Optimal Unemployment Insurance with Directed Search.

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Very Preliminary

Abstract

In a labor market characterized by directed search, unemployed workers search for jobs that are flexible with respect to how much effort they require. Assuming separable preferences, we characterize the optimal unemployment contract for both the case in which savings are observed and, if preferences are of the CARA-GHH type, for the case in which they are hidden. It is always optimal for the government to distort downwards effort through positive marginal tax rates on labor earnings. In the case with hidden savings, we show that optimal contracts take a very simple form, thus showing that Shimer and Werning's (2007) findings for a McCall search model have a counterpart in a directed search environment with intensive margin adjustments. ***Keywords:*** *Unemployment Insurance; Directed Search; Intensive Margin; Hidden Savings.* ***JEL Codes:*** *H21; J64.*

1 Introduction

Unemployment insurance programs must strike a balance between insurance provision and disincentives for work. An important literature has been developed to address its optimal design – Hopenhayn and Nicolini (1997); Shimer and Werning (2007, 2008). By focusing on the extensive margin of labor supply, this line of work has emphasized how insurance reduces the incentives for searching for and/or accepting job offers.

In this paper, we study the problem of a government that offers optimal unemployment insurance financed with income taxes in a dynamic environment of directed search. We depart from most of the literature by taking into account the intensive margin of labor supply. The main consequence of allowing for intensive margin adjustments is that firms may expand the supply of vacancies by requiring agents to increase their output conditional on lending a job. Or, taking it from the workers' perspective, they can reduce the expected unemployment spell if they are willing to look for jobs that will require more effort for the same earnings.

While framed as increased effort, our model may also be interpreted as capturing the notion of equalizing differences. The many different non-pecuniary dimensions by which one job may be different from another, have been recently shown to be quantitatively relevant for unemployment insurance analysis — Hall and Mueller (2018); Luo and Mongey (2019). What makes them important for our analysis is that these are adjustments in work conditions that are neither observed nor controlled by the planner.[1]

We fully characterize the optimum for general separable preferences when that the planner controls agents' savings. We prove that, at the optimum, there is a positive wedge at the intensive margin of effort. This is somewhat surprising since the planner can use non-distortionary (with respect to the effort margin) instruments, and we assume no distributive motive. The logic is as follows. The planner observes earnings but not effort or the non-wage characteristics of a job. Hence, if an agent decides to deviate by becoming more selective, which we prove to be the relevant deviation, he will do so by choosing a job that requires less effort but which she has a lower probability of getting. Then, conditional on landing the job, she will have a higher marginal willingness to work than someone who abides by the rules and follows the strategy prescribed by the planner. Slightly distorting downwards effort

[1] In the real world, a worker may adjust his search not only by becoming more selective with regards to his wages but also with respect to the amount of effort he will have to make in case he lands the job and the quality of his prospective work environment, neither of which is within the reach of policy. These equalizing differences surveyed by Rosen (1987) have been shown to be quantitatively in recent work by Mas and Pallais (2017); Sorkin (2018); Hall and Mueller (2018).

has second order utility costs for equilibrium choices but first order costs for one who deviates, as a deviator enjoys more leisure thus attributing a lower marginal value to it. This relaxes incentive constrains and lowers the cost of providing insurance.

Unemployment benefits and net earnings decline with the length of the unemployment spell. The repeated moral hazard nature of the problem implies that, at the optimum, the stochastic process governing consumption satisfies the inverse Euler Equation. In the long run, unemployment benefits converge to zero.

To implement the optimal allocation described above, the planner must control agents' savings which need not be possible in practice. We take the possibility of hidden savings and borrowing in perfect capital markets into account. For this case, we restrict our analysis to preferences of the Greenwood et al. (1988) type specialized to the case of Constant Absolute Risk Aversion, henceforth CARA-GHH preferences. We show that the optimal allocation can be implemented by a very simple stationary contract: an upfront unemployment installment, a_0, constant gross earnings, y^e, and taxes, T, for the case in which the agent is able to land a job. The pattern of declining consumption in both the employment and unemployment state is achieved by the worker's (dis)savings along the unemployment spell. We also prove an immiseration result for this hidden savings environment.

Importantly, in this case too, a positive wedge on the intensive margin characterizes the optimum.

The rest of the paper is organized as follows. After a brief literature review, in Section 2, we describe the environment and offer a one period account of the forces explaining our findings. We derive the properties of an optimal system under the assumption that the planner controls agents' savings in Section 3 and use Section 4 to do the same for the case of hidden savings. In Section 5 we assess the quantitative relevance of our theoretical results and offer a conclusion in Section 6.

Literature Review

The modern treatment of unemployment insurance program design has its roots in Shavel and Weiss (1979), and found its first canonical treatment in Hopenhayn and Nicolini (1997). We add to this classic paper by focusing on directed search and by introducing the possibility of selecting jobs according to how hard one has to work conditional on being employed.

Acemoglu and Shimer (1999) consider a general equilibrium model of directed search with risk aversion. The static version of our model generalizes theirs by considering the possibility of adjusting the effort requirements of different jobs. Our is, however, a component planner's program whilst their focus is on the general equilibrium aspects of unemployment insurance.

Shimer and Werning (2007, 2008) evaluate the consequences of allowing agents to borrow and save in perfect capital markets using McCall's (1970) model of sequential job search. Under CARA preferences, they prove that a policy comprised of a constant benefit during unemployment, a constant tax during employment and free access to a riskless asset is optimal. We consider a directed search environment with the possibility of intensive margin adjustments in effort. We also find that simple stationary policies are optimal under CARA, thus extending their findings to this alternative environment. We add to the prescription, the optimality of distorting downwards effort.

A burgeoning literature – Golosov et al. (2013); Kroft et al. (2020); da Costa et al. (2022) – investigates redistributive policy in the presence of labor market frictions.

Golosov et al. (2013) consider the redistribution of residual income, i.e., the share of agents incomes which cannot be explained by fundamentals but are rather the consequences of randomness in the search process. Assuming directed search they prove that the optimal redistribution of residual income can be attained with positive unemployment benefits and an increasing and regressive income tax schedule. As in our case, a positive wedge obtains despite worker's homogeneity, albeit for a different reason.

Kroft et al. (2020) focus on finding sufficient statistics for the optimal combination of income taxes and unemployment benefits using perturbation methods. They do not consider intensive margin adjustments.

da Costa et al. (2022) study optimal distributive policies in the presence of labor market frictions. While they focus on intensive margin choices, their model is static and focused on the interaction between distributive motives and unemployment insurance design. They offer an approach to quantify the gains from using information on a worker's output whenever labor contracts are observable. Here, we abstract from redistribution and focus on the dynamics of insurance when contracts are not observed and there is scope for adjustments in the intensive margin. The results herein shed light of their findings regarding the consequences of not having full information on contracts.

2 Environment

Time runs for $t = 0, 1, ...$, and is discounted by $\beta \in (0, 1)$. We assume that preferences are separable, across time, states and between consumption, c, and effort, n. The flow utility generated by (c, n) is given by $U(c, n) = \varphi(c) - \eta(n)$, for $\varphi', -\varphi'', \eta', \eta'' > 0$, satisfying the Inada conditions $\lim_{c \downarrow 0} \varphi'(c) = \infty$ and $\lim_{n \downarrow 0} \eta'(n) = 0$.

One unit of effort, n, produces one unit of the consumption good, c, which price is normalized to one.

The economy starts with the worker in an unemployment state. A job offer is a contract specifying how much effort, n, the worker must make if he is hire and the earnings, y, to which he is entitled if he supplies the specified amount, n. A labor contract, i.e., a pair, (n, y), defines a (sub)market. The probability, p, of receiving a job offer in any market depends on the market tightness, with the implied relationship captured by the function, $\varrho : [0, 1] \to [0, \infty)$. This function associates to every employment probability, p, the vacancy-to-workers ratio that generates it. That is, with some abuse in notation, let $p(\lambda)$ denote the probability that an agent gets an offer when the workers-to-vacancy ratio is λ. If $\lambda : [0, 1] \to [0, \infty)$ is its inverse, then, for all p, we define the ϱ function by $\varrho(p) := 1/\lambda(p)$. We follow usual directed search specifications and assume that ϱ is strictly increasing, twice differentiable and strictly convex, satisfies $\varrho(0) = 0$, $\lim_{p \uparrow 1} \varrho(p) = \infty$. We remark that this implies that $\varrho(p)/p$ is strictly increasing and assume that[2] $\lim_{p \downarrow 0} \varrho(p)/p > 0$.

To model a firm's hiring decision, we normalize the cost of posting a vacancy to $\kappa/(1 - \beta)$. We assume that an unemployed worker who applies for a job at time t receives the answer at the beginning of the same period, before taking the

[2]One example is $\lambda(p) = p^{-1} - 1$.

unemployment insurance.

We study the cost minimization of a government that must guarantee a lifetime utility, W_0, to the worker. In its attempt to offer insurance for the agent, the planner faces the following informational restrictions. First, the planner does not know whether the agent received a job offer and rejected it or whether he did not. Second, conditional on the worker landing a job, the planner does not know the type of contract that the agent was offered. More precisely, the planner observes earnings, y, but not n. Whereas the first source of informational asymmetry has been extensively studied, the second one is novel. To highlight its role we start by presenting a one period version of our economy in which the heuristics for our main findings are simpler to convey.

2.1 A one-period economy

Consider a simplified version of our model in which an agent lives a single period split in two sub-periods. In the first sub-period the agent decides where to search, i.e., chooses p. If he lends a job, he earns y^e in exchange for producing $\kappa\varrho(p)/p$. If not, he is entitled to an unemployment benefit c^u.

Two things about our description of a worker's problem are worth highlighting. First, we assume that contracts are not observable. What this means in our setting is that the planner observes how much an employed worker is paid, y^e, but it cannot monitor how much effort, n^e, a job demands. Second, from the zero profit condition, we have $p[n^e - y^e] = \kappa\varrho(p)$, a condition that must hold for any contract on and off the equilibrium path.

We model tax policy as follows. For an employed worker to consume c^e he must earn $y^e = c^e + T$ and pay total taxes, T, to the government. Note that since the output he produces must also cover the vacancy-related expenditures, we must have $n^e > y^e$. So, in what follows we leave n^e and T in the background and write the planner's program with the controls c^e, c^u and y^e.

Since the planner observes both c^e and y^e, the only margin in which deviation is possible is on the choice of p. Second, the choice of effort, $n^e = y^e + \kappa\varrho(p)/p$ is made when the worker chooses the job to which apply, a seemingly relevant feature of actual labor markets. Since the government observes neither p nor n^e, it can only condition policy on the employment status and on y^e. Note that a worker who chooses a higher matching probability must provide higher effort for the same level of earnings.

Noting that (c^e, c^u, y^e) is controlled by the planner, define, for any $\hat{p}$,

$$U\left(\hat{p}, c^u, c^e, y^e\right) := (1-\hat{p})\varphi(c^u) + \hat{p}\left[\varphi(c^e) - \eta\left(y^e + \kappa\frac{\varrho(\hat{p})}{\hat{p}}\right)\right],$$

and define the agent's optimal choice of $\hat{p}$ by

$$p \in \arg\max_{\hat{p}} U\left(\hat{p}, c^u, c^e, y^e\right). \tag{1}$$

Under the assumption that the solution to the agents' problem is interior, that is, that the worker actively searches for a job at the optimal solution, it must satisfy the first order condition,

$$\varphi(c^e) - \varphi(c^u) - \eta\left(y^e + \frac{\varrho(p)}{p}\right) - p\eta'\left(y^e + \frac{\varrho(p)}{p}\right)\left(\frac{\varrho(p)}{p}\right)' = 0, \tag{2}$$

where the notation,

$$\left(\frac{\vartheta(p)}{p}\right)' = \frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right),$$

is used to simplify the expressions.

The Pareto frontier can be obtained by maximizing the planner's expected revenue

$$-(1-p)c^u + p\left(y^e - c^e\right), \tag{3}$$

subject to delivering utility, U^*, to the agent,

$$U\left(p, c^u, c^e, y^e\right) \geq U^*, \tag{4}$$

and to respecting the incentive-compatibility constraint (1). Due to concavity of the problem, the latter can be replaced by (2) whenever it is desirable to induce positive search. Clearly, if it is not desirable to induce positive search, the solution displays

$c^u = c^e = \varphi^{-1}(U^*)$. Let us focus on the case in which the optimal amount of search is positive.

Let μ be the Lagrangian multiplier associated with the constraint (4) and λ, the one associated with (2). The first-order conditions are

$$\varphi'(c^u) = \left(\mu - \frac{\lambda}{1-p}\right)^{-1}, \qquad [c^u]$$

$$\varphi'(c^e) = \left(\mu + \frac{\lambda}{p}\right)^{-1}, \qquad [c^e]$$

$$\eta'\left(y^e + \frac{\varrho(p)}{p}\right) = \frac{1}{p\mu+\lambda}\left[p - \eta''\left(y^e + \frac{\varrho(p)}{p}\right)\left(\frac{\varrho(p)}{p}\right)'\right], \qquad [y^e]$$

and

$$(c^e - y^e) - c^u = -\lambda \frac{\partial^2}{\partial p^2} U(p, c^u, c^e, y^e). \qquad [p]$$

Clearly, to incentivize effort, one must ensure that $c^e > c^u$, which implies $\lambda > 0$ and $\mu > 0$. This fact coupled with the concavity of the worker's problem with respect to p, $\partial^2 U(p, c^u, c^e, y^e)/\partial p^2 < 0$, confirm that $(c^e - y^e) - c^u > 0$. Hence, the planner raises more revenues when the worker finds employment. This is the source of moral-hazard of our model. The worker does not internalize the insurance provided by the government. Accordingly, the government finds higher matching probability desirable.

Let us now investigate how this impacts the marginal rate of substitution between consumption and leisure. From the first-order conditions with respect to c^e and y^e we immediately see that

$$\varphi'(c^e) - \eta'\left(y^e + \frac{\varrho(p)}{p}\right) = \frac{1}{p\mu+\lambda}\eta''\left(y^e + \frac{\varrho(p)}{p}\right)\left(\frac{\varrho(p)}{p}\right)' > 0.$$

At the optimal allocation, labor effort choice is distorted downward. To better understand this property, consider a putative optimal allocation in which this margin is not distorted: $\varphi'(c^e) - \eta'(y^e + \varrho(p)/p) = 0$. A small perturbation in which consumption when employed, c^e, and earnings, y^e, are both decreased by some small

$\varepsilon > 0$ has no direct fiscal effect and only a second-order effect on the worker's utility. However, it changes the marginal incentive to search for a job. The convexity of the cost of labor and the fact that workers who intend to find a job with higher probability must provide higher effort once employed imply that this perturbation makes relatively more attractive searching for a job. The relaxes the moral-hazard constraint and allows the planner to improve policy.

3 Optimal Unemployment Insurance

The one-period version of our model was useful to highlight the extra margin for deviation when not all aspects of jobs can be controlled by the planner. Yet it precludes an important dimension of real world unemployment insurance: the time dimension of optimal policy. We start our investigation of optimal unemployment insurance policy by describing what would be possible if contract offers were observed. That is, we characterize the benchmark case in which a firm's posted contract is observable, but one cannot observe whether the worker received or not an offer.

3.1 Observable Contracts

Assume that the planner observes all the details of contracts that are offered. We can think of a job contract as a pair, (c^e, n^e), where c^e denotes the consumption to which an employed person is entitled, and n^e the level of effort which is required from him.

Clearly, one cannot force the agent to find a job such that

$$\underline{u} > \varphi(c^e) - \eta\,(n^e)\,.$$

Hence, letting $c\,(\cdot) := \varphi^{-1}(\cdot)$, one solves

$$C(W) = \max_{p,c^e,c^u,n,\tilde{W}} \frac{p}{1-\beta}\left[n - c - \kappa\frac{\vartheta(p)}{p}\right] + (1-p)\left[-c^u + \beta C(\tilde{W})\right],$$

subject to the promise keeping,

$$W = \frac{p}{1-\beta}\left[\varphi(c) - \eta\left(n\right)\right] + (1-p)\left[\varphi(c^u) + \beta\tilde{W}\right],$$

and the incentive constraint,

$$\frac{\varphi(c) - \eta\left(n\right)}{1-\beta} \geq \varphi(c^u) + \beta\tilde{W}. \tag{5}$$

We can rely on Lemma 5 to write the following Khun-Tucker problem,

$$\begin{aligned} C(W) = \max_{p,c^e,c^u,n,\tilde{W}} \frac{p}{1-\beta}\left[n - c^e - \kappa\frac{\vartheta(p)}{p}\right] + (1-p)\left[-c^u + \beta C(\tilde{W})\right] + \\ \mu\left[\frac{p}{1-\beta}\left[\varphi(c^e) - \eta\left(n\right)\right] + (1-p)\left[\varphi(c^u) + \beta\tilde{W}\right] - W\right] + \\ \psi\left[\frac{\varphi(c^e) - \eta\left(n\right)}{1-\beta} - \varphi(c^u) - \beta\tilde{W}\right] \end{aligned}$$

The first-order conditions for the problem above are

$$\begin{aligned} n - c^e - \kappa\frac{\vartheta(p)}{p} + c^u - \beta C(\tilde{W}) + \frac{\mu}{1-\beta}\Big\{\varphi(c^e) - \eta\left(n\right) \\ -\left(1-\beta\right)\left[\varphi(c^u) + \beta\tilde{W}\right]\Big\} - \kappa\vartheta'(p) = 0, \quad [\ p\ ] \end{aligned}$$

$$\frac{p}{1-\beta}\left(1 - \mu\eta'\left(n\right) - \frac{\psi\eta'\left(n\right)}{p}\right) = 0, \qquad [\ n\ ]$$

$$\frac{p}{1-\beta}\left[-1 + \mu\varphi'(c^e) + \frac{\psi\varphi'(c^e)}{p}\right] = 0, \qquad [\ c^e\ ]$$

$$-\left(1-p\right) + \mu\left(1-p\right)\varphi'(c^u) - \psi\varphi'(c^u) = 0, \qquad [\ c^u\ ]$$

$$C'(W) = -\mu, \qquad [\ W\ ]$$

and

$$(1-p)\left[\beta C'(\tilde{W}) + \beta\mu - \frac{\beta\psi}{1-p}\right] = 0. \qquad [\ \tilde{W}\ ]$$

To evaluate these first-order conditions we must first assess whether the moral hazard and the promise keeping constraints bind at the optimum. Lemma 1, below, states that, whenever agents search for a job they are indifferent between doing so and remaining unemployed for another period.

Lemma 1 *In any period in which there is positive search, the moral-hazard constraint binds,* $[\varphi(c^e) - \eta(n)][1-\beta]^{-1} = \varphi(c^u) + \beta\tilde{W}$, *and* $\psi > 0$.

Note that in each period t in which the moral-hazard constraint binds we have

$$\mu_{t+1} = \mu_t - \frac{\psi_t}{1-p_t},$$

which implies that

$$c_{t-1}^u = (\varphi')^{-1}\left(\mu_t^{-1}\right) > (\varphi')^{-1}\left(\mu_{t+1}^{-1}\right) = c_t^u.$$

The consequence is that unemployment consumption decreases over time.

Also,

$$\frac{1}{\varphi'\left(c_{t-1}^u\right)} = \mu_t = p_t\left[\mu_t + \psi_t p_t^{-1}\right] + (1-p_t)\left[\mu_t + \psi_t p_t^{-1}\right] = \frac{p_t}{\varphi'\left(c_t^e\right)} + \frac{1-p_t}{\varphi'\left(c_t^u\right)}.$$

The consumption process is described by an inverse Euler equation. In contrast with our one-period model with non-observed contracts, effort is not distorted at the optimum in the dynamic model with observed contract according to Proposition 1.

Proposition 1 *The solution for the planner's problem when contracts are observable has the following properties:*

[i)]

1. *It entails a zero marginal income tax rate.*
2. *The unemployment insurance is decreasing over time. Moreover, if there is search at period* t *then the unemployment insurance is strictly lower than the one from the previous period.*

3. *The consumption process is described by an inverse Euler equation.*

First, the government observes contracts offered by firms. Because of that, the relevant incentive-compatibility constraint (19) depends only on the agent's utility when employed, not on how it is generated. Therefore, given any utility level, there is no reason to distort effort, which implies i). Second, unemployment insurance should decrease over time in order to make more costly to turn down employment opportunities, which is the content of ii). Finally, similar to several dynamic moral-hazard models (Rogerson (1985)), the consumption process is described by an inverse Euler equation.

This model assumes that the government observes the contracts chosen by workers and hence the disutility of effort from a particular job. The next section studies optimal contract under non-observable contracts.

3.2 Non-observable Contracts

Assume that the planner does not observe the contracts that are offered to agents. Policy must, therefore be based on whether the agent is employed or not, his earnings and on the length of his unemployment spell only. To characterize the optimal unemployment insurance program in this case we rely on a first-order approach. With Lemma 6, we prove that the solution for this relaxed problem is the solution to the original program.

Start by noting that the planner's program has a recursive structure,

$$C\left(W_0\right) = \max \frac{p}{1-\beta}\left(y^e - c^e\right) + (1-p)\left[-c(\underline{u}) + \beta C\left(W_1\right)\right],$$

subject to

$$\frac{p}{1-\beta}\left[\varphi(c^e) - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\right] + (1-p)\left[\underline{u} + \beta W_1\right] - W_0 \geq 0, \tag{6}$$

and

$$\frac{1}{1-\beta}\left[\varphi(c^e)-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]-\underline{u}-\beta W_1=\frac{1}{1-\beta}p\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'. \tag{7}$$

We show that the planner's problem is differentiable, and hence the optimal must satisfy a constraint optimization maximization in which we write μ and λ for the multipliers relative to the constraints (6) and (7) respectively.

We associate to (6) the Lagrange multiplier μ and to (7), λ. Both are strictly positive. μ is strictly positive for otherwise the planner could save by giving less utility to the agent in both states with no consequences for incentives. λ is strictly positive because the worker does not internalize the fiscal externality when unemployed.

Combining the first order conditions with respect to y^e and c^e one obtains

$$\varphi'(c^e)-\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)=\frac{\lambda}{\mu p+\lambda}p\eta''\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'>0.$$

The optimal allocation now displays a positive wedge at the intensive margin. The static is inherited by the dynamic model. If a firm deviates by offering higher quality work, i.e., those requiring less effort for the same earnings then will attract more workers which, in turn, will find it harder to land a job, thus remaining unemployed for longer periods. Conditional on getting one of these jobs a workers would have a higher willingness to make effort, than someone who got one of the jobs offered by firms along the equilibrium path. To make these deviations less attractive the planner distort effort downwards by taxing earnings at the margin.

As for the inverse Euler equation. The fact that preferences are separable in consumption and effort means that it is always feasible to vary the unemployment consumption utility in a period and compensate it by varying the consumption utility in all states of nature in the subsequent period changing neither incentives nor expected utility. This being the case, these perturbations cannot save resources at the optimum. Because the marginal cost of delivering utility is the $1/\varphi'$, the inverse Euler equation ensues.

These findings are summarized in Theorem 1, proved in the appendix.

Theorem 1 *At the optimum, in every period in which there is positive search,*

[i)]

1. *the moral-hazard constraint (7) binds, and the government benefits from strictly increasing p;*

2. *the marginal income tax rate is always positive, and;*

3. *conditional on not finding a job at period t, the worker's marginal utility of consumption satisfies the inverse Euler equation,*

$$\frac{1}{\varphi'\left(c_t^u\right)} = \mathbb{E}\left[\frac{1}{\varphi'\left(c_{t+1}\right)}\right].$$

It is important to emphasize that the planner may avoid distorting the effort margin. Taxes may be based on employment, independently of earnings. Moreover, the utility conditional on finding a job depends on $\varphi(c^e) - \eta(n^e)$, regardless of whether c^e and n^e are efficiently chosen. What is then the rationale for distorting the intensive margin prescribed in Proposition 1? It is the same that we have seen in a static setting. Consider a worker deciding whether to apply for a job in a slightly tighter sub-market, $\hat{p} > p$. The planner controls y^e and c^e, but not the amount of effort the agent is making. Upon lending a job in a tighter market, the worker is required to supply effort $\hat{n} = y^e + \kappa\varrho(\hat{p})/\hat{p} < y^e + \kappa\varrho(p)/p = n$, while receiving the same c^e. This worker, therefore, has a lower marginal disutility of effort than agents who abode by the prescription. To make this downward deviation less valuable – this is the relevant deviation according to *(i)* – the planner distorts downward effort by introducing a positive wedge.

A little less surprising is the fact that, as in Rogerson (1985); Atkeson and Lucas (1995), the Inverse Euler Equation characterizes the dynamics of consumption for the unemployed.

Proposition 2 *The unemployment benefit is decreasing over time with $c_t^u > c_{t+1}^u$ whenever the worker searches in period $t+1$.*

Moreover, whenever the worker searches in period $t+1$, his consumption from employment is strictly greater than the unemployment benefit from any period $\tau \geq t$.

Hence, according to Proposition 2, provided that agents keep searching, the unemployment insurance declines over time. The question is whether agents stop searching as the unemployment spell becomes very long. We show — Lemma 9 — that this is not the case; if there is search in one period then there is an infinite number of periods in which there is search. Hence, at the optimum, for all t, $c_t^u > c_{t+1}^u$.

Finally, we ask what happens to the unemployment benefit as the unemployment spell goes to infinity. Since $c_t^u > c_{t+1}^u$ for all t and $c_t^u \geq 0$, the unemployment converges to a non-negative number. With Proposition 3 we show that this number is 0.

Proposition 3 *The unemployment benefits converge to zero.*

In this section, we have followed most of the literature, and Hopenhayn and Nicolini (1997), in particular, in assuming that the planner controls the worker's savings. This allowed us to define a one-to-one mapping from after-tax earnings, $y^e - T$, to consumption, c^e. What happens if this is not the case? If the government does not control savings, do these results remain valid? We answer these questions next.

4 Hidden Savings

As we know from Allen (1985); Cole and Kocherlakota (2001), allowing for hidden savings represents an important constraint for the design of optimal policies. In this section, we consider the consequence of this constraint on optimal policy.

We address the case in which the government does not observe agents' savings. In the context of unemployment insurance, Shimer and Werning (2007) highlight the distinction between the optimal consumption path and the optimal transfer path when consumption and earnings need not coincide due to the possibility of borrowing and saving.

We follow several papers in the literature and restrict ourselves to preferences that do not exhibit income-effects. Concretely, we assume that preferences are of the

GHH-CARA type,[3]

$$\mathcal{U}(c,n) = -\exp\left\{-\alpha\big[c-\eta(n)\big]\right\}.$$

We also assume perfect capital markets with an interest rate $r = \beta^{-1} - 1$.

Assume that the worker starts with assets A_0. In a deterministic mechanism, the government adds liquidity $a_0 - A_0$ at time 0 and transfers b_t to the unemployed in period t. If a job is found at period t the government demands the amount of work y_t^e in every future period and makes a net transfer T_t^e (which may be negative) in every future period.

The planner's program is to minimize the expected cost of delivering utility W_0 for the unemployed agent subject to providing incentives for him to follow the optimal search strategy.

Recall that $p \to (\varrho(p)/p)$ is strictly increasing and strictly convex. We make the following assumption, which guarantees that it is optimal for the agent to search for a job.

Assumption H1: *There exists y and $p > 0$ such that*

$$y > \eta\left(y + \kappa\frac{\vartheta(p)}{p}\right).$$

Intuitively, if H1 were violated, the disutility of effort would not compensate its benefit and optimal programs would entail no vacancy creation.

Definition 2 *A **simple policy** is a triple (a_0, y^e, T^e) in which the earnings, y_t, of an employed agent, are constant, $y_t^e = y^e$ for all t, and the transfers, T_t^e, that the agent makes to the government once employed are also constant, $T_t^e = T^e$.*

When facing a simple contract the worker's problem can be written in a recursive

[3]The constant absolute risk aversion (CARA) case is the only one for which Shimer and Werning (2007) have theoretical results. They offer numeric explorations for the constant relative risk aversion (CRRA) case. Because we are also interested in understanding choices at the intensive margin, we suppress income effects through the assumption of quasi-linearity as in Greenwood et al. (1988).

form as

$$W_t(a_t) = \max_{a_{t+1}\in R,\ p\in[0,1]} -\exp\left\{-\alpha\left(a_t - \beta^{-1}a_{t+1}\right)\right\} \\ + \beta\left\{pW^e_{t+1}\left(a_{t+1},p\right) + (1-p)W_{t+1}\left(a_{t+1}\right)\right)\right\},$$

s.t.,

$$W^e_{t+1}\left(a_{t+1},p\right) = -\frac{1}{1-\beta}\exp\left\{-\alpha\left[y^e - T^e + \left(\beta^{-1}-1\right)a_{t+1} - \eta\left(y^e - \kappa\frac{\varrho(p)}{p}\right)\right]\right\}$$

For short, we write $W_t = W_t(a_t)$, $W^e_t = W^e_{t+1}\left(a_{t+1},p_t\right)$, $W^u_t = W_{t+1}\left(a_{t+1}\right)$, at the optimal (a_{t+1},p_t).

Let $\hat{c}^e_{t,\tau}$ be the consumption at τ fo an agent who found a job in period $t<\tau$. It is immediate to see that $\hat{c}^e_{t,\tau} = c^e_t$ for all $\tau > t$. Hence, in what follows we omit the current period subscript, τ, and write c^e_t to denote the time-invariant consumption of an agent who found a job in period t.

The next lemma links best response to simple policies.

Lemma 2 *Assume that the planner offers a simple contract to the agent. In this case, the agent chooses,*

1. *a stationary p;*

2. $c^e_{t+1} - c^e_t = -\Delta_c$ *and* $c^u_{t+1} - c^u_t = -\Delta_c$*, for a constant,* $\Delta_c > 0$*;*

3. $$\frac{W^e_t}{W_t} = \frac{1}{1+\alpha p(1-p)\eta'\left(y^e + \kappa\vartheta(p)/p\right)\kappa\left(\vartheta(p)/p\right)'} = k_e > 1,$$

   and

   $$\frac{W^u_t}{W_t} = \left(1 + \frac{\alpha p^2\eta'\left(y^e + \kappa\vartheta(p)/p\right)\kappa\left(\vartheta(p)/p\right)'}{1+\alpha p(1-p)\alpha p\eta'\left(y^e + \kappa\vartheta(p)/p\right)\kappa\left(\vartheta(p)/p\right)'}\right) = k_u < 1;$$

4. $$\frac{p}{(1-\beta)W_t}\exp\left\{-\alpha\left[c_e - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\right]\right\} = 1-(1-p)\,k_u.$$

That is, under a simple contract the agent chooses a constant p and, for every extra period in which he is unemployed, he reduces both the consumption while unemployed, c_t^u, and the consumption while employed, c_t^e, by the same amount Δ_c.[4] As a result, the ratios W_t^e/W_t and W_t^u/W_t remain constant at k_e and k_u, respectively.

Theorem 3 *There exists a simple policy that implements the optimal allocation.*

Under Assumption H1, GHH-CARA preferences imply that changes in asset positions have no impact on agents' search choices while producing an easily characterized adjustment in consumption during the unemployment spell and a simple scaling of expected utility. Given this invariance to asset position a Ricardian-equivalence result obtains. Alternative paths are fully characterized by the time in which the worker finds a job. The worker's decision only depends on the present value of transfers associated with each path. By performing simple changes in the timing of payments one can show that simple insurance schemes are optimal.

The optimal unemployment contract, therefore, implements an allocation characterized by a constant search effort, p, and a constant effort, $n^e = y^e + \kappa\varrho(p)/p$. To provide incentives for agents to keep searching one must guarantee that $c_t^e > c_t^u$ in every period t. Spreading consumption across the two states, unemployment and employment, is, however, a costly way of delivering promised utility. To reduce this cost, incentives are back-loaded; promised utility is reduced every time an agent fails to find a job, as stated in *(iii)*. For the preference specification we adopt, a lower utility promise with the same p and y^e can be made incentive compatible by an equal reduction in c^e and u^e, which in Theorem 3 we prove to be optimal.

Next, we show how the optimal allocation can be implemented with a very simple contract where the agent is given assets a_0 and is promised a labor contract (y^e, c^e) if he manages to land a job. Agents (dis)savings choices guarantee that c_t^e and c_t^u will follow the path prescribed in Theorem 3.

Finally, we prove the following counterpart of the immiseration result that applies here for hidden savings.

[4] Note that consumption is kept constant after the

4.1 The Optimal Policy

Now that we have established that the optimal contract is stationary and of the form (a_0, c^e, y^e), we rely on this simple structure to provide its complete characterization.

We can restrict the search for the optimal contract to that of finding a triple (a_0, c^e, y^e) that solves the problem

$$\max_{(a_0, c^e, y^e)} \left\{ \frac{p\left(y^e, c^e\right)}{1 - \left(1 - p\left(y^e, c^e\right)\right)\beta} \left(\frac{y^e - c^e}{1 - \beta} \right) - a_0 \right\},$$

subject to

$$U\left(y^e, c^e, a_0\right) \geq U_0.$$

Note that the incentive constraint is summarized by the dependence of p on y^e and c^e. Also, the promise keeping constraint can equivalently be written as

$$U\left(y^e, c^e\right) \exp\left\{-\alpha\left(1 - \beta\right) a_0\right\} \geq U_0,$$

where we use the simplified form $U\left(y^e, c^e\right)$ for $U\left(y^e, c^e, 0\right)$.

The first order condition with respect to a_0 allows us to eliminate the Lagrange multiplier and the promise-keeping constraint. We may, then, write the planner's objective as

$$\mathcal{L} = \frac{p\left(y^e, c^e\right)}{1 - \left(1 - p\left(y^e, c^e\right)\right)\beta} \left(\frac{y^e - c^e}{1 - \beta} \right) - a_0 + \frac{U\left(y^e, c^e\right)}{\exp\left\{-\alpha\left(1 - \beta\right) a_0\right\} \alpha \left(1 - \beta\right) \left|U_0\right|}.$$

The first-order condition with respect to c^e is

$$- \frac{p\left(y^e, c^e\right)}{1 - \left(1 - p\left(y^e, c^e\right)\right)\beta} + \frac{\partial U\left(y^e, c^e\right) / \partial c^e}{\exp\left\{-\alpha\left(1 - \beta\right) a_0\right\} \alpha \left(1 - \beta\right) \left|U_0\right|} + \frac{\partial}{\partial p} \left[\frac{p}{1 - \left(1 - p\right)\beta} \right] \left(\frac{y^e - c^e}{1 - \beta} \right) \frac{\partial p}{\partial c^e} = 0, \quad (8)$$

whereas the one with respect to y^e is

$$\frac{p\left(y^e,c^e\right)}{1-\left(1-p\left(y^e,c^e\right)\right)\beta}+\frac{\partial U\left(y^e,c^e\right)/\partial y^e}{\exp\left\{-\alpha\left(1-\beta\right)a_0\right\}\alpha\left(1-\beta\right)\left|U_0\right|} \\ +\frac{\partial}{\partial p}\left[\frac{p}{1-\left(1-p\right)\beta}\right]\left(\frac{y^e-c^e}{1-\beta}\right)\frac{\partial p}{\partial y^e}=0. \quad (9)$$

Consider the optimality conditions above. The first term regards the direct fiscal cost of an increase in y^e, the second term concerns the impact on the worker´s utility, while the third terms regard indirect fiscal effects which are present because p is not observable. Since, as we show, $\partial p/\partial c^e > 0$ and $\partial p/\partial y^e < 0$, the worker best responds to a higher income by increasing the job-finding probability and decrease the job-finding probability when he is offered. Of course, the fiscal effect depends on the sign of $y^e - c^e$,that is, whether the fiscal effect is positive or negative. This is shown in the following theorem.

Theorem 4 *The efficient allocation is characterized by:*

1. $y^e - c^e$ *is strictly positive;*
2. *The labor wedge,*

$$1+\frac{\partial U(y^e,c^e)/\partial y^e}{\partial U(y^e,c^e)/\partial c^e},$$

   is strictly positive;

3. *The utility of the agent who does not get a job by period t diverges to minus infinity as well as the utility of the agent who gets a job at period t.*

According to item 1., $y^e - c^e > 0$; when the worker finds a job he pays a net tax. Using this finding one concludes that an increase in y^e increases the job-finding probability, while an increase in c^e decreases it. A worker that provides lower effort responds better to incentives, being more prone to increase his job-finding rate due to an increase in employment consumption. As a result, we obtain 2. from the Theorem above. Hence, as in the model with observable savings, the moral-hazard problem implies that effort should be desincentivized at the margin.

It is important to emphasize that the moral hazard problem does not arise because of positive marginal tax rate. On the contrary, a positive wedge is imposed to lessen it. Indeed, the planner can make taxes dependent only on whether the agent is employed regardless of how much he or she earns thus avoiding the distortions at the work effort margin. The fiscal externality is important because it makes desirable for the planner to induce agents to search harder. What ultimately makes it optimal for the government to distort the effort margin is the fact that a positive effort wedge increases the cost of downward deviation of the search margin.

Finally, the last point of the theorem. The worker always expects to find a job with a constant probability in every period. Because of that, he dis-saves and hence his unemployment consumption decreases along the duration of the unemployment spell. The absence of income effects in our specification implies that his consumption diverges to minus infinity as the unemployment spell becomes arbitrarily long.

5 Quantitative Exploration

[TO BE DONE]

6 Conclusion

We have explored the consequences for optimal unemployment insurance design of adding to an otherwise standard search problem the real world feature that an important dimension of labor contracts is not observed by the policy maker: how hard an agent is required to work in each job.

We found that it is always optimal to distort the intensive margin by imposing a positive marginal income tax rate. This is true regardless of whether savings are controlled by the planner or not.

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A Lemmata

A.1 Observable Savings

Lemma 3 *If there exists W^* such that $z(W^*) = c^u(W^*)$, then $z(W) > c^u(W)$, for all $W > W^*$, and $z(W) < c^u(W)$, for all $W < W^*$. Otherwise, $z(W) > c^u(W)$ for every W. Moreover, both mappings, $z(\cdot)$ and $c^u(\cdot)$, are strictly increasing, twice differentiable, and strictly convex.*

Proof. [Proof of Lemma 3] Let $y^e(W)$ be given by

$$\underset{y^e}{\operatorname{argmax}} \left[\varphi\left(y^e + z(W)\right) - \eta\left(y^e + \phi\right)\right],$$

and notice that if

$$\varphi'\left(z(W)\right) - \eta'(\phi) \leq 0,$$

then $y^e(W) = 0$. Otherwise, $y^e(W)$ is given by

$$\varphi'\left(y^e + z(W)\right) - \eta'\left(y^e + \phi\right) = 0.$$

Hence, we have

$$z'(W) = \frac{1}{\varphi'\left(z(W) + y^e(W)\right)} > \frac{1}{\varphi'\left(c^u(W)\right)} = c^{u\prime}(W),$$

because $z(W) + y^e(W) > c^u(W)$. This implies that if $z(\cdot)$ and $c^u(\cdot)$ cross at most once, and $z(W) > c^u(W)$ (resp. $z(W) < c^u(W)$) for every utility greater (resp. lower) than this utility.

Next, since $Z(W) \to \infty$ as $W \to \infty$, we have $y^e(W) = 0$ for W large enough, which implies $z(W) > c^u(W)$, and hence we may also have $Z(W) > c^u(W)$ for every W.

It remains to show that both mappings are strictly convex. Since $c^e(W) := z(W) + y^e(W)$ is strictly increasing with positive derivative, we have

$$z''(W) = \frac{-\varphi''\left(c^e(W)\right)}{\varphi'\left(c^e(W)\right)^2} c^{e\prime}(W) > 0,$$

and

$$z''(W) = \frac{-\varphi''(c^u(W))}{\varphi'(c^u(W))^2} c^{u\prime}(W) > 0.$$

■

Lemma 4 *Assume that $\varphi(c_0^u) < W^*$, which means that the worker is better off working than consuming the first unemployment insurance. Then, there is positive search in every period.*

Proof. [Proof of Lemma 4] Since unemployment benefits are weakly decreasing, it suffices to show that if there is no search in period t then the planner could profitably deviate by offering a contract in which the worker also searches at t. Notice that the (normalized) utility $(1-\beta)W_t$ can be written as a convex combination of the terms:

[i.]

1. $$\varphi\left(c_{t+k}^e\right) - \eta\left(y_{t+k}^e + \frac{\varrho(p_{t+k})}{p_{t+k}}\right),$$

   which are obtained if the worker finds a job at period $t+k$, and;

2. $\varphi\left(c_{t+k}^u\right)$, which are obtained if the worker does not get a job by period $t+k$.

Since $c_{t+k}^u \leq c_0^u$, this implies that $\varphi\left(c_{t+k}^u\right) < W^*$. Hence, the cost of delivering $\varphi\left(c_{t+k}^u\right)$ is less than $Z\left(\varphi\left(c_{t+k}^u\right)\right)$. Notice also that the cost of providing utility,

$$\varphi\left(c_{t+k}^e\right) - \eta\left(y_{t+k}^e + \frac{\varrho(p_{t+k})}{p_{t+k}}\right),$$

is less than

$$Z\left(\varphi\left(c_{t+k}^e\right) - \eta\left(y_{t+k}^e + \frac{\varrho(p_{t+k})}{p_{t+k}}\right)\right).$$

Since the function Z is strictly convex, by Jensen's inequality and a continuity argument, there exists $\varepsilon > 0$ such that, if the planner offers the contract in which demands payments $y^e(W_t+\varepsilon)$ from the worker and delivers consumption $c^e(W_t+\varepsilon)$, the worker strictly searches for a job with positive probability, obtains a utility $\tilde{W} > W_t$ from this search ,and the government has a strictly lower cost.

This strategy makes both the worker as well as the planner better-off at period t, but may decrease worker's incentives at period $t-1$. To avoid that, the planner decreases the worker's unemployment consumption at period $t-1$ until a point at which the worker is indifferent at period $t-1$. This further improves the planner's utility at $t-1$, showing a strictly more profitable contract. ■

Lemma 5 *The value function, $W \rightarrow C(W)$, is differentiable.*

Proof. [Proof of Lemma 5] Consider the function $\tilde{W} \rightarrow \hat{C}\left(\tilde{W}\right)$ define for $\tilde{W} \in (W-\varepsilon, W+\varepsilon)$ and for every $t \geq 0$ by

$$\varphi\left(c_t^{e(t)}\left(\tilde{W}\right)\right) = \varphi\left(c_t^{e(t)}(W)\right) + (1-\beta)\left(\tilde{W} - W\right)$$

and

$$\varphi\left(c_t^{u}\left(\tilde{W}\right)\right) = \varphi\left(c_t^{u}(W)\right) + (1-\beta)\left(\tilde{W} - W\right),$$

where $\left\{c_t^{e(t)}(W), c_t^{e(t)}(W) ...,\right\}$ is the consumption stream if employment is found at t and $c_t^u(W)$ is the unemployment insurance at t. Clearly this allocation is incentive compatible and yields utility W. Its cost is

$$C(\tilde{W}) - \sum_{t \geq 0} \beta^t \mathbb{E}\left[\varphi^{-1}\left(\varphi\left(c_t^{\varpi}(W)\right) + (1-\beta)\left(\tilde{W} - W\right)\right)\right],$$

where $\varpi \in \{e(t)\}_{t=0}^{\infty} \cup \{u\}$. The function $-\varphi^{-1}$ is concave and hence so is the sum above. The result follows from Benveniste and Scheinkman (1979). ■

Lemma 6 *Suppose that if a worker gets a job then he must earn $c^e + T$, paying T to the government, to consume c^e whereas if the worker fails to get a job then he obtains the continuation utility W. Then this problem admits a unique solution. If the solution is interior then it is given by the associated first order conditions.*

Proof. [Proof of Lemma 6] Consider the problem

$$\max p \left[\varphi(c^e) - \eta\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right) - W\right]$$

This problem admits an interior solution if and only if

$$\varphi(c^e) - \eta\left(c^e + T\right) > W.$$

Assume that this is the case and consider p that makes its derivative equal to zero:

$$\varphi(c^e) - \eta\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right) - W - p\eta'\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right)\kappa\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right) = 0$$

Differentiate the left hand side again to obtain

$$\begin{aligned}
&-2\eta'\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right)\kappa\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right)\\
&\qquad\qquad - p\eta'\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right)\kappa\frac{d^2}{dp^2}\left(\frac{\vartheta(p)}{p}\right)\\
&\qquad\qquad\qquad\qquad - p\eta''\left(c^e + T + \kappa\frac{\vartheta(p)}{p}\right)\kappa\left[\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right)\right]^2.
\end{aligned}$$

To show that the expression above is negative, it suffices to show that

$$\begin{aligned}
&-2\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right) - p\frac{d^2}{dp^2}\left(\frac{\vartheta(p)}{p}\right) < 0 \Leftrightarrow\\
&\qquad -2\left(\frac{\vartheta'(p)p - \vartheta(p)}{p^2}\right) - p\frac{d}{dp}\left(\frac{\vartheta'(p)p - \vartheta(p)}{p^2}\right) < 0 \Leftrightarrow\\
&-2\left(\frac{\vartheta'(p)p - \vartheta(p)}{p^2}\right) - \left(\frac{\frac{d}{dp}\left[\vartheta'(p)p - \vartheta(p)\right]p^2 - 2p\left[\vartheta'(p)p - \vartheta(p)\right]}{p^3}\right) < 0 \Leftrightarrow\\
&\qquad\qquad\qquad\qquad -\left(\frac{\vartheta''(p)}{p}\right) < 0.
\end{aligned}$$

■

Lemma 7 *1) If there is a random-correlation device, C is concave, then it is differentiable.*

2) If there is no random-correlation device, then letting W_t be the promised utility in period t, C is differentiable at W_t for every $t > 0$.

Proof. [Proof of Lemma 7] We first prove 1). Assume that the following equations hold (the case in which $p = 0$ is analogous and omitted)

$$\frac{p}{1-\beta}\varphi(c^e) - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right) + (1-p)\left[\underline{u} + \beta W_1\right] - W_0 = 0,$$

and

$$\frac{1}{1-\beta}\varphi(c^e) - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right) - \underline{u} - \beta W_1 - \frac{1}{1-\beta}p\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\kappa\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right) = 0$$

Let

$$\begin{aligned} A &:= \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right) - (1-p)\beta W_1, \\ B &:= \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right) + \beta W_1 + \frac{1}{1-\beta}p\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\kappa\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right) \end{aligned}$$

Then, we have

$$\frac{p}{1-\beta}\varphi(c^e) + (1-p)\,\underline{u} = A + W_0,$$

and

$$\frac{1}{1-\beta}\varphi(c^e) - \underline{u} = B.$$

Substituting one equation into the other we get

$$\frac{p}{1-\beta}\left[B + \underline{u}\right] + (1-p)\,\underline{u} = A + W_0.$$

Hence,

$$\underline{u} = W_0 + D_1,$$

and

$$\varphi(c^e) = (1-\beta)\,(W_0 + D_2)\,,$$

implying that the function,

$$\tilde{C}\,(W_0+\epsilon) = p\frac{y^e - \varphi^{-1}\big(\,(1-\beta)\,(W_0+D_2)+\epsilon\big)}{1-\beta} + (1-p)\,[\beta - c(W_0+D_1+\epsilon)]\,,$$

is a strictly concave function of ϵ and lies below $C\,(W_0+\epsilon)$. Apply Benveniste-Scheinkman.

Now we prove 2). More precisely, we prove that C is differentiable at W_1. For that we assume that $p_1 > 0$ as the other case is analogous. Consider any small $\epsilon \in \mathbb{R}$ and notice that the following perturbation is feasible:

$$(\underline{\tilde{u}}_0, \underline{\tilde{u}}_1, \tilde{c}_1^e) = \big(\underline{u}_0 + \epsilon, \underline{u}_1 - \epsilon\beta^{-1}, \varphi^{-1}\left(\varphi(c_1^e)+\epsilon\right)\big)\,.$$

One can thus apply the argument in Clausen and Strub (2020) to conclude that

$$C'\,(W_1) = -c'(\underline{u}_0) = \frac{1}{\varphi'(\underline{u}_0)}.$$

■

Lemma 8 *The multipliers μ and λ are strictly positive if there is search.*

Proof. [Proof of Lemma 8] First notice that

$$\begin{aligned}
[\mu(1-p)-\lambda]\,\varphi'(c^u) &= (1-p) \\
\frac{p\mu+\lambda}{1-\beta}\varphi'(c^e) &= \frac{p}{1-\beta}
\end{aligned}$$

hence $\mu = 0$ implies $\varphi'(c^u)\varphi'(c^e) \leq 0$ which is an absurd.

Hence assume towards a contradiction that $\lambda_0 \leq 0$. Clearly, there is a last period at which $\lambda_t \leq 0$ and $\lambda_{t+1} > 0$, otherwise, as we will verify below, $c_t^u \geq c_t^e$ for every t, and hence there is no search. Assume that $\lambda_1 > 0$ (case in which $\lambda_s \leq 0$ for all $s < t$ and $\lambda_t > 0$ for some $t > 1$ can be analogously handled).

From the first-order condition with respect to p we get

$$\varphi'(c^u) = \frac{1}{\mu - \lambda(1-p)^{-1}} \leq \frac{1}{\mu + \lambda p^{-1}} = \varphi'(c^e),$$

hence $c^u \geq c^e$.

Moreover, notice that from the first order condition we have

$$C'\left(W_0\right) = -\mu_0,$$

and

$$C'\left(W_1\right) = -\mu_0 + \frac{\lambda_0}{(1-p)} = -\mu_1,$$

which implies

$$\mu_1 = \mu_0 - \frac{\lambda_0}{(1-p)} \geq \mu_0.$$

This, and $\lambda_0 \leq 0 < \lambda_1$ imply

$$\varphi'(c_1^e) = \frac{1}{\mu_1 + p_1^{-1}\lambda_1} < \frac{1}{\mu_0 + p_0^{-1}\lambda_0} = \varphi'(c_0^e).$$

Hence,

$$c_1^e > c_0^e. \tag{10}$$

We can rearrange the first order condition with respect to y^e to get

$$\mu\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right) = 1 - \lambda\eta''\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\kappa\frac{d}{dp}\left(\frac{\vartheta(p)}{p}\right) - \frac{\lambda}{p}\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right).$$

Therefore, $\lambda_0 \leq 0 < \lambda_1$ imply

$$\eta'\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right) < \mu_1^{-1}.$$

Similarly,

$$\eta'\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right) \geq \mu_0^{-1}.$$

Since $\mu_1 \geq \mu_0$, this implies

$$y_1^e + \kappa \frac{\vartheta(p_1)}{p_1} < y_0^e + \kappa \frac{\vartheta(p_0)}{p_0},$$

and

$$\eta \left(y_1^e + \kappa \frac{\vartheta(p_1)}{p_1} \right) < \eta \left(y_0^e + \kappa \frac{\vartheta(p_0)}{p_0} \right),$$

because η is strictly convex.

Since $p_0 > 0$, by assumption of the lemma, we have

$$\begin{aligned}
0 &< \frac{1}{1-\beta} \left[\varphi(c_0^e) - \eta \left(y_0^e + \kappa \frac{\vartheta(p_0)}{p_0} \right) \right] - \left[\varphi(c_0^u) + \beta W_1 \right] \\
&= \frac{1}{1-\beta} \left[\varphi(c_0^e) - \eta \left(y_0^e + \kappa \frac{\vartheta(p_0)}{p_0} \right) \right] - \varphi(c_0^u) \\
&\quad - \beta \left[p_1 \frac{1}{1-\beta} \left[\varphi(c_1^e) - \eta \left(y_1^e + \kappa \frac{\vartheta(p_1)}{p_1} \right) \right] + (1-p_1) \left[\varphi(c_1^u) + \beta W_2 \right] \right] \\
&= \varphi(c_0^e) - \eta \left(y_0^e + \kappa \frac{\vartheta(p_0)}{p_0} \right) - \varphi(c_0^u) + \beta \left[\frac{1}{1-\beta} \left[\varphi(c_0^e) - \eta \left(y_0^e + \kappa \frac{\vartheta(p_0)}{p_0} \right) \right] \right. \\
&\quad \left. - p_1 \frac{1}{1-\beta} \left[\varphi(c_1^e) - \eta \left(y_1^e + \kappa \frac{\vartheta(p_1)}{p_1} \right) \right] - (1-p_1) \left[\varphi(c_1^u) + \beta W_2 \right] \right] \\
&= \frac{1}{1-\beta} \left[\varphi(c_0^e) - \eta \left(y_0^e + \kappa \frac{\vartheta(p_0)}{p_0} \right) \right] - \varphi(c_0^u) \\
&\quad - \beta \left[p_1 \frac{1}{1-\beta} \left[\varphi(c_1^e) - \eta \left(y_1^e + \kappa \frac{\vartheta(p_1)}{p_1} \right) \right] + (1-p_1) \left[\varphi(c_1^u) + \beta W_2 \right] \right] \qquad (11)
\end{aligned}$$

Since $p_1 > 0$, due to $\lambda_1 > 0$, we have

$$\frac{1}{1-\beta} \left[\varphi(c_1^e) - \eta \left(y_1^e + \kappa \frac{\vartheta(p_1)}{p_1} \right) \right] > \varphi(c_1^u) + \beta W_2$$

Hence,

$$
\begin{aligned}
\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right) - \varphi(c_0^u) + \beta\Bigg\{ \frac{1}{1-\beta}\left[\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\right] \\
- p_1\frac{1}{1-\beta}\left[\varphi(c_1^e) - \eta\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\right] - (1-p_1)\left[\varphi(c_1^u) + \beta W_2\right] \Bigg\} \\
< \varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right) - \varphi(c_0^u) \\
+ \beta\left[\frac{1}{1-\beta}\left[\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\right] - \left[\varphi(c_1^u) + \beta W_2\right]\right]
\end{aligned}
$$

Since the first line from the last term is negative, the entire term is less than

$$
\beta\left[\frac{1}{1-\beta}\left[\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\right] - \left[\varphi(c_1^u) + \beta W_2\right]\right],
$$

which is less than,

$$
\frac{1}{1-\beta}\left[\varphi(c_0^e) - \eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\right] - \left[\varphi(c_1^u) + \beta W_2\right],
$$

since the term is positive.

Since $\varphi(c_0^e) < \varphi(c_1^e)$, and

$$
\eta\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right) > \eta\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right),
$$

this is less than

$$
\frac{1}{1-\beta}\left[\varphi(c_1^e) - \eta\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\right] - \left[\varphi(c_1^u) + \beta W_2\right].
$$

Hence, using the first-order conditions with respect to p, the algebra just performed means that

$$
\frac{p_1}{1-\beta}\eta'\left(y_1^e + \kappa\frac{\vartheta(p_1)}{p_1}\right)\kappa\left(\frac{\vartheta(p_1)}{p_1}\right) > \frac{p_0}{1-\beta}\eta'\left(y_0^e + \kappa\frac{\vartheta(p_0)}{p_0}\right)\kappa\left(\frac{\vartheta(p_0)}{p_0}\right). \tag{12}
$$

Since

$$y_1^e + \kappa \frac{\vartheta(p_1)}{p_1} < y_0^e + \kappa \frac{\vartheta(p_0)}{p_0},$$

if $y_1^e \geq y_0^e$, we will have $p_1 < p_0$ which together contradict (12). We conclude that

$$y_1^e < y_0^e. \tag{13}$$

Finally, notice that $\lambda_1 > 0$ and the first order condition with respect to p and the fact that p is a local maximum imply

$$\frac{y_0^e - c_0^e}{1-\beta} \leq -c_0^u + \beta C(W_1). \tag{14}$$

Analogously, in period 1, using $\lambda_0 \leq 0$, the first order condition with respect to p implies

$$\frac{y_1^e - c_1^e}{1-\beta} \geq -c_1^u + \beta C(W_2).$$

But notice that

$$\begin{aligned} C(W_1) &= p_1 \frac{y_1^e - c_1^e}{1-\beta} + (1-p_1)\left[-c^u + \beta C(W_2)\right] \\ &\leq \frac{y_1^e - c_1^e}{1-\beta} \end{aligned} \tag{15}$$

Hence, using (14), we have

$$\begin{aligned} c_0^u &\leq \beta C(W_1) - \frac{y_0^e - c_0^e}{1-\beta} \\ &\Leftrightarrow c_0^u + (y_0^e - c_0^e) \leq \beta C(W_1) - \frac{\beta(y_0^e - c_0^e)}{1-\beta} \end{aligned}$$

Since $c_0^u \geq c_0^e$ and $y_0^e \geq 0$ we have

$$0 \leq \beta \left[C(W_1) - \frac{y_0^e - c_0^e}{1-\beta} \right].$$

Using (15), the last term is less than

$$\beta\left[\frac{(y_1^e - c_1^e)}{1-\beta} - \frac{(y_0^e - c_0^e)}{1-\beta}\right] = \beta\left[\frac{(y_1^e - y_0^e)}{1-\beta} + \frac{(c_0^e - c_1^e)}{1-\beta}\right].$$

Hence, using (12) and (13), we see that $(y_1^e - y_0^e) + (c_0^e - c_1^e) < 0$.

A contradiction. ■

Lemma 9 *If there is search in one period, then there is an infinite number of periods in which there is search.*

Proof. [Proof of Lemma 9] Suppose towards a contradiction that there is a last period t^* at which search happens. Lemma 8 implies that $\lambda_{t^*} > 0$ and hence the first order condition of the government with respect to p_t implies

$$\frac{y_{t^*}^e - c_{t^*}^e}{1-\beta} > \frac{-c_{t^*}^u}{1-\beta},$$

where we have used the fact that if there is a last period in which there is search then $c_s^u = c_{t^*}^u$ for every $s \geq t^*$.

Moreover, the fact that the worker actively searches at t^* implies that

$$\varphi(c_{t^*}^e) - \eta\left(y_{t^*}^e + \kappa\frac{\vartheta(p_{t^*})}{p_{t^*}}\right) > \varphi(c_{t^*}^u).$$

With this at hand, we can propose a deviation in which the government offers $(c_{t^*}^e, y_{t^*}^e)$ at period $t^* + 1$. The worker searches with the same intensity (because $c_s^u = c_{t^*}^u$ for every $s \geq t^* + 1$) and the government decreases $c_{t^*}^u$ to make the worker indifferent. ■

A.2 Non-observable Savings

Lemma 10 *Consider any deterministic mechanism. Assume that the agent starts with income a_0 and let (c_t^u, p_t) be his optimal choices at period t. Then, an agent who* *starts with income* $\tilde{a}_0$ *optimally chooses* $(c_t^u + (\tilde{a}_0 - a_0)(1-\beta), p_t)$ *in every period* t *and obtains* $\exp\{-\alpha(1-\beta)(\tilde{a}_0 - a_0)\}W_t$ *where* W_t *is the utility obtained at* t *by the agent who starts the game with assets* a_0.

Proof. [Proof of Lemma 10] The proof will be based on the optimality principle. We will guess and verify that if W_t is the agent's continuation utility when period t is started with income a_t^1, then $\exp\{-\alpha(1-\beta)(a_t^2 - a_t^1)\}W_t$ is the continuation utility when starts period t with a_t^2.

Take any optimal strategy $\{(c_\tau^u(a_t), p_\tau(a_t))\}_{\tau \geq t}$ when period t game starts with income $a_t \in \{a_t^1, a_t^2\}$ and let W_t^i, $i = 1, 2.$, be its value. Notice that the strategy,

$$\left\{\left(c_\tau^u\left(a_t^1\right) + \left(a_t^1 - a_t^2\right)(1-\beta), p_\tau\left(a_t^1\right)\right)\right\}_{\tau \geq t},$$

is feasible for a worker who starts with assets a_t^2. Hence, by revealed preference,

$$W_t^1 \geq \exp\{-\alpha(1-\beta)(a_t^2 - a_t^1)\}W_t^2.$$

Analogously,

$$W_t^2 \geq \exp\{-\alpha(1-\beta)(a_t^1 - a_t^2)\}W_t^1.$$

Thus,

$$W_t^1 = \exp\{-\alpha(1-\beta)(a_t^2 - a_t^1)\}W_t^2.$$

Finally, let W_0 be the value from following the optimal strategy when the initial asset is a_0 and observe that strategy $(c_t^u + (\tilde{a}_0 - a_0)(1-\beta), p_t)$ is feasible and it leads to $\exp\{-\alpha(1-\beta)(\tilde{a}_0 - a_0)\}W_0$. This strategy is, therefore, optimal. ■

Lemma 11 *Consider any deterministic mechanism. This mechanism can be implemented by an initial endowment* a_0 *and unemployment benefits* $b_t = 0$ *for every* t.

Proof. [Proof of Lemma 11] Let $\mathbf{c}_t = (c_t^u, c_t^e)_{t=0}^\infty$ the consumption stream under the original mechanism. Choose $\tilde{a}_0 = \sum_{t=0}^{\infty} \beta^t c_t$ and transfers $\left(\tilde{T}_e\right)$ so that

$$\left(\beta^{-t}\tilde{a}_0 - \sum_{t=0}^{t-1} \beta^{-t} c_t^u\right)(1-\beta) + \tilde{T}_e = c_t^e, \tag{16}$$

In this new mechanism, the initial wealth is $\tilde{a}_0$, unemployment benefits are zero, $y_t^e = \tilde{y}_t^e$ for every t and $\tilde{T}_e$ is given by the equation above. Moreover, (16) implies that the same strategy profile is feasible. Using Lemma 10 one can check that there is no profitable deviation. ■

Lemma 12 *In every period, t, $p_t > 0$.*

Proof. [Proof of Lemma 12] Let W_1 be the value from starting the first period with zero assets. Consider the problem

$$C(W_0) = \max_{W_1, c_e, y_e} \left\{ p\frac{y^e - c_e}{1-\beta} + (1-p)\,\beta C\Big(\exp\{-\alpha a\,(1-\beta)\}W_1\Big)\right\}$$

subject to

$$\begin{aligned} p = \operatorname*{argmax}_{\tilde{p}} \Bigg\{ & \tilde{p}\left[-\frac{1}{1-\beta}\exp\left\{-\alpha\left\{c_e - \eta\left(y^e + \kappa\frac{\vartheta(\tilde{p})}{\tilde{p}}\right)\right\}\right\}\right] \\ & - (1-\tilde{p})\max_{a'}\left[-\exp\{\alpha a'\beta\} + \exp\{-\alpha a'(1-\beta)\}\beta W_1\right]\Bigg\} \end{aligned}$$

and

$$\begin{aligned} \max \Bigg\{ & \tilde{p}\left[-\frac{1}{1-\beta}\exp\left\{-\alpha\left[c_e - \eta\left(y^e + \kappa\frac{\vartheta(\tilde{p})}{\tilde{p}}\right)\right]\right\}\right] \\ & - (1-\tilde{p})\max_{a'}\left[-\exp\{\alpha a'\beta\} + \exp\{-\alpha a'(1-\beta)\}\beta W_1\right]\Bigg\} = W_0 \end{aligned}$$

Notice that $p = 0$ implies

$$W_0 = \max_{a'} \left[-\exp\{\alpha a' \beta\} + \exp\{-\alpha a' (1-\beta)\} \beta W_1 \right],$$

hence,

$$\beta \exp\{\alpha a' \beta\} = (1-\beta) \exp\{-\alpha a' (1-\beta)\} \beta W_1.$$

Therefore,

$$W_0 = \max_{a'} \left[(1-\beta) \exp\{-\alpha a' (1-\beta)\} W_1 + \exp\{-\alpha a' (1-\beta)\} \beta W_1 \right] = W_1,$$

implying that there exists an optimal contract that is stationary and involves no employment. By assumption H1, there exists y_e, c_e and p such that $y^e > c^e$ and $c^e - \eta \left(y^e + \kappa \frac{\vartheta(p)}{p} \right) > 0$. This shows that there exists a profitable deviation. ■

Lemma 13 *Consider an equilibrium generating* $(c_t^u, c_t^e, y_t^e, p_t)_{t=0}^{\infty}$ *from assets* a_0 *and* $b_t = 0$ *for every* t. *The same equilibrium can be generated from any level of assets* $\tilde{a}_0 < a_0$.

Proof. [Proof of Lemma 13] Take the initial level of assets $\tilde{a}_0$ and set asset holdings $(\tilde{a}_t)$ to generate the unemployment consumption profile $(c_t^u)_{t=0}^{\infty}$. At each point of time define y_t^e and T_t so that

$$\tilde{a}_t (1-\beta) + T_t = c_t^e.$$

It follows that if the agent chooses (c_t^u, p_t) at every time t then the original allocation $(c_t^u, c_t^e, y_t^e, p_t)_{t=0}^{\infty}$ is implemented. We must therefore show that this allocation satisfies the intertemporal budget constraint and that it is optimal. Budget constraint feasibility follows from lemma 12 and the fact that consumption is decreasing over time (see Lemma 14 below). ■

Lemma 14 W_t *and* c_t^u *are strictly decreasing in* t.

Proof. [Proof of Lemma 14] The agent's first-order condition with respect to p implies

$$-\frac{1}{1-\beta}\exp\left\{-\alpha\left\{c_e-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right\}\right\}\left[1+\alpha p\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'\right]$$
$$=\max_{a'}\left[-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\right].$$

Hence,

$$W_0>-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1.$$

Next notice that

$$\max_{a'}\left[-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\right]$$

implies

$$-\beta\exp\{\alpha a'\beta\}=(1-\beta)\exp\{-\alpha a'(1-\beta)\}\beta W_1,$$

and thus,

$$-\exp\{\alpha a'\beta\}=\max_{a'}\left[-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\right].=W_1.$$

Therefore, $W_0>W_1$ and similarly $W_1>W_2$. Since $-\exp\{-\alpha c_0^u\}=W_1$, we have

$$c_0^u=\frac{\ln(-W_1)}{-\alpha}>\frac{\ln(-W_2)}{-\alpha}=c_1^u\Longleftrightarrow W_1>W_2,$$

which was just proved. ■

Lemma 15 *Let $(c_t^u,c_t^e,y_t^e,p_t)_{t=0}^\infty$ solve the government' s problem when the agent starts with utility W_0. Then $(c_t^u+\Delta,c_t^e+\Delta,y_t^e,p_t)_{t=0}^\infty$ solves the government' s problem when the agent who starts with utility $\tilde{W}=e^{-\alpha\Delta}W_0$.*

Proof. [Proof of Lemma 15] We claim that $(c_t^u+\Delta,c_t^e+\Delta,y_t^e,p_t)_{t=0}^\infty$ is at least as good as any allocation $(\tilde{c}_t^u,\tilde{c}_t^e,\tilde{y}_t^e,\tilde{p}_t)_{t=0}^\infty$ that yields utility $\tilde{W}$. Indeed, take $(\tilde{c}_t^u,\tilde{c}_t^e,\tilde{y}_t^e,\tilde{p}_t)_{t=0}^\infty$ and notice that $(\tilde{c}_t^u-\Delta,\tilde{c}_t^e-\Delta,\tilde{y}_t^e,\tilde{p}_t)_{t=0}^\infty$ generates utility W_0. Hence, the optimality of $(c_t^u, c_t^e, y_t^e, p_t)_{t=0}^{\infty}$ implies

$$\sum_{t=0}^{\infty} \beta^t \Bigg\{ \sum_{\tau \leq t} \left(\Pi_{s<\tau}\left(1-p_s\right)\right) p_\tau \left[y_\tau^e - c_\tau^e\right] + \left(1 - \sum_{\tau \leq t} \left(\Pi_{s<\tau}\left(1-p_s\right)\right) p_\tau\right) \left[-c_t^u\right] \Bigg\}$$

$$\geq \sum_{t=0}^{\infty} \beta^t \Bigg\{ \sum_{\tau \leq t} \left(\Pi_{s<\tau}\left(1-\tilde{p}_s\right)\right) \tilde{p}_\tau \left[\tilde{y}_\tau^e - \left(\tilde{c}_\tau^e - \Delta\right)\right] + \left(1 - \sum_{\tau \leq t} \left(\Pi_{s<\tau}\left(1-\tilde{p}_s\right)\right) \tilde{p}_\tau\right) \left[-\left(\tilde{c}_t^u - \Delta\right)\right] \Bigg\},$$

which holds if and only if

$$\sum_{t=0}^{\infty} \beta^t \Bigg\{ \sum_{\tau \leq t} \left(\Pi_{s<\tau}\left(1-p_s\right)\right) p_\tau \left[y_\tau^e - \left(c_\tau^e + \Delta\right)\right] + \left(1 - \sum_{\tau \leq t} \left(\Pi_{s<\tau}\left(1-p_s\right)\right) p_\tau\right) \left[-\left(c_t^u + \Delta\right)\right] \Bigg\}$$

$$\geq \sum_{t=0}^{\infty} \beta^t \Bigg\{ \sum_{\tau \leq t} \left(\Pi_{s<\tau}\left(1-\tilde{p}_s\right)\right) \tilde{p}_\tau \left[\tilde{y}_\tau^e - \tilde{c}_\tau^e\right] + \left(1 - \sum_{\tau \leq t} \left(\Pi_{s<\tau}\left(1-\tilde{p}_s\right)\right) \tilde{p}_\tau\right) \left[-\tilde{c}_t^u\right] \Bigg\},$$

which proves the optimality of $(c_t^u + \Delta, c_t^e + \Delta, y_t^e, p_t)_{t=0}^{\infty}$ when the promised utility is $\tilde{W}$. ■

Proof. [Proof of Lemma **??**] Start with the the first order condition with respect to

p

$$
\begin{aligned}
&-\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta} \\
&\qquad\qquad -\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'\left(1-\beta\right)\}\beta W_1\Big\} \\
&-\alpha p\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}\eta'\left(y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'=0. \quad (17)
\end{aligned}
$$

Next we remark that the problem is strictly concave in p and hence the derivative of (17) with respect to p is strictly negative. Differentiating the left hand side of (17) condition with respect to c_e we obtain

$$
\begin{aligned}
&\alpha\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta} \\
&\qquad\qquad -\frac{d}{dc_e}\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'\left(1-\beta\right)\}\beta W_1\Big\} \\
&\qquad +\alpha^2 p\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}\eta'\left(y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'
\end{aligned}
$$

Now notice that

$$
\begin{aligned}
&\frac{d}{dc_e}\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'\left(1-\beta\right)\}\beta W_1\Big\} \\
&\qquad\qquad < \alpha\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'\left(1-\beta\right)\}\beta W_1\Big\}
\end{aligned}
$$

as the last number is obtained by the derivative of an increase in c in every state of nature.

Therefore, we have

$$\alpha\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}$$
$$-\frac{d}{dc_e}\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'\left(1-\beta\right)\}\beta W_1\Big\}$$
$$+\alpha^2 p\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}\eta'\left(y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'$$
$$=\alpha\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}+\alpha\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'\left(1-\beta\right)\}\beta W_1\Big\}$$
$$+\alpha^2 p\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}\eta'\left(y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'$$
$$-\alpha\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'\left(1-\beta\right)\}\beta W_1\Big\}$$
$$-\frac{d}{dc_e}\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'\left(1-\beta\right)\}\beta W_1\Big\}$$
$$=-\alpha\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'\left(1-\beta\right)\}\beta W_1\Big\}$$
$$-\frac{d}{dc_e}\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'\left(1-\beta\right)\}\beta W_1\Big\}>0,$$

where we have used (17) and (**??**). Therefore, $\partial p/\partial c^e>0$.

Next, differentiating the first order condition with respect to y^e we get

$$-\alpha\eta'\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}$$
$$-\frac{d}{dy_e}\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'\left(1-\beta\right)\}\beta W_1\Big\}$$
$$-\alpha^2 p\eta'\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}\eta'\left(y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'$$
$$-\alpha p\eta''\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}\eta'\left(y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'.$$

Notice that

$$\frac{d}{dy_e}\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\Big\} > \\ \alpha\eta'\frac{d}{dy_e}\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\Big\}. \quad (18)$$

Hence,

$$\begin{aligned}
&-\alpha\eta'\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta} \\
&\quad-\frac{d}{dy_e}\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\Big\} \\
&-\alpha^2 p\eta'\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}\eta'\left(y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right)\kappa\left(\frac{\vartheta(p)}{p}\right)' \\
&-\alpha p\eta''\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}\eta'\left(y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right)\kappa\left(\frac{\vartheta(p)}{p}\right)' = \\
&\quad-\alpha\eta'\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta} \\
&\quad-\alpha\eta'\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\Big\} \\
&-\alpha^2 p\eta'\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}\eta'\left(y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right)\kappa\left(\frac{\vartheta(p)}{p}\right)' \\
&\quad\alpha\eta'\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\Big\} \\
&\quad-\frac{d}{dy_e}\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\Big\} \\
&-\alpha p\eta''\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}\eta'\left(y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right)\kappa\left(\frac{\vartheta(p)}{p}\right)' = \\
&\quad\alpha\eta'\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\Big\} \\
&\quad-\frac{d}{dy_e}\max_{a'}\Big\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\Big\} \\
&-\alpha p\eta''\frac{\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\left(\vartheta(p)/p\right)\right)\right]\right\}}{1-\beta}\eta'\left(y^e+\kappa\left(\frac{\vartheta(p)}{p}\right)\right)\kappa\left(\frac{\vartheta(p)}{p}\right)' < 0,
\end{aligned}$$

where we have used (17) and (18). ■

Lemma 16 *It is the case that*

$$\sum_{t=0}^{\infty} p\beta^t (1-p)^{t-1} \left[\frac{\exp\{-\alpha(1-\beta) a_t\}}{(1-\beta)|U_0|} \exp\left\{-\alpha \left\{c_e - \eta \left(y^e + \kappa \frac{\vartheta(p)}{p}\right)\right\}\right\} - 1 \right] < 0.$$

Proof. [Proof of Lemma 16] We have

$$\sum_{t=0}^{\infty} p\beta^t (1-p)^{t-1} \left[\frac{\exp\{-\alpha(1-\beta) a_t\}}{(1-\beta)|U_0|} \exp\left\{-\alpha \left\{c_e - \eta \left(y^e + \kappa \frac{\vartheta(p)}{p}\right)\right\}\right\} - 1 \right] < 0 \Leftrightarrow$$
$$\sum_{t=0}^{\infty} p\beta^t (1-p)^{t-1} \left[\frac{-\exp\{-\alpha(1-\beta) a_t\}}{(1-\beta)\sum_{t=0}^{\infty} p\beta^t (1-p)^{t-1}} \exp\left\{-\alpha \left\{c_e - \eta \left(y^e + \kappa \frac{\vartheta(p)}{p}\right)\right\}\right\} \right] > U_0.$$

Since $z \to -e^{-\alpha z}$ is strictly increasing. Notice that U_0 is the mixture of the distribution F^e above and the distribution over $-e^{-\alpha c_t^u}$, which we call F^u. It follows that if F^e first-order stochastic dominates F^u :

$$\int x dF^e(x) > \int x dF^u(x)$$

and hence for any $\lambda \in (0,1)$,

$$\int x d\left[\lambda F^e(x) + (1-\lambda) F^e(x)\right] < \int x dF^e(x).$$

■

B Proofs

B.1 Observable Savings

Proof. [Proof of Lemma 1] Assume towards a contradiction that, without loss of generality, the constraint does not bind at $t = 0$,

$$\varphi'(c_0^u) = \varphi'(c_0^e) = \mu_0^{-1} = \eta'(n_0).$$

In this case,

$$\varphi(c_0^e) - \eta(n_0) < \varphi(c_0^u). \tag{19}$$

Notice that the moral-hazard constraint must bind for some $t > 0$, otherwise,

$$\varphi(c_t^u) = \mu_0^{-1},$$

for every t. This means that getting a job in period zero is worse than being unemployed forever.

Assume that the first period in which the constraint binds is $t = 1$ (the other case is analogous). We have $\mu_1 = \mu_0$, $\psi_1 > 0$ and, hence,

$$\varphi'(c_1^e) = \frac{1}{\mu_0 + \frac{\psi_1}{p_1}} = \eta'(n_1).$$

Therefore,

$$\varphi(c_0^e) - \eta(n_0) < \varphi(c_1^e) - \eta(n_1) \tag{20}$$

Hence, using (19) and (20) we obtain

$$\frac{\varphi(c_0^e) - \eta(n_0)}{1-\beta} < \varphi(c_0^u) + \beta\frac{\varphi(c_1^e) - \eta(n_1)}{1-\beta},$$

which, using the fact that the moral-hazard constraint was binding in the second period, implies that worker strictly prefers being unemployed than getting a job at zero, a contradiction. ■

Proof. [Proof of Proposition 1] ***Part i)*** Immediate from the first-order condition.

Part ii) We have

$$\begin{aligned}
(\varphi'^e) &= \frac{p}{\mu p+\lambda}\\
\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right) &= \frac{p}{\mu p+\lambda}-\frac{p\lambda}{\mu p+\lambda}\eta''\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'
\end{aligned}$$

Hence,

$$1-\frac{\eta'\Big(y^e+\kappa\vartheta(p)/p\Big)}{(\varphi'^e)}=\lambda\eta''\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'>0$$

Part iii) Notice that

$$\begin{aligned}
\frac{1-p_t}{\varphi'(c_t^u)} &= [\mu_t(1-p_t)-\lambda_t]\\
\frac{1-p_{t+1}}{\varphi'(c_{t+1}^u)} &= [\mu_{t+1}(1-p_{t+1})-\lambda_{t+1}]\,,
\end{aligned}$$

If there is no search at period $t+1$, then

$$\frac{1}{\varphi'(c_{t+1}^u)}=\mu_{t+1}=\mu_t-\frac{\lambda_t}{(1-p_t)}\frac{1}{\varphi'(c_t^u)}.$$

If there is search in period $t+1$, then we have

$$\mu_{t+1}p_{t+1}+\lambda_{t+1}=\frac{p_{t+1}}{\varphi(c_{t+1}^e)},$$

Thus

$$\frac{p_{t+1}}{\varphi(c_{t+1}^e)}+\frac{1-p_{t+1}}{\varphi'(c_{t+1}^u)}=\mu_{t+1}=\frac{1}{\varphi'(c_t^u)}.$$

■

Proof. [Proof of Proposition 2] Notice that

$$\varphi'(c_t^u)=\frac{1}{\mu_t-\frac{\lambda_t}{1-p_t}}$$

and

$$\mu_{t+1} = \mu_t - \frac{\lambda_t}{(1-p_t)},$$

hence

$$\varphi'(c^u_{t+1}) - \varphi'(c^u_t) = \frac{1}{\mu_{t+1} - \frac{\lambda_{t+1}}{1-p_{t+1}}} - \frac{1}{\mu_t - \frac{\lambda_t}{1-p_t}} = \frac{1}{\mu_{t+1} - \frac{\lambda_{t+1}}{1-p_{t+1}}} - \frac{1}{\mu_{t+1}} \geq 0,$$

with strict inequality whenever the worker searches in $t+1$ and hence $\frac{\lambda_{t+1}}{1-p_{t+1}} > 0$.

Finally, for the last claim assume that the worker actively searches in period $t+1$, use

$$\frac{p_{t+1}}{\varphi'(c^e_{t+1})} + \frac{1-p_{t+1}}{\varphi'(c^u_{t+1})} = \frac{1}{\varphi'(c^u_t)}$$

and $c^u_{t+1} > c^u_t$ to conclude that $c^e_{t+1} > c^u_\tau$ for every $\tau \geq t$. ■

Proof. [Proof of Proposition 3] Notice that the unemployment insurance is decreasing. Suppose towards a contradiction that it converges to $c^u_\infty > 0$. If it does not converge to zero, since $\varphi'(c^u_t) = \frac{1}{\mu_{t+1}}$, we conclude that $\mu_t \to (\varphi'(c^u_\infty))^{-1}$. Therefore,

$$\frac{\lambda_t}{1-p_t} \to 0.$$

We claim that $p_t \to 0$. Suppose towards a contradiction that there is a subsequence $p_{t_r} \to \hat{p} > 0$ and notice that, since

$$\varphi'(c^e_t) = \frac{1}{\mu_t + p_t^{-1}\lambda_t},$$

we have along the subsequence $\varphi'(c^e_{t_r}) \to \varphi'(c^u_\infty)$, implying $c^e_{t_r} \to c^u_\infty$. By incentive compatibility,

$$\eta\left(y^e_{t_r} + \kappa\frac{\vartheta(p_{t_r})}{p_{t_r}}\right) \to 0,$$

which implies $p_{t_r} \to 0$, a contradiction.

But then by a continuity argument, for every $\varepsilon > 0$, there exists a period t^* such that $t \geq t^*$ implies that the government's utility is ε away from $-c^u_\infty/(1-\beta)$ while the worker's utility is ε away from $\varphi(c^u_\infty)/(1-\beta)$.

Noting that

$$\lim \eta' \left(y_{t^*}^e + \kappa \frac{\vartheta(p_{t^*})}{p_{t^*}} \right) < \lim \varphi'(c_t^u), \tag{21}$$

it is easy to show that there exist $\alpha \in (0,1)$ and $\chi > 0$ and a (large) period $\tilde{t}$ at which there is active search.

The following change thus improves the government's payoff. The worker is asked to produce $y_{\tilde{t}}^e + \chi$ and his consumption is increased by $\chi\alpha$, while the government increases his revenue by $\chi (1 - \alpha)$. Using (21), this can be done so that the worker chooses a higher search intensity and increases his payoff. Recall that the first-order condition for the government with respect to p_t implies

$$\frac{y_{\tilde{t}}^e - c_{\tilde{t}}^e}{1 - \beta} > \frac{-c_\infty^u}{1 - \beta} - \varepsilon.$$

Since ε is arbitrary we can, therefore, guarantee that

$$\frac{y_{\tilde{t}}^e - c_{\tilde{t}}^e + \chi (1 - \alpha)}{1 - \beta} > \frac{-c_\infty^u}{1 - \beta} + \varepsilon.$$

In summary, this increases both the government as well as the worker's payoff at $\tilde{t}$. Finally, to keep previous periods search incentives as given, one can then decrease $c_{\tilde{t}-1}^u$ in order to keep the worker's utility conditional on not obtaining a job at $\tilde{t} - 1$ constant.

This is a profitable deviation. A contradiction. ■

B.2 Non-observable Savings

Claim 1 *If the agent has chosen c_τ^u at every $\tau < t$ for some $t \geq 0$ and W_{t+1} is his continuation payoff under consumption choices $(c_0^u, ..., c_t^u)$ in the new mechanism, then, it is optimal for him to choose c_t^u and p_t at period t.*

Proof. [Proof of Claim 1] To prove Claim 1, assume first that the agent did not get a job at t and that his previous consumption choices were $\left(c_0^u, ..., c_{t-1}^u\right)$, up to period

t. The agent chooses

$$\max_{\tilde{c}_t^u}\{-\exp\{-\alpha\,(1-\beta)\,\tilde{c}_t^u\}+\beta\exp\{-\alpha\,(1-\beta)\,\beta^{-1}\,(\tilde{c}_t^u-c_t^u)\}W_{t+1}\},$$

where we used the fact that if the agent chooses $\tilde{c}_t^u = c_t^u$, then his continuation payoff is W_{t+1} and hence, by Lemma 10, it would be $\exp\{-\alpha\,(1-\beta)\,(\tilde{c}_t^u - c_t^u)\}W_{t+1}$ if his choice were $\tilde{c}_t^u$, instead.

Since this problem is strictly concave, the optimality condition is

$$-\exp\{-\alpha\,(1-\beta)\,\tilde{c}_t^u\}=\exp\{-\alpha\,(1-\beta)\,\beta^{-1}\,(\tilde{c}_t^u-c_t^u)\}W_{t+1}.$$

However, since c_t^u was optimal in the original mechanism and W_{t+1} was his continuation utility, the respective optimality condition in the original mechanism reads

$$-\exp\{-\alpha\,(1-\beta)\,c_t^u\}=W_{t+1},$$

which then implies $\tilde{c}_t^u = c_t^u$.

Next, to verify that p_t is optimal, just notice that the algebra above implies that the agent obtains continuation value W_{t+1} if he does not get a job at t. Since his employment utility is the same as in the original mechanism, the optimality of p_t in the original mechanism implies that p_t is also optimal here. ■

Claim 2 *If the agent starts period t with assets $\hat{a}_t$, then it is optimal for him to choose $c_t^u + (\hat{a}_t - \tilde{a}_t)\,(1-\beta)$ and p_t if W_{t+1} is his continuation payoff under consumption levels $(c_0^u, ..., c_t^u)$ in the new mechanism.*

Proof. [Proof of Claim 2] The proof of Claim 2 follows directly from Lemma 10. ■

Proof. [Proof of Theorem 3] i) According to Lemma 15, p and y^e are stationary.

ii) Consider an optimal mechanism for promised utility W_0. Notice that the first-order condition reads

$$
\begin{aligned}
&-\frac{1}{1-\beta}\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\} \\
&\qquad\qquad -\max_{a'}\left\{-\exp\{\alpha a'\beta\}+\exp\{-\alpha a'(1-\beta)\}\beta W_1\right\} \\
&-\alpha p\frac{1}{1-\beta}\exp\left\{-\alpha\left[c_e-\eta\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\right]\right\}\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'=0.
\end{aligned}
$$

We have

$$W_0=pW_0^e+(1-p)W_0^u,$$

and

$$W_0^e-W_0^u=-\alpha pW_0^e\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)',$$

solving these two equations for W_0^e and W_0^u we get

$$W_0=pW_0^e+(1-p)\left[W_0^e+\alpha pW_0^e\eta'\left(y^e+\kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'\right]$$

Thus

$$W_0^e=\frac{W_0}{1+\alpha p(1-p)\eta'\Big(y^e+\kappa\vartheta(p)/p\Big)\kappa\left(\frac{\vartheta(p)}{p}\right)'}>W_0,$$

and

$$W_0^u=\frac{1+\alpha p\eta'\Big(y^e+\kappa\vartheta(p)/p\Big)\kappa\left(\frac{\vartheta(p)}{p}\right)'}{1+\alpha p(1-p)\eta'\Big(y^e+\kappa\vartheta(p)/p\Big)\kappa\left(\frac{\vartheta(p)}{p}\right)'}W_0<W_0.$$

iii) Notice that $W_0^u=e^{\alpha\Delta_c}W_0$ and

$$e^{\alpha\Delta_c}=\frac{W_0^u}{W_0}=\frac{1+\alpha p\eta'\Big(y^e+\kappa\vartheta(p)/p\Big)\kappa\left(\frac{\vartheta(p)}{p}\right)'}{1+\alpha p(1-p)\eta'\Big(y^e+\kappa\vartheta(p)/p\Big)\kappa\left(\frac{\vartheta(p)}{p}\right)'}.$$

Hence,

$$\Delta_c = \alpha^{-1} \log \left(\frac{1 + \alpha p \eta' \Big(y^e + \kappa \vartheta(p)/p \Big) \kappa \left(\frac{\vartheta(p)}{p} \right)'}{1 + \alpha p (1-p) \eta' \Big(y^e + \kappa \vartheta(p)/p \Big) \kappa \left(\frac{\vartheta(p)}{p} \right)'} \right).$$

iv) Consider the problem

$$C(W_0) = \max_{W_1, c_e, y_e} p \left[\frac{y^e - c_e}{1-\beta} \right] + (1-p)\, \beta C(e^{-\alpha a (1-\beta)} W_1),$$

subject to

$$- \frac{p}{1-\beta} \exp \left\{ -\alpha \left[c_e - \eta \left(y^e + \kappa \frac{\vartheta(p)}{p} \right) \right] \right\} + \\ (1-p) \max_{a'} \Big\{ - \exp\{\alpha a' \beta\} + \exp\{-\alpha a' (1-\beta)\} \beta W_1 \Big\} - W_0 = 0$$

and

$$- \frac{1}{1-\beta} \exp \left\{ -\alpha \left[c_e - \eta \left(y^e + \kappa \frac{\vartheta(p)}{p} \right) \right] \right\} \\ - \max_{a'} \Big\{ - \exp\{\alpha a' \beta\} + \exp\{-\alpha a' (1-\beta)\} \beta W_1 \Big\} \\ - \alpha p \frac{1}{1-\beta} \exp \left\{ -\alpha \left[c_e - \eta \left(y^e + \kappa \frac{\vartheta(p)}{p} \right) \right] \right\} \eta' \left(y^e + \kappa \frac{\vartheta(p)}{p} \right) \kappa \left(\frac{\vartheta(p)}{p} \right)' = 0.$$

Plugging the last constraint into the problem, one obtains the following Lagrangian

$$C(W_0) = \max_{W_1, c_e, y_e} p \left[\frac{y^e - c_e}{1-\beta} \right] + (1-p)\, \beta C(e^{-\alpha a (1-\beta)} W_1) + \\ \mu \Bigg[- \frac{1}{1-\beta} \exp \left\{ -\alpha \left[c_e - \eta \left(y^e + \kappa \frac{\vartheta(p)}{p} \right) \right] \right\} - \\ \alpha (1-p)\, p \frac{1}{1-\beta} \exp \left\{ -\alpha \left[c_e - \eta \left(y^e + \kappa \frac{\vartheta(p)}{p} \right) \right] \right\} \eta' \left(y^e + \kappa \frac{\vartheta(p)}{p} \right) \kappa \left(\frac{\vartheta(p)}{p} \right)' - W_0 \Bigg]$$

Therefore, we have the first-order conditions with respect to c^e,

$$p = \mu\alpha \exp\left\{-\alpha\left[c_e - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\right]\right\}\left[1 + \alpha(1-p)p\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)'\right]$$

with respect to y^e,

$$\begin{aligned} p = \mu\alpha \exp\left\{-\alpha\left[c_e - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\right]\right\}&\left[\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right) + \right.\\ &\left.\alpha(1-p)p\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)^2\kappa\left(\frac{\vartheta(p)}{p}\right)'\right] \\ + \mu(1-p)p\exp\left\{-\alpha\left[c_e - \eta\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\right]\right\}&\eta''\left(y^e + \kappa\frac{\vartheta(p)}{p}\right)\kappa\left(\frac{\vartheta(p)}{p}\right)' \end{aligned}$$

Therefore, we have

$$\eta'\left(y^e + \kappa\frac{\vartheta(p)}{p}\right) = 1 - \frac{(1-p)p\eta''\Big(y^e + \kappa\vartheta(p)/p\Big)\kappa\left(\frac{\vartheta(p)}{p}\right)'}{\alpha\left[1 + \alpha(1-p)p\eta'\Big(y^e + \kappa\vartheta(p)/p\Big)\kappa\left(\frac{\vartheta(p)}{p}\right)'\right]}.$$

■

Proof. [Proof of Theorem **??**] Let $(p^*, y^{e*}, \{(c_t^{u*}, c_t^{e*})\}_{t=0}^{\infty})$ be the optimal allocation. The initial income of the agent when employed is c_0^{e*} and the desired utilities are W_0^*, W_0^{e*} and W_0^{u*} satisfying $W_0^* = pW_0^{e*} + (1-p)W_0^{u*}$. We claim that there exists (a_0, c^e) that solves the system:

$$\begin{aligned} c_0^{e*} &= (1-\beta)a_0 + c^e \\ W_0^{u*} &= \max_c -e^{-\alpha c} + \beta U\left(\beta^{-1}(a_0 - c), y^e, c^e\right), \end{aligned}$$

where $U(a, y^{e*}, c^e)$ is the utility of a worker who starts a period with assets a and has the option to search for (y^{e*}, c^e) in this and every future period.

The sequence of efforts that give rise to $U(\beta^{-1}(a_0 - c), y^{e*}, c^{e*})$ is stationary.

Indeed, take $a_0 = c_0^{e*}(1-\beta)^{-1}, c^e = 0$, and note that

$$W_0^{u*} < \max_c -e^{-\alpha c} + \beta U\left(\beta^{-1}(a_0 - c), y^{e*}, 0\right).$$

When $a_0 = c_0^{e*}(1-\beta)^{-1}, c^e = 0$, the agent can keep consumption constant at c^{e*} even without taking a job. In fact, he best responds to that contract by choosing $p = 0$ in every period.

Next, notice that if we decrease a_0 by $-\frac{\varepsilon}{1-\beta}$ and increase c^e by ε the planner's payoff is increased by

$$\frac{\varepsilon}{1-\beta} - \frac{p(a_0, c^e)\,\varepsilon}{1-(1-p(a_0, c^e))(1-\beta)} > 0. \tag{22}$$

Assume towards a contradiction that if we take a_0 to $-\infty$ and increase c^e to infinity respecting $c_0^{e*} = (1-\beta)a_0 + c^e$ we do not reach a point at which

$$W_0^{u*} = \max_c -e^{-\alpha c} + \beta U\left(\beta^{-1}(a_0 - c), y^{e*}, c^e\right).$$

From the first order condition, we know that p remains bounded below p^* (and by lemma 10 this holds in every future period) and the principal obtains infinite profits because of (22). At the same time the agent' s utility is greater than $pW_0^{e*} + (1-p)W_0^{u*}$. A contradiction.

It follows that there is $(\bar{a}_0, \bar{c}^e)$ that satisfies the system above. Notice that

$$W_0^{u*} = \max_c -e^{-\alpha c} + \beta U\left(\beta^{-1}(\bar{a}_0 - c), y^{e*}, \bar{c}^e\right).$$

Hence, from the agent' s optimality condition, if we let

$$\bar{c}_1 := \arg\max -e^{-\alpha c} + \beta U\left(\beta^{-1}(\bar{a}_0 - c), y^{e*}, \bar{c}^e\right)$$

and $\bar{a}_1 := \beta^{-1}(\bar{a}_0 - c)$, then, by construction,

$$W_0^* = \max_p \Bigg[-p\frac{\exp\left\{-\alpha\left[\bar{c}^e + (1-\beta)\bar{a}_0 - \eta\left[y^{e*} + \kappa\left(\frac{\vartheta(p)}{p}\right)\right]\right]\right\}}{1-\beta} + (1-p)\left[\max_c -e^{-\alpha c} + \beta U\left(\beta^{-1}(\bar{a}_0 - c), y^{e*}, \bar{c}^e\right)\right]\Bigg] = U(\bar{a}_0, y^{e*}, \bar{c}^e),$$

which coupled with $\bar{c}^e + (1-\beta)\bar{a}_0 = c_t^{e*}$ implies that the chosen p is p^*.

Moreover, by construction,

$$U\left(\beta^{-1}(\bar{a}_0 - c_0^{u*}), y^{e*}, \bar{c}^e\right) = U(\bar{a}_1, y^{e*}, \bar{c}^e) = \exp\{-\alpha(1-\beta)(\bar{a}_1 - \bar{a}_0)\}U(\bar{a}_0, y^{e*}, \bar{c}^e),$$

where we have used Lemma 10.

Inductively the utility in the beginning of period $t+1$ satisfies

$$W_{t+1}^* = W_t^{u*} = \frac{W_0^{u*}}{W_0^*}W_{t-1}^{u*},$$

as desired. Finally, since the employment utility W_t^e in period t satisfies

$$W_t^{u*} = p^*W_t^e + (1-p^*)W_{t+1}^{u*},$$

and hence

$$W_t^e = \frac{W_t^{u*} - (1-p^*)W_{t+1}^{u*}}{p^*},$$

as desired, and hence the employment consumption satisfies $c_t^e = c_t^{e*}$ for every t. Finally, since $-\exp\{-\alpha c_t^u\} = W_{t+1}^{u*}$ we have $c_t^u = c_t^{u*}$ as desired. This completes the proof. ■

Proof. [Proof of Proposition ??] Notice that $W_t^* < W_t^{e*}$ and hence it suffices to show that $\lim W_t^{e*} = -\infty$. We have

$$\lim(1-\beta)W_t^{e*} = \\ -\lim\exp\left\{-\alpha\left[c_0^* + (1-\beta)\,\bar{a}_0 - \eta\left[y^{e*} + \kappa\left(\frac{\vartheta(p^*)}{p^*}\right)\right]\right]\alpha\,(t-1)\,\Delta_c\right\} = -\infty.$$

■

Proof. [Proof of Proposition **??**] Recall from (8)

$$\frac{\partial}{\partial p}\left[\frac{p}{1-(1-p)\,\beta}\right]\left(\frac{y^e - c^e}{1-\beta}\right)\frac{\partial p}{\partial c^e} = \frac{p\,(y^e, c^e)}{1-(1-p\,(y^e, c^e))\,\beta} - \frac{U_{c^e}\,(y^e, c^e)}{e^{\alpha(1-\beta)a_0}\alpha\,(1-\beta)\,|U_0|}$$

Since

$$\frac{\partial}{\partial p}\left[\frac{p}{1-(1-p)\,\beta}\right] > 0 \quad \text{and} \quad \frac{\partial p}{\partial c^e} > 0,$$

$y^e - c^e$ has the same sign as

$$-\sum_{t=0}^{\infty} p\beta^t\,(1-p)^{t-1}\left[\frac{-1}{1-\beta} + \frac{\exp\left\{-\alpha\left\{c_e - \eta\big(y^e + \kappa\vartheta(p)/p\big)\right\}\right\}}{(1-\beta)^2\,|U_0|}\right],$$

by Lemma **??**, which is strictly positive by Lemma 16. ■