(z^\star)\mathbb{1}\{z^\star \in \mathcal{I}^K\} \int \frac{\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^K}{\sigma_b}\right)\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right) f_{Z_{t+1}^\star|Z^T;\Theta}(\tilde{z}^\star|z^T;\theta)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^K}{\sigma_{b+1}}\right)\beta_{b+1}^K(\tilde{z}^\star)} d\tilde{z}^\star + o(z^\star) \tag{P4.2}$$

where $b = t - \tau^K$. Assuming (18) holds for $t + 1 \in (\tau^K, T]$, the induction step simply requires showing (18) solves (P4.2) . Substituting (18) in the smoothed pdf in (P4.2) yields,

$$\begin{aligned} f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) &= \tilde{C}_t \, \check{C}_{t+1}\beta_b^K(z^\star)\mathbb{1}\{z^\star \in \mathcal{I}^K\} \\ &\times \int_{\underline{Z}^K}^{\bar{Z}^K} \frac{\frac{1}{\sigma_b}\phi\left(\frac{z^\star-\mu_b^K}{\sigma_b}\right)\frac{1}{\sigma_\varepsilon}\phi\left(\frac{\tilde{z}^\star-(\mu+z^\star)}{\sigma_\varepsilon}\right)\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^K}{\sigma_{b+1}}\right)\beta_{b+1}^K(\tilde{z}^\star)\iota_{b+1}^K(\tilde{z}^\star)}{\frac{1}{\sigma_{b+1}}\phi\left(\frac{\tilde{z}^\star-\mu_{b+1}^K}{\sigma_{b+1}}\right)\beta_{b+1}^K(\tilde{z}^\star)} d\tilde{z}^\star + o(z^\star) \end{aligned} \tag{P4.3}$$

where $\check{C}_{t+1}$ is the normalisation constant of the smoothed pdf at $t + 1$. Define $\check{C}_t \equiv \tilde{C}_t \, \check{C}_{t+1}$, cancel out terms inside the integral, bring the first pdf outside the integral and use the definition of $\iota_b^K(x)$ in (19) to write (P4.3) as,

$$f_{Z_t^\star|Z^T;\Theta}(z^\star|z^T;\theta) = \check{C}_t \frac{1}{\sigma_b}\phi\left(\frac{z^\star - \mu_b^K}{\sigma_b}\right) \beta_b^K(z^\star)\, \iota_b^K(z^\star)\, \mathbb{1}\{z^\star \in \mathcal{I}^K\} + o(z^\star) \tag{P4.4}$$

Therefore (18) solves the backward smoothing recursion in (P4.2). This completes the proof. □

A.2.7 Proof of corollary 1

Proof of corollary 1. For the cases where $\exists k \in \mathbb{Z}_{[0,K]}$ such that $t = \tau^k$, use the definition (21) along with the smoothed density in (11) to obtain,

$$\widehat{Z}_t = \mathbb{E}\left[Z^\star|z^T;\widehat{\Theta}^c\right] = \int z^\star \delta(z^\star - c^k) dz^\star = c^k \tag{C1.1}$$

This follows from the properties of the Dirac delta function.

For the remaining cases, the key fact to notice is that if $\underline{x} = -\infty$ and $\bar{x} = +\infty$ then $\underline{Z}^k \equiv \mathbb{1}\{k=0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \bar{x} = -\infty$ and $\bar{Z}^k \equiv \mathbb{1}\{k=0\}x_0 + \mathbb{1}\{k \geqslant 1\}c^k - \underline{x} = \infty$. If the boundaries of the inaction region diverge, then for a given $k \in \mathbb{Z}_{[0,K-1]}$ and $b \in [1, \Delta^k - 1]$ or $k = K$ and $b \in [1, \Delta^K]$ it holds that $\beta_b^k(x) = 1$, $\chi_b^k(x) = 1$ and $\iota_b^k(x) = 1$ $\forall x \in \mathbb{R}$. To formally verify this consider first the recursion $\beta_b^k(x)$ as defined in (13) and verify this by induction. Fix k equal to some value in $\mathbb{Z}_{[0,K]}$. Consider the base case $b = 1$. By definition, $\beta_1^k(x) = 1, \forall x \in \mathbb{R}$ so the base case is trivially satisfied. For the induction step, suppose $\beta_b^k(x) = 1, \forall x$ for some $b > 1$then using the definition in (13),

$$\beta_{b+1}^k(x) = \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\tilde{\sigma}_b}\,\phi\left(\frac{y - \tilde{\mu}_b^k(x)}{\tilde{\sigma}_b}\right)\, \beta_b^k\,(y)\, dy = \int_{-\infty}^{\infty} \frac{1}{\tilde{\sigma}_b}\,\phi\left(\frac{y - \tilde{\mu}_b^k(x)}{\tilde{\sigma}_b}\right)\, dy = 1 \tag{C1.2}$$

where the last equality follows from the fact that it is an integral of a normal density from $-\infty$ to ∞. Similarly, for a given k equal to some value in $\mathbb{Z}_{[0,K]}$, it is the case that $\chi_{\Delta^k-1}^k(x) = 1$ by definition. For the induction step, suppose that $\chi_{b+1}^k(x) = 1$ for some $b < \Delta^k - 2$. Then using the definition in (14) and similarly to above,

$$\chi_b^k(x) = \int_{\underline{Z}^k}^{\bar{Z}^k} \frac{1}{\ddot{\sigma}_{b+1}^k}\,\phi\left(\frac{y - \ddot{\mu}_{b+1}^k(x)}{\ddot{\sigma}_{b+1}^k}\right)\, \chi_{b+1}^k\,(y)\, dy = \int_{-\infty}^{\infty} \frac{1}{\ddot{\sigma}_{b+1}^k}\,\phi\left(\frac{y - \ddot{\mu}_{b+1}^k(x)}{\ddot{\sigma}_{b+1}^k}\right)\, dy = 1 \tag{C1.3}$$

where again the last equality follows from the fact that it is an integral of a normal density from $-\infty$ to ∞. The proof for $\iota_b^k(x)$ is analogous to that of $\chi_b^k(x)$ in (C1.3).

For t such that $\exists k \in \mathbb{Z}_{[0,K-1]}$ such that $t \in (\tau^k, \tau^{k+1})$ it is the case that,

$$\widehat{Z}_t = \mathbb{E}\left[Z^\star | z^T; \widehat{\Theta}^c\right] = \int z^\star \frac{1}{\breve{\sigma}_b^k} \phi\left(\frac{z^\star - \breve{\mu}_b^k}{\breve{\sigma}_b^k}\right) \beta_b^k(z^\star)\, \chi_b^k(z^\star)\, \mathbb{1}\{z^\star \in \mathcal{I}^k\} dz^\star$$

$$= \int z^\star \frac{1}{\breve{\sigma}_b^k} \phi\left(\frac{z^\star - \breve{\mu}_b^k}{\breve{\sigma}_b^k}\right) dz^\star$$

$$= \breve{\mu}_b^k \tag{C1.4}$$

For t such that $t \in (\tau^k, T]$ it is the case that,

$$\widehat{Z}_t = \mathbb{E}\left[Z^\star | z^T; \widehat{\Theta}^c\right] = \int z^\star \frac{1}{\sigma_b} \phi\left(\frac{z^\star - \mu_b^K}{\sigma_b}\right) \beta_b^K(z^\star)\, \iota_b^K(z^\star)\, \mathbb{1}\{z^\star \in \mathcal{I}^K\} dz^\star$$

$$= \int z^\star \frac{1}{\sigma_b} \phi\left(\frac{z^\star - \mu_b^K}{\sigma_b}\right) dz^\star$$

$$= \breve{\mu}_b^k \tag{C1.5}$$

where in both (C1.4) and (C1.5) the first equality just uses the definition of smoothed estimates in (21) with the smoothed pdf in (12) and (18), respectively. The second equality holds since, under the special case with $\underline{x} = -\infty$ and $\bar{x} = \infty$, it is the case that $\beta_b^k(x) = 1$, $\chi_b^k(x) = 1$ and $\iota_b^k(x) = 1\ \forall x \in \mathbb{R}$ and $\mathcal{I}^k = \mathbb{R}$ for any $k \in \mathbb{Z}_{[0,K]}$.

This completes the proof. □

B Computational details for estimation and smoothing

This appendix contains computational details for parameter estimation and for the computation of the smoothed density estimates as described in section 3.All the procedures here described are implemented in MATLAB R2015b.[49]

B.1 Details for the parameter estimation

Parameter estimation is done in two stages as described in section 3. In this section, I briefly describe the algorithms used to obtain parameter estimates for common parameters in the first stage and to obtain the estimates of initial price gaps in the second stage.

B.1.1 Algorithm for estimation of common parameters

In general, the vector of simulated price changes $\Delta p^s_{j|t>\tau^1}$ depends on the vector of common parameters θ_j and the vector of primitive shocks used to generate that panel of simulated data $\xi^s = \{\{\varepsilon_{i,j,t}, \nu_{i,j,t}\}^{T_{i,j}}_{\tau^0_{i,j}}\}^{\mathrm{N}_j}_{i=1}$.[50] More precisely, one should write $\Delta p^s_{j|t>\tau^1} = f(\xi^s, \theta_j)$ where the function $f(\cdot)$ is implicitly defined by the state-space representation in (5) to (7). Moreover, for a sequence of S vectors of shocks $\Xi = [\xi^1, \ldots, \xi^S]$, let $G(\Delta p_{j|t>\tau^1}, \theta_j, \Xi)$ denote the value of (9) where $\Delta p_{j|t>\tau^1}$ be the vector of price changes excluding the first observed in the data.

Algorithm 1 The algorithm for minimisation of $G(\Delta p_{j|t>\tau^1}, \theta_j, \Xi)$ is as follows:

1. Draw 50 vectors of shocks conform with the data template. Let Ξ^0 denote that vector.
2. Choose an initial value for the vector of parameters, say $\theta^{(0)}_j$.
3. Use a global search algorithm to search for the minimiser of $G(\Delta p_{j|t>\tau^1}, \theta_j, \Xi^0)$.[51]
4. The search is subject to the restrictions: $\underline{x}_j \leqslant 0$, $\bar{x}_j \geqslant 0$, $\sigma_{\varepsilon,j} \geqslant 0$ and $\lambda_j \in [0, 1]$.

Some points about algorithm 1 above are worth emphasising. First, in step 3 I use global search methods since in preliminary simulations gradient based methods failed to converge in many instances. Second, the simulated data is generated according to the same data template observed in the actual data in accordance with the general principle in simulated based estimation of treating real and simulated data as similarly as possible. Third, the vectors of primitive shocks in Ξ^0 are drawn only *once* at the beginning of the algorithm and kept fixed when searching for a minimum. Otherwise, the algorithm would not numerically converge and the asymptotic statistical properties would no longer be valid.[52] Fourth, for each product I run

[49]Codes used for parameter estimation and computation of smoothed estimates are available from the author upon request.

[50]Note that the drawings of the shocks are done such that in simulated data the number of quote lines and their respective starting and ending dates exactly match those that are observed in actual data.

[51]I use the algorithm *patternsearch* in MATLAB R2015b with the default options

[52]See, for instance, p. 29 in Gouriéroux and Monfort (1996).

steps 2 to 4 twice starting two different initial conditions in step 2.[53] In case the results differ, I choose the value of parameters that results in the smallest objective function.

B.1.2 Algorithm for estimation of initial price gaps

For the estimation of initial conditions of quote-lines of a given product, the minimisation problem in (10) has to be solved once for each quote-line that has at least one non-zero price change. Given the large number of such quote-lines performing separate global search methods for each quote line individually as for the estimation of common parameters (algorithm 1) would be infeasible. Instead, I perform the minimisation on a grid of possible values of $x_{i,j,0}$. More precisely, to estimate the initial conditions for all the quote-lines of a given product I implement the following algorithm:

Algorithm 2 The algorithm for estimating initial conditions is as follows:

1. Determine the number of panels to be simulated as $S = \lceil 10^4/N_j \rceil$.[54]

2. Generate S vectors of primitive shocks conform with the data template, $\Xi = [\xi^1, \ldots, \xi^S]$.

3. Create a grid $\mathcal{G} = [x^{(1)}_{i,j,0}, \ldots, x^{(50)}_{i,j,0}]$, where $x^{(k)}_{i,j,0} = \hat{\underline{x}}_j + \frac{k}{(50+1)}(\hat{\bar{x}}_j - \hat{\underline{x}}_j)$.

4. Set $\theta = \hat{\theta}$ and $x_{i,j,0} = x^{(k)}_{i,j,0}, \forall i$ and use (5) to (7) and Ξ to generate S panels of data.

5. For a given collection of panels compute $f(x^{(k)}_{i,j,0}) = (S\, N_j)^{-1} \sum_{s=1}^{S} \sum_{i=1}^{N_j} h(\Delta p^s_{i,j}(x^{(k)}_{i,j,0}, \hat{\theta}_j))$

6. Repeat steps 4 and 5 for each $x^{(k)}_{i,j,0} \in \mathcal{G}$ and store $\mathcal{F} = \{f(x^{(k)}_{i,j,0})\}_{k=1}^{50}$

7. Create a new grid $\tilde{\mathcal{G}} = [\tilde{x}^{(1)}_{i,j,0}, \ldots, \tilde{x}^{(50,000)}_{i,j,0}]$ where $\tilde{x}^{(i)}_{i,j,0} = x^{(1)}_{i,j,0} + \frac{(i-1)}{(49,999)}(x^{(50)}_{i,j,0} - x^{(1)}_{i,j,0})$

8. For each $\tilde{x}^{(k)}_{i,j,0} \in \tilde{\mathcal{G}}$ use a cubic spline on the values in $\mathcal{F}$ to approximate $f(\tilde{x}^{(i)}_{i,j,0})$.[55]

9. For each $\tilde{x}^{(k)}_{i,j,0} \in \tilde{\mathcal{G}}$ compute $\tilde{H}(\Delta p_{i,j}, \tilde{x}^{(k)}_{i,j,0}, \hat{\theta}_j) = \left\| \Big(h(\Delta p_{i,j}) - f(\tilde{x}^{(k)}_{i,j,0}) \Big) \oslash h(\Delta p_{i,j}) \right\|^2$

10. For a given quote-line with at least one price change take $\hat{x}_{i,j,0} = \underset{a \in \tilde{\mathcal{G}}}{\arg\min}\, \tilde{H}(\Delta p_{i,j}, a, \hat{\theta}_j)$

11. Repeat steps 9 and 10 for each quote-line with at least one non-zero price change.

12. For quote-lines without price changes, set $\hat{x}_{i,j,0}$ equal to the average of values in step 11.

Some points about algorithm 2 are worth emphasising. First, in step 9 the deviations of data moments from their simulated counterparts are expressed as percentage deviations of the data

[53] The first set of initial conditions is designed to be an educated guess for a model that is closer to a pure menu cost model. In that case, I set the initial values for $-\underline{x}_j$ and $-\bar{x}_j$ to be equal to the average values of positive and negative log price changes, respectively, whereas initial value for λ_j is equal to 25% of the frequency of price changes in the data. The second initial condition is designed to be an educated guess for a model that is closer to a pure Calvo model. In that case, I set $-\underline{x}_j$ and $-\bar{x}_j$ to be equal to the 95th and 5th percentiles of the distribution of log price changes, respectively, and λ_j equal to 75% of the frequency of price changes observed in the data.

[54] This ensures the simulator is based on at least 10 thousand individual price trajectories.

[55] I use the function *interp1* in MATLAB R2015b with the option 'spline'.

moments and that is necessary with equally weighted moments to ensure that one moment condition does not receive a disproportional weight simply due to differences in scale. Second, once the grid for the approximation of the moment is constructed (steps 1 to 8) it can be used as the moment simulator for other quote-lines, in practice, this implies that only steps 9 and 10 need to be repeated at the quote-line each speeds up the calculations. Third, the initial grid $\mathcal{F}$ could be equivalently generated by simply generating 10,000 separate individual price trajectories all starting from a given initial condition and computing average over those. The construction in steps 1 to 6 simply takes advantage of some functions used to generate simulated data for the estimation of common parameters.

B.2 Details on the computation of smoothed estimates

From a purely computational perspective, once a vector of parameters is estimated the key challenge to compute smoothed estimates is to be able to numerically evaluate integrals sufficiently fast so that such estimates can be computed for the millions of observations in micro price data. In order to do that, I numerically approximate integrals using *Gauss-Legendre quadrature* methods.[56] As in the main text, and without loss of generality, henceforth I consider an arbitrary quote-line with $K \geqslant 0$ non-zero price changes and fix k equal to some value $\mathbb{Z}_{[0,K]}$. For that quote-line, τ^k denotes the period on which the k-th non-zero price change is observed and τ^0 denotes the initial period. In addition, let n denote the number of Gauss-Legendre nodes used in the approximation, let $\mathbf{z}$ be a $n \times 1$ vector of Gauss-Legendre nodes in the interval $[\underline{Z}^k, \bar{Z}^k]$ and $\boldsymbol{\omega}$ be the associated $n \times 1$ vector of Gauss-Legendre weights. Finally, let z_i and ω_i denote the i-th elements of $\mathbf{z}$ and $\boldsymbol{\omega}$, respectively.[57]

Matrix notation Specifically to describe the algorithms in this subsection, I use: T as a superscript to denote matrix transposition; $a_{*,j}$ to denote the j-th column of the matrix $\mathbf{A}$; $a_{i,*}$ to denote the i-th row of $\mathbf{A}$; $\mathbf{I}_n$ denotes the identity matrix of order n; $\mathbb{1}_{m\times n}$ denotes an $m \times n$ matrix of ones; $\otimes$ denotes the Kronecker product; $\odot$ denotes the Hadamard product; $\oslash$ denotes the Hadamard division and the exponent $^{\circ -1}$ the Hadamard inverse. Finally, for a given function $f : \mathbb{R} \to \mathbb{R}$ and a matrix $\mathbb{A} \in \mathbb{R}^{m\times n}$, the notation $B = f \circ (\mathbf{A})$ is equivalent to $b_{i,j} = f(a_{i,j})$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Any other notation is as defined in the main text.

B.2.1 Smoothed estimates based on the the pdf in Proposition 3

Consider first the case where $k < K$. For $i = 1, \ldots, n$ and $j = 1, \ldots, \Delta^k - 1$, define:

[56] See, for example, Judd (1998, section 7.2).

[57] To compute the Gauss-Legendre nodes and associated weights on an arbitrary interval $[a, b]$, I use the *lgwt* function provided by Greg von Winckel on File Exchange and available for download here. Here I describe the computations for a general number of Gauss-Legendre notes n. In practice, in any integral numerical evaluation I use 50 Gauss-Legendre nodes.

$$a_{i,j} \equiv \frac{1}{\breve{\sigma}_j^k}\,\phi\left(\frac{z_i - \breve{\mu}_j^k}{\breve{\sigma}_j^k}\right) \tag{B.2.1.1}$$

$$b_{i,j} \equiv \mathbb{1}\{j > 1\}\left[\sum\nolimits_{l=1}^{n} \omega_l\,\frac{1}{\tilde{\sigma}_{j-1}}\phi\left(\frac{z_l - \tilde{\mu}_{j-1}^k(z_i)}{\tilde{\sigma}_{j-1}}\right) b_{l,j-1}\right] + \mathbb{1}\{j = 1\} \tag{B.2.1.2}$$

$$c_{i,j} \equiv \mathbb{1}\{j < \Delta^k - 1\}\left[\sum\nolimits_{l=1}^{n} \omega_l\,\frac{1}{\ddot{\sigma}_{j+1}}\phi\left(\frac{z_l - \ddot{\mu}_{j+1}^k(z_i)}{\ddot{\sigma}_{j+1}}\right) c_{l,j+1}\right] + \mathbb{1}\{j = \Delta^k - 1\} \tag{B.2.1.3}$$

For a given $j \in \mathbb{Z}_{[1,\Delta^k-1]}$, the smoothed estimates as defined in (21) can be obtained as:

$$\widehat{Z}^\star_{\tau^k+j} = \frac{\sum\limits_{l=1}^{n} \omega_l\, a_{l,j}\, b_{l,j}\, c_{l,j}\, z_l}{\sum\limits_{l=1}^{n} \omega_l\, a_{l,j}\, b_{l,j}\, c_{l,j}} \tag{B.2.1.4}$$

Matrix form For computational efficiency it is preferable to implement the calculation in (B.2.1.4) in matrix form. For that purpose, first define $\breve{\mu}_*^k = [\breve{\mu}_1^k, \dots, \breve{\mu}_{\Delta^k-1}^k]$ and $\breve{\sigma}_*^k = [\breve{\sigma}_1^k, \dots, \breve{\sigma}_{\Delta^k-1}^k]$. Let $\mathbf{A}$ be a $n \times (\Delta^k - 1)$ matrix given by,

$$\mathbf{A} = \left[\mathbb{1}_{n\times 1} \otimes (\breve{\sigma}_*^k)^{\circ -1}\right] \odot \left[\phi \circ \left(\,(\mathbb{1}_{1\times(\Delta^k-1)} \otimes \mathbf{z} - \mathbb{1}_{n\times 1} \otimes \breve{\mu}_*^k) \oslash (\mathbb{1}_{n\times 1} \otimes \breve{\sigma}_*^k)\,\right)\right] \tag{B.2.1.5}$$

Note that the (i, j) element of $\mathbf{A}$ in (B.2.1.5) is equal to (B.2.1.1). Moreover, let $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times(\Delta^k-1)}$ be such that $b_{*,1} = \mathbb{1}_{n\times 1}$ and $c_{*,\Delta^k-1} = \mathbb{1}_{n\times 1}$ and the remaining columns are defined recursively according to:

$$b_{*,j} = \left[\,\mathbf{I}_n \otimes \omega^T\,\right]\left[\frac{1}{\tilde{\sigma}_{j-1}} \times \phi \circ \left(\frac{1}{\tilde{\sigma}_{j-1}}\left(\mathbb{1}_{n\times 1} \otimes \mathbf{z} - \tilde{\mu}_j^k \circ (\mathbf{z} \otimes \mathbb{1}_{n\times 1})\right)\right) \odot (\mathbb{1}_{n\times 1} \otimes b_{*,j-1})\right] \tag{B.2.1.6}$$

$$c_{*,j} = \left[\,\mathbf{I}_n \otimes \omega^T\,\right]\left[\frac{1}{\ddot{\sigma}_{j+1}^k} \times \phi \circ \left(\frac{1}{\ddot{\sigma}_{j+1}^k}\left(\mathbb{1}_{n\times 1} \otimes \mathbf{z} - \ddot{\mu}_j^k \circ (\mathbf{z} \otimes \mathbb{1}_{n\times 1})\right)\right) \odot (\mathbb{1}_{n\times 1} \otimes c_{*,j+1})\right] \tag{B.2.1.7}$$

The (i, j) element of $\mathbf{B}$ and $\mathbf{C}$ are equal to those defined in (B.2.1.2) and (B.2.1.3). Finally, the smoothed estimates can be obtained as:

$$\widehat{\mathbf{Z}}^k = \left[\omega^T \left((\mathbb{1}_{1\times(\Delta^k-1)} \otimes \mathbf{z}) \odot \mathbf{A} \odot \mathbf{B} \odot \mathbf{C} \right)\right] \oslash \left[\omega^T \left(\mathbf{A} \odot \mathbf{B} \odot \mathbf{C}\right)\right] \tag{B.2.1.8}$$

where $\widehat{\mathbf{Z}}^k \in \mathbb{R}^{1\times(\Delta^k-1)}$ and the j-th element of $\widehat{\mathbf{Z}}^k$ is equal to $\widehat{Z}^{\star}_{\tau^k+j}$ as given in (B.2.1.4).

B.2.2 Smoothed estimates based on the the pdf in Proposition 4

For $k = K$ the smoothed pdf is given by (18). Numerically, the computation of the smoothed estimates is similar to that in the previous section. For $i = 1, \ldots, n$ and $j = 1, \ldots, \Delta^K$, define:

$$d_{i,j} \equiv \frac{1}{\sigma_j} \phi \left(\frac{z_i - \mu_j^k}{\sigma_j} \right) \tag{B.2.2.1}$$

$$e_{i,j} \equiv \mathbb{1}\{j < \Delta^K\} \left[\sum\nolimits_{l=1}^{n} \omega_l \frac{1}{\sigma_\varepsilon} \phi \left(\frac{z_l - (\mu + z_i)}{\sigma_\varepsilon} \right) e_{l,j+1} \right] + \mathbb{1}\{j = \Delta^K\} \tag{B.2.2.2}$$

For a given $j \in \mathbb{Z}_{(\tau^K, T]}$, the smoothed estimates as defined in (21) can be obtained as:

$$\widehat{Z}^{\star}_{\tau^k+j} = \frac{\sum_{l=1}^{n} \omega_l \, d_{l,j} \, b_{l,j} \, e_{l,j} \, z_l}{\sum_{l=1}^{n} \omega_l \, d_{l,j} \, b_{l,j} \, e_{l,j}} \tag{B.2.2.3}$$

Matrix form Again, for computational efficiency I implement the computation of (B.2.2.3) in matrix form. First, define $\mu_* = [\mu_1, \ldots, \mu_{\Delta^K}]$ and $\sigma_* = [\sigma_1, \ldots, \sigma_{\Delta^K}]$. Let $\mathbf{D}$ be a $n \times \Delta^K$ matrix given by,

$$\mathbf{D} = \left[\mathbb{1}_{n\times1} \otimes (\sigma_*)^{\circ -1}\right] \odot \left[\phi \circ \left((\mathbb{1}_{1\times\Delta^K} \otimes \mathbf{z} - \mathbb{1}_{n\times1} \otimes \mu_*) \oslash (\mathbb{1}_{n\times1} \otimes \sigma_*) \right)\right] \tag{B.2.2.4}$$

Note that the (i,j) element of $\mathbf{D}$ is equal to (B.2.2.1). Let $\tilde{\mathbf{B}}, \mathbf{E} \in \mathbb{R}^{n\times\Delta^K}$ be such that $\tilde{b}_{*,1} = \mathbb{1}_{n\times1}$ and $e_{*,\Delta^K} = \mathbb{1}_{n\times1}$. The remaining columns of $\tilde{\mathbf{B}}$ are defined recursively according to (B.2.1.6) and the remaining columns of $\mathbf{E}$ according to:

$$e_{*,j} = \left[\mathbf{I}_n \otimes \omega^T\right] \left[\frac{1}{\sigma_\varepsilon} \times \phi \circ \left(\frac{1}{\ddot{\sigma}^k_{j+1}} \left(\mathbb{1}_{n\times1} \otimes \mathbf{z} - (\mu \mathbb{1}_{n\times1} + \mathbf{z}) \otimes \mathbb{1}_{n\times1} \right) \right) \odot \left(\mathbb{1}_{n\times1} \otimes e_{*,j+1} \right) \right] \tag{B.2.2.5}$$

Finally, the smoothed estimated after the last non-zero price change are computed from:

$$\widehat{\mathbf{Z}}^K = \left[\omega^T \left((\mathbb{1}_{1\times(\Delta^k-1)} \otimes \mathbf{z}) \odot \mathbf{D} \odot \tilde{\mathbf{B}} \odot \mathbf{E} \right)\right] \oslash \left[\omega^T \left(\mathbf{D} \odot \tilde{\mathbf{B}} \odot \mathbf{E}\right)\right] \tag{B.2.2.6}$$

where $\widehat{\mathbf{Z}}^K \in \mathbb{R}^{1\times\Delta^K}$ and the j-th element of $\widehat{\mathbf{Z}}^K$ is equal to $\widehat{Z}^\star_{\tau^K+j}$ as given in (B.2.2.3).

C Frictionless inflation in the basic new Keynesian model

The underlying model and all the notation are identical to the basic new Keynesian model in Galí (2008, chapter 3). I focus only on the key equations to derive: (i) the relation between frictionless inflation and the output gap; (ii) the difference in the responses of inflation and frictionless inflation to a monetary policy shock.

C.1 Relationship between the inflation, frictionless inflation and the output gap

Under monopolistic competition and a demand function arising from a CES aggregator with elasticity of substitution ε, frictionless prices satisfy:

$$P_t^\star = \mathcal{M}\,\psi_{t|t} \tag{C.1.1}$$

where $\psi_{t|t}$ are the nominal marginal costs of a firm changing prices at time t and $\mathcal{M} \equiv \varepsilon/(\varepsilon-1)$ is a constant markup that monopolist would charge at every time period in the absence of constraints on the frequency of price adjustment, also referred to as the desired or frictionless markup. Dividing both sides of (C.1.1) by P_t and taking logs yields,

$$p_t^\star - p_t = mc_t - mc \tag{C.1.2}$$

where $mc = -\log(\mathcal{M})$ is the steady state value of marginal cost and mc_t is the log of the economy's average real marginal cost. Since the log of deviation of real marginal cost from steady state is proportional to the log deviation of output from its flexible price counterpart, I use equation (20) in Galí (2008, p. 48) to obtain,

$$p_t^\star - p_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\tilde{y}_t \tag{C.1.3}$$

where $\tilde{y}_t$ is the *output gap*, defined as the log deviation of output from its flexible part counterpart, σ is the elasticity of intertemporal substitution, φ is the Frisch elasticity of labor supply and $1-\alpha \in [0,1]$ is the exponent of labor in the production function. Lagging (C.1.3) by one period and subtracting from (C.1.3),

$$\pi_t^\star - \pi_t = C\,\Delta\tilde{y}_t \tag{C.1.4}$$

where $C \equiv \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) > 0$. Taking the covariance between $\pi_t - \pi_t^\star$ and $\Delta\tilde{y}_t$ gives equation (26), that is,

$$\frac{\mathrm{Cov}(\pi_t - \pi_t^\star, \tilde{y}_t)}{\mathrm{Var}(\tilde{y}_t)} = -C < 0 \tag{C.1.5}$$

Tests for this prediction in the UK data are presented in table 3.

C.2 Impulse responses to a monetary policy shock

I consider the responses of inflation and frictionless inflation under an interest rate rule as in Galí (2008, section 3.4.1). The stochastic component in the interest rate is v_t and it is assumed to follow an AR(1) process, that is, $v_t = \rho_v v_{t-1} + \varepsilon_t^m$ where $\rho_v \in [0, 1)$ and ε_t^m is the monetary policy shock. From (C.1.4) and any given horizon $h \geqslant 0$ we have that,

$$\frac{\partial \pi^\star_{t+h}}{\partial \varepsilon_t^v} - \frac{\partial \pi_{t+h}}{\partial \varepsilon_t^m} = C\,\frac{\partial \Delta \tilde{y}_{t+h}}{\partial \varepsilon_t^m} \tag{C.2.6}$$

So the relationship between the impulse responses of frictionless inflation and inflation can be inferred from the sign of the impulse response of the changes in the output gap. Using the method of undetermined coefficients, the solution for the output gap is given by,

$$\tilde{y}_{t+h} = -(1-\beta\rho_v)\,\Lambda_v\, v_{t+h} \tag{C.2.7}$$

where β is the representative household discount factor and Λ_v is a convolution of structural parameters that and takes only positive values.[58] Finally, in terms of impulse responses it is the case that,

$$\frac{\partial \Delta \tilde{y}_{t+h}}{\partial \varepsilon_t^m} = -(1-\beta\rho_v)\,\Lambda_v \left(\frac{\partial v_{t+h}}{\partial \varepsilon_t^m} - \frac{\partial v_{t+h-1}}{\partial \varepsilon_t^m} \right) \tag{C.2.8}$$

Note that for the case $h = 0$, the last term in brackets in (C.2.9) is equal to one since $\partial v_{t-1}/\partial \varepsilon_t^m = 0$. Since the term $(1-\beta\rho_v)\Lambda_v$ is positive, it follows that the expression above is *negative* and, hence, $\partial \pi_t^\star/\partial \varepsilon_t^v < \partial \pi_t/\partial \varepsilon_t^m$. Moreover, since the solution for inflation is given by $\pi_t = -\kappa\,\Lambda_v\, v_t$ where $\kappa > 0$ is the slope of the new Keynesian Phillips curve and, hence, $\partial \pi_t/\partial \varepsilon_t^m < 0$. Therefore, (28) holds. Finally, for any $h > 0$ expression (C.2.9) simplifies to,

$$\frac{\partial \Delta \tilde{y}_{t+h}}{\partial \varepsilon_t^m} = -\underbrace{(1-\beta\rho_v)\,\Lambda_v \rho_v^{h-1}}_{>0}\;\underbrace{(\rho_v - 1)}_{<0} > 0 \tag{C.2.9}$$

Therefore, $\partial \pi^\star_{t+h}/\partial \varepsilon_t^v > \partial \pi_{t+h}/\partial \varepsilon_t^m$ as stated in (29). As an endnote, notice that the sign of $\pi_{t+h}/\partial \varepsilon_t^m$ cannot be determined unambiguously. More precisely using the solutions for π_t and $\Delta \tilde{y}_t$ and the relationship in (C.1.4),

$$\frac{\partial \pi^\star_{t+h}}{\partial \varepsilon_t^v} = \kappa \Lambda_v \rho_v^{h-1} \left[\frac{(1-\beta\rho_v)(1-\rho_v)}{\lambda} - \rho_v \right] \tag{C.2.10}$$

[58]For values of structural parameters that ensure equilibrium uniqueness which is maintained assumption, see Galí (2008, equation 27).

where λ is a positive constant that is a convolution of structural parameters. The sign of the last term in square brackets will depend on the specific calibration of structural parameters.