Brazilian Business Cycles Chronology: Asymmetry, Heterogeneous Duration, and Time-varying Transition Probabilities

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Abstract

This paper develops a generalization of the Markov-Switching model with evolving regime-specific means and stochastic volatility proposed Eo and Kim (2016). The model allows for R regimes, B breaks, and time-varying transition probabilities, where the latter are modeled as a multinomial logistic function. We apply our flexible methodology to model the Brazilian GDP growth, which features very complicated dynamics over the last four decades. Our results point to the presence of 3 regimes, at least one break in the long-run trend, and substantial time-variation in the volatility process. Moreover, the selected model features time-varying probability driven by domestic (real interest rate and real exchange rate) and international (commodity prices and global uncertainty) factors. Finally, the results indicate a significant reduction in the Brazilian long-run growth trend.

Keywords:

JEL - Classification: C11, C32, E44, E32, E44.

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1 Introduction

From a macroeconomic perspective, the history of the Brazilian economy during the last four decades could be divided into two sub-periods, according to Ayres et al. (2021): the first, from 1981 to 1994, featuring slow growth, hyperinflation, and high debt, and the second, from 1995 to 2016, with moderate growth, relatively low inflation, and low debt. Extending their analysis to the present day, it seems plausible to include yet another sub-period, from 2016 to the present, featuring low growth, moderate inflation, and relatively high debt.

These three different sub-periods reflect the fact that, from 1980 to the present day, the Brazilian economy has undergone remarkable changes and experienced many events that might have affected its long-run growth trend. During 1981-94, six major unsuccessful stabilization plans were implemented until the introduction of the successful stabilization program, the Real Plan in 1994. Thus, such a sub-period is marked by frequent policy swings that, as shown by Aguiar and Gopinath (2007), may affect the economy's productivity, ultimately leading to sudden changes in the growth trend path. Moreover, most of these stabilization plans featured price freezing policies combined with fiscal debts and loose monetary policies (see, e.g., Ayres et al., 2021), which may be seen as "bad policies" in the spirit of Calvo (2005). Furthermore, the period is also characterized by external debt renegotiation events (1984, 1985, and 1993) and a default episode in 1982.

The sub-period from 1995 to 2016 is marked by a variety of institutional reforms, such as a series of bank sector reforms between 1995 and 1996,[1] the introduction of inflation targeting in 1999, the introduction of The Fiscal Responsibility Law in 2000, and, more recently, a ceiling for government spending, introduced as a fiscal rule in 2016. Again, some of such reforms could, in principle, affect the economy's productivity.

Finally, the last four decades in Brazil are a period also characterized by several episodes of political tensions, including a re-democratization process in 1985 and two events of presidential impeachments, in 1993 and 2016.

Given this background about the recent Brazilian economic history, it is not surprising that the Brazilian GDP presents a somewhat erratic behavior from 1980 to the present day. Figure 1 shows the level and the log-difference of the index of quarterly GDP of the Brazilian economy from 1980Q1 to 2019Q4. Clearly, the trend associated with the level of the GDP presents significant shifts, both up and down, through time. At the same time, the growth rate of GDP, captured by the log-difference of the GDP, suggests different patterns of GDP volatility over time but also a substantial time-varying behavior in the average growth rate. Another important aspect of the Brazilian business cycles emerges from

[1]Three programs were the Incentive Program for Restructuring and Strengthening of the Financial System (PROER), in 1995, the Incentive Program for the Reduction of the State Public Sector in Banking Activity (PROES), and the Credit Guarantee Fund (FGC), in 1996.

a visual inspection of the GDP growth: the duration of the cycles seems to feature a time-varying pattern. This last observation suggests that booms and busts in Brazil are not alike.

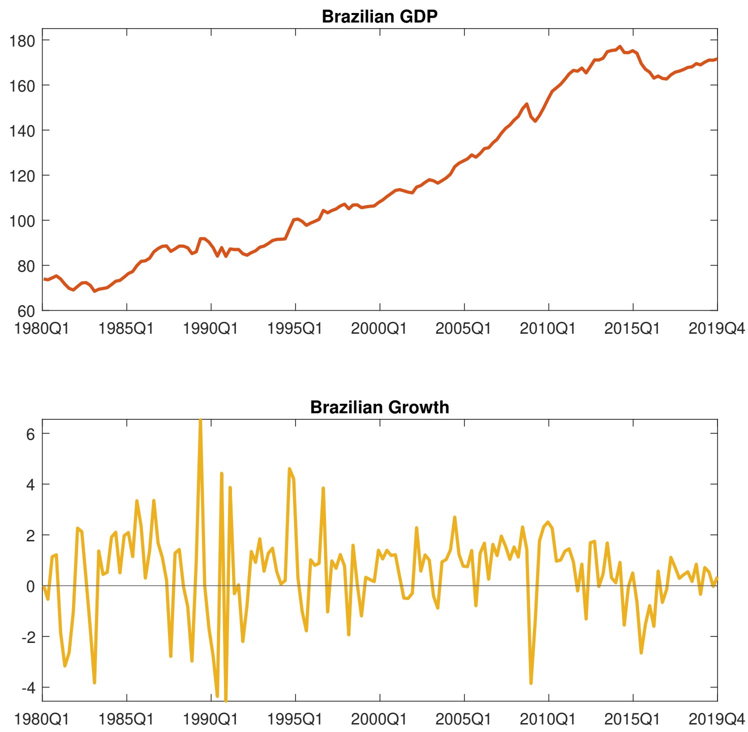

Figure 1: Brazilian GDP: level (red line) and growth rate (yellow line).

This paper aims to model the Brazilian business cycles from 1980Q1 to 2019Q4, taking into account the complicated behavior of Brazil's growth. To do so, we first developed a generalization of the Markov-Switching model with evolving regime-specific means and stochastic volatility proposed Eo and Kim (2016). Our model, referred to as Markov-Switching Model with Evolving Regime-Specific Parameters (MS-ERSP), allows for $1 < R < \infty$ regimes, $0 \leq B < \infty$ breaks in the long-run growth trend, and time-varying probabilities in the spirit of Filardo and Gordon (1998). Eo and Kim (2016) proposed a model in which the properties of booms and busts may vary over time. Our model can have an arbitrary number of intermediary regimes, both of them with their own properties over time. Moreover, time-varying probabilities, modeled as a multinomial logistic function as in Kaufmann (2015), allow the transition between regimes to vary through time and to depend on selected factors, e.g., domestic and international conditions. Thus, the way that the economy goes from a peak to a trough, for example, can have a time-varying pattern. A final important feature is that the model decomposes the GDP growth rate into an unobserved smooth, long-run trend and the unobserved regime-dependent mean growth rates, so we can analyze the economy's long-run growth rate evolution.

Our results suggest that the Brazilian business cycles are better characterized by 3 regimes: recession, fast recovery, and boom. Moreover, we find substantial variation in the time-varying volatility process, with an initial increase in volatility until 1990Q1, followed by a consistent reduction until 2019Q4. Furthermore, the Brazilian economy seems to feature one break in its long-run growth trend. The average quarterly growth rate reduced from 0.5% during the first half of the 1980s to about 0.22% nowadays. Such a result is consistent with the notion that the Brazilian economy has been experiencing a reduction in its average productivity over the last five decades.

Our paper is related to the literature on modeling business cycles through Markov-Switching models as in Hamilton (1989), Albert and Chib (1993), Filardo and Gordon (1998), Kaufmann (2015), and Eo and Kim (2016). It is also related to the literature modeling business cycles in emerging markets, as in Phillips and Shi (2021). Finally, it is related to the literature on dating the Brazilian business cycles, as Céspedes et al. (2006).

The paper is organized as follows: Section 2 discusses the chronologies of the Brazilian economy. Section 3 presents our flexible model. Section 4 presents the priors and posteriors distributions and the Bayesian algorithms used to estimate the posteriors distributions. Section 5 discusses parsimonious models for the Brazilian economy. Section 6 presents the results, and Section 6 concludes.

2 Brazilian business cycles: previous chronologies

In 2004 the Business Cycle Dating Committee (CODACE) was created by Fundação Getulio Vargas (FGV) to establish reference chronologies for Brazilian business cycles from 1980 onward. The approach of such a committee is like the one from NBER. Thus, the chronology is established by alternating dates of peaks and troughs in the level of economic activity. The recession is the phase marked by the decline in economic activity disseminated among different economic sectors. The expansions are given by the phase between trough and peak in the cycles. Table 1 presents the quarterly and monthly chronologies of the business cycle from the last release from CODACE. As can be seen, CODACE has identified ten recessions, but the end of the last one has not yet been defined. Despite that, recessions would be frequent in the Brazilian economy, and their duration would be considerably heterogeneous.

From 1980Q1 to 2019Q4, Figure 2 presents the recession periods from CODACE and either the log of GDP or the GDP growth rate.[2] The visual inspection confirms that recessions are frequent with heterogeneous duration. Furthermore, the GDP growth rate does

[2] Data is taken from IBGE and Ipeada and is seasonally adjusted.

Period	Duration (quarters)	Period	Duration (quarters)
1981Q1 - 1983Q1	9	2001Q2 - 2001Q4	3
1987Q3 - 1988Q4	6	2003Q1 - 2003Q2	2
1989Q3 - 1992Q1	11	2008Q4 - 2009Q1	2
1995Q2 - 1995Q3	2	2014Q2 - 2016Q4	11
1998Q1 - 1999Q1	5		

Table 1: CODACE chronology for Brazilian recessions

not seem to be constant across the recessions, as suggested by the time-varying, episode-specific mean growth. Such time-varying means also suggest a decline in the unconditional average growth rate of the economy.

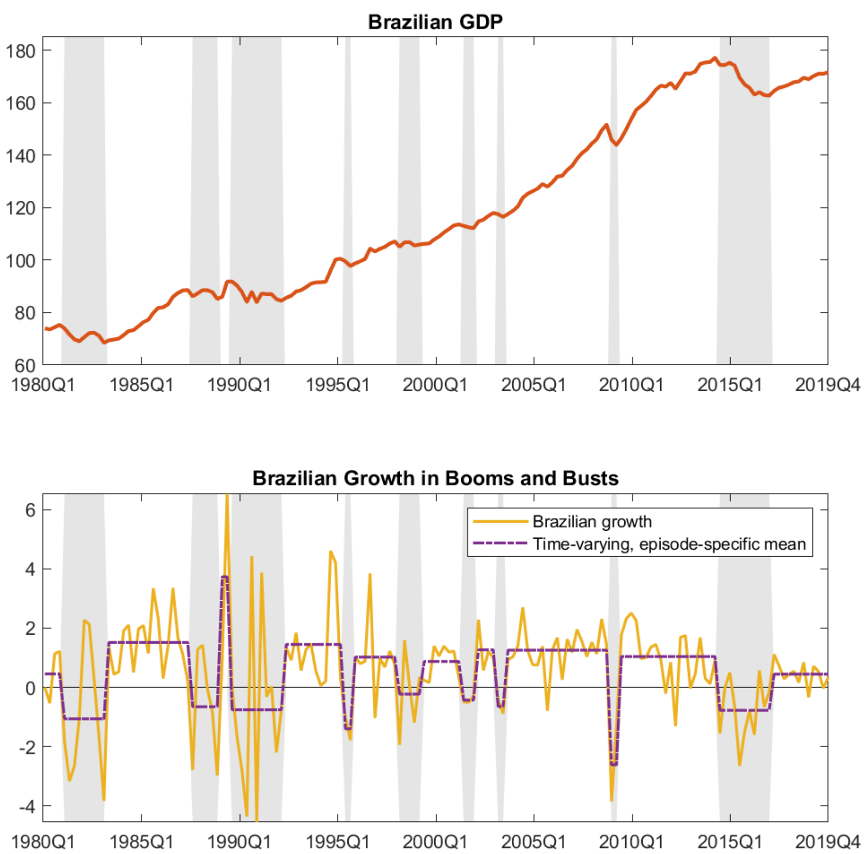

Figure 2: Brazilian Business Cycles under CODACE recession chronology

The 1980s were marked by GDP stagnation and acceleration of inflation. Not by chance, this decade became known as 'The Lost Decade'. According to CODACE, there were three recessions in such decade. Two of them were very long, and the last one extended into the early 1990s when president Collor put forward the failed heterodox stabilization plans Collor I (1990) and Collor II (1991). Previously, other heterodox plans took place, but no one reached success, such as Cruzado (1986) and Bresser (1987). A common feature of these heterodox plans was the price freezing; however, at a certain degree, Collor I went further by freezing a fraction of the savings accounts. This strategy caused large GDP instability.

The Real Plan was implemented in 1994, and it can be considered a turning point in the Brazilian economy, once it was the first stabilization package that really managed to bring down inflation in the country. However, one of the elements of this plan was the adoption of an appreciated fixed exchange rate, which culminated in serious consequences to the trade balance account, the level of international reserves, and the unemployment rates. Furthermore, in the 1990s, the Brazilian economy was hit by the contagion effect of emerging countries that went through serious crises. It is worth mentioning the Mexican (Tequila) Crisis of 1994, the Asian financial crisis of 1997, and the Russian financial crisis of 1998. The internal problems and the international crises forced the country to adopt a flexible exchange rate in January 1999, after a substantial loss of international reserves. According to CODACE, Brazil went through two recessions in this decade, one shortly after the implementation of the Real Plan and the occurrence of the Mexican crisis (1995Q2 - 1995Q3), another after the Asian crisis and during the Russian crisis (1998Q1 - 1999Q1).

After the abandonment of the fixed exchange rate regime, the Brazilian Central Bank adopted the inflation target system by mid-1999 to anchor the inflation expectations. Since then, to keep inflation under control, the central bank has resorted to high interest rates, which has impacted both the economic performance and the government debt. Nevertheless, the beginning of the 2000s was marked by external and internal problems. In 2000 the dot-com bubble came to an ending. In 2001, Brazil went through an energy crisis as a consequence of both the demand for electricity grew faster than the supply and there was a long period of drought that severely reduced the levels of water reservoirs. The government implemented a rationing scheme to reduce national electricity consumption to avoid blackouts. Indeed, according to the CODACE chronology, Brazil went through a recession during the year 2001. The subsequent recession in the first half of 2003 coincides with the beginning of President Lula's first term. In fact, his election in the previous year was marked by great apprehension, evidenced by the devaluation of the Brazilian currency against the US dollar. Finally, the last recession of the decade, according to CODACE, occurred in the period 2008Q4-2009Q1, being clearly related to the international financial crisis 2007–2008 that began years earlier with cheap credit and lax lending standards that fueled a housing bubble.

In the 2010s, there would have been only one recession, however, an extremely long one, from the second quarter of 2014 to the fourth quarter of 2016. After the re-election of President Dilma Roussef in 2014, monitored prices increased substantially in 2015, putting pressure on costs for the entire economy.[3] Furthermore, the budget operations carried out by the National Treasury to mask the Brazilian government's terrible fiscal situation became public. After all, president Dilma Roussef was impeached in mid-2016. Finally, as mentioned, CODACE has identified the beginning of the recession due to the covid-19 pandemic but has not yet commented on its end.

The previous analyses suggest that a flexible model may be necessary to describe Brazilian GDP growth and to date its recessions. The heterogeneity in the duration of the recession suggests that assuming constant transition probabilities over time would not be appropriate. Inasmuch as the recessions came from internal and external factors, the (time-varying) transition probabilities should depend on these factors. Indeed, the visual inspection of Figure 2 also suggests that the recessions' intensity and volatility are heterogeneous. A heteroscedastic model with more than two regimes may be needed to accommodate these features.

To accommodate all these possibilities, Section 3 put forward a flexible Markov- Switching model that generalizes the two-regimes Markov-Switching model with evolving regime-specific means and stochastic volatility proposed by Eo and Kim (2016). In order to model the Brazilian business cycles, our model allows $0 < R < \infty$ regimes and adds sudden changes in the long-run growth rate ($0 \leq B < \infty$) and time-varying transitions probabilities.

Céspedes et al. (2006) applied Markov switching models to quarterly Brazilian GDP growth rate from 1975Q1 to 2003Q4, allowing for breaks at the Collor Plans. The authors argued that the MS-AR(2) specification, with such breaks, presents the best fit of the data. This model leads to recession periods similar to those from CODACE, but it does not identify a crisis in the period 1998-1999. However, given the similarity of the recession chronologies from Céspedes et al. (2006) and CODACE, from now on, we use the latter as a reference to compare with our results.

3 A Flexible Markov-Switching Model with Evolving Regime-Specific Parameters

Let y_t be the GDP growth rate and consider the following parametrization:

$$y_t = \delta_t + x_t' \beta_{s,t} + e_t, \tag{1}$$

[3] In Brazil, the term monitored (or regulated) prices refers to prices that are established by contract or set by a public entity, being less sensitive to supply and demand conditions.

in which term $\beta_{s,t}$ captures the possibility that the coefficients associated with covariates x_t can vary either between states s, $s \in \{1, \cdots, R\}$, and across time, for $t = 1, \cdots, T$, and where $e_t \sim N(0, \sigma^2_{e,t})$ is allowed to feature time-varying volatility. In this case, $\sigma^2_{e,t}$ is governed by the following stochastic process:

$$ln(\sigma^2_{e,t}) = ln(\sigma^2_{e,t-1}) + \zeta_t \tag{2}$$

$\zeta_t \sim N(0, \sigma^2_\zeta)$.

Similar to Albert and Chib (1993) and Eo and Kim (2016), we abstract from the autoregressive terms, as the model with no such parameters seems to fit the data well, so we only consider deviations of y_t from its mean.[4] Therefore, we substitute the term $x'_t\beta_{s,t}$ in equation (1) by the following expression:

$$y_t = \delta_t + S_{0,t}\bar{\mu}_{0,\tau_0} + \cdots + S_{R-1,t}\bar{\mu}_{R-1,\tau_i} + e_t \tag{3}$$

$$= \delta_t + \sum_{i=0}^{R-1} S_{i,t}\bar{\mu}_{i,t} + e_t \tag{4}$$

in which $\tau_i = 1, \cdots, T_i$, for $T_i < T$, possibly having $\tau_j \neq \tau_i, \forall j, i \in \{0, \cdots, R-1\}$, with $j \neq i$, and where $\sum_{i=0}^{R-1} T_i = T$. Thus, $\bar{\mu}_{i,\tau_i}$ is the mean growth rate of the GDP during regime i at time t, where state i have T_i random realizations. The latent variables $S_{i,t},\ i \in \{0, \cdots, R-1\}$, equal 1 if the state in place at t is i, and 0 otherwise. At this point, it may be useful to use such latent variables to construct the following auxiliary variable:

$$S_t = \begin{cases} i, & \text{if } S_{i,t} = 1 \\ 0 & \text{otherwise.} \end{cases},$$

so that we can let $\pi_i = Pr[S_t = i],\ i = 0, 1, ...R-1$, be the unconditional probability of regime i.

The model in (4) is a more general version of the two-regimes, Markov-Switching Model with Evolving Regime-Specific Parameters (MS-ERSP) proposed by Eo and Kim (2016). In the present paper, the model can have $1 < R < \infty$ regimes. Moreover, we allow the long-run (stochastic) trend, δ_t, to have an arbitrarily large number of structural breaks B, $0 \leq B < \infty$, and we refer to such general model as MS-ERSP(R, B).

We also allow the dependence of the mean growth rates across regimes by assuming hierarchical priors in the spirit of Eo and Kim (2016); Koop and Potter (2007), with the

[4] However, it would be straightforward to introduce an autoregressive structure into our model.

following law of motion:

$$\bar{\mu}_{i,\tau_i} = \bar{\mu}_{i,\tau_i-1} + \omega_{i,\tau_i}, \quad \omega_{i,\tau_i} \sim \text{i.i.d. } N(0, \sigma^2_{\omega_{i,\tau_i}}) \tag{5}$$

where $\tau_i = 1, \cdots, T_i$, with $\omega_{i,\tau_i} \perp \omega_{i,\tau_j}$ for $i \neq j$ and $E(e_{\tau_i}\omega_{i,\tau_i}) = 0, \forall i$. As discussed by Eo and Kim (2016) in a two-regimes context, if $\sigma^2_{\omega_{i,\tau_i}} = 0, \forall i$, the model collapses to that of Hamilton (1989) without autoregressive terms as in Albert and Chib (1993).

However, as shown by Eo and Kim (2016), the long-run trend in the model in terms of mean growth rates as in (4) may not exists. Thus, following the authors, for the model to have a well-defined long-run (stochastic) trend, we redefine the model as

$$y_t = \delta_t + \sum_{i=0}^{R-1} S_{i,t}\mu_{i,t} + e_t, \tag{6}$$

in which $\mu_{i,t}$ denotes the deviations from the mean growth rates during regime i from the long-run mean growth rate δ_t.

By isolating the summation in (6) and applying the unconditional expectation on both sides of the resulting equation, we can get:

$$E[y_t - \delta_t] = E\left[\sum_{i=0}^{R-1} S_{i,t}\mu_{i,t}\right] = 0,$$

which, by the law of iterated expectations and the independence of S_t from $\mu_{i,\tau_i}, \forall i, t, \tau$, results in the following long-run restriction that has to be satisfied:

$$E\left[\sum_{i=0}^{R-1} \pi_i \mu_{i,\tau_i}\right] = 0, \tag{7}$$

where $\pi_i = \text{Pr}(S_{t+1} = i)$ is the unconditional probability of regime i.

Given such a restriction, combined the random walk assumption for μ_i, analogous to Eo and Kim (2016) there is a cointegrating vector, $[\pi_0 \cdots \pi_{R-1}]'$, connecting $[\mu_{0,\tau} \cdots \mu_{R-1,\tau}]$ so that changes in $\mu_{i,\tau}$ have no long-run effects on y_t. This set of restrictions can be summarized by the following system of equations:

$$
\begin{aligned}
\mu_{0,\tau} &= \mu_{0,\tau-1} + \theta_0 \left(\sum_{i=0}^{R-1} \pi_i \mu_{i,\tau-1} \right) + \omega_{0,\tau} \\
\vdots &= \vdots \\
\mu_{R-1,\tau} &= \mu_{R-1,\tau-1} + \theta_{R-1} \left(\sum_{i=0}^{R-1} \pi_i \mu_{i,\tau-1} \right) + \omega_{R-1,\tau}
\end{aligned} \tag{8}
$$

where $\omega_{i,\tau} \sim N(0, \sigma^2_{\omega_i}), \forall i \in \{0, \cdots, R-1\}$

Note that the above system has an error-correction term given by $\theta_i \left(\sum_{i=0}^{R-1} \pi_i \mu_{i,\tau-1} \right),\ i = 0, \cdots, R-1$, which, in order to have stability, must satisfy the following condition:

$$
-1 < 1 + \sum_{i=0}^{R-1} \pi_i \theta_i < 1 \tag{9}
$$

Thus, we can cast the model into the state-space representation:

$$
\mu^*_{0,t} = \mu^*_{0,t-1} + \theta_0 d_t \left(\sum_{i=0}^{R-1} \pi_i \mu^*_{i,t-1} \right) + \omega_{0,t} \tag{10}
$$

$$
\vdots \quad = \quad \vdots \tag{11}
$$

$$
\mu^*_{R-1,t} = \mu^*_{R-1,t-1} + \theta_{R-1} d_t \left(\sum_{i=0}^{R-1} \pi_i \mu^*_{i,t-1} \right) + \omega_{R-1,t} \tag{12}
$$

where $d_t = \sum_{j=0}^{R-1} \sum_{i=0}^{R-1} d_{ij,t}$, with

$$
d_{ij,t} = \begin{cases} 1, & \text{if } S_{t-1} = i, i \neq j \\ 0, & \text{otherwise} \end{cases}
$$

Aguiar and Gopinath (2007) shows that a typical emerging economy subject to frequent policy regime switches may experience substantial variation in its long-run trend. Thus, we consider two alternatives to model the long-run trend to account for potential time-varying volatility in the Brazilian growth trend. First, suppose the long-run growth trend grows steadily, changing smoothly over time. In this case, we assume that δ_t evolves as a random walk with time-varying volatility given by:

$$
\delta_t = \delta_{t-1} + \varepsilon_t \tag{13}
$$

in which $\varepsilon_t \sim N(0, \sigma^2_\varepsilon)$.

Now suppose that, after some structural reforms or a sequence of permanent shocks to the economy's productivity, the long-run growth suddenly changes. Given its smooth hallmark, specification in (13) may not capture well such sudden jumps. Thus, similar to Eo and Kim (2016), we assume that δ_t may be decomposed by the following:

$$\delta_t = \delta_t^b$$

in which

$$\delta_t^b = \begin{cases} \delta_1 & \text{if } 1 \leq t \leq \tau_{\delta_1} \\ \delta_2 & \text{if } \tau_{\delta_1} < t \leq \tau_{\delta_2} \\ \vdots & \\ \delta_B & \text{if } \tau_{\delta_{B-1}} < t \leq \tau_{\delta_Q} \\ \delta_{B+1} & \text{if } \tau_{\delta_B} < t \leq T \end{cases}$$

We model breaks following the change-point strategy proposed by Chib (1998), in which the breaks are interpreted as a regime-switching process associated with a latent regime variable, S_t^b. Such a variable is governed by a first-order Markov process with the following transition probabilities:

$$Pr[S_t^b = i, S_{t-1}^b = i] = p_{i,i}^b \quad \text{and} \quad Pr[S_t^b = i+1, S_{t-1}^b = i] = 1 - p_{i,i}^b \tag{14}$$

for $i = 1, \cdots, B$, and

$$Pr[S_t^b = B+1, S_{t-1}^b = B+1] = 1 \tag{15}$$

with $B + 1$ being an absorbing state. Such restrictions imply the following transition matrix:

$$P^b = \begin{bmatrix} p_{1,1}^b & p_{1,2}^b & 0 & \cdots & \cdots & 0 \\ 0 & p_{2,2}^b & p_{2,3}^b & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & p_{B,B}^b & p_{B,B+1}^b \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

3.1 Modeling state transition

Fixed Transition Probabilities (FTP) Following the typical parameterization for the time-invariant switching case (Albert and Chib, 1993; Eo and Kim, 2016; Hamilton, 1989), we assume that the probabilities are given by:

$$p_{ij,t} = p_{ij} = Pr[S_t = i | S_{t-1} = j], \quad \forall t \tag{16}$$

where it is ensured that $0 < p_{ij} < 1$ so that there will be no absorbing state. Thus, p_{ij} depends only on S_{t-1} and S_t.

Time-Varying Transition Probabilities (TVTP) For models with TVTP in the sense of Filardo and Gordon (1998), we follow Kaufmann (2015) by assuming a (multinomial) logistic functional form given by:

$$p_{ij,t} = \frac{\exp\left(\tilde{Z}_t \gamma_{ji}\right)}{\sum_{j=0}^{R-1} \exp\left(\tilde{Z}_t \gamma_{ji}\right)} \tag{17}$$

where the transition distribution is assumed to depend on a set of predetermined variables Z_t, including a constant term, past realizations of $S_{i,t}$ –potentially as interactive terms –, and some dependent variables that may affect the probability of a regime change. As in Kaufmann (2015), such variables are expressed in deviations from their means. Moreover, for identification purposes, we impose a set of restrictions. First, we assume that $\mu_{0,\tau} < \mu_{1,\tau} < \cdots < \mu_{R-1,\tau}$, where $\mu_{0,\tau} < 0$ and $\mu_{R-1,\tau} > 0$, which is an identification requirement of the Kaufmann (2015)'s procedure, but which is consistent with our modeling strategy. Moreover, if a model is assumed to have $G > 2$ regimes, we impose that all G regimes have to be a minimum number of observations. Eventually, to better identify the states, we may also impose that some observations in the sample belong to a particular state in the assumed state space. In contrast, the rest of the observations are left to be labeled by the Bayesian algorithm. Finally, we normalize regime 0 to be the reference regime, which is a requirement for identification from Kaufmann's procedure. Such normalization implies the following transition probability:

$$P(S_t = 0 | S_{t-1} = j, Z_t) = \frac{1}{1 + \sum_{l \in K_{-0}} \exp(Z_t \gamma_{jl})}$$

4 Priors and Posterior Simulation

Before eliciting the priors, it is useful to express the model in a more convenient way. Let δ_t be the (time-varying) long-run growth trend. Similar to Koop and Potter (2007) and Eo and Kim (2016), we can rewrite the model into the following state-space form:

$$y_t = \delta_t + \begin{bmatrix} S_{0,t} & \cdots & S_{R-1,t} \end{bmatrix} \begin{bmatrix} \mu^*_{0,t} \\ \vdots \\ \mu^*_{R-1,t} \end{bmatrix} + e_t \tag{18}$$

with the associated transition equation given by:

$$\begin{bmatrix} \mu^*_{0,t} \\ \mu^*_{1,t} \\ \vdots \\ \mu^*_{R-1,t} \end{bmatrix} = \begin{bmatrix} 1+\theta_0\pi_0 d_t & \theta_0\pi_1 d_t & \cdots & \theta_0\pi_{R-1}d_t \\ \theta_1\pi_0 d_t & 1+\theta_1\pi_1 d_t & \cdots & \theta_0\pi_{R-1}d_t \\ \vdots & \cdots & \ddots & \vdots \\ \theta_{R-1}\pi_0 d_t & \theta_{R-1}\pi_1 d_t & \cdots & 1+\theta_{R-1}\pi_{R-1}d_t \end{bmatrix} + \begin{bmatrix} \mu^*_{0,t-1} \\ \mu^*_{1,t-1} \\ \vdots \\ \mu^*_{R-1,t-1} \end{bmatrix} + \begin{bmatrix} \omega^*_{0,t} \\ \omega^*_{1,t} \\ \vdots \\ \omega^*_{R-1,t} \end{bmatrix} \quad (19)$$

along with the error-correction mechanism restriction implicit in condition (9).

Equivalently, we can represent equations (18) and (19) by the following compact form:

$$y_t - \delta_t = \bar{y}_t = H_t\mu^*_t + e_t \quad (20)$$

$$\mu^*_t = F_t\mu^*_{t-1} + \omega^*_t \quad (21)$$

where $\omega^*_t \sim N(0, \Omega_t)$, with

$$\Omega_t = diag\Big(\sigma_{\omega,0}\sum_{j\in K_{-0}} d_{j0,t}, \sigma_{\omega,1}\sum_{j\in K_{-1}} d_{j1,t}, \cdots, \sigma_{\omega,R-1}\sum_{j\in K_{-(R-1)}} d_{(R-1)0,t}\Big),$$

where $\sum_{j\in K_{-i}}$ denotes summation over all regimes but i.

4.1 Prior Scheme

Given (20), (21), $\tilde{S}_T = [S_1 \;\; S_2 \;\; \cdots \;\; S_T]'$, and the assumed form of error term e_t, the model is conditionally linear and may feature heteroscedasticity. Defining further $\tilde{y}_T = [y_1 \;\; \cdots \;\; y_T]'$, $\tilde{\delta}_T = [\delta_1 \;\; \cdots \;\; \delta_T]'$, $\tilde{\mu}^*_T = [\mu^*_1 \;\; \cdots \;\; \mu^*_T]'$, and $\tilde{\sigma}_T = [\sigma_1 \;\; \cdots \;\; \sigma_T]'$, where $\tilde{\mu}^*_t = [\mu^*_{0,t} \cdots \mu^*_{R-1,t}]'$, the likelihood function associated with the model is given by

$$L(\tilde{y}_T|\tilde{\delta}_T, \tilde{\mu}^*_T, \tilde{S}_T, \tilde{\sigma}_T) = \prod_{t=1}^{T} f(y_t|\delta_t, \mu^*_t, S_t, \sigma_t) \quad (22)$$

where

$$f(y_t|\delta, \mu^*_t, S_t, \sigma_t) = \frac{1}{\sqrt{2\pi\sigma_t}}\exp\left(-\frac{1}{2\sigma_t}(y_t - \delta_t - \mu^*_t)^2\right) \quad (23)$$

In the Bayesian paradigm, we assume that the mean deviations and the unobserved counterfactual means are normally distributed around hierarchical priors given by[5]:

$$\pi(\tilde{\mu}_t) = \prod_{i=0}^{R-1} \tilde{\mu}_{i,t} = \prod_{i=0}^{R-1} N(\tilde{\mu}_{i,t-1}, B_0) \tag{24}$$

We assume normal priors for the error-correction terms,

$$\pi(\theta) = \prod_{i=0}^{R-1} \theta_i = \prod_{i=0}^{R-1} N(\theta_{0,i}, \Theta_{0,i}) \tag{25}$$

The parameters associated with the long-run trend are also normal, given by:

$$\pi(\delta) = \prod_{l=1}^{B+1} \delta_l = \prod_{l=1}^{B+1} N(\delta_{0,l}, \Delta_{0,l}) \tag{26}$$

For models with FTP, a Dirichlet conjugate prior is assumed for the transition probabilities, represented by $(p_{0,j}, \cdots, p_{R-1,j}) \sim D(\kappa_{0,j}, \cdots, \kappa_{R-1,j})$, which has the following state-invariant prior:

$$\pi(P) = \prod_{i}^{R-1} D(\kappa_{0,i}, \cdots, \kappa_{R-1,i}) \tag{27}$$

In the models with TVTP, we follow Kaufmann (2015) by assuming the following prior scheme:

$$\pi(\gamma) = \prod_{k \in K_{-k_0}} \pi(\gamma_j) = \prod_{k \in K_{-k_0}} N(\gamma_{0,k}, \Gamma_{0,k}), \tag{28}$$

for some $k \in K$ but those in the reference state, k_0, since they are normalized to zero for identification purposes (Kaufmann, 2015).

The constant proportion of the standard deviations associated with the GDP growth is assumed to have an inverse gamma distribution,

$$\pi(\sigma_e^2) = IG(s_{0,e}, s_{1,e}), \tag{29}$$

as well as for σ_ω^2:

[5] Details are presented in Appendix A

$$\pi(\sigma_\omega^2) = \prod_{i=0}^{R-1} \pi(\sigma_{\omega,i}^2) = \prod_{i=0}^{R-1} IG(s_{0,\omega_i}, s_{1,\omega_i}), \tag{30}$$

Moreover, in models with constant transition probability, we assume that

$$\pi(\tilde{S}_T) = \prod_{t=1}^{T} \pi(S_t|S_{t-1}, P), \tag{31}$$

while in the case of TVTP, in which the transition depends on variables Z_t, the assumed distribution is summarized by

$$\pi(\tilde{S}_T|Z^T, \gamma) = \prod_{t=1}^{T} \pi(S_t|S_{t-1}, Z_t, \gamma), \tag{32}$$

Finally, for the transition matrix associated with breaks, we follow Chib (1998) by assuming a beta distribution, such that

$$p_{l,h}^b \sim Beta(b_0, b_1) \tag{33}$$

Thus, given initial conditions gathered by the vector $\Phi_0 = [\mu_0^* \;\; S_0 \;\; \delta_0 \;\; \sigma_0]'$, the full posterior of the model is given by

$$p(\Theta_1|\Phi_0) = L(\tilde{y}_T|\tilde{\delta}_T, \tilde{\mu}_T^*, \tilde{S}_T, \tilde{\sigma}_T)\pi(\tilde{\mu}_T)\pi(\theta)\pi(\delta)\pi(\sigma_e^2)\pi(\sigma_\omega^2)\pi(\tilde{S}_T)\pi(P) \tag{34}$$

where $\Theta_1 = [\tilde{\delta}_T, \tilde{\mu}_T^*, \tilde{S}_T, \tilde{\sigma}_T, \tilde{\mu}_T, \theta, \delta, \sigma_e^2, \sigma_\omega^2, \tilde{S}_T, P]$. In the case of TVTP, the posterior of the model is given by

$$\begin{aligned} p(\Theta_2|\Phi_0) = L(\tilde{y}_T|\tilde{\delta}_T, \tilde{\mu}_T^*, \tilde{S}_T, \tilde{\sigma}_T)\pi(\tilde{\mu}_T)\pi(\theta)\pi(\delta) \\ \pi(\sigma_e^2)\pi(\sigma_\omega^2)\pi(\tilde{S}_T)\pi(P)\pi(P^b)\pi(\gamma) \end{aligned} \tag{35}$$

where $\Theta_2 = [\tilde{\delta}_T, \tilde{\mu}_T^*, \tilde{S}_T, \tilde{\sigma}_T, \tilde{\mu}_T, \theta, \delta, \sigma_e^2, \sigma_\omega^2, \tilde{S}_T, P, P^b, \gamma]$.

4.2 Posterior Estimation

Given the posterior distributions (34) and (35), we describe two Bayesian algorithms to estimate the parameters of the models.

4.2.1 Bayesian algorithm under constant transition probability

The algorithms to sample from the posterior probabilities for models with FTP or TVTP ope through equations (18) and (19) in both cases. For the FTP model, the posterior probability is given by (34) whereas, in the case of the TVTP model, the posterior probability is given by (35).

Algorithm 1

1. Draw μ_t^* using the modified Carter and Kohn (1994)'s algorithm as used by Eo and Kim (2016). The posterior distribution is normal, centered around a hierarchical mean, and is presented in Appendix A.

2. Draw S_T using a multi-move strategy similar to Carter and Kohn (1994) combined with the Hamilton (1989)'s filter, in a way similar to that followed by Kim and Nelson (1998).

3. Draw $\theta_i, \sigma_{\omega,i}, i = 0, \cdots, R-1$, given the normal priors for θ_i and inverse gamma priors for $\sigma_{\omega,i}$ presented in the previous section. This step is more challenging in the context of R>2 regimes because it involves counterfactual means. Departing from Eo and Kim (2016), we use a Random Walk Metropolis step in which we evaluate a likelihood function derived from the Kalman filter. Thus, we treat the counterfactual means as unobservables. In this step, also presented in Appendix A, we apply the adaptive scheme proposed by Atchadé and Rosenthal (2005) to automatically tune the Metropolis-Hastings scale parameters. Note that the draws for θ_i must satisfy condition (9).

4. Draw $\sigma_{e,t}$ using the stochastic volatility algorithm proposed by Kim et al. (1998).

5. Draw the fixed proportion of the volatility process. The posterior is inverted gamma, given by: $\sigma_e^2 \sim IG\left(\frac{s_{0,e}+T}{2}, \frac{s_{1,e}+\sum_{t=1}^{T}(y_t-\delta_t-\mu_t^*)^2}{2}\right)$

6. Draw δ_t:

   - For $B = 0$: Define $y_t^* = y_t - \sum_{i=0}^{R-1} S_{i,t}\mu_{i,t}$ and write the model:

     $$y_t^* = \delta_t + e_t$$

     where δ_t is given by (13) and the volatility process is given by (2). We recast the model for y_t^* in state-space representation and draw δ_t using the Kalman filter within the Carter and Kohn (1994) algorithm for time-varying parameters, taking as given the entire path of the stochastic volatility process.

   - For $B > 0$: Draw δ_l, $\forall l \in \{1, \cdots, B+1\}$, taking into account potential breaks in the long-run growth trend. The posterior is normal, given by: $\delta_l \sim$

$N(\bar{\delta}_l, \bar{\Sigma}_{\delta_l})$, where $\bar{\delta}_l = \bar{\Sigma}_{\delta l}\left(\Delta_{0,l}^{-1}\delta_{0,l} + \frac{1}{\sigma_e^2}\sum_{\tau_{\delta_{l-1}}}^{\tau_{\delta_l}}(y_t - \sum_{i=0}^{R-1} S_{i,t}\mu_{i,t})\right)$, with $\bar{\Sigma}_{\delta_l} = \left(\Delta_{0,l}^{-1}\delta_{0,l} + \frac{T^*}{\sigma_e^2}\right)$, and where $T^* < T$ is the number of observations in which δ_l is in place.

For $B = 0$, the algorithm ends here. Otherwise,

7. Draw p_{ij}. The posterior distribution is a Dirichlet distribution with hyper-parameters $\kappa_{0,j} + n_{0,j}, \cdots, \kappa_{R-1,j} + n_{R-1,j}$, where n_{ij} refers to the total number of transitions from state i to j.
8. Draw $p_{l,h}^b$. The posterior distribution is given by $p_{l,h}^b \propto Beta(b_0 + n_{l,h}, b_1 + n_{l,h})$.

4.2.2 Bayesian algorithm under time-varying probability

Algorithm 2

1. Follow steps 1 to 6 of Algorithm 1, where steps 1 to 3 have to be modified to account for the TVTP.
2. Draw γ. To do so, we follow the strategy developed by Kaufmann (2015) by introducing latent state-specific random utilities for all but the reference state 0.

$$z_{jt}^u = Z_t'\gamma_j + \eta_{jt}, \forall j \in K_{-0},$$

   where η_{jt} is i.i.d., Type I extremes value.

   The procedure is based on a partial representation of the model, in which the latent utilities are expressed in difference from the maximum utility of the other states:

$$z_{jt}^\star := z_{jt}^u - z_{-jt}^u = c + Z_t'\gamma_j + \zeta_{jt}, \forall j \in K_{-0},$$

   where ζ_{jt} is i.i.d. Logistic, and where $z_{-jt}^u = \max_{l \in K_{-0}} z_{lt}^u$.
3. Compute P_t given S_t and γ.

5 Parsimonious models for the Brazilian economy

Until now, we have discussed very general models, including models with no empirical appeal. Let us now focus on modeling the Brazilian economy. To do so, we consider a variety of flexible models, considering MS-ERSP(R,B) and MS-ERSP-TVTP(R,B) models with R $\in \{2, 3\}$ and B $\in \{0, 1, 2\}$. We begin by using an MS-ERSP-TVTP(3,1) model to illustrate the procedures. In this case, the system of equations governing the state-dependent means is given by:

$$\mu^*_{0,t} = \mu^*_{0,t-0} + \theta_0 d_t(\pi_{0,t}\mu^*_{0,t-1} + \pi_{1,t}\mu^*_{1,t-1} + \pi_{2,t}\mu^*_{2,t-1}) + \omega_{0,t} \quad (36)$$

$$\mu^*_{1,t} = \mu^*_{1,t-0} + \theta_1 d_t(\pi_{0,t}\mu^*_{0,t-1} + \pi_{1,t}\mu^*_{1,t-1} + \pi_{2,t}\mu^*_{2,t-1}) + \omega_{1,t} \quad (37)$$

$$\mu^*_{2,t} = \mu^*_{2,t-0} + \theta_2 d_t(\pi_{0,t}\mu^*_{0,t-1} + \pi_{1,t}\mu^*_{1,t-1} + \pi_{2,t}\mu^*_{2,t-1}) + \omega_{2,t} \quad (38)$$

$d_t = \sum_{j=0}^{2}\sum_{i=0}^{2} d_{ij,t}$,

$$d_{ij,t} = \begin{cases} 1, \text{ if } S_{t-1} = i, i \neq j \\ 0, \text{ otherwise} \end{cases}$$

Thus, we can cast the model into the following state-space form:

$$y_t = \delta_t + \begin{bmatrix} S_{0,t} & S_{1,t} & S_{2,t} \end{bmatrix} \begin{bmatrix} \mu^*_{0,t} \\ \mu^*_{1,t} \\ \mu^*_{2,t} \end{bmatrix} + e_t \quad (39)$$

$$\begin{bmatrix} \mu^*_{0,t} \\ \mu^*_{1,t} \\ \mu^*_{2,t} \end{bmatrix} = \begin{bmatrix} 1 + \theta_0\pi_{0,t}d_t & \theta_0\pi_{1,t}d_t & \theta_0\pi_{2,t}d_t \\ \theta_1\pi_{0,t}d_t & 1 + \theta_1\pi_{1,t}d_t & \theta_1\pi_{2,t}d_t \\ \theta_2\pi_{0,t}d_t & \theta_2\pi_{1,t}d_t & 1 + \theta_2\pi_{2,t}d_t \end{bmatrix} + \begin{bmatrix} \mu^*_{0,t-1} \\ \mu^*_{1,t-1} \\ \mu^*_{2,t-1} \end{bmatrix} + \begin{bmatrix} \omega^*_{0,t} \\ \omega^*_{1,t} \\ \omega^*_{2,t} \end{bmatrix} \quad (40)$$

or

$$y_t - \delta_t = H_t\mu^*_t + e_t \quad (41)$$

$$\mu^*_t = F_t\mu^*_{t-1} + \omega^*_t \quad (42)$$

where $\omega^*_t \sim N(0, \Omega_t)$, with $\Omega_t = diag((d_{10,t} + d_{20,t})\sigma_{\omega,0}, (d_{01,t} + d_{21,t})\sigma_{\omega,1}, (d_{02,t} + d_{12,t})\sigma_{\omega,1})$.

Therefore, the above system has a time-varying error-correction term given by

$$\theta_i(\pi_{0,t}\mu_{0,\tau-1} + \pi_{1,t}\mu_{1,\tau-1} + \pi_{2,t}\mu_{2,\tau-1}), \quad i = 0, 1, 2$$

and it must satisfy the following restriction in every point in time:

$$-1 < 1 + \pi_{0,t}\theta_0 + \pi_{1,t}\theta_1 + \pi_{2,t}\theta_2 < 1$$

The associated time-varying probability matrix is represented by:

$$P_t = \begin{pmatrix} p_{11}(Z_t) & p_{12}(Z_t) & p_{13}(Z_t) \\ p_{21}(Z_t) & p_{22}(Z_t) & p_{23}(Z_t) \\ p_{31}(Z_t) & p_{32}(Z_t) & p_{33}(Z_t) \end{pmatrix}$$

where Z_t is a set of predetermined variables that potentially affect the probability of regime-switching. Indeed, the evolution of the state variable S_t is assumed to be governed by a multinomial logit (Kaufmann, 2015) given by (17), with regime 0, the "recession" regime, expressing the reference regime.[6] In all applications with TVTP, Z_t has the following variables, expressed in $t-1$:

1. Quarterly growth rate of the Brazilian government debt;
2. Standardized changes in reserves;
3. Standardized changes in the real exchange rate;
4. Standardized changes in real interest rate (annualized overnight rate minus CPI inflation);
5. G7 quarterly growth rate;
6. A measure of exportable commodity prices, proxied by the standardized World Bank Commodity Price Index (energy + non-energy) divided by the US consumer price index.

Variable 1 captures the fiscal instance of the Brazilian economy; variables 2 to 4 aim to capture pressures in the exchange market in the sense of Eichengreen et al. (1996); and, finally, variables 5 to 7 capture the global conditions that may affect an emerging economy.

In our empirical application, $\tilde{Z}_t$ is defined as follows:

$$\tilde{Z}_t = \left(\hat{Z}_t \odot S_{0,t}, \cdots, \hat{Z}_t \odot S_{R-1,t}, S_{0,t}, \cdots, S_{R-1,t}\right)$$

with $\hat{Z}_t = Z_t - \bar{Z}_t$, where $\bar{Z}_t$ is the mean of Z_t, and in which $\odot$ is the element-wise product.

In the TVTP, we use algorithm 2 to draw the parameters. However, in the FTP case, the transition matrix is given by:

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix},$$

[6] The same restriction is applied in the TVTP model with two regimes.

and we apply algorithm 1 to draw the parameters of the model.

Recall that, for identification purposes, one has to impose a series of restrictions on the models, especially in models with time-varying probabilities. Thus, in models with 2 regimes, we simple require $\mu^*_{0,t} < 0$ and $\mu^*_{1,t} > 0$. In case of models with R $= 3$, we impose that $\mu^*_{0,t} < 0$, $\mu^*_{1,t} > 0$, and $\mu^*_{2,t} > 1.5\% > \mu^*_{1,t}$. In our empirical applications, regime three can be interpreted as a regime of relatively fast recovery. As suggested by Figure 2, a typical feature in the Brazilian business is that crises are followed by fast but short recovery periods. Thus, we expect that regime 3 features a relatively high mean but with a very low persistence profile. The only exception would be the period just after the global financial crisis of 2007-2008 when the Brazilian economy experienced a slightly more persistent recovery period. For these reasons, we impose that such a period should be labeled regime 3.

Note that our regime labeling scheme for the three-states models differs from that followed by Boldin (1996), which assumes a restricted matrix of transition probabilities so that the economy evolves from a mature expansion to recession and then to post-recession expansion. Previous experiments suggested that such a restriction scheme seems inconsistent with the Brazilian data.

Models are evaluated through the Deviance information criterion (DIC) (Spiegelhalter et al., 2002), which is the standard choice for models featuring stochastic volatility. The DIC criterion, which can be viewed as a generalization of the Akaike information criterion (AIC), penalizes model complexity while rewarding the model's fit to the data. The preferred model is the one with the smaller DIC.

6 Results

We use data from 1980Q1 to 2019Q4, which is the same period covered by CODACE. Inference is based on 300.000 simulations after discarding 50.000 draws as burn-in. Considering $R \in \{2, 3\}$, $B \in \{0, 1, 2\}$, and constant and time-varying transition probabilities, we estimated 12 specifications. Table 2 presents the DIC for each model, suggesting that the preferred model is MS-ERSP-TVTP(3,1). Thus, the selected model requires three regimes, one break in the long-run growth, and TVTP transition probabilities to describe the Brazilian data well.

Table 3 presents the prior and posterior moments for the MS-ERSP-TVTP(3,1) model. The posterior mean for speed of adjustment coefficients, $\theta_i, i = 0, 1, 2$, associated with the error-correction term suggests that the mean growth rates of regimes 0 and 2 converge to its long-run equilibrium at a very similar speed, whereas the convergence to regime 1 seems relatively faster.

Note that the upper bound for 90% error band for θ_0 is higher than that for θ_2. This re-

Table 2: Results

	Regimes (R)	Breaks (B)	DIC
MS-ERSP	2	0	494.232
		1	488.004
		2	532.175
	3	0	546.207
		1	517.043
		2	544.445
MS-ERSP-TVTP	2	0	480.886
		1	478.136
		2	492.014
	3	0	482.093
		1	**473.753**
		2	495.628

sult suggests that, on average, the economy may go faster from a bad regime to a better one than the converse, which is the opposite of the so-called bounce-back effect documented by the literature as a salient feature of the US business cycles (see, for example, Kim et al., 2005).

Table 3: Prior and Posterior Moments

	Prior		Posterior		
	Mean	SD	Mean	SD	90% Error Band
θ_0	-0.2500	0.5000	-0.5406	0.4790	[-1.343, 0.2605]
θ_1	-0.5000	0.5000	-0.6481	0.3715	[-1.3161, -0.0985]
θ_2	-0.2500	0.5000	-0.5604	0.4589	[-1.3516, 0.1384]
$\sigma_{\omega,0}$	0.3000	0.1500	0.5439	0.5340	[0.1626, 1.4217]
$\sigma_{\omega,1}$	0.3000	0.1500	0.4347	0.3262	[0.1532, 0.9952]
$\sigma_{\omega,2}$	0.3000	0.1500	0.6633	0.6159	[0.1767, 1.7671]
σ_e	0.0100	0.2000	0.0144	0.0088	[0.0059, 0.0302]
σ_ζ	1.0000	10.000	1.2541	0.4723	[0.6635, 2.1504]

The posterior probabilities from the MS-ERSP-TVTP(3,1) model are displayed in Figure 3. In general, the posterior probabilities of recessions is higher in the recessions dated by CODACE. However, such probabilities are not larger than 0.5 in two of them, 2001Q1-2001Q4 and 2003Q1-2003Q2. Furthermore, our selected model identifies a novel recession in the period 2012. Regarding the expansionary regimes, the posterior probabilities suggest that the second regime is more frequent than the third one, which has a more significant growth rate. Given the complicated path of the Brazilian business cycles, we see these results suggesting that the model does a reasonable job in replicating the CODACE recession dating.

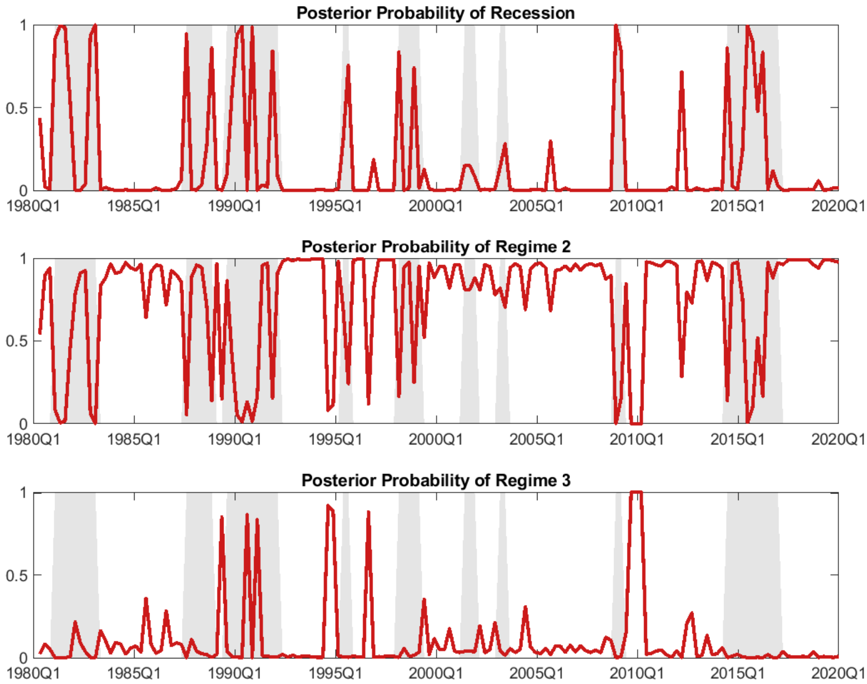

Figure 3: Posterior Probabilities for the MS-ERSP-TVTP(3,1) model

Figure 4 displays the real GDP growth and its posterior mean, given by $\delta_t + \sum_{i=0}^{2} \pi_i \mu_{i,t}$. The visual inspection suggests that the overall model fit is good, indeed. In particular, the significant changes in the GDP growth seem to be well captured by the MS-ERSP-TVTP(3,1) model. Note that the pattern of Brazilian growth has changed over time. First, there is a clear reduction in the volatility of the posterior mean from 1999 onwards, with a substantial reduction in the incidence of periods of fast recovery. However, crises seem to become much more persistent in recent years than in the 1980s. Given the three regimes indicated by the MS-ERSP-TVTP(3,1) model, we can summarize the results from Figure 4 as follows: (i) the Brazilian business cycles are asymmetrical, in the sense that recessions and booms do not occur with similar probabilities nor with equivalent properties; (ii) the duration of the regimes seem to be changing over time, with recession becoming more persistent and not followed by a fast recovery regime; (iii) given that the selected model features TVTP,

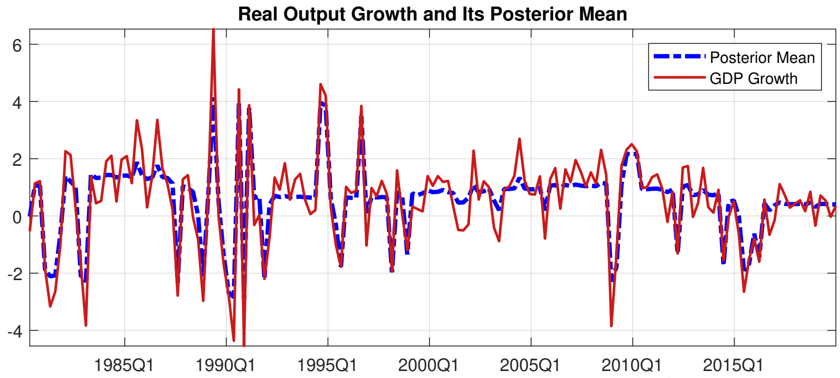

Figure 4: Posterior Mean for the MS-ERSP-TVTP(3,1) model

the state of the economy seems to depend on a series of factors. In particular, the MS-ERSP-TVTP(3,1) features time-varying probability driven by domestic and international factors, captured by the real interest rate, real exchange rate, commodity prices, and global uncertainty.

Figure 5 presents the posterior probability of a break associated with the MS-ERSP-TVTP(3,1) model. Such probability increases over time, reaching 50% by 1995Q1 and a unitary value by 2015Q2. Given this estimation, we can plot a time-varying long-run growth rate given by the following:

$$\tilde{\delta}_T = [\pi^b_{1,1}\delta_1 + (1 - \pi^b_{1,1})\delta_2, \cdots, \pi^b_{1,T}\delta_1 + (1 - \pi^b_{1,T})\delta_2]' \tag{43}$$

where $\pi^b_{j,t} = 1 - \Pr(S^b_{t+1} = j)$. This is represented in Figure 6, which shows that the long-run growth rate of the Brazilian economy has decreased systematically over time. While the average long-run growth rate was about 0.5% between 1980Q1 to 1984Q4, it has slowed down to about 0.22% in recent years. This result is consistent with the findings that the Brazilian economy has experienced decreasing productivity during the last four decades.

Finally, Figure 6 presents the posterior distribution of the stochastic volatility for the MS-ERSP-TVTP(3,1) model. It peaked around 1989 and, since then, has decreased. Therefore, it seems that the Brazilian economy has stabilized around a lower growth trend.

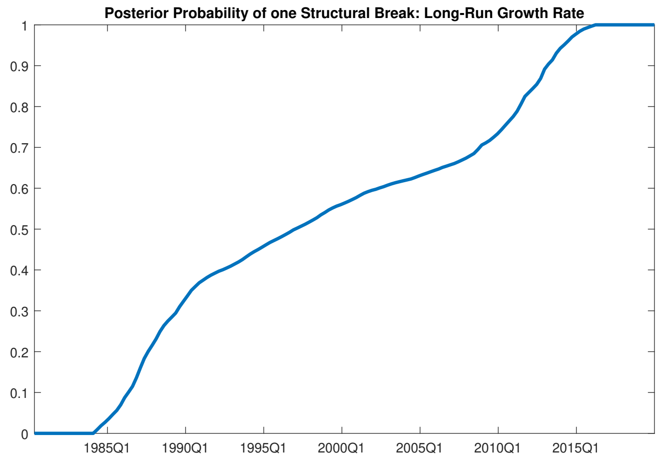

Figure 5: Posterior Probability Of a Break in the MS-ERSP-TVTP(3,1) model

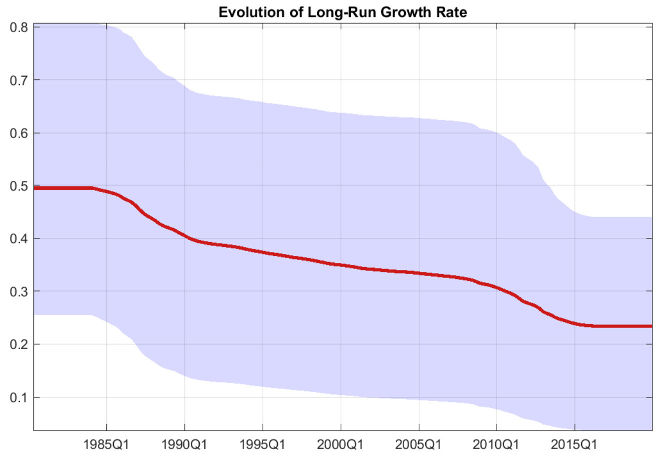

Figure 6: Estimated Long-run trend for the MS-ERSP-TVTP(3,1) model

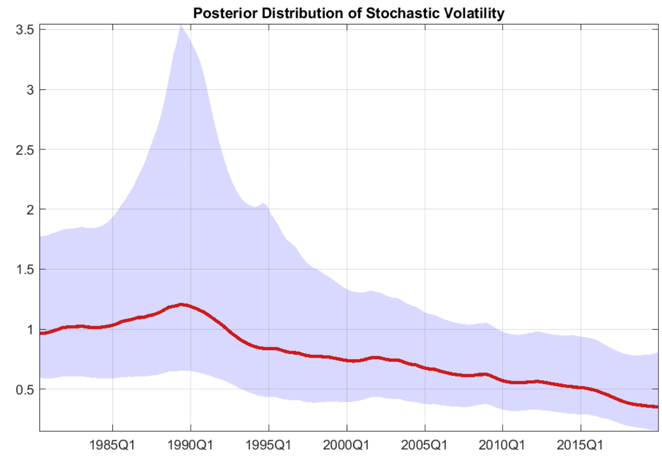

Figure 7: Stochastic Volatility for the MS-ERSP-TVTP(3,1) model

7 Conclusion

This paper generalizes the Markov-Switching model with evolving regime-specific means and stochastic volatility by Eo and Kim (2016) to allow for R regimes, B breaks, and time-varying transition probabilities. We applied our flexible methodology to model the Brazilian business cycles, a large emerging economy that has experienced many events during the last four decades that may have permanently affected its long-run growth trend.

Our results suggest that the Brazilian business cycles are better characterized by 3 regimes: recession, fast recovery, and boom. Moreover, we found that the regime-dependent average growth rates vary over our sample, besides presenting heterogeneous duration. Furthermore, we found evidence for one break in the Brazilian long-run growth trend, suggesting that the average long-run growth has reduced from 0.5% at a quarterly frequency to about 0.22%. Finally, given our finding that the stochastic volatility of the GDP growth has substantially reduced in recent years, our results suggest that the Brazilian economy has stabilized at a slow growth rate.

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A Appendix

This section presents details on the algorithms used in the paper. Let us first redefine the likelihood function associated with the model. Collect all the parameters of the model through Θ. Then, we can rewrite the likelihood function by

$$f(y_t|\Theta, S_t) = \prod_{i=0}^{R-1} \frac{1}{\sqrt{2\pi\sigma_t}}\exp\left(-\frac{1}{2\sigma_t}(y_t - \delta_t - \mu^*_{i,t})^2\right) \mathrm{Pr}(S_t = i) \tag{44}$$

where $\mathrm{Pr}(S_t = i)$ is the probability of $S_t = i$. Thus, the log-likelihood of the model is given by

$$\ell = \sum_{t=1}^{T} \ln f(y_t|\Theta, S_t) \tag{45}$$

A.1 Derivation of Step 1

Let $\Theta_{-\mu^*}$ be the set of all parameters in the model but μ^*_T. As shown by Eo and Kim (2016), conditional on $\tilde{S}_T$ the model has a linear state-space representation given by equations

(20) and (20). In this case, we can apply the modified Carter and Kohn (1994)'s algorithm as described in Eo and Kim (2016).

For identification purposes, we impose the restriction $\mu^*_{0,t} < \cdots < \mu^*_{R-1,t}$, where $\mu^*_{0,t} < 0$ and $\mu^*_{R-1,t} > 0$.

For $t = T$, the posterior distribution of μ^*_T is normal, given by:

$$\mu^*_T | \tilde{Y}_T, \tilde{S}_t, \Theta_{-\mu^*} \sim N(\tilde{\mu}_{T|T}, P_{T|T}) \tag{46}$$

where $\tilde{\mu}_{T|T}$ and $P_{T|T}$ are obtained from the Kalman filter.

For $t = T-1, \cdots, 1$, μ^*_t is drawn conditional on μ^*_{t+1}, where

$$\mu^*_{t|t,\mu^*_{t+1}} = \mu^*_{t|t} + P_{t|t}F'_{t+1}(F_{t+1}P_{t|t}F'_{t+1} + \Omega_{t+1})^{-1}(\mu^*_{t+1}F_{t+1}\mu^*_{t|t}) \tag{47}$$

and

$$P_{t|t,\mu^*_{t+1}} = \mu^*_{t|t} - P_{t|t}F'_{t+1}(F_{t+1}P_{t|t}F'_{t+1} + \Omega_{t+1})^{-1}F_{t+1}P_{t|t} \tag{48}$$

Finally, we draw the counterfactual means by the following procedure: if $S_t = i$ and $S_{t+1} = j$, for all $j \neq i$, draw μ_j, t from

$$\mu^*_{j,t,\mu^*_{t+1}} \sim N(\mu^*_{t+1}|\mu^*_{t+1}(j+1,1), P_{t,t|\mu^*_{t+1}}(j+1,j+1))$$

Otherwise, set

$$\mu^*_{j,t} = \mu^*_{t|t,\mu_{t+1}}(j+1,1),$$

where $\mu^*_{j,t+1}|\mu^*_{t+1}(j+1,1)$ is the $j+1$-th element of $\mu^*_{t+1}|\mu^*_{t+1}$ and $P_{t,t|\mu^*_{t+1}}(j+1,j+1)$ is the $j+1$-th element of the diagonal of the matrix $P_{t,t|\mu^*_{t+1}}$, and $i,j \in [0, R-1]$.

A.2 Derivation of Step 2

Define $\tilde{\Theta}_T = [\tilde{\delta}_T \ \tilde{\mu}^*_T \ \tilde{\sigma}_T]$ and let $\Theta_t = [\delta_t \ \mu^*_t \ \sigma_t]$. Given the state-space representation of the model, let $p(\tilde{S}_T|\tilde{\Theta}_T, \tilde{Y}_T)$ be the joint distribution of $S_t, t = 1, \cdots, T$. This density can be decomposed as the following:

$$p(\tilde{S}_T|\tilde{\Theta}_T, \tilde{Y}_T) = p(S_T|\Theta_T, y_T) \prod_{t=1}^{T-1} p(S_t|S_{t+1}, \Theta_t, \tilde{Y}_t)$$

1. Using (44), run Hamilton (1989)'s filter to draw S_T. The final iteration of the Kalman filter delivers S_T.

2. Generate $S_t|\tilde{\Theta}_T, \tilde{Y}_T, S_{t+1}$ for $t = T-1, \cdots, 1$ using the fact that:

$$p(S_t|\Theta_t, y_t, S_{t+1}) \propto p(S_{t+1}|S_t, \Theta_t)p(S_t|\tilde{Y}_t, \Theta_t),$$

   where $p(S_t|\tilde{Y}_t, \Theta_t)$ denotes the transition probability. We can now generate S_t using the following probability:

$$\Pr(S_t = i|S_{t+1}, \tilde{\Theta}_t, \tilde{Y}_t) = \frac{p(S_{t+1}|S_t = i, \Theta_t)p(S_t = i|\tilde{Y}_t, \Theta_t)}{\sum_{j=0}^{R-1} p(S_{t+1}|S_t = j, \Theta_t)p(S_t = j|\tilde{Y}_t, \Theta_t)}$$

A.3 Derivation of Step 3

Given the priors for θ_i and $\sigma_{\omega,i}$, $i = 0, \cdots, R-1$, we again use equations (20) and (20) to evaluate the likelihood of the model using the Kalman filter, conditional on $\tilde{S}_T, \tilde{\delta}_T, \tilde{\sigma}_T$ and $\tilde{\mu}^*_T$.

The parameters in this step are drawn using a random walk Metropolis-Hasting step, where employ the adaptive scheme proposed by Atchadé and Rosenthal (2005) to automatically tune the Metropolis-Hastings scale parameters.

Draws for θ_i should be consistent with the restriction:

$$-1 < 1 + \sum_{i=0}^{R-1} \pi_i \theta_i < 1$$

B Testing the Algorithm

We perform a simple Monte Carlo experiment to evaluate the performance of the algorithm. In doing so, we consider a model with $R = 3$, $B = 0$, CPS, and we use the following data generating process:

$$ln(\sigma^2_{e_t}) = ln(\sigma^2_{e_{t-1}}) + \zeta^{1/2} e^{\zeta}_t \quad (49)$$

$e^{\zeta}_t \sim N(0, 1)$ and $\zeta = 0.02$. and

$$y_t = \delta_t + \sum_{i=0}^{2} S_{i,t}\mu_{i,t} + e_t, \quad (50)$$

where $\mu_0 \approx -1.5$, $\mu_1 \approx 0.5$, and $\mu_2 \approx 2.0$, which are simulated from the following system:

$$\begin{bmatrix} \mu^*_{0,t} \\ \mu^*_{1,t} \\ \mu^*_{2,t} \end{bmatrix} = \begin{bmatrix} 1+\theta_0\pi_{0,t}d_t & \theta_0\pi_{1,t}d_t & \theta_0\pi_{2,t}d_t \\ \theta_1\pi_{0,t}d_t & 1+\theta_1\pi_{1,t}d_t & \theta_1\pi_{2,t}d_t \\ \theta_2\pi_{0,t}d_t & \theta_2\pi_{1,t}d_t & 1+\theta_2\pi_{2,t}d_t \end{bmatrix} + \begin{bmatrix} \mu^*_{0,t-1} \\ \mu^*_{1,t-1} \\ \mu^*_{2,t-1} \end{bmatrix} + \begin{bmatrix} \omega^*_{0,t} \\ \omega^*_{1,t} \\ \omega^*_{2,t} \end{bmatrix} \tag{51}$$

where $\theta_0 = -0.55, \theta_1 = -0.15$ and $\theta_2 = -0.35$. Moreover, we assume $\omega^*_t \sim N(0, \Omega_t)$, with $\Omega_t = diag((d_{10,t} + d_{20,t})\sigma_\omega, (d_{01,t} + d_{21,t})\sigma_\omega, (d_{02,t} + d_{12,t})\sigma_\omega)$, in which $\sigma_\omega = 0.02$.

Moreover, the process for δ_t is given by:

$$\delta_t = \delta_{t-1} + \sigma_\delta^{1/2} e^\delta_t$$

where $e^\delta_t \sim N(0,1)$, $\sigma_\delta = 0.02$ and the starting value is $\delta_0 = 0.8$.

Finally, we assume the following transition matrix:

$$P = \begin{pmatrix} 0.950 & 0.010 & 0.015 \\ 0.025 & 0.975 & 0.010 \\ 0.025 & 0.015 & 0.975 \end{pmatrix}$$

We first simulate a synthetic GDP growth path with 500 observations. We then estimate the model using an MCMC run of 10000 iterations, with a burn-in of 5000 iterations. Figure 8 presents the simulated series and its associated estimated posterior mean. The visual inspection suggests that the algorithm works well.

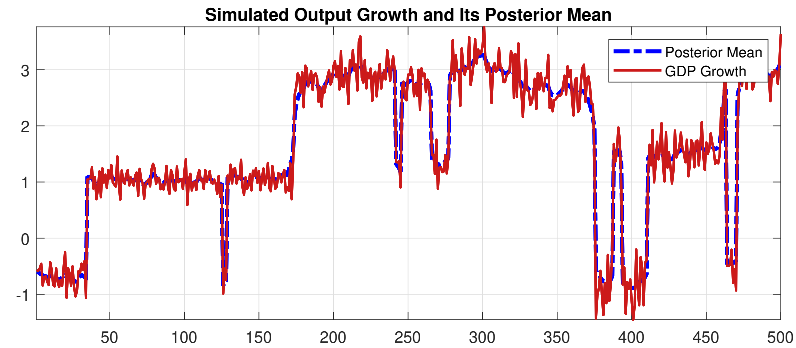

Figure 8: Monte Carlo Experiment: Estimation of the posterior mean ($\delta_t + \sum_{i=0}^{2} \pi_i \mu_{i,t}$).