Unobserved Competition in First-Price Auctions: A Bayesian Approach for Estimation and Inference

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Abstract

The degree of competition is a key feature in the empirical analysis of auctions, since it directly affect the estimation of the structural parameters. Following the framework of Guerre and Luo (2022), we propose a reasonably simple estimation procedure to first-price auctions under independent and private values paradigm using minimal information. By focusing on the location-scale family, we estimate the participation distribution and the private values distribution as well as the pair of minimal and maximal number of bidders. Due to the nonregularities in the econometric model, we adopt a Bayesian procedure. We find that the procedure performs relatively well in a certain region of the support of the participation probabilities. As expected, its performance improves with the sample size.

Keywords: first-price auctions, unobserved competition, nonregular models, discontinuity, Bayesian estimation.

JEL Codes: C13, C57, D44.

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1 Introduction

Auctions have long been used as a mechanism to sell and exchange objects[1]. These goods range from art & antiques objects, commodities to bonds. Governments worldwide also usually perform auctions to sell public properties to private individuals and organisations (Krishna, 2009). They play an important role on the allocation of public resources. For instance, a large share of government contracts are allocated through procurement auctions, being it from highway constructions to purchase of hospital equipment, to mention a few.

Since the seminal papers of Vickrey (1961) and Harsanyi (1968), much effort have been directed to further understand the properties of auction as allocation mechanism as Myerson (1981), Riley and Samuelson (1981), and Milgrom and Weber (1982) in early stages of the literature[2]. In particular, first-price auctions, which will be the focus of this paper, has received a lot of attention with respect to the establishment and characterisation of its equilibrium as in Riley and Samuelson (1981) under symmetric Independent Private Value (IPV, hereafter) paradigm; Lebrun (1999) for the asymmetric IPV case with an arbitrary number of bidders and single reserve price for all bidders; Maskin and Riley (2000) with the affiliated value paradigm without reserve prices; and Athey (2001) that shows existence of equilibrium in a more general framework[3]. Furthermore, Lebrun (1999), Bajari (2001), and Lebrun (2006) tackle the issue of uniqueness.

Despite of the rapid expansion of the theoretical literature and its solid background, in the early stages the empirical analysis of auctions did not seem to evolve at the same speed, even in face of the ample availability of data. As argued in Perrigne and Vuong (1999), much of it could be credited to the complex form of equilibrium strategies derived from complicated differential equations that are highly nonlinear with respect to the parameters of interest.

[1]It became widely spread with the advent of the internet and the rise of online auctions in the late 90s, see Lucking-Reiley (2000) and Bajari and Hortaçsu (2004).

[2]See Milgrom (1989) and Klemperer (1999) for an early development of the auction literature as well as Krishna (2009) for a summary of the main types of auctions and its basic properties.

[3]Shen et al. (2019) studies the characterisation of Bayesian Nash Equilibrium (BNE) with discrete value distribution.

The upsurge of the structural approach aimed to recover auctions' primitives from observed data enabling the analyst to provide rich policy insights[4]. With that in mind, Paarsch (1992) aimed to develop a maximum likelihood based method (MLE) to estimate the model. However, the author faced numerical issues given the complex structure of the equilibrium strategies. Much of the approaches developed thereafter made use of direct procedures, i.e., by assuming a certain specification for the valuation distribution and computing the BNE as Donald and Paarsch (1993); Donald and Paarsch (1996); and Laffont, Ossard and Vuong (1995) which propose a simulated nonlinear least square procedure that avoids the computation of the equilibrium strategy.

To overcome these issues, Guerre, Perrigne and Vuong (2000) show that for the symmetric IPV case the valuation distribution can be nonparametrically identified by the observed bids and number of bidders. In addition, they propose a two-step nonparametric procedure to estimate the valuation distribution from observed bids[5]. Nevertheless, note that all the aforementioned approaches require to observe all bids and the effective number of bidders, which may not always be feasible as Slattery (2020) that observes only winning bids in its analyse on subsidy competition for firms in the U.S. between state and local government, adopting an auction framework.

The degree of competition is a key feature in empirical analyse of auctions since it significantly impacts the estimation of the structural parameters. As argued in Guerre and Luo (2022), whether it is the presence of collusion placing phantom bids or inexperienced bidders, the difference between number of actual buyers and observed ones is an important issue to be considered since participation is by itself an object of major interest. Some attempts to detect collusion in auction models can be found in the literature, see Bajari and Summers (2002) and references therein. Schurter (2020) develops a nonparametric approach to detect the presence of collusion in first-price auctions and estimate its effect on the seller's expected

[4]Until late 1980s, the empirical literature was constrained to analyse the implications of the game theoretical model using experimental or field data as explained by Perrigne and Vuong (1999).

[5]Gimenes and Guerre (2022) propose a quantile regression (QR) framework for estimation and inference of the private value QR from the bid QR.

revenue. Even so, its approach is not immune to the curse of dimensionality.

It is important pointing out that the problem of unobserved competition is not new. Early papers in the literature have, even if in different degrees, considered this matter. Laffont, Ossard and Vuong (1995) consider the actual number of bidders, constant across auctions, as an additional parameter to be estimated. Paarsch (1997) for the case of English auctions, where the degree of competition is treated as a nuisance parameter to be eliminated afterwards in the estimation step. An, Hu and Shum (2010) study the identification of first-price auctions when the degree of competition is unknown by the analyst due to the presence of binding reserve price rule. The observed number of bidders is used as a proxy for the degree of competition along with an instrument that could be defined as a discretised second bid. Shneyerov and Wong (2011) study the identification of first-price and Dutch auctions when the set of active bidders is not observed due to the presence of reserve price. Their result relies on the fact that all submitted bids are observable.

In a different note, Guerre and Luo (2022) develop a density discontinuity approach to deal with unobserved competition, including both collusion and uninformed bidders. Their framework allow the degree of competition to vary across auctions and requires only the winning bid to be observed, which is likely to offer better econometric properties. However, they restrain their analysis to the identification of the participation and private value distribution, so estimation and inference procedures remain to be developed.

The contribution of this paper is twofold. First, we propose a reasonably simple estimation procedure to first-price auctions using minimal information, following the framework of Guerre and Luo (2022). By focusing on the location-scale family, we estimate the participation distribution and the private values distribution. Second, we contribute to the literature that concerns to the empirical analysis of finite mixture via Bayesian methods. In particular, when it presents nonregularities in the form of discontinuities in the mixture density as well as when the support of each mixture component depends on the parameter of interest. We find that the procedure performs better in a certain region of the support of the participation probabilities. Nonetheless its performance improves with the sample size, as expected.

This paper is organised as following. In section 2 we present the auction model. In section 3 we describe the implications of the economic theory to the econometric model, concluding with the identification of the participation distribution and the private values distribution. In section 4 we discuss the Bayesian estimation procedure. In section 5 we present simulations to assess the behavior of the proposed estimator. Finally, we conclude with section 6.

2 The symmetric and independent private values framework

This section starts by describing the first-price auction model under the symmetric and independent private values (IPV, hereon) paradigm. Next, we derive the two quantile equilibrium mappings that will be used in the identification exercise. Assume that there are $N \in \mathcal{N}$ symmetric risk neutral buyers in which N is observed by all buyers and $\mathcal{N}$ is the set of active buyers. There is a single object for sale and no reserve price, where each bidder i values the object by an amount V_i that comes from a distribution F independent of the level of competition N. Notice that in this set up each buyer places a sealed bid and the buyer with the highest bid wins the auction. She then pays a price equal to her bid and enjoys V_i. Also, assume that for each i and $N = n$, the bidding strategy B_i is formed by a mapping $s_i(V_i \mid n)$. Therefore, the expected utility of i will be given by $(V_i - s_i(V_i \mid n))\mathbb{P}(s_i(V_i \mid n) \geq s_j(V_j \mid n) \mid V_i)$ with $j \neq i$ and $i, j = 1, ..., n$. Below we state the set of additional assumptions on the private value distribution and briefly discuss its implications.

Assumption IPV: Private values V_i are iid copies from a common knowledge distribution F, known by bidder i but not by the competitors. Moreover, F has compact support on $[\underline{v}, \overline{v}]$ and its density function f is continuous and strictly positive over $[\underline{v}, \overline{v}]$.

Maskin and Riley (1984) prove that under Assumption IPV, the mapping $s_i(V_i \mid N)$ is strictly increasing and continuously differentiable with symmetric strategies, i.e, $s_i(V_i \mid N) = s(V_i \mid N) = s_N(V_i)$ for all i[6]. In particular, Maskin and Riley (2003) show that the compact support of F with positive density over the whole support, including the boundaries, rule out the existence of multiple equilibria.

2.1 Quantile equilibrium mappings

Following Liu and Luo (2017); Gimenes and Guerre (2022); Guerre and Luo (2022), we can write the buyers problem using a quantile framework. Let $V(\alpha) = F^{-1}(\alpha)$ represent the private value quantile function with quantile level $\alpha \in [0, 1]$. Let $B_n(\alpha)$ be the bid quantile function for a given auction with n bidders. The bidder i private value rank $A_i = F(V_i)$ follows a uniform distribution on $[0, 1]$. Denote G_n and g_n, respectively, as the winning bid cumulative distribution and density function given an auction with $N = n$ buyers. Let the conditional bid lower and upper bound be defined as $\underline{b}_n \equiv B_n(0)$ and $\overline{b}_n \equiv B_n(1)$, then the support of the conditional bid distribution will then be given by $\left[\underline{b}_n, \overline{b}_n\right]$. Remember that $B_i = s_n(V_i)$ for all i and number of buyers n, where $s_n(\cdot)$ is strictly increasing and continuously differentiable. As a result of that, since $B_n(A_i) = s_n(V(A_i))$, the best-response strategy is the bid quantile function $B_n(\alpha) = s_n(\alpha)$ for all $\alpha \in [0, 1]$.

Assume that only bidder i with rank $A_i = \alpha$ places a suboptimal bid $B_n(a)$ with $a \neq \alpha$ meanwhile the competitors bid optimally. Notice then, that the probablity of winning the auction with n bidders is given by $\mathbb{P}(\max_{j \neq i} B_n(A_j) \leq B_n(a)) = \mathbb{P}(\max_{j \neq i} A_j \leq a) = a^{n-1}$. The expected payoff can be written as $(V(\alpha) - B_n(a))a^{n-1}$ in which $\alpha =$

[6] The change of notation from $s(V_i \mid N)$ to $s_N(V_i)$ is purely for the sake of convenience. The strategy mapping $s_N(\cdot)$ will depend on the level of competition N, but not on the individual buyer i beyond her private value V_i.

$\arg\max\{(V(\alpha) - B_n(a))a^{n-1}\}$. Since the best-response strategy is the bid quantile function, it follows that $B_n(.)$ is differentiable and therefore, the first order condition (FOC) of the expected payoff optimization gives

$$\frac{\partial(V(\alpha) - B_n(a))a^{n-1}}{\partial a}\,|_{a=\alpha}= (n-1)\alpha^{n-2}\left[V(\alpha) - B_n(\alpha) - \frac{\alpha B_n^{(1)}(\alpha)}{n-1}\right] = 0,$$

which in its turn, gives the private quantile function as

$$V(\alpha) = B_n(\alpha) + \alpha\frac{B_n^{(1)}(\alpha)}{n-1}. \tag{1}$$

Equation (1) provides the equilibrium mapping from the bid quantile function to the private value quantile function. The inverse equilibrium mapping from the private value quantile function to the bid quantile function is also attainable. From the FOC, we have that $\frac{\partial[B_n(\alpha)\alpha^{n-1}]}{\partial\alpha} = (n-1)V(\alpha)\alpha^{n-2}$ which yields

$$B_n(\alpha) = \frac{n-1}{\alpha^{n-1}}\int_0^{\alpha} t^{n-2}V(t)dt = V(\alpha) - \int_0^{\alpha} t^{n-2}V^{(1)}(t)dt \tag{2}$$

with $V^{(1)}(\cdot) > 0$. Equation (2) provides an equilibrium mapping from the private value quantile function to the bid quantile function. Also, note that from (2) we get that the quantile equilibrium bid is increasing with respect to n, meaning that buyers place bid more aggressively when facing higher competition, therefore increasing sellers expected revenue as noted by Bulow and Klemperer (1996). See example 2.1 for the illustration of the main concepts using a uniform model for the private values. The uniform model is a powerful tool since it is simple enough to be analytically treated and still exhibit the main challenges as the location-scale model, which is the focus of this paper.

Example 2.1 (Uniform model) Assume that there can be N_l bidders in each auction $l = 1, ..., L$. Let $V_{il} \sim U[0, \theta]$ for some parameter $\theta \in \Theta$. Then, the equilibrium strategy will be $B_{il} = \frac{N_l-1}{N_l}V_{il}$, implying that $B_{il} \mid N_l = n \sim U[0, \frac{n-1}{n}\theta]$.

3 The econometric model

In this section, we present the restrictions that the model imposes on the winning bid structure. Since the primitives of the model are the distribution of the number of bidders $\mathbb{P}_N$ and the private values distribution F, we describe the discontinuity approach that enable the nonparametric identification of these quantities, making use of the quantile equilibrium mappings, following the arguments of Guerre and Luo (2022). As we explain later, these discontinuities in the winning bid distribution are caused by a positive bid density at the largest bid. Indeed, as the largest bid is smaller than the largest private value, many bidders would be able to bid at this level, resulting in a positive density yielding a positive profit. Finally, the fact that bids increase with competition will also play an essential role in this identification argument.

3.1 Winning bid distribution

Now, assume that N is not observed by the econometrician and only the winning bid defined by $W = \max_{i\in\mathcal{N}} B_i$ is available. As it turns out, the analyst will then observes draws from the unconditional winning bid distribution given by

$$G(b) = \sum_{n=2}^{\infty} \mathbb{P}(N = n) \times \mathbb{P}(\max_{1\leq i\leq n} B_i \leq b) = \sum_{n=2}^{\infty} G_n^n(b) \times \mathbb{P}(N = n) \tag{3}$$

where $G_n(\cdot)$ stands for winning bid cumulative distribution function conditional to the number of bidders equal to n. The RHS of equation (3) holds due to the fact that participation is considered exogenous, which is reflected by the assumption of the private values distribution F being independent of the level of competition N. The exogenous participation feature is part of Guerre and Luo (2022)'s benchmark model to illustrate the identification procedure without further complications. Nonetheless, the authors relax this assumption and show that the primitives of the model are still identified. In this paper we will focus on the estimation of their benchmark model with exogenous participation. Lets focus on the structure of the winning bid distribution. For that, assume that N is a discrete random variable with distribution $\mathbb{P}_N$ as per assumption N below.

Assumption N: The number of active buyers N is a discrete random variable with support $\{\underline{n}, \underline{n}+1, ..., \overline{n}\}$ in which $2 \leq \underline{n} \leq \overline{n} < \infty$, with $\pi_n = P(N = n) > 0$ and $\sum_{n=\underline{n}}^{\overline{n}} \pi_n = 1$.

Following Guerre and Luo (2022)[7], the authors show that when assumptions IPV and N hold, it follows that a c.d.f $G(\cdot)$ is rationalised by a first-price auction if and only if two conditions are met. The first condition is that $G(\cdot)$ has a finite mixture structure as below

$$G(\cdot) = \sum_{n=\underline{n}}^{\overline{n}} G_n^n(\cdot) \times \pi_n \tag{4}$$

where $G_n(\cdot)$'s are c.d.f's defined on the support $\left[\underline{b}_n, \overline{b}_n\right]$and $2 \leq \underline{n} \leq \overline{n} < \infty$, with $\pi_n > 0$ and $\sum_{n=\underline{n}}^{\overline{n}} \pi_n = 1$. The second condition is that the bid quantile function $B_n(\cdot) = G_n^{-1}(\cdot)$ is continuously differentiable and the equilibrium mapping defined by (1), $V(\alpha) = B_n(\alpha) + \alpha \frac{B_n^{(1)}(\alpha)}{n-1}$, is invariante in the level of competition $N = n$ for any fixed $\alpha \in [0, 1]$, along with the fact that it is continuously differentiable over $[0, 1]$ with $V^{(1)}(\cdot) > 0$.

Besides that, given that the components of the mixture defined by (4) are generated by the same private value distribution provided that assumptions IPV and N hold, it imposes some constraints on the extremities of the conditional winning bid density $g_n(\cdot)$. These constraints are expressed as

$$g_n(\underline{v}) = \frac{n}{n-1} f(\underline{v}), \tag{5}$$

$$g_n(\overline{b}_n) = \frac{1}{(n-1)(\overline{v} - \overline{b}_n)} \tag{6}$$

where $\underline{v} = \underline{b}_n = V(0)$ and $\overline{v} = V(1)$ with $\overline{v} > \overline{b}_n$ for all n in the support of the number of buyers. It turns out that under assumptions IPV and N, the upper bound of the support

[7] Proposition 2.1 in Guerre and Luo (2022).

of the conditional bid distribution, $\bar{b}_n$, increases with the level of competition $n = \underline{n}, \cdots, \overline{n}$ as can be seen by (2)[8]. Furthermore, Guerre and Luo (2022) show that (6) implies that the unconditional winning bid density function $g(.)$ will be discontinuous at each $\bar{b}_n$ with jumps of size

$$\triangle_n = \frac{n\pi_n}{(n-1)(\overline{v} - \bar{b}_n)}. \tag{7}$$

Given that $g(.)$ will be discontinuous at each $\bar{b}_n$, it follows that $\bar{b}_n$ is identified for each n from the location of each jump. As a result, the jump size $\triangle_n$ is as well identified up to a constant $\overline{v}$.

Example 3.1 (Uniform model continuation.) The uniform model introduced in the example 2.1 has unconditional winning bid c.d.f and density denoted by $G(b \mid \theta)$ and $g(b \mid \theta)$, which depends on a certain parameter θ. As a matter of fact, the support of each component as well depends on θ and it is given by $[0, \frac{n-1}{n}\theta]$.

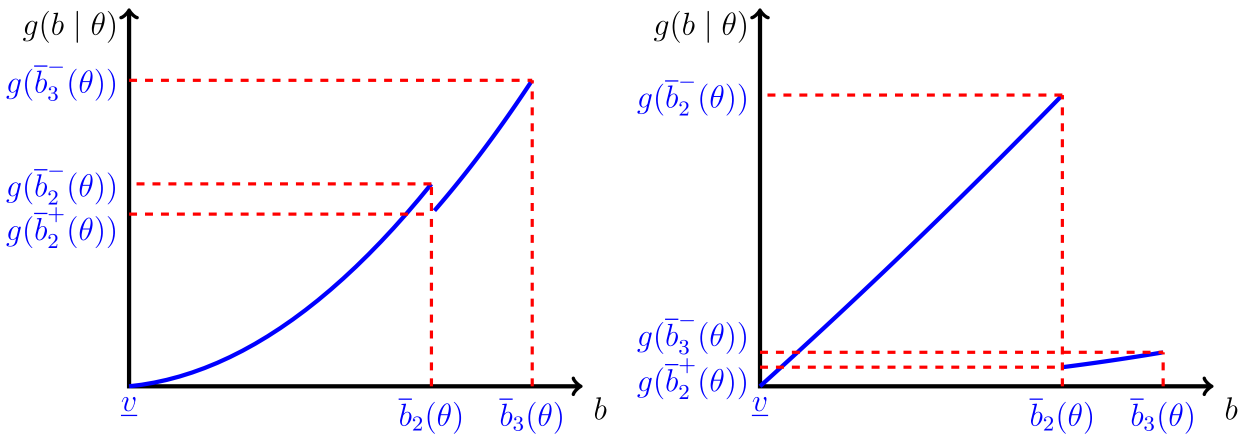
$$G(b \mid \theta) = \sum_{n=\underline{n}}^{\overline{n}} \pi_n \left(\frac{b}{\frac{n-1}{n}\theta} \right)^n \mathbb{I}\{b \leq \frac{n-1}{n}\theta\} \tag{8}$$

and

$$g(b \mid \theta) = \sum_{n=\underline{n}}^{\overline{n}} \pi_n \left(\frac{b}{\frac{n-1}{n}\theta} \right)^{n-1} \frac{n^2}{(n-1)\theta} \mathbb{I}\{b \leq \frac{n-1}{n}\theta\}. \tag{9}$$

Notice that by equation 9 it is clear that the location of the discontinuities depend upon θ, namely $\bar{b}_n(\theta) = \frac{n-1}{n}\theta$. This feature characterises nonregularities in the econometric model if θ is of main interest. Figure 1b illustrates the unconditional density $g(\cdot \mid \theta)$ for $(\underline{n}, \overline{n}) = (2, 3)$, $\theta = 2$, and for values of $\pi_2 = 0.1$ and $\pi_2 = 0.9$. In this case, $\bar{b}_2(2) = 1$ and $\bar{b}_3(2) = \frac{4}{3}$ regardless of π_2.

[8]Lemma 2.1-(i) in Guerre and Luo (2022).



(a) $\pi_2 = 0.1$.

(b) $\pi_2 = 0.9$.

Figure 1: Unconditional winning bid density function - Uniform model.

As per figure 1, the size of the jumps can be expressed as $\triangle_2 = g(\overline{b}_2^-(\theta)) - g(\overline{b}_2^+(\theta))$ and $\triangle_3 = g(\overline{b}_3^-(\theta)) - 0$. The value of π_2 does not affect the location of the discontinuity, but rather the size of the jumps.

3.2 Participation and the private values distribution

In this section we describe the identification of the main quantities of interest, i.e, the participation distribution and the private values distribution.

The lower bound of the support of the participation distribution is identified from the lower tail of the winning bid distribution as in Hill and Shneyerov (2013) where $\underline{n} = \lim_{t\to 0^+} = \frac{\log G(\underline{v}+t)}{\log t}$. Also, it is important to point out that assumption IPV ensures the existence of discontinuities in the unconditional winning bid distribution and given that $\overline{b}_n < \overline{v}$, it places a positive mass of $g_n(.)$ at $\overline{b}_n$. Consequently, the upper bound $\overline{n}$ will also be identified through $\underline{n}$ and the number of discontinuity points of $g(\cdot)$, i.e, $\overline{n} = \underline{n} + \#\{b; g(\cdot) \text{ is discontinuous at } b\} - 1$. In conclusion, due to the fact that each n generates a discontinuity in the unconditional winning bid density $g(\cdot)$, the entire support of the number of buyers is identified. The jump size defined in (7) play a key role to identify the participation probabilities. First, notice that from (7) we have that $\pi_n = \frac{n-1}{n}(\overline{v} - \overline{b}_n)\triangle_n$. In addition to that, the fact that $\Sigma_{\underline{n}}^{\overline{n}}\pi_n = 1$

yields

$$\bar{v} = \frac{1 + \sum_{n=\underline{n}}^{\bar{n}} \frac{n-1}{n} \Delta_n \bar{b}_n}{\sum_{\underline{n}}^{\bar{n}} \frac{n-1}{n} \Delta_n} \tag{10}$$

and by replacing (10) into the expression of π_n found above gives

$$\pi_n = \frac{\frac{n-1}{n}\Delta_n}{\sum_{k=\underline{n}}^{\bar{n}} \frac{k-1}{k}\Delta_k} + \frac{n-1}{n}\Delta_n \left(\frac{\sum_{k=\underline{n}}^{\bar{n}} \frac{k-1}{k}\Delta_k \bar{b}_k}{\sum_{k=\underline{n}}^{\bar{n}} \frac{k-1}{k}\Delta_k} - \bar{b}_n \right) \tag{11}$$

with $n = \underline{n}, ..., \overline{n}$. Hence, since $\overline{b}_n$ and Δ_n are identified for each n, we conclude that π_n is also identified for each n. In summary, $\mathbb{P}_N$ and its support are identified.

With respect to the private values distribution, the identification of F relies upon the fact that the support of $g_n(\cdot)$ monotonically increases with respect to n and the iterative relation between the two equilibrium mappings described above. The upper bound of $B_n(\alpha)$ can be written, using (12), as $\overline{b}_n = B_n(1) = (n-1)\int_0^1 t^{n-2}V(t)dt$ which gives

$$B_n(\alpha) = \frac{n-1}{\alpha^{n-1}} \left[\frac{\overline{b}_n}{n-1} - \int_{\alpha}^{1} t^{n-2} V(t) dt \right]. \tag{12}$$

Also, remember that equation (1) is given by $V(\alpha) = B_n(\alpha) + \alpha \frac{B_n^{(1)}(\alpha)}{n-1}$. As stated in Gimenes and Guerre (2022), expression (1) is a quantile version of the corresponding mapping in Guerre et al. (2000) and, similarly, is the key ingredient to indetify $V(.)$ with knowledge of $B_n(\cdot)$[9]. Above we explained that the competition distribution in (4) is identified along with its support. Additionally, the fact that the conditional winning bid distribution has support $[\underline{v}, \overline{b}_n]$ for each n, we have that $G_{\overline{n}-1}(\cdot) = 1$ on $[\overline{b}_{\overline{n}-1}, \overline{b}_{\overline{n}}]$ and from (4) we get that

$$G(b) = \left(1 - \pi_{\overline{n}} + \pi_{\overline{n}} G_{\overline{n}}^{\overline{n}}(b)\right).$$

Therefore, on $[\overline{b}_{\overline{n}-1}, \overline{b}_{\overline{n}}]$

[9]Equation 3 in Guerre et al. (2000).

$$G_{\overline{n}}(b) = \left(\frac{G(b)}{\pi_{\overline{n}}} + \frac{\pi_{\overline{n}} - 1}{\pi_{\overline{n}}} \right)^{\frac{1}{\overline{n}}}.$$

It follows then that $B_{\overline{n}}(.) = G_{\overline{n}}^{-1}(\cdot)$ is identified on $[\alpha_1, 1]$ with $\alpha_1 = G_{\overline{n}}(\overline{b}_{\overline{n}-1})$ using the top portion of the winning bid distribution as noted by Guerre and Luo (2022). From (1) we find that $V(.)$ is also identified on $[\alpha_1, 1]$. The mapping (12) gives

$$B_{\overline{n}-1}(\cdot) = (\overline{n} - 2) \left[\frac{\overline{b}_{\overline{n}-1}}{\overline{n} - 2} - \int_{\alpha_1}^{1} t^{\overline{n}-3} V(t) dt \right]$$

which results in the identification of $B_{\overline{n}-1}(\cdot)$ on $[\alpha_1, 1]$. Hence, $B_{\overline{n}-1}(\cdot)$, $B_{\overline{n}}(\cdot)$, and $V(\cdot)$ are identified on $[\alpha_1, 1]$. Let $\overline{b}_{\overline{n}-1} < \beta_1 = B_{\overline{n}-1}(\alpha_1) < \overline{b}_{\overline{n}-1}$ where $G_{\overline{n}-1}(b)$ is identified for $b > \beta_1$. Note that (4) then becomes

$$G(b) = \left(1 - \pi_{\overline{n}-1} - \pi_{\overline{n}} + \pi_{\overline{n}-1} G_{\overline{n}-1}^{\overline{n}-1}(b) + \pi_{\overline{n}} G_{\overline{n}}^{\overline{n}}(b)\right)$$

and

$$G_{\overline{n}}(b) = \left(\frac{\pi_{\overline{n}-1} G_{\overline{n}-1}^{\overline{n}-1}(b)}{\pi_{\overline{n}}} + \frac{\pi_{\overline{n}-1} + \pi_{\overline{n}} - 1}{\pi_{\overline{n}}} \right)^{\frac{1}{\overline{n}}}$$

which results in $G_{\overline{n}}(b)$ being identified on $b > \beta_1$. Consequently, $B_{\overline{n}}(.)$ will then be identified on $\alpha \geq \alpha_2 = G_{\overline{n}}(\beta_1)$. Similarly as before, $B_{\overline{n}-1}(\cdot)$ and $V(\cdot)$ will also be identified on $\alpha \geq \alpha_2$ using the mappings (1) and (12). Considering that $B_n(\cdot)$ is monotonically increasing with n, the remaining part of the identification exercise constitutes of applying this iterative procedure so that $V(\cdot)$ is identified on its whole support.

4 The estimation strategy

As seen from the uniform example, the private value and participation parameters affect the support, the discontinuity location and jumps of the winning bid distribution. It is the signature of irregular econometric models, characterised in particular by a discontinuous likelihood function. Consequently, standard frequentist methods might not be the most appropriated to estimate the parameters of interest. Some papers in the literature have contributed to the understanding of nonregularities as Ibragimov and Has'minskii (1981) and Ghosal and Samanta (1997) that study the single parameter density with jumps; Ghosal and Samanta (1995) extends the result of Ibragimov and Has'minskii (1981) for the mixed case of nonregular and regular parameters; Hirano and Porter (2003) study models with parameter-dependent support; and Chernozhukov and Hong (2004) for when the location of the discontinuity depends on the parameter of interest through a regression curve. In a different type of nonregularity, Bochkina and Green (2014) study the asymptotic behaviour when the posterior distribution concentrates at the boundary of the support.

The common conclusion of the studies mentioned above is that Bayesian methods outperform gold standard frequentist methods as MLE, both in terms of efficiency as in what concerns the computational cost involved. With that in mind, we opt to implement a Bayesian approach. In this section we describe the Bayesian estimation procedure of the components parameters along with the components weights of the finite mixture derived from the model with unobserved competition presented in section 3.1. We focus on the scale-location model for the private values. In this paper, we make use of an importance sampling algorithm to estimate the location, scale, and the competition distribution.

As per the uniform model example and the density function in (9), it is evident that different pairs $\left(\underline{N}, \overline{N}\right)$ yield different density functions, either by exhibiting different number of components or by displaying different components itself. We illustrate it using the uniform model as below.

Example 4.1 (Uniform model continuation.) Lets consider the following candidate pairs: $(2,3), (3,4)$, and $(2,4)$. The equivalent of equation (9) for each pair is, respectively,

$$g(b \mid \theta) = \pi_2 \left(\frac{b}{\frac{\theta}{2}}\right) \frac{4}{\theta} \mathbb{I}\{b \leq \frac{\theta}{2}\} + (1-\pi_2) \left(\frac{b}{\frac{2}{3}\theta}\right)^2 \frac{9}{2\theta} \mathbb{I}\{b \leq \frac{2}{3}\theta\}, \tag{13}$$

$$g(b \mid \theta) = \pi_3 \left(\frac{b}{\frac{2}{3}\theta}\right)^2 \frac{9}{2\theta}\mathbb{I}\{b \leq \frac{2}{3}\theta\} + (1-\pi_3)\left(\frac{b}{\frac{3}{4}\theta}\right)^3 \frac{16}{3\theta}\mathbb{I}\{b \leq \frac{3}{4}\theta\}, \tag{14}$$

and

$$g(b \mid \theta) = \pi_2 \left(\frac{b}{\frac{\theta}{2}}\right) \frac{4}{\theta}\mathbb{I}\{b \leq \frac{\theta}{2}\} + \pi_3 \left(\frac{b}{\frac{2}{3}\theta}\right)^2 \frac{9}{2\theta}\mathbb{I}\{b \leq \frac{2}{3}\theta\} + \pi_4 \left(\frac{b}{\frac{3}{4}\theta}\right)^3 \frac{16}{3\theta}\mathbb{I}\{b \leq \frac{3}{4}\theta\} \tag{15}$$

with $\pi_4 = 1 - \pi_2 - \pi_3$. Notice that each of the candidate pair defines a different model.

So the fact that $\left(\underline{N}, \overline{N}\right)$ is unknown and the estimator of θ will depend on the pair, leads to the necessity of estimating $\left(\underline{N}, \overline{N}\right)$ or, equivalently, to the choice of the most appropriate model. Take, for instance, the density function (13) and assume that $\pi_2 = 1$ so that the correct model has only one component. Nonetheless, assume that we are trying to estimate the model considering the pair $(2, 3)$, using (13). In this example, the posterior distribution of theta, say $\mathbb{P}_\theta$, will have a support $[\frac{3}{2}\max w_l, \infty)$, where it should be given by $[2 \max w_l, \infty)$. In the case of a Pareto prior, for instance, the distribution will concentrate around its lower bound. Therefore, the Bayesian estimator of θ will converge to $\frac{3}{4}\theta$, thus inconsistent. This is mostly due to the choice of continuous prior for π_2. In conclusion, taking into account the correct pair is of utmost importance to estimate θ and π. We aim to tackle this issue by considering $\overline{N}$ as parameter to be estimated using the Bayesian approach that will be presented later on.

Finally, we derived the priors from a GMM estimator developed in a first step. In section 4.1 we describe the preliminary GMM estimation procedure and the derivation of its asymptotic distribution; and in section 4.2 we present in details the derivation of the priors and the Bayesian estimation using an importance sampling approach.

4.1 GMM preliminary estimation

Define $\left(\underline{N}, \overline{N}\right)$ as the candidate pair of minimum and maximum number of bidders. This step concerns to the computation of the GMM estimator of $\theta = (\mu,\ \sigma,\ \pi')'$ conditional to the pair $(\underline{N}, \overline{N})$. First, lets focus on to describe the set of moments that do identify θ and the iterative procedure to compute the estimates. Then, we derive the asymptotic distribution of the GMM estimator and explain how it can be used to calibrate the priors to perform the Bayesian estimation of section 4.2.

Let $W = (W_1, \cdots, W_L)$ be the vector with size L of winning bids in which W_l are iid copies from $g(w \mid \theta)$, where

$$g(w \mid \theta) = \sum_{n=\underline{N}}^{\overline{N}} \pi_n n G_n^{n-1}\left(\frac{w-\mu}{\sigma}\right)\frac{g_n\left(\frac{w-\mu}{\sigma}\right)}{\sigma}. \tag{16}$$

As explained in the previous section, $g_n(\cdot)$ is define on the support $[\underline{b}, \bar{b}_n]$ with $g_n(\bar{b}_n) > 0$.

Assume $\underline{N} < \overline{N}$. Moreover, define $I_n\mu, \sigma) \equiv \mathbb{E}\left[\mathbb{I}\left(\bar{b}_{n-1} \leq \frac{W_l-\mu}{\sigma} \leq \bar{b}_n\right)\right]$ with n in $\{\underline{N}+1, \cdots, \overline{N}\}$. Therefore, from (19) we can write this set of $\overline{N} - \underline{N}$ moments as

$$I_n(\mu,\sigma) = \sum_{k=n}^{\bar{N}} \pi_k \left[G_k^k\left(\bar{b}_n\right) - G_k^k\left(\bar{b}_{n-1}\right)\right]. \tag{17}$$

In addition to that, there are two more moments derived from the maximum winning bid and the expected winning bid as described below.

$$G^{-1}(1 \mid \theta) = \mu + \sigma \bar{b}_{\bar{N}}, \tag{18}$$

$$\mathbb{E}[W \mid \theta] = \mu + \sigma \sum_{n=\underline{N}}^{\bar{N}} \pi_n m_n \tag{19}$$

with $m_n = \int b \cdot n G_n^{n-1}(b) \cdot g_n(b) db$. Altogether, (17)-(19) and the fact that $\pi'\mathbf{1} = 1$, with $\mathbf{1}' = (1, \cdots, 1)$, enable us to just identify θ. Let $I(\mu,\sigma) \equiv (I_{\underline{N}+1}(\mu,\sigma), \cdots, I_{\overline{N}}(\mu,\sigma))'$ and T an upper triangular matrix, so that the sample analogue of (17)-(19) can be written as

$$\hat{I}(\hat{\mu}, \hat{\sigma}) - T \begin{bmatrix} \hat{\pi}_{\underline{N}+1} \\ \vdots \\ \hat{\pi}_{\overline{N}} \end{bmatrix} = 0, \tag{20}$$

$$\hat{\pi}_{\underline{N}} - \left(1 - \sum_{n=\underline{N}+1}^{\overline{N}} \hat{\pi}_n\right) = 0, \tag{21}$$

$$\overline{W} - \hat{\mu} - \hat{\sigma} \sum_{n=\underline{N}}^{\bar{N}} \hat{\pi}_n m_n = 0, \tag{22}$$

$$\max_{1 \leq l \leq L} W_l - \hat{\mu} - \hat{\sigma} \bar{b}_{\overline{N}} = 0 \tag{23}$$

where $\overline{W}$ is the sample average of W and $\hat{I}_n(\hat{\mu}, \hat{\sigma}) = \sum_{l=1}^{L} \mathbb{I}\left(\bar{b}_{n-1} \leq \frac{W_l - \hat{\mu}}{\hat{\sigma}} \leq \bar{b}_n\right)$. To implement the iterative procedure we make use of (20)-(23). Let $\tilde{\mathbf{1}}_{(\overline{N}-\underline{N}+1)}$ be the first column of an identity matrix of order $(\overline{N} - \underline{N} + 1)$ and $J_{(\overline{N}-\underline{N}+1)\times(\overline{N}-\underline{N})}$ be an upper triangular matrix as below

$$J = \begin{bmatrix} -\mathbf{1}_{1\times(\overline{N}-\underline{N})} \\ I_{(\overline{N}-\underline{N})} \end{bmatrix} \tag{24}$$

with a row vector $\mathbf{1}_{1\times(\overline{N}-\underline{N})} = (1, \cdots, 1)'$ and $I_{(\overline{N}-\underline{N})}$ being an identity matrix of order $(\overline{N} - \underline{N})$. From (20) and (21) we can write the competition probabilities as

$$\hat{\pi} = \tilde{\mathbf{1}}_{(\overline{N}-\underline{N}+1)} + J \times T^{-1} \times \hat{I}(\hat{\mu}, \hat{\sigma}). \tag{25}$$

Moreover, the equations (22) and (23) yield

$$\hat{\sigma} = \frac{\max_l W_l - \overline{W}}{\bar{b}_{\overline{N}} - \sum_{n=\underline{N}}^{\bar{N}} \hat{\pi}_n m_n}, \tag{26}$$

$$\hat{\mu} = \max_l W_l - \hat{\sigma}\bar{b}_{\overline{N}}. \tag{27}$$

The iterative procedure implied by (25), (26), and (27) is described below.

Algorithm 1 GMM iterative procedure

---

1. Initialize with
$$\hat{\sigma}_0 = \frac{\max_l W_l - \overline{W}}{\bar{b}_{\overline{N}} - \sum_{n=\underline{N}}^{\bar{N}} \hat{\pi}_0 m_n},$$
$$\hat{\mu}_0 = \max_l W_l - \hat{\sigma}_0\bar{b}_{\overline{N}}.$$
2. Given $(\hat{\mu}_{k-1}, \hat{\sigma}_{k-1})$, compute
$$\hat{\pi}_k = \tilde{\mathbf{1}}_{(\overline{N}-\underline{N}+1)} + J \times T^{-1} \times \hat{I}(\hat{\mu}_k, \hat{\sigma}_k).$$
3. Given $\hat{\pi}_k$, compute
$$\hat{\sigma}_k = \frac{\max_l W_l - \overline{W}}{\bar{b}_{\overline{N}} - \sum_{n=\underline{N}}^{\bar{N}} \hat{\pi}_k m_n},$$
$$\hat{\mu}_k = \max_l W_l - \hat{\sigma}_k\bar{b}_{\overline{N}}.$$
4. Iterate until converges.

4.1.1 Asymptotic distribution of the GMM estimator

Notice that in equation (23) we use $\max_{1\leq l\leq L} W_l$ as an analogue for $G^{-1}(1 \mid \theta)$. As it turns out, it does not affect the asymptotic distribution of the estimator of θ. We start with following result.

Proposition 1 *Let $\bar{b}_{\overline{N}}(\theta) = \mu + \sigma \bar{b}_{\overline{N}}$ and $W = \max_{i \in \mathcal{N}} B_i$. Under assumptions N and IPV, we have that*

$$L(\max_{1 \le l \le L} W_l - \mu - \sigma \bar{b}_{\overline{N}}) \xrightarrow{d} \frac{\sigma}{\overline{N} \pi_{\overline{N}} g_{\overline{N}}(\bar{b}_{\overline{N}})} \mathcal{E} \tag{28}$$

with $\mathrm{AsVar}(\max_l W_l) = \frac{1}{L^2} \left(\frac{\sigma}{\overline{N} \pi_{\overline{N}} g_{\overline{N}}(\bar{b}_{\overline{N}})} \right)$.

Proof.

$$\begin{aligned}
\mathbb{P}\left[L(\max_{1 \le l \le L} W_l - \bar{b}_{\overline{N}}(\theta)) \le -t \mid \theta \right] &= \mathbb{P}\left[\max_{1 \le l \le L} W_l \le \bar{b}_{\overline{N}}(\theta) - \frac{t}{L} \mid \theta \right] \\
&= \mathbb{P}^L \left(W \le \bar{b}_{\overline{N}}(\theta) - \frac{t}{L} \mid \theta \right) \\
&= \exp \left\{ L \log \left[1 - \mathbb{P} \left(\bar{b}_{\overline{N}}(\theta) - \frac{t}{L} \le W \le \bar{b}_{\overline{N}}(\theta) \right) \right] \right\} \\
&= \exp \left\{ L \log \left[1 - \frac{t}{L} g \left(\bar{b}_{\overline{N}}(\theta) \mid \theta \right) + o(1) \right] \right\} \\
&\to \exp \left[-t \frac{\overline{N} \pi_{\overline{N}} g_{\overline{N}}(\bar{b}_{\overline{N}})}{\sigma} \right]
\end{aligned}$$

■

As a result, $\max_l W_l = \mu + \sigma \bar{b}_{\overline{N}} + O_{\mathbb{P}}\left(\frac{1}{L}\right)$, which in its turn along with the fact that the estimator of θ converges at rate $\sqrt{L}$, guarantees that $\max_l W_l$ can be replaced by $\mu + \sigma \bar{b}_{\overline{N}}$ without any effect on the asymptotic distribution. Let $\hat{M}(\hat{\theta}) = 0$ be the set of equations defined by (20)-(23). Thus, under consistency of $\hat{\theta}$ we get that

$$\sqrt{L}(\hat{\theta} - \theta) = (\hat{M}^{(1)}(\theta) + o_{\mathbb{P}}(1))^{-1} \sqrt{L} \hat{M}(\theta) \xrightarrow{d} (M^{(1)}(\theta))^{-1} \times \mathcal{N}\left(0, \lim_{L \to \infty} \mathrm{Var}(\sqrt{L} \hat{M}(\theta)) \right).$$

Notice that $\hat{I}(\mu, \sigma)$ is discontinuous and not differentiable, meaning that one should pay close attention to it when deriving the asymptotic distribution of $\hat{\theta}$. Nonetheless, the expansion of $\hat{M}(\hat{\theta})$ can be obtained through the partial derivatives of the mapping $(\mu, \sigma) \to I(\mu, \sigma)$ as shown by Chen et al. (2003). Denote $I_\mu(\mu, \sigma) \equiv \frac{\partial I(\mu, \sigma)}{\partial \mu}$ and $I_\sigma(\mu, \sigma) \equiv \frac{\partial I(\mu, \sigma)}{\partial \sigma}$, then

$$\hat{I}(\hat{\mu}, \hat{\sigma}) = \hat{I}(\mu, \sigma) + (I_\mu(\mu, \sigma) + o_{\mathbb{P}}(1))(\hat{\mu} - \mu) + (I_\sigma(\mu, \sigma) + o_{\mathbb{P}}(1)(\hat{\sigma} - \sigma).$$

Therefore, the $M^{(1)}(\theta)$ matrix and the asymptotic variance $\lim_{L\to\infty}\mathrm{Var}(\sqrt{L}\hat{M}(\theta))$ can be expressed, respectively, as

$$M^{(1)}(\theta) = \begin{bmatrix} I_\mu(\mu,\sigma) & I_\sigma(\mu,\sigma) & \mathbf{0}_{(\overline{N}-\underline{N})\times 1} & -T \\ 0 & 0 & 1 & \mathbf{1}_{(\overline{N}-\underline{N})\times 1} \\ -1 & -\sum_{n=\underline{N}}^{\bar{N}} \pi_n m_n & -\sigma[m_{\underline{N}}] & \sigma[m_{\underline{N}+1}\cdots m_{\overline{N}}] \\ -1 & \bar{b}_{\underline{N}} & 0 & \mathbf{0}_{(\overline{N}-\underline{N})\times 1} \end{bmatrix}$$

and

$$\mathrm{AsVar}(\sqrt{L}\hat{M}(\theta)) = \lim_{L\to\infty} \begin{bmatrix} L\mathrm{Var}(\hat{I}(\mu,\sigma)) & \mathbf{0}_{(\overline{N}-\underline{N})\times 1} & L\mathrm{Cov}(\overline{W},\hat{I}(\mu,\sigma)) & \mathbf{0}_{(\overline{N}-\underline{N})\times 1} \\ \mathbf{0}_{1\times(\overline{N}-\underline{N})} & 0 & 0 & 0 \\ L\mathrm{Cov}(\overline{W},\hat{I}(\mu,\sigma))' & 0 & \mathrm{Var}(W) & 0 \\ \mathbf{0}_{1\times(\overline{N}-\underline{N})} & 0 & 0 & 0 \end{bmatrix}$$

where

$$L\mathrm{Var}(\hat{I}(\mu,\sigma)) = Diag(I(\mu,\sigma)) - I(\mu,\sigma)I(\mu,\sigma)'$$

and

$$L\mathrm{Cov}(\overline{W},\hat{I}(\mu,\sigma)) = \mathbb{E}\left[\mathbb{I}\left(\bar{b}_{\underline{N}} \le \frac{W_l-\mu}{\sigma} \le \bar{b}_{\underline{N}+1}\right)\right] - \mathbb{E}[W_l]I(\mu,\sigma).$$

4.2 Bayesian estimation via importance sampling

In this section, we describe the Bayesian estimation method using an importance sampling algorithm with focus on the scale-location model for the private values. Similarly as before, let $(\underline{N},\overline{N})$ be the pair of possible minimum and maximum number of bidders and $W = (W_1,\cdots,W_L)'$ an iid sample of winning bids defined by $W_l = \max(B_{1l},\cdots,B_{N_l l})$ where

$$B_{il} \mid N_l \overset{iid}{\sim} \frac{1}{\sigma} g_{N_l}\left(\frac{b-\mu}{\sigma}\right)$$

with support of $g_n(\cdot)$ being $[\underline{v}, \bar{b}_n]$. Having that in mind, we get that the winning bid, W_l, conditional to a particular level of competition, $N_l = n$, follows the distribution $h(w \mid \theta, n)$ with $\theta = (\mu, \sigma, \pi')'$ where

$$h(w \mid \theta, n) = nG_n^{n-1}\left(\frac{w-u}{\sigma}\right)\frac{1}{\sigma}g_n\left(\frac{w-u}{\sigma}\right).$$

Moreover, we have that

$$\mathbb{P}(N_l = n \mid \pi, \underline{N}, \overline{N}) = \pi_n > 0$$

with $n = \underline{N}, \cdots, \overline{N}$ so that $\sum_{n=\underline{N}}^{\overline{N}} \pi_n = 1$. Therefore, the unconditional winning bid distribution is given by

$$h(w \mid \theta, \underline{n}, \overline{n}) = \sum_{n=\underline{n}}^{\overline{n}} \pi_n h(w \mid \theta, n). \tag{29}$$

4.2.1 The choice of Priors

Before introducing the sampling algorithm, it is important to describe how we use the preliminary GMM estimator and its asymptotic variance for the choice of priors. Remember that each of the the pairs of candidates for minimum and maximum number of active buyers $(\underline{N}, \overline{N})$ define a model. We set a flat prior on the set of candidate pairs. The GMM estimation then is used to calibrate the hyper-parameters of the prior of θ conditional to each pair. Therefore, the priors are written conditional to the data W. Moreover, conditional to the pair $(\underline{N}, \overline{N})$, (μ, σ) are independent of π, so the prior can be written as

$$p(\theta \mid W, \underline{N}, \overline{N}) = p(\mu \mid W, \underline{N}, \overline{N})p(\sigma \mid W, \underline{N}, \overline{N})p(\pi \mid W, \underline{N}, \overline{N}).$$

We choose the prior of π, conditional to $(\underline{N}, \overline{N})$, to be a Dirichlet distribution with parameters $\hat{e}_{(\underline{N}, \overline{N})} = \gamma \times \mathbf{1}_{1\times(\overline{N}-\underline{N}+1)}$, $\gamma > 0$ as

$$\pi \mid \underline{N}, \overline{N} \sim D(\hat{e}_{(\underline{N},\overline{N})}). \tag{30}$$

Furthermore, we set

$$\mu + \sigma \bar{b}_{\overline{N}} \mid \underline{N}, \overline{N} \sim \text{Pareto}\left(\max_{1\le l\le L} W_l,\ \hat{s}_{\overline{N}}\right). \tag{31}$$

We calibrate $\hat{s}_{\overline{N}}$ to match the estimate of the variance in (28) up to a multiplicative constant ρ as

$$\rho^2 \hat{\text{Asvar}}\left(\max_{1\le l\le L} W_l\right) = \frac{\rho^2}{L^2}\left(\frac{\hat{\sigma}}{\overline{N}\hat{\pi}_{\overline{N}} g_{\overline{N}}(\bar{b}_{\overline{N}})}\right)^2 = \left(\max_{1\le l\le L} W_l\right)^2 \frac{\hat{s}_{\overline{N}}}{(\hat{s}_{\overline{N}}-1)^2(\hat{s}_{\overline{N}}-2)}$$

so that, for each $\overline{N}$,

$$\hat{s}_{\overline{N}} = \left(\frac{\rho}{L}\right)^{-1}\left(\frac{\hat{\sigma}}{\overline{N}\hat{\pi}_{\overline{N}} g_{\overline{N}}(\bar{b}_{\overline{N}})}\right)^{-1}(1+o(1)).$$

Thus, we have that

$$\sigma \mid \mu + \sigma \bar{b}_{\overline{N}} \sim \text{IG}\,(a, b) \tag{32}$$

and $\mu = \mu + \sigma \bar{b}_{\overline{N}} - \sigma \bar{b}_{\overline{N}}$. The mean and variance in (32) are given, respectively, by $\frac{b}{a-1}$ and $\left(\frac{b}{a-1}\right)^2 \frac{1}{a-2}$ where the latter is calibrated to match the variance of the GMM estimator.

4.2.2 Posterior distributions and the Bayesian estimation algorithm

The marginal posterior distribution of the pair $(\underline{N}, \overline{N})$ given by

$$\begin{aligned}
\mathbb{P}((\underline{N}, \overline{N}) = (\underline{n}, \overline{n}) \mid W = w) &= \frac{\mathbb{P}((\underline{N}, \overline{N}) = (\underline{n}, \overline{n}), W = w)}{h(w)} \\
&= \frac{h(w \mid (\underline{N}, \overline{N}) = (\underline{n}, \overline{n}))p(\underline{n}, \overline{n})}{\sum_{\underline{k},\overline{k}} h(w \mid (\underline{N}, \overline{N}) = (\underline{k}, \overline{k}))p(\underline{k}, \overline{k})}
\end{aligned} \tag{33}$$

with

$$
\begin{aligned}
h(w \mid \underline{n}, \overline{n}) &= \mathbb{E}[h(w \mid \theta, \underline{N}, \overline{N}) \mid \underline{N} = \underline{n}, \overline{N} = \overline{n}] \\
&= \int h(w \mid \theta, \underline{n}, \overline{n}) p(\theta \mid \underline{n}, \overline{n}) d\theta
\end{aligned}
\tag{34}
$$

is a key ingredient to estimate the pair $(\underline{N}, \overline{N})$ (i.e. to select the model defined by the pair $(\underline{N}, \overline{N})$) as we explain later. To estimate (33) we first need to estimate the marginal likelihood (34) for each candidate pair $(\underline{n}, \overline{n})$. To do so, note that (34) is an expectation with respect to the prior of θ conditional to the pair $(\underline{n}, \overline{n})$, then it can be estimated using

$$
\hat{h}_S(W \mid \underline{n}, \overline{n}) = \frac{1}{S} \sum_{s=1}^{S} h(W \mid \theta_s, \underline{n}, \overline{n}) \tag{35}
$$

with θ_s as iid copies from the prior $p(\theta \mid W, \underline{n}, \overline{n})$.

Finally, we can estimate moments of the posterior distribution of θ, namely $\mathbb{E}[\psi(\theta) \mid W, \underline{N}, \overline{N}]$, using importance sampling [10]. For the case of the posterior mean (i.e., $\psi(\theta) = \theta$) we have

$$
\begin{aligned}
\mathbb{E}[\theta \mid W, \underline{N}, \overline{N}] &= \int \theta dP_\theta \\
&= \int \theta \frac{h(W \mid \theta, \underline{N}, \overline{N})}{h(W \mid \underline{N}, \overline{N})} p(\theta \mid \underline{N}, \overline{N}) d\theta
\end{aligned}
\tag{36}
$$

where P_θ denotes the posterior distribution of θ conditional to the pair $(\underline{N}, \overline{N})$ and $p(\theta \mid \underline{N}, \overline{N})$ is the prior distribution of θ for a given pair $(\underline{N}, \overline{N})$. Define $\hat{\theta}_{\underline{N}, \overline{N}} = \mathbb{E}[\theta \mid W, \underline{N}, \overline{N}]$, which can be estimated as

$$
\frac{1}{S \hat{h}_S(W \mid \underline{n}, \overline{n})} \sum_{s=1}^{S} \theta_s h(W \mid \theta_s, \underline{n}, \overline{n}) \tag{37}
$$

with θ_s being iid copies from the prior distribution $p(\cdot \mid W, \underline{n}, \overline{n})$. The estimation algorithm is described in details below.

[10] An importance sampling algorithm is a Monte Carlo method that enable the analyst to compute expectations of a target distribution using a given alternative distribution, as explain Robert and Casella (2005, Chapter 3).

Algorithm 2 Estimation of θ and $(\underline{N}, \overline{N})$.

1. Conditional to each pair $(\underline{N}, \overline{N})$, compute the GMM estimator of $\theta = (\mu, \sigma, \pi)$ and of its corresponding asymptotic variance according to 4.1.

2. Set the priors as (30), (31), and (32) so that their parameters match the GMM estimation performed in the previous step.

3. For a given possible pair $(\underline{N}, \overline{N}) = (\underline{n}, \overline{n})$, draw S iid copies from the priors and approximate the marginal likelihood using (35). Repeat the procedure for each possible pair $(\underline{N}, \overline{N})$ to obtain an estimate of the marginal posterior pair using (33).

4. Estimate $(\hat{\underline{N}}, \hat{\overline{N}})$ as below

   (a) $\hat{\underline{N}} = \left\lfloor \sum_{\underline{n}, \overline{n}} \underline{n} \hat{\mathbb{P}}((\underline{n}, \overline{n}) \mid W) \right\rfloor$, where $\lfloor * \rfloor$ stands for the integer part of $*$.

   (b) $\hat{\overline{N}} = \left\lfloor \frac{\sum_{\overline{n}} \overline{n} \hat{\mathbb{P}}((\underline{N}, \overline{n}) \mid W) \mathbb{I}(\underline{N} = \hat{\underline{N}})}{\sum_{\overline{n}} \hat{\mathbb{P}}((\underline{N}, \overline{n}) \mid W) \mathbb{I}(\underline{N} = \hat{\underline{N}})} \right\rfloor$, where $\hat{\mathbb{P}}(\cdot \mid W)$ denotes the estimated values of the marginal posterior probabilities of the corresponding pair; and $\mathbb{I}(\cdot)$ is the indicator function.

5. Given a pair $(\hat{\underline{N}}, \hat{\overline{N}})$, estimate $\hat{\theta}_{\underline{N}, \overline{N}}$ using (37).

5 Simulation Exercise

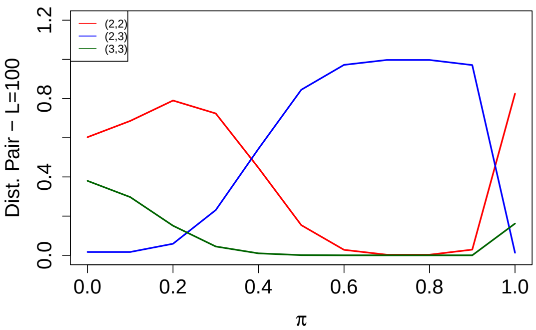
For the simulation exercise, set $V_i = \mu + \sigma v_i$, where $(\mu, \sigma) = (10, 1)$ and $v_i \sim N(0, 1)$ with truncated support at the 5th and 95th quantile so that assumption IPV is satisfied. Moreover, define $(\underline{N}, \overline{N}) = (2, 3)$. Therefore, the candidate pairs are $(2, 2), (2, 3)$, and $(3, 3)$. In addition, we implement the simulations for values of $\pi_2 = 0, 0.1, 0.2, ..., 1$. The sample size is set to $100, 500$, and 1000 auctions. The priors for (μ, σ) are set according to the GMM procedure described in the section 4.1. For the mixture weights π, we set it to follow a Dirichlet distribution as in (30) with $\gamma = 0.5$. Finally, a flat prior is defined for the candidate pairs so that each candidate has equal probability. We follow the algorithm 2.

Figure 2 shows the performance of our model selection approach, i.e, the performance of the estimator of the pair denoted by $(\hat{\underline{N}}, \hat{\overline{N}})$. We evaluate the performance by computing the simulation distribution of the pair estimation. when $\pi_2 = 0$ and small sample size the method tends to underestimate $\underline{N}$, selecting the wrong pair often as figure 2 illustrates. This phenomenon diminishes as the sample size increase. A possible explanation lies on the fact that the size of the jumps are monotonically increasing with respect to the magnitude of π_2[11]. So for low values of π_2 it becomes harder to detect $\underline{N} < \overline{N}$[12].

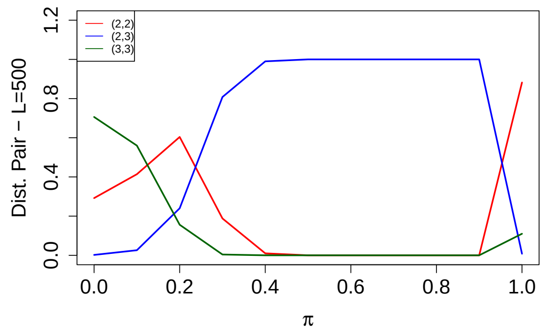
Furthermore, to assess the performance of $\hat{\theta}$ we compute the root mean square error (RMSE). Remember that the pair $(\underline{N}, \overline{N})$ directly impact the estimation of the component parameters as well as the participation distribution. As shown by figure 3, for low values of π_2, the RMSE spikes as the algorithm tends to perform relatively worse regarding the estimation of $(\hat{\underline{N}}, \hat{\overline{N}})$. This highlights the importance of a reasonable estimation procedure for $(\underline{N}, \overline{N})$ and, in particular, further investigation regarding the estimation of $\underline{N}$.

[11]See equation (7).

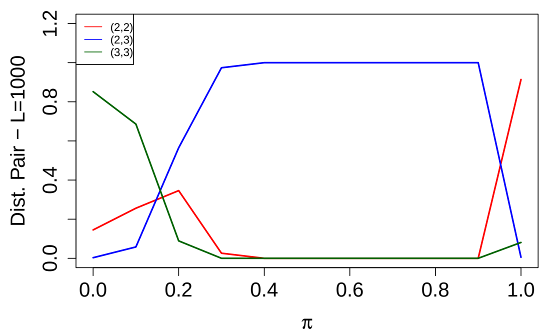
[12]See figure 1.



(a) L=100.

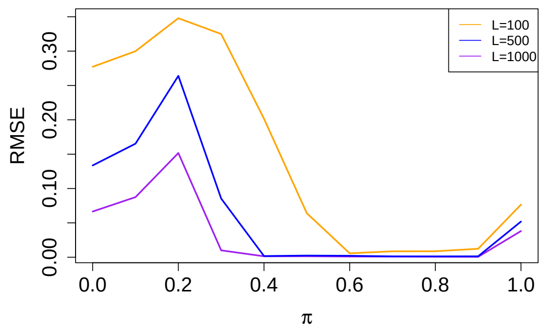


(b) L=500.

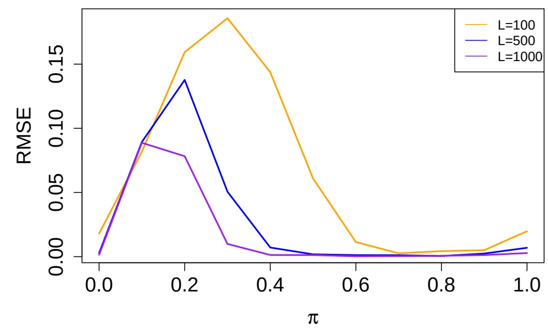


(c) L=1000.

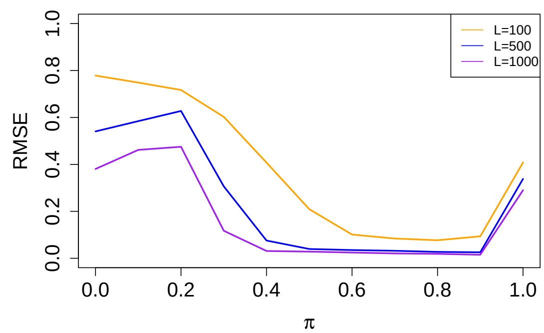
Figure 2: Distribution of the choice of pairs for different values of π_2.



(a) $RMSE_{\mu}$.



(b) $RMSE_{\sigma}$.



(c) $RMSE_{\pi}$.

Figure 3: Performance based on the RMSE.

6 Conclusion

In this paper, we have developed a Bayesian approach to estimate the parameters of first-price auctions with unobserved competition. Our model incorporates the uncertainty in the number of bidders and leverages the non-regularity of the econometric model to provide a more accurate estimation method.

The results of our simulations the overall performance of our estimator confirms the effectiveness of Bayesian techniques in handling the complexities inherent in auction models with unobserved competition. The method tends to be less accurate for some values of π, highlighting the need for further refinement in these cases. Nonetheless, the performance improves as the sample size increases.

Future research could extend our framework to more complex settings, including the addition of covariates as well as the presence of unobserved heterogeneity. Additionally, exploring alternative prior distributions and alternative sampling procedures can be useful to determine the estimation accuracy and could provide further insights into the flexibility and applicability of Bayesian methods to the model in question.

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