Traffic Flow Optimization Study Using the QUBO Formulation

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Abstract: Traffic congestion is a major issue in large cities and urban areas, with impacts that are not only social but also environmental. Its solution is complex and involves processing large volumes of data, which can exceed the capabilities of classical systems. Therefore, the advent of quantum computing offers new tools to address this challenge. In this paper, we present a traffic optimization model based on the Quadratic Unconstrained Binary Optimization (QUBO) framework, which can be solved using quantum annealing algorithms.
Keywords: Traffic congestion. Optimization problems. QUBO. Quantum annealing.
Abbreviations: QUBO, Quadratic Unconstrained Binary Optimization.

1. Introduction

From the early 2000s to the end of the first decade of the 21st century, Brazil experienced an unprecedented increase in its vehicle fleet, with 86% growth in the number of cars, 97% in trucks, and a 314% increase in motorcycles [1]. A direct reflection of improvements in the population's economic conditions, increased credit availability, the decline of public transportation, and changes in population dynamics, the dream of millions of people to own their means of transportation earned the national industry the title of sixth-largest automobile producer in 2010 [1, 2].

Through heavy investment in advertising, the automotive sector has come to play a critical role in the Brazilian economy [1]. However, the expansion of the private vehicle fleet has come at a very high social and environmental cost [1, 2]. The increase in average commute times from home to work, caused by growing traffic congestion, highlights that even with significant investments, road expansion has been insufficient [1, 2]. At the same time, Braess' paradox demonstrates that simply increasing the number of streets is not enough to reduce travel time [3].

In addition to increasing traffic congestion in virtually every city across the country, road transport has become the leading contributor to greenhouse gas emissions, far surpassing the role of industry [1]. Air pollution (which not only damages vegetation but also irritates the human respiratory system) tends to be higher during traffic jams, as stationary or slow-moving vehicles require their engines to work harder to keep the car running.

Several strategies have been developed to reduce traffic congestion problems; however, processing vast volumes of data remains a significant challenge for current technology. As such, recent studies have focused on the feasibility of quantum solutions to address this issue. In this initial study, we initiate the investigation and analysis of traffic flow solutions based on the Quadratic Unconstrained Binary Optimization (QUBO) formula for traffic optimization, utilizing quantum annealing algorithms, quantum computers, or digital annealers to achieve efficient results. The QUBO formalism has a natural equivalence to the Ising model and can be described by "a graph with qubits as vertices and couples as edges connecting qubits" [4, 5].

The basic QUBO formulation is a model expressed as an optimization problem—minimizing or maximizing the following objective function:

$$Obj(x,Q) = x^T.Q.x, \tag{1}$$

where x is a binary decision variable vector of size N, and Q is a square constant matrix of size $N \times N$ [6, 7, 5]. Given the matrix Q, finding the binary variable assignments that minimize the objective function in Equation 1 is equivalent to solving an Ising model minimization problem, which is known to be NP-hard for classical computers [8].

This work analyzes the optimization model proposed by Tambunam (2023) *et al* [5], which aims to reduce travel time for a group of vehicles. In Section 2, we introduce the graph representing the streets under analysis, explaining some elements relevant to the model, such as the cost function, constraint equations, and both unweighted and weighted segments. In Section 3, we begin by simulating a model without weighted segments, and later analyze models with arbitrary street weight values. Section 4 concludes the discussions initiated in the third section.

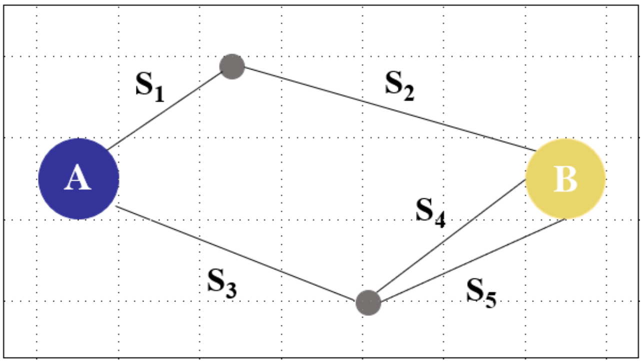
2. Method for traffic flow optimization

This work analyzes an optimization model for traffic flow between points A and B, aiming to minimize the total travel time for a group of vehicles across the available routes [5]. A major factor contributing to increased travel time is congestion, which arises as the number of vehicles on a given route grows. To estimate congestion effects, a simplifying assumption is adopted: the time required to traverse a street is modeled as a function proportional to the number of vehicles currently on that street [7]. The objective, therefore, is to minimize the total travel time along the trajectory by reducing congestion across all possible routes.

To simulate the traffic scenario, the graph shown in Figure 1 is considered. This graph is composed of four vertices and five edges, where each edge represents a street. Vertex A serves as the starting point, while vertex B marks the destination. Within this structure, three distinct paths can be identified between A and B: the first traverses segments s_1 and s_2, the second passes through s_3 and s_4, and the third follows s_3 and s_5.

In this work, we consider a scenario with three vehicles, each of which can select between two alternative routes, both composed of two street segments s. The assignment of a vehicle to a route is represented by a binary variable (which can also be interpreted as a qubit) q_{ij}, where $i \in 1,2,3$ denotes the vehicle and $j \in 1,2$ denotes the chosen route. Each route associated with a vehicle corresponds to a set of street segments, formally defined as $S = s_1, s_2, s_3, \ldots, s_k$, where k represents the number of segments in the route.

Figure 1: Model graph for optimization problem



First, we examine the vehicle routing optimization model on unweighted graph segments, which provides insight into the formulation of the cost, constraint, and objective equations. We then extend the model by introducing weights to the segments, representing factors such as distance, road priority, or traffic signal duration [5]. This extension incorporates the necessary modifications to the equations to account for these additional variables.

2.1. Model with unweighted segments

To describe a function proportional to the number of cars occupying that segment, we need to identify every street $s \in S$ that is shared by the variables q_{ij}. Table 1 presents the routes q_{ij}, each defined as a set of street segments s_k for vehicle i and route options j. Car $i = 1$ traverses segments s_1 and s_2 when taking route $j = 1$, or segments s_3 and s_4 when taking route $j = 2$. Additionally, we observe that the car $i = 1$ on route $j = 1$ and car $i = 3$ on route $j = 2$ share segments s_1 and s_2. Since we are interested in modeling congestion, we introduce a variable B_s to identify all binary variables q_{ij} corresponding to routes that include the same street segments. For example, $B_{s_1} = \{q_{11}, q_{32}\}$, as both vehicle-route combinations include s_1, note that the same reasoning would apply to route s_2 resulting in $B_{s_2} = \{q_{11}, q_{32}\}$.

Table 1: Selection of street segments S_{ij} for each vehicle i and route option j.

Vehicle	**Route $j = 1$**	**Route $j = 2$**
$i = 1$	$S_{11} = \{s_1, s_2\}$	$S_{12} = \{s_3, s_4\}$
$i = 2$	$S_{21} = \{s_3, s_4\}$	$S_{22} = \{s_3, s_5\}$
$i = 3$	$S_{31} = \{s_3, s_5\}$	$S_{32} = \{s_1, s_2\}$

Using Table 1, it is possible to determine all B_s for the five segments. To model the congestion, we define a cost function $cost(s)$ that penalizes the simultaneous use of a street segment by multiple vehicles. The idea in Equation 2 is to sum the binary decision variables $q_{ij} \in B_s$, and square the result to increase the penalty.

$$cost(s_m) = \left(\sum_{q_{ij} \in B_s} q_{ij} \right)^2 \tag{2}$$

For example, we can apply the cost equation to the segment s_1, obtaining the cost $cost(s_1)$.

$$
\begin{aligned}
cost(s_1) &= \left(\sum_{q_{ij} \in B_{s_1}} q_{ij}\right)^2 = (q_{11}+q_{32})^2 \\
&= q_{11}+q_{32}+2q_{11}q_{32}
\end{aligned} \tag{3}
$$

The Equation 3 produces linear terms due to the binary property $x^2 = x$, with each term having a coefficient of $+1$. Additionally, it generates quadratic (mixed) terms for each pair of distinct binary variables, each with a coefficient of $+2$. Since multiple vehicle–route variables q_{ij} can share the same segment s, the number of linear and quadratic terms in the cost function depends on the specific configuration under consideration. Table 2 presents the cost functions for all five segments, which are used to evaluate congestion across the network.

Table 2: function cost for every segment s_k.

Segment s_k	**Function cost**
s_1	$(q_{11}+q_{32})^2$
s_2	$(q_{11}+q_{32})^2$
s_3	$(q_{12}+q_{21}+q_{22}+q_{31})^2$
s_4	$(q_{12}+q_{21})^2$
s_5	$(q_{22}+q_{31})^2$

For the QUBO formulation, we require a constraint function that evaluates to 1 only when exactly one route option for a vehicle is active, and to 0 otherwise, as shown in Equation 4. By expanding that equation, we obtain Equation 6, which defines the relationship between the binary variables in the constraint.

$$
1 = \left(\sum_{j=1}^{2} q_{ij}\right) \tag{4}
$$

$$
0 = \left(\sum_{j=i}^{2} q_{ij} - 1\right)^2 \tag{5}
$$

$$
0 = -q_{i1} - q_{i2} + 2q_{i1}q_{i2} + 1 \tag{6}
$$

We can define the optimization model in a QUBO form by combining the cost functions for all segments (Equation 2) with the constraint equation (Equation 5) for all vehicles, which together form the objective function shown in Equation 7. The parameter λ, introduced as a Lagrange multiplier (also known as a scaling parameter), is used to ensure a valid priority solution while meeting the optimal objective value.

$$
\begin{aligned}
Obj &= \sum_{s \in S} cost(s) + \\
&\lambda \sum_i \left(\sum_j q_{ij} - 1\right)^2 \\
&= \sum_{s \in S} \left(\sum_{q_{ij} \in B_{s_m}} q_{ij}\right) + \lambda \sum_i \left(\sum_j q_{ij} - 1\right)^2
\end{aligned} \tag{7}
$$

We can define the value of the Lagrange multiplier λ based on Equation 2 by finding the maximum number of times any vehicle i appears in the cost terms, and assigning this value to λ.

2.2. Model with weighted segments

A routing model defined on weighted segments provides a more general framework, as the edge weights can encode diverse physical characteristics of the streets, such as speed bumps, construction work, or other localized conditions. Within this formulation, the weight w_k is specified by the study parameters and may assume arbitrary real values. To incorporate these factors into the objective, we augment the cost function by explicitly introducing the segment weight w_k. Accordingly, the cost function is expressed as a function of both the vehicle density parameter s_k and the edge weight w_k, thereby capturing their joint contribution to the overall congestion cost. The resulting formulation serves as the basis for evaluating and optimizing traffic flow under heterogeneous street conditions. The following cost function accounts for these two factors to evaluate the overall congestion impact.

$$cost(s_m) = \left(\sum_{q_{ij} \in B_{s_m}} w_m q_{ij} \right)^2 \tag{8}$$

For example, we can calculated the cost function for the segment s_1 using the Equation 8. In the Equation 9, we assume that the weight in the segment has a value of four. We can see that the weight acts as a multiplicative coefficient in the cost function.

$$\begin{aligned} cost(s_1) &= \left(\sum_{q_{ij} \in B_{S_m}} w_m q_{ij} \right)^2 \\ &= (4q_{11} + 4q_{32})^2 \\ &= 16q_{11} + 16q_{32} + 32q_{11}q_{32} \end{aligned} \tag{9}$$

The constraint function will not be altered by the addition of the weighted segments; therefore only the cost function part in the objective function will be altered, resulting in the following equation.

$$\begin{aligned} Obj = \sum_{s \in S} \left(\sum_{q_{ij} \in B_{s_m}} w_m q_{ij} \right) + \\ \lambda \sum_i \left(\sum_j q_{ij} - 1 \right)^2 \end{aligned} \tag{10}$$

This objective function can represent the overall time in traffic for all vehicles that are considered in the problem. Therefore, achieving lower values means reducing traffic time and optimizing traffic flow.

3. Results and Discussion

Initially, to calculate the objective function, we define the λ parameter as value two. Using Equation 10, we can simulate a model without weighted segments by setting the weight of all street segments to one. Using this condition, the results obtained for the optimization problem for the model defined in Figure 1 are shown in Table 3 in ascending order.

From Table 3, we can observe that the two best configurations for traffic flow are (q_{11}, q_{21}, q_{31})

Table 3: Results for objective function by the routes q_{ij} for vehicle i and route option j for unweighted segments.

Binary variables						Function
q_{11}	q_{12}	q_{21}	q_{22}	q_{31}	q_{32}	Obj
1	0	1	0	1	0	8
0	1	0	1	0	1	8
1	0	1	0	0	1	10
1	0	0	1	1	0	10
1	0	0	1	0	1	10
0	1	1	0	0	1	10
0	1	1	0	0	1	14
0	1	0	1	1	0	14

and (q_{12}, q_{22}, q_{32}). These configurations prioritize minimizing the number of street segments shared by cars, thereby reducing congestion time. Since all streets have the same weight ($w_k = 1$), the only factor to consider is ensuring that vehicles do not share segments, and in both cases, only the street s_3 is shared by two cars.

The worst-case scenario is presented in the configuration (q_{12}, q_{21}, q_{32}) and (q_{12}, q_{22}, q_{31}), where the three cars pass through street s_3, increasing the congestion time. Since the cost function is a quadratic operation on the sum of the binary variables, having all cars travel along the same street segment maximizes its value.

In this work, we will define arbitrary values for the weights of the streets in the weighted model, as we are not interested in specific physical conditions for the segments. The first values defined are shown in Table 4, and we call this set a. We can see that the heaviest segment is s_5, therefore is the worst street for cars to choose.

Table 4: weight for the segments of the streets defined for set a.

Segment s_k	weight w_k
s_1	1
s_2	2
s_3	2
s_4	2
s_5	3

This selection of segment weights favors the trajectory consisting of streets s_1 and s_2, since the total weight of this route is three. In contrast, the trajectory consisting of segments s_3 and s_5 is the most penalized, with a total route weight of five. Even so, the configuration of car routes that minimizes the total time of the trajectory is undefined. Thus, optimization of the problem reveals the best existing route selection configuration for each vehicle, which can be impractical to determine with an analytical method in more difficult scenarios.

The results for the objective function of the problem in set a are shown in Table 5, with the values arranged in ascending order. The best configuration for set a is (q_{11}, q_{21}, q_{32}), where cars $i = 1,3$ travel through segments s_1 and s_2, while car $i = 2$ go trough segments s_3 and s_4. In other words, the segment s_5 proves unnecessary to optimize the traffic flow. This outcome is in line with Braess' Paradox, which states that adding an extra street between an origin and a destination can increase the overall travel cost for network users [3].

Table 5: Results for objective function by the routes q_{ij} for vehicle i and route option j in set a.

Binary variables						Function
q_{11}	q_{12}	q_{21}	q_{22}	q_{31}	q_{32}	Obj
1	0	1	0	0	1	28
1	0	0	1	0	1	33
1	0	1	0	1	0	34
0	1	0	1	0	1	34
0	1	1	0	0	1	37
1	0	0	1	1	0	57
0	1	1	0	1	0	61
0	1	0	1	1	0	76

To test the Braess' Paradox in this problem, we will alter the weight in the segments, defining s_1 as the heaviest segment. This new set is called as set b and is presented in Table 6. In this new configuration, the optimal route is defined as go through segments s_4 and s_5.

Table 6: weight for the segments of the streets defined for set a.

Segment s_k	weight w_k
s_1	3
s_2	2
s_3	2
s_4	2
s_5	1

The results for the objective function of the problem in set b are shown in Table 7, with the values arranged in ascending order. The best configuration for set b are (q_{11}, q_{22}, q_{31}), where cars $i = 2,3$ share the streets s_3 and s_5, while car $i = 1$ go trough segments s_1 and s_2.

Once again a segment will not be necessary to optimize the traffic flow. In this configuration of weight (Table 6), the existence and use of street s_4 increase the overall time. Therefore, the Braess' Paradox reappears, showing that in certain configurations, the existence of an extra segment in the routes is unnecessary.

Table 7: Results for objective function by the routes q_{ij} for vehicle i and route option j in set b.

Binary variables						Function
q_{11}	q_{12}	q_{21}	q_{22}	q_{31}	q_{32}	Obj
1	0	0	1	1	0	33
1	0	1	0	1	0	34
0	1	0	1	0	1	34
0	1	0	1	1	0	44
0	1	1	0	0	1	45
0	1	1	0	1	0	53
1	0	0	1	0	1	57
1	0	1	0	0	1	60

4. Conclusion

With this work, we observe that models based on Quadratic Unconstrained Binary Optimization (QUBO) represent a promising strategy for developing techniques to analyze traffic and congestion problems and to propose formal solutions.

The weighted scenarios addressed in this study revealed that the presence of an extra segment does not always lead to an improvement in traffic flow, as showed in the Brass' Paradox.

For scenarios with a limited number of vehicles, routes, and segments, the QUBO functions presented in this work can be solved using simple programs and minimal computational time. The model used in this study is not directly applicable to real traffic scenarios; however, the equations can be applied to other graph compositions with more complex and realistic structures. In such cases, the use of quantum computational optimization methods, such as quantum annealing, proves to be essential.

For future research, we propose refining the segment weight assignment to represent real-world physical and operational conditions, such as road length, capacity, speed limits, and dynamic traffic factors. In addition, we intend to employ quantum annealing to perform large-scale traffic flow optimization, leveraging its potential for solving combinatorial problems more efficiently than classical approaches. Finally, the proposed formulation should be adapted for direct implementation on quantum annealing hardware, ensuring full compatibility with quantum computational architectures.

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